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Lorentzian 4 Dimensional Manifolds with "Local Isotropy"

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Abstract. We define "locally isotropic" spaces, as spaces in which there exists, in the tangent space at each point P, a subgroup $A(P)$ (of dimension at least 1) of the Lorentz group $L_+^\uparrow$, leaving the Riemann tensor and its 2 first covariant derivatives invariant; the subgroups $A(P)$ are assumed to be conjugate in $L_+^\uparrow$. These spaces admit a group of local isometries G. If I_P denotes the subgroup of G leaving P fixed, then $dA(P) = I_P$. All spaces of petrov type D, admitting local isotropy are determined.

1. Introduction

A 4 dimensional Lorentzian manifold V_4 is a differentiable and orientable manifold on which is everywhere defined a regular metric of hyperbolic normal type. One generally assumes also that a coherent time orientation exists; this is equivalent to the existence of a continuous nowhere vanishing time like vector-field; we make this assumption here. In general relativity it is customary to consider local coordinate transformations which are defined by functions of class C^2, piecewise C^4 [1]. We shall need here slightly stronger assumptions: the second derivatives of the Riemann tensor must be continuous at least piecewise.

V_4 is said to admit an isotropy group at the point P if: 1) there exists a locally compact effective transformation group G of isometries of V_4 operating differentiably on V_4.

2) There exists a subgroup I_P of G which leaves the point P fixed. I_P is called the isotropy group at P. A manifold is said to have local isotropy if in each point P it admits an isotropy group I_P; the I_P's are conjugate subgroups of G.

The transformations of I_P induce linear transformations in the tangent space T_P at P. The set of these linear transformations is a subgroup $A^q(P)$ of $L_+^\uparrow$ of dimension $q \geqq 1$; the $A^q(P)$'s are conjugate subgroups of $L_+^\uparrow$.

It has been shown that, in a C^∞ locally isotropic V_4, $A^q(P)$ leaves the Riemann tensor and all its covariant derivatives invariant [2, 3, 4]. Two problems thus arise:

1) to determine all locally isotropic V_4;

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2) to see if the existence of $A^q(P)$ in each point does not imply the existence of an isotropy group I_P and thus of an isometry group.

In view of the differentiability requirements of general relativity, we shall redefine local isotropy in terms of certain subgroups of $L_+^\uparrow$. If V_4 is a lorentzian manifold of class C^2, C^5 piecewise, it is said to be locally isotropic if:

(1′) There exists in each tangent space T_P a subgroup $A^q(P)$ of dimension $q \geqq 1$ leaving the Riemann tensor and its 2 first covariant derivatives invariant; all the $A^q(P)$'s are conjugate subgroups of $L_+^\uparrow$.

It is immediately clear that the Petrov type of such a V_4 must be D, N or O. We shall consider here only the type D. If one assumes that the Petrov type is constant on V_4, it is easy to show that the $A^q(P)$'s are automatically conjugate subgroups of $L_+^\uparrow$.

We plan to show that the 2 definitions of local isotropy are equivalent and to determine all locally isotropic V_4. The proof of equivalence is in fact done by explicit construction of the metric.

It is highly probable that the assumption (1′) is a little too restrictive and that the invariance of the Riemann tensor and only its first covariant derivative would be sufficient to derive the equivalence; we have in fact proven this generalised result in all but one exceptional case. We would have very much liked to stick to the usual C^2, C^4 differentiability requirements; this we could not achieve.

Let us note that G. Ellis [3] has proved, in the case of spaces describing a perfect fluid, that the existence of $A^q(P)$ implies the existence of $I_q(P)$.

2. Description of Results

The group $A^q(P)$ can be of dimension 2 or of dimension 1. If the dimension is 2, the group A^2 is the product of a space like rotation by a time like rotation (4-screw) [5]. If the group is of dimension 1, the ratio of the rate of space like rotation to the rate of time like rotation is constant. It can furthermore be shown that, when this ratio is not 0 or infinite, the space admits necessarily a group of isotropy A_2. There are thus only 2 cases: space like rotations (B_1), or time like rotations (C_1).

The manifolds admitting an A^2 were determined by Bertotti and Robinson [6, 7]. The metric can be written:

$$ds^2 = \frac{du\,dv}{(1-\alpha uv)^2} - \frac{d\zeta\,d\zeta^*}{(1-\beta\zeta\zeta^*)^2} \tag{2.1}$$

where α and β are real constants. The group G of isometries is the direct product of 2 "rotation" groups. The Lie algebra of G can be written as:

$$\begin{cases} [X_1, X_2] = X_3 & [X_2, X_3] = -\alpha X_1 & [X_3, X_1] = -X_2 \\ [X_4, X_5] = X_6 & [X_6, X_4] = X_5 & [X_5, X_6] = -\beta X_4 \\ & [X_a, X_\alpha] = 0 & a = 1,2,3;\ \alpha = 4,5,6. \end{cases} \tag{2.2}$$

The manifolds admitting the isotropy group B^1 are of 5 different types. Let us denote $(t\,\bar{t})$ the 2-plane passing through the origin and parallel to the orbits of B^1 in the tangent space. If this family of 2 planes is integrable the metric can be written as:

$$ds^2 = 2p^2(x, y)\,dx\,dy - 2A^2(x, y)\frac{d\zeta\,d\zeta^*}{\left(1 - \frac{K}{4}\zeta\zeta^*\right)^2}\,. \tag{2.3}$$

This metric has also been indicated by many authors. Various subcases must be considered. They are distinguished by the sign of the square of the gradient of A.

$$A_x A_y \gtreqless 0\,. \tag{2.4}$$

If $A_x \cdot A_y < 0$ one can choose A as a space like variable x, and t as a time like variable, the gradient of which is orthogonal to grad A and belongs to the (x, y) plane. The metric can then be written as:

$$ds^2 = \beta^2(x, t)\,dt^2 - \alpha^2(x, t)\,dx^2 - (\lambda x + 1)^2\frac{d\zeta\,d\zeta^*}{\left(1 - \frac{K}{4}\zeta\zeta^*\right)^2}\,. \tag{2.3a}$$

We have written A in the form $(1 + \lambda x)$ where λ can take the values 0,1 to include the limiting case where $A =$ Cte.

If $A_x \cdot A_y > 0$ one can choose A as a time like variable x and t as a space like "orthogonal" variable and write the metric as

$$ds^2 = \alpha^2(x, t)\,dx^2 - \beta^2(x, t)\,dt^2 - (\lambda x + 1)^2\frac{d\zeta\,d\zeta^*}{\left(1 - \frac{K}{4}\zeta\zeta^*\right)^2} \tag{2.3b}$$

where λ is 1 or 0; the limiting case $\lambda = 0$ belongs to the class of metrics (2.3a).

If $A_x \cdot A_y = 0$ one can, by a scale transformation, reduce (2.3) to:

$$ds^2 = 2p^2(x, y)\,dx\,dy - 2(1 + \lambda x)^2\frac{d\zeta\,d\zeta^*}{\left(1 - \frac{K}{4}\zeta\zeta^*\right)^2}\,. \tag{2.3c}$$

The limiting case $\lambda = 0$ is the one we had already encountered twice. These 3 metrics admit the same group of isometries G, which is a "rotation" group with Lie algebra:

$$[X_1, X_2] = X_3 \qquad [X_2, X_3] = -\frac{K}{4}X_1 \qquad [X_3, X_1] = X_2\,. \tag{2.5}$$

The two remaining B^1-type metrics correspond to the case where the $(t\,\bar{t})$ family of 2-planes is not integrable. They both admit a 4-parametric group of motions, the orbits of which are 3-dimensional submanifolds.

If these orbits are time-like the metric is:

$$ds^2 = \frac{1}{4} f^2(u) \left[dt - \frac{i C_1 \zeta^* d\zeta}{1 - \frac{K}{4}\zeta\zeta^*} + \frac{i C_1 \zeta\, d\zeta^*}{1 - \frac{K}{4}\zeta\zeta^*} \right]^2 - \frac{du^2}{f^2(u)} - q^2(u) \frac{d\zeta\, d\zeta^*}{\left(1 - \frac{K}{4}\zeta\zeta^*\right)^2}. \tag{2.6}$$

The group of isometries G is the direct product of a "rotation" group and the one-parametric group. The Lie algebra is:

$$\begin{cases} [X_1, X_2] = X_3 \quad [X_2, X_3] = -\frac{K}{4} X_1 \quad [X_3, X_1] = X_2 \\ \qquad\qquad [X_i, X_4] = 0 \qquad\qquad i = 1, 2, 3. \end{cases} \tag{2.7}$$

When the orbits are space like one has:

$$ds^2 = \frac{du^2}{f^2(u)} - \frac{1}{4} f^2(u) \left[dt - \frac{i C_1 \zeta^* d\zeta}{1 - \frac{K}{4}\zeta\zeta^*} + \frac{i C_1 \zeta d\zeta^*}{1 - \frac{K}{4}\zeta\zeta^*} \right]^2 - q^2(u) \frac{d\zeta\, d\zeta^*}{\left(1 - \frac{K}{4}\zeta\zeta^*\right)^2}. \tag{2.8}$$

The group of isometries G is again the direct product of a "rotation" group and the one parametric group. Its Lie algebra is given by (2.7). The metrics (2.6, 8) are particular cases of the metrics given by D. B. Carter [8]; (2.6) is a special case of the b_+ metrics and (2.8) of the b_-.

There are 2 different types of C^1-manifolds. Let us denote by (k, m) the 2-plane passing through the origin and parallel to the orbits of C^1 in the tangent space. If this family of 2-planes is integrable one obtains the metrics:

$$ds^2 = (1 + \lambda x)^2 \frac{du\, dv}{\left(1 - \frac{K}{4} uv\right)^2} - \alpha^2(x, t)\, dt^2 - \beta^2(x, t)\, dx^2 \tag{2.9}$$

where λ is 1 or 0. There exists a 3-parametric isometry group G which is again a "rotation" group with Lie algebra:

$$[X_1, X_2] = X_3 \quad [X_3, X_1] = -X_2 \quad [X_2, X_3] = -\frac{K}{4} X_1. \tag{2.10}$$

When the (k, m) family is non integrable one finds:

$$ds^2 = q^2(x) \frac{du\, dv}{\left(1 - \frac{K}{4} uv\right)^2} - \frac{f^2(x)}{4} \left[dt - \frac{C_1 v du}{1 - \frac{K}{4} uv} + \frac{C_1 u dv}{1 - \frac{K}{4} uv} \right]^2 - \frac{dx^2}{f^2(x)}. \tag{2.11}$$

The group G of isometries is the direct product of a "rotation" group and the one parametric group; its Lie algebra is given by (2.7). The metric (2.11) is a particular case of the Carter's metrics [8]; they correspond to his c case.

3. Sketch of the Proof of Equivalence Between the Definitions of Local Isotropy

To prove this equivalence we shall show how the definition (1′) of local isotropy allows us effectively to build the metrics listed in § 2. We shall limit ourselves to the metrics (2.1, 3, 6) as the other cases can be done in much the same way.

In each point P of V_4 we choose in the dual T_p^* of the tangent space a cobase $\theta^\alpha (\alpha = 1, 2, 3, 4)$ such that the metric is:

$$ds^2 = 2(\theta^1\theta^4 - \theta^2\theta^3) . \tag{3.1}$$

The forms θ^1 and θ^4 are real; $\theta^3 = (\theta^2)^*$. The orientation is fixed by:

$$\theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 = i . \tag{3.2}$$

If $\overset{(\alpha)}{h}_\lambda$ are the components of the θ^α's in a local coordinate system, $\underset{(\alpha)}{h}^\lambda$ the components of the base vectors of T_p canonically associated with θ^α, the time orientation is fixed by:

$$\underset{(1)}{h}^4 > 0 . \tag{3.3}$$

We choose in the space of the self dual 2-forms E_p^3 a base Z^i ($i = 1, 2, 3$) given by:

$$\begin{cases} Z^1 = \theta^3 \wedge \theta^4 \qquad Z^2 = \theta^1 \wedge \theta^2 \\ Z^3 = \frac{1}{2}(\theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^3) . \end{cases} \tag{3.4}$$

The connexion form is defined as usual by:

$$dZ^i = -\varepsilon^i{}_j{}^k \sigma_k \wedge Z^j . \tag{3.5}$$

Indices are raised and lowered by means of the metric γ of E_p^3. The explicit relation between the components $\sigma_{k\alpha}$ of the connexion form and the rotation coefficients $\gamma_{\alpha\beta\lambda} = -\overset{(\alpha)}{h}_{\varrho;\sigma}\underset{(\beta)}{h}^\varrho \underset{(\lambda)}{h}^\sigma$ are given in [9]. The curvature form:

$$\Sigma_k = d\sigma_k - \frac{1}{2}\varepsilon_k^{st}\sigma_s \wedge \sigma_t \tag{3.6}$$

can be expressed in terms of the Z's and the Z^*'s:

$$\Sigma_k = \left(C_{kj} - \frac{R}{6}\gamma_{kj}\right) Z^j + E_{k\bar{j}} Z^{j*} . \tag{3.7}$$

The symmetric matrix C_{kj} is the self dual Weyl tensor; R is the scalar curvature and the hermitian matrix $E_{k\bar{j}}$ corresponds to the trace free Ricci tensor. Bianchi identities can be written as:

$$d\Sigma_k = -\frac{1}{2}\varepsilon_k^{st}(\Sigma_s \wedge \sigma_t - \sigma_s \wedge \Sigma_t)\,. \tag{3.8}$$

To express most conveniently in this formalism the assumption (1′) of invariance of the covariant derivatives of the Riemann tensor it is useful to introduce a new kind of covariant derivative which we will denote $\overset{*}{;}$ (also 10, 11, 12). $\overset{*}{;}$ is an operation, defined on objects with $\binom{p}{p'}\binom{q}{q'}$ indices [p contravariant latin indices, . . ., q' covariant greek indices] having the following properties:

(1) linearity

(2) $\overset{*}{;} : \binom{p}{q}\binom{p'}{q'} \to \binom{p}{q}\binom{p'}{q'+1}$

(3) the Leibnitz rule

(4) $Z^i_{\alpha\beta\overset{*}{;}\nu} = 0$

(5) It coincides with covariant differentiation on $\binom{0}{0}\binom{p'}{q'}$.

One can check very easily that the $*$ derivative of a "vector" v_i is:

$$\begin{aligned} v_{i\overset{*}{;}\nu} &= v_{i,\nu} - v_j\sigma^j_{i\nu} \\ &= \frac{1}{2}v_{\alpha\beta;\nu}Z_i^{\alpha\beta}\,. \end{aligned} \tag{3.9}$$

As the Petrov class of V_4 is D we can choose our cobase θ^α in such a way that:

$$C_{ij} = \gamma(\delta_i^1\delta_j^2 + \delta_i^2\delta_j^1 + 4\delta_i^3\delta_j^3)\,. \tag{3.10}$$

The vectors $\underset{(1)}{h}$, $\underset{(4)}{h}$ coincide with the 2 eigen-vectors of the Weyl tensor.

Let us assume that there exists a 2 dimensional subgroup A^2 of $L_+^\uparrow$ leaving the Riemann tensor invariant [this is clearly the highest q admissible]. One has then:

$$\begin{cases} E_{i\bar{j}} = 2(c+e)\,\delta_i^3\delta_{\bar{j}}^{\bar{3}} \\ R \;\;= 2(c-e) \end{cases} \tag{3.11}$$

where c and e are arbitrary functions.

If A^2 leaves the first covariant derivative of the Riemann tensor invariant, one finds:

$$\begin{cases} R_{,\nu} = \gamma_{,\nu} = (c+e)_{,\nu} = 0 & \text{(3.12a)} \\ \sigma_{1\nu} = \sigma_{2\nu} = 0\,. & \text{(3.12b)} \end{cases}$$

The conditions (3.12b) are in fact the expression of the invariance of $C_{13\overset{*}{;}\nu}$ and $C_{23\overset{*}{;}\nu}$ respectively; the conditions (3.12a) can be deduced from (3.12b) and Bianchi identities.

The metric equations [13] can be integrated to prove the existence of a coordinate system (x, y, ζ, ζ^*) such that:

$$\begin{cases} \theta^1 = r\,dx \\ \theta^2 = \alpha\, d\zeta \\ \theta^4 = s\,dy\,. \end{cases}$$

One can obviously choose tetrads such that $r = s$, $\alpha = \alpha^*$. In these tetrads the conditions (3.11, 12) imply that:

$$r = r(x, y) \qquad \alpha = \alpha(\zeta, \zeta^*)$$

and furthermore the 2 surfaces (x, y), (ζ, ζ^*) have constant curvature. Thus the metric

$$ds^2 = \frac{dx\,dy}{(1-\alpha xy)^2} - \frac{d\zeta\,d\zeta^*}{(1-\beta\zeta\zeta^*)^2} \tag{3.13}$$

which has been given in (2.1). It is to be noted that in this case only the invariance of the first covariant derivative was used and an assumption of differentiability C^2, C^4 piecewise would have been sufficient. The group A^2 operates in the tangent space by:

$$\theta^1 \to e^p\theta^1;\; \theta^2 \to e^{iq}\theta^2;\; \theta^4 \to e^{-p}\theta^4\,. \tag{3.14}$$

If we assume the existence of a 1-dimensional subgroup B^1 of $L_+^\uparrow$ leaving the Riemann tensor invariant one finds:

$$\begin{cases} E_{i\bar{j}} = g\,\delta_i^1\delta_{\bar{j}}^{\bar{1}} + a\,\delta_j^2\delta_{\bar{j}}^{\bar{2}} + 2(c+e)\,\delta_i^3\delta_{\bar{j}}^{\bar{3}} \\ R \;\;= 2(c-e)\,. \end{cases} \tag{3.15}$$

The invariance of the first covariant derivative of the Riemann tensor implies that:

$$\left.\begin{aligned}
&R_2 = R_3 = \gamma_2 = \gamma_3 = (c+e)_2 = (c+e)_3 = 0 && \text{(a)}\\
&\sigma_{11} = \sigma_{13} = \sigma_{14} = 0 && \text{(b)}\\
&\sigma_{21} = \sigma_{22} = \sigma_{24} = 0 && \text{(c)}\\
&g_2 = \frac{g}{2}(\sigma_{32} + \sigma_{32}^*) \qquad g_3 = \frac{g}{2}(\sigma_{33} + \sigma_{33}^*) && \text{(d)}\\
&a_2 = -\frac{a}{2}(\sigma_{32} + \sigma_{32}^*) \qquad a_3 = -\frac{a}{2}(\sigma_{33} + \sigma_{33}^*) && \text{(e)}
\end{aligned}\right\} \tag{3.16}$$

The crucial relations (3.16, b, c) come again from the invariance of $C_{13;\nu}^{*}$ and $C_{23;\nu}^{*}$. Bianchi identities reduce to the 4 equations:

$$\left.\begin{aligned}
&-\frac{1}{6}R_4 + \gamma_4 + g_1 = \frac{3\gamma}{2}\sigma_{12} + \frac{g}{2}(\sigma_{31} + \sigma_{31}^* - \sigma_{22}^*) + \frac{c+e}{2}\sigma_{12} && \text{(a)}\\
&-\frac{1}{6}R_4 - 2\gamma_4 + (c+e)_4 = -\,3\gamma\sigma_{12} + (c+e)\,\sigma_{31}^* - g\sigma_{23} && \text{(b)}\\
&\frac{1}{6}R_1 - \gamma_1 - a_4 = \frac{3\gamma}{2}\sigma_{23} + \frac{a}{2}(\sigma_{34} + \sigma_{34}^* - \sigma_{13}^*) + \frac{c+e}{2}\sigma_{23} && \text{(c)}\\
&-\frac{1}{6}R_1 - 2\gamma_1 + (c+e)_1 = 3\gamma\sigma_{23} - (c+e)\,\sigma_{22}^* + a\sigma_{12} && \text{(d)}
\end{aligned}\right\} \tag{3.17}$$

We have now to consider separately 3 cases:

1) $\sigma_{12} - o^*_{13} = \sigma_{23} - \sigma^*_{22} = 0$; that is the 2 vectors $\underset{(1)}{\boldsymbol{h}}$ and $\underset{(4)}{\boldsymbol{h}}$ are parallel to gradients. It is then clear that there exists a coordinate system and a family of tetrads such that:

$$\begin{cases} \theta^1 = p\,dx \\ \theta^2 = q\,d\zeta \\ \theta^4 = p\,dy\,. \end{cases} \tag{3.18}$$

The conditions $\sigma_{11} = \sigma_{24} = 0$ imply that $p = p(x, y)$. From the vanishing of the C_{13}, C_{23} components of the Weyl tensor one deduces that:

$$q = A(x, y)\, B(\zeta, \zeta^*)\,. \tag{3.19}$$

Finally combining linearly (3.16a) one shows that the metric $B^2 d\zeta\, d\zeta^*$ has constant curvature. Thus the metric:

$$ds^2 = 2p^2(x, y)\, dx\, dy - 2A^2(x, y) \frac{d\zeta\, d\zeta^*}{\left(1 - \frac{K}{4}\zeta\zeta^*\right)^2}\,. \tag{3.20}$$

It is worth noticing that once again we used only the invariance of the first covariant derivative of the Riemann tensor and that we did need the continuity of only the first covariant derivative of the Riemann tensor.

2) $\sigma_{12} - \sigma^*_{13} = 0$; $\sigma_{23} - \sigma^*_{22} \neq 0$. If one now assumes the existence and continuity of the second derivatives of the Riemann tensor one can compute commutation relations between the conditions (3.16a). This gives the alternative:

$$\text{a)}\quad \sigma_{23} - \sigma^*_{22} = 0 \qquad \text{or} \qquad \text{b)}\quad R_4 = \gamma_4 = (c + e)_4 = 0\,. \tag{3.21}$$

The first part of the alternative brings us back to case 1). In the second part b) we deduce from (3.17b) that:

$$-3\gamma\sigma_{12} + (c + e)\,\sigma_{12} - g\sigma_{22} = 0\,.$$

The imaginary part of this relation is:

$$3(\gamma^* - \gamma)\,\sigma_{12} = g(\sigma_{23} - \sigma^*_{22})\,. \tag{3.22}$$

The explicit expression of C_{12} allows us to compute $(\gamma - \gamma^*)$ and thus obtain a value for g:

$$g = \frac{3}{2}\sigma^2_{12}\,. \tag{3.23}$$

From the expression of g we get, in a tetrad for which $\sigma_{34} + \sigma^*_{34} = 0$:

$$\sigma_{12/4} = -\,\sigma^2_{12}\,. \tag{3.24}$$

The vanishing of C_{13} and the rality of σ_{12} allows us to compute the commutation relation $\sigma_{12[23]}$ which again gives the alternative $\sigma_{23} - \sigma^*_{22}$

$= 0$ or $\sigma_{12/4} = 0$. The second half contradicts (3.24) as σ_{12} may not vanish. Thus this second case is reduced to the first. To obtain (3.21) we used second covariant derivatives of the Riemann tensor but not their invariance under B^1.

3) $(\sigma_{12} - \sigma^*_{13})(\sigma_{23} - \sigma^*_{22}) \neq 0$. Let us consider certain tetrad transformations leaving (3.10, 15, 16) invariant:

$$\theta^1 \to e^p \theta^1 \qquad \theta^2 \to \theta^2 \qquad \theta^4 \to e^{-p}\theta^4 .$$

There always exists a function p such that:

$$\sigma_{12} - \sigma^*_{13} = \varepsilon(\sigma_{23} - \sigma^*_{22}) \qquad \varepsilon^2 = 1 . \tag{3.25}$$

Commuting the relations (3.16a) one gets:

$$\varepsilon R_1 + R_4 = \varepsilon\gamma_1 + \gamma_4 = \varepsilon(c + e)_1 + (c + e)_4 = 0 \tag{3.26}$$

which shows that these components of the curvature depend of only one variable.

The reality of $E_{1\bar{1}}$ gives:

$$(\sigma_{12} - \sigma^*_{13})_4 = \frac{1}{2}(\sigma^2_{12} - \sigma^{*2}_{13}) + \frac{1}{4}(\sigma_{34} + \sigma^*_{34})(\sigma_{12} - \sigma^*_{13}) . \tag{3.27}$$

On the other hand if one computes $C_{21} - C^*_{21}$ one gets:

$$\begin{aligned}\gamma - \gamma^* = &- (\sigma_{23} - \sigma^*_{22})_4 \\ &+ \frac{1}{2}(\sigma_{23}\sigma^*_{13} - \sigma^*_{22}\sigma_{12}) - \frac{1}{4}(\sigma_{34} + \sigma^*_{34})(\sigma_{23} - \sigma^*_{22}) .\end{aligned} \tag{3.28}$$

Substituting (3.27) in (3.28) by virtue of (3.25) one finds a value of $\gamma - \gamma^*$ in terms of the $\sigma_{k\alpha}$. A similar calculation using the reality of $E_{2\bar{2}}$ and the imaginary part of C_{12} gives another expression for $(\gamma - \gamma^*)$. The equality between these 2 expressions implies that:

$$\sigma_{31} + \sigma^*_{31} = \varepsilon(\sigma_{34} + \sigma^*_{34}) . \tag{3.29}$$

The second covariant derivatives of C_{ij} are given by:

$$C_{ij;\varrho\sigma}^{*} = C_{ij;\varrho/\sigma}^{*} - C_{sj;\varrho}^{*}\sigma^s_{i\sigma} - C_{is;\varrho}^{*}\sigma^s_{j\sigma} + C_{ij;\xi}^{*}\gamma^\xi_{\varrho\sigma} .$$

The conditions expressing their invariance in B^1 imply that:

$$\begin{cases}\sigma_{12/2} = \dfrac{1}{4}\sigma_{12}(\sigma_{32} + \sigma^*_{32}) \\ \sigma_{23/3} = -\dfrac{1}{4}\sigma_{23}(\sigma_{33} + \sigma^*_{33})\end{cases} \tag{3.30}$$

and give the values of certain second derivatives of γ in terms of the $\sigma_{k\alpha}$. The vanishing of C_{13} and C_{23} gives us:

$$\begin{cases}\sigma_{12/3} = \dfrac{1}{4}\sigma_{12}(\sigma_{32} + \sigma^*_{32}) \\ \sigma_{23/2} = -\dfrac{1}{4}\sigma_{23}(\sigma_{32} + \sigma^*_{32}) .\end{cases} \tag{3.31}$$

Commuting the equations (3.30) and (3.31) we obtain:

$$\begin{aligned} \gamma - \frac{R}{6} - \varepsilon g &= \frac{1}{2}\sigma_{12}\sigma_{22}^* - \frac{\varepsilon}{2}\sigma_{12}^2 - \frac{1}{2}(\sigma_{31}+\sigma_{31}^*)\,\sigma_{12} \quad &\text{(a)} \\ \gamma - \frac{R}{6} - \varepsilon a &= \frac{1}{2}\sigma_{13}^*\sigma_{23} - \frac{\varepsilon}{2}\sigma_{23}^2 - \frac{\varepsilon}{2}(\sigma_{31}+\sigma_{31}^*)\,\sigma_{23}\,. \quad &\text{(b)} \end{aligned} \tag{3.32}$$

The difference of these 2 equations is:

$$\begin{aligned} \varepsilon(a-g) &= \frac{1}{2}(\sigma_{12}\sigma_{22}^* - \sigma_{13}^*\sigma_{23}) \\ &\quad - \frac{\varepsilon}{2}(\sigma_{12}^2 - \sigma_{23}^2) - \frac{1}{2}(\sigma_{31}+\sigma_{31}^*)(\sigma_{12} - \varepsilon\sigma_{23})\,. \end{aligned} \tag{3.33}$$

The imaginary part of this equation and (3.25) show that:

$$\sigma_{12} = \varepsilon\sigma_{23} \tag{3.34}$$

and thus also:

$$a = g\,. \tag{3.35}$$

Comparing (3.30, 31, 34) one gets:

$$\sigma_{32} + \sigma_{32}^* = 0\,. \tag{3.36}$$

The crucial coefficients σ_{12} and σ_{23} are thus functions of only one variable. The metric equations show that:

$$\theta^1 + \varepsilon\theta^4 = du \tag{3.37}$$

and the various quantities are functions of u alone. One must separate the 2 cases corresponding to a space like or a time like variable u. It is then possible by a straightforward, although somewhat lengthy integration, to obtain the 2 metrics:

$$ds^2 = \frac{f^2(u)}{4}\left[dt + iC_1\frac{\zeta d\zeta^* - \zeta^* d\zeta}{1 - \frac{K}{4}\zeta\zeta^*}\right]^2 - \frac{du^2}{f^2(u)} - q^2(u)\frac{d\zeta\, d\zeta^*}{\left(1 - \frac{K}{4}\zeta\zeta^*\right)^2} \tag{3.38}$$

and

$$ds^2 = \frac{du^2}{f^2(u)} - \frac{f^2(u)}{4}\left[dt + iC_1\frac{\zeta d\zeta^* - \zeta^* d\zeta}{1 - \frac{K}{4}\zeta\zeta^*}\right]^2 - q^2(u)\frac{d\zeta\, d\zeta^*}{\left(1 - \frac{K}{4}\zeta\zeta^*\right)^2}\,. \tag{3.39}$$

These B^1 metrics all admit a group of isometries G and an isotropy group A^1. In the tangent space the transformations have the form

$$\theta^1 \to \theta^1 \qquad \theta^2 \to e^{iq}\theta^2 \qquad \theta^4 \to \theta^4\,. \tag{3.40}$$

In the case C^1, a similar construction can be made. The explicit form of the metrics show that the definition (1′) of isotropy always implies the existence of a group of isometries G which is multiply transitive on its orbits.

It is maybe worth remarking that in the non integrable case an alternative derivation of the metric can be done, which does not use the invariance of the second covariant derivative of the Riemann tensor in B_1. This approach however seems to fail in one exceptional case.

4. Properties of the Locally Isotropic Spaces

a) Metric (2.1)

$$ds^2 = \frac{du\,dv}{(1-\alpha u v)^2} - \frac{d\zeta\,d\zeta^*}{(1-\beta\zeta\zeta^*)^2}\,. \tag{4.1}$$

The change of variables leaving this form of the metric invariant are:

$$\begin{aligned} &\text{a)}\ u \to \frac{u + \frac{\lambda_2\lambda_3}{\alpha}}{\lambda_1 u + \lambda_3} \\ &\text{b)}\ v \to \frac{v + \frac{\lambda_1\lambda_3^{-1}}{\alpha}}{\lambda_2 v + \lambda_3^{-1}} \\ &\text{c)}\ \zeta \to \frac{\zeta + \mu\beta^{-1}e^{-i\varrho}}{\bar{\mu}\zeta + e^{-i\varrho}}\,. \end{aligned} \tag{4.2}$$

This space contains a non singular constant electromagnetic field:

$$\overset{+}{F} = \sqrt{-2(\alpha+\beta)}\,e^{i\varphi}\left[\frac{du\wedge dv}{(1-\alpha u v)^2} - \frac{d\zeta\wedge d\zeta^*}{(1-\beta\zeta\zeta^*)^2}\right]. \tag{4.3}$$

The cosmological constant is:

$$\Lambda = 2(\alpha-\beta)\,. \tag{4.4}$$

The components of the Killing vectors are:

$$\begin{cases} X_1 = [u, 0, 0, -v] & \text{(a)} \\ X_2 = \left[\frac{1-\alpha u^2}{2}, 0, 0, \frac{1-\alpha v^2}{2}\right] & \text{(b)} \\ X_3 = \left[-\frac{1+\alpha u^2}{2}, 0, 0, \frac{1+\alpha v^2}{2}\right] & \text{(c)} \\ X_4 = [0, i\zeta, -i\zeta^*, 0] & \text{(d)} \\ X_5 = [0, 2(1-\beta\zeta^2), 2(1-\beta\zeta^{*2}), 0] & \text{(e)} \\ X_6 = [0, -2i(1+\beta\zeta^2), 2i(1+\beta\zeta^{*2}), 0]\,. & \text{(f)} \end{cases} \tag{4.5}$$

The pair of surfaces $u = v = c^{te}$, and $\zeta = c^{te}$ are orbits of the isometry group G_6; the second fundamental forms of these surfaces vanish identically. It is worth noticing that the Riemann Christoffel tensor of (4.1) is covariantly constant and that (4.1) is thus a symmetric space.

b) Metric (2.3a)

$$ds^2 = \beta^2(x, t)\, dt^2 - \alpha^2(x, t)\, dx^2 - (\lambda x + 1)^2 \frac{d\zeta\, d\zeta^*}{\left(1 - \frac{K}{4}\zeta\zeta^*\right)^2}. \tag{4.6}$$

The admissible coordinate transformations are:

$$t \to \varphi(t) \tag{4.7}$$

and (4.2c).

The components of the curvature tensor are:

$$\begin{aligned}
&2\gamma - \frac{R}{6} = \frac{\lambda}{2\alpha^2(1+\lambda x)}\left[\ln\frac{\beta}{\alpha}\right]_x && \text{(a)}\\
&8\gamma + \frac{R}{3} = \frac{1}{\beta}\left[\frac{\alpha_t}{\alpha\beta}\right]_t - \frac{1}{\alpha}\left[\frac{\beta_x}{\alpha\beta}\right]_x - \frac{1}{\alpha^2\beta^2}[\beta_x^2 - \alpha_t^2] && \\
&\qquad - \frac{\lambda^2}{\alpha^2[1+\lambda x]^2} - \frac{K}{[1+\lambda x]^2} && \text{(b)}\\
&2E_{1\bar{1}} = \frac{-\lambda}{2\alpha^2(1+\lambda x)}[\ln(\alpha\beta)]_x + \frac{\lambda}{\alpha\beta(1+\lambda x)}\frac{\alpha_t}{\alpha} && \text{(c)}\\
&2E_{2\bar{2}} = \frac{-\lambda}{2\alpha^2(1+\lambda x)}[\ln\alpha\beta]_x - \frac{\lambda}{\alpha\beta(1+\lambda x)}\frac{\alpha_t}{\alpha} && \text{(d)}\\
&2E_{3\bar{3}} = \frac{1}{\beta}\left(\frac{\alpha_t}{\alpha\beta}\right)_t - \frac{1}{\alpha}\left(\frac{\beta_x}{\alpha\beta}\right)_x - \frac{1}{\alpha^2\beta^2}(\beta_x^2 - \alpha_t^2) && \\
&\qquad + \frac{\lambda^2}{\alpha^2(1+\lambda x)^2} + \frac{K}{(1+\lambda x)^2}. && \text{(e)}
\end{aligned} \tag{4.8}$$

The components of the Killing vectors are given by (4.5d, e, f). The orbits of the group of isometries are given by:

$$x = x_0, \quad t = t_0.$$

The surface $x = t = c^{te}$ is an orbit of the isometry group G; its 2 second fundamental forms are:

$$\begin{cases} \Omega^{(1)}_{ab} = -\dfrac{\lambda}{\alpha}(\delta^2_a\delta^3_b + \delta^3_a\delta^2_b) \\ \Omega^{(2)}_{ab} = 0. \end{cases} \tag{4.9}$$

If one assumes (4.6) to be a solution of the field equations:

$$R_{\alpha\beta} = 0$$

one finds in the case $\lambda = 1$ $[x + 1 \to x]$

$$\beta^2 = \frac{1}{\alpha^2} = \frac{m - Kx}{x}. \tag{4.10}$$

The case $\lambda = 0$ is impossible.

If one considers the equations:

$$R_{\alpha\beta} - \frac{R + \Lambda}{2} g_{\alpha\beta} = 0$$

one finds in the case $\lambda = 0$ $[x + 1 \to x]$

$$\beta^2 = \frac{1}{\alpha^2} = \frac{m - Kx}{x} + \frac{\Lambda x^2}{3}\,. \tag{4.11}$$

Cosmological models with perfect fluid and vanishing cosmological constant are solutions of

$$E_{1\bar{1}} E_{2\bar{2}} = E^2_{3\bar{3}}\,. \tag{4.12}$$

Up to now we have been able to determine only particular solutions of (4.12). Cosmological models with dust are characterized by (4.12) and

$$2E_{3\bar{3}} + R = 0\,. \tag{4.12'}$$

Spaces (4.6) admitting a non singular electromagnetic field solution of the source free Maxwell equations are determined $[\lambda = 1; x + 1 \to x]$ by:

$$\beta^2 = \frac{1}{\alpha^2} = -K + \frac{m}{x} - \frac{\sigma\sigma^*}{2x^2}\,. \tag{4.13}$$

The electromagnetic field is

$$\overset{+}{F} = \frac{\sigma}{x^2} Z^3\,. \tag{4.14}$$

There are no spaces of this type containing a null electromagnetic field.

If $\alpha = \alpha(x)$, $\beta = \beta(x)$ the space admits a 4-parametric isometry group which is the direct product of the "rotation" group by a one parametric group; if $\alpha = \alpha(t)$, $\beta = \beta(t)$ [which then can be made equal to 1] and $\lambda = 0$ one also has a 4-parametric isometry group.

A five parametric isometry group occurs for the metric:

$$ds^2 = x^{2B}\, dt^2 - \frac{dx^2}{x^2} - x^2 d\zeta\, d\zeta^* \tag{4.15}$$

where B is a constant. If $B = 1$ this space is a solution of:

$$R_{\alpha\beta} - \frac{R + \Lambda}{2} g_{\alpha\beta} = 0$$

with $R = -5$, $\gamma = -1/3$. There are no models with dust, perfect fluid or electromagnetic field and no empty space. The components of the Killing vectors of (4.15) are:

$$\begin{cases} X_1 = [x, -\zeta, -\zeta^*, -Bt] \\ X_2 = [0, -2i\zeta, 2i\zeta^*, 0] \\ X_3 = [0, -2, -2, 0] \\ X_4 = [0, -2i, 2i, 0] \\ X_5 = [0, 0, 0, 1]\,. \end{cases} \tag{4.16}$$

The group of isometries is solvable; its Lie algebra is:

$$\begin{cases} [X_1, X_2] = 0 \quad [X_1, X_3] = X_3 \quad [X_1, X_4] = X_4 \quad [X_1, X_5] = B X_5 \\ [X_2, X_3] = 2 X_4 \quad [X_2, X_4] = -2 X_3 \quad [X_2, X_5] = 0 \\ [X_3, X_4] = [X_4, X_5] = [X_5, X_3] = 0 . \end{cases} \tag{4.17}$$

The metrics (2.3a) have been studied by PLEBANSKI and STACHEL [14].

c) Metric (2.3b)

$$ds^2 = \alpha^2(x, t)\, dx^2 - \beta^2(x, t)\, dt^2 - (\lambda x + 1)^2 \frac{d\zeta\, d\zeta^*}{\left(1 - \frac{K}{4} \zeta \zeta^*\right)^2} . \tag{4.18}$$

As previously the admissible coordinate transformations are:

$$t \to \varphi(t) \tag{4.19}$$

and (4.2c).

The components of the curvature tensor are those given in (4.8) with a change of sign in (4.8a) and in the first four terms of (4.8b and c). The components of the Killing vectors are unchanged. Also unmodified are the second fundamental forms of the orbits $x = t = c^{te}$. In empty space ($R_{\alpha\beta} = 0$) one has:

$$\beta^2 = \frac{1}{\alpha^2} = K + \frac{m}{x} . \tag{4.20}$$

If there is a cosmological constant

$$\beta^2 = \frac{1}{\alpha^2} = K + \frac{m}{x} - \frac{\Lambda x^2}{3} . \tag{4.21}$$

The equations for a perfect fluid or dust are again (4.12) or (4.12 and 12′).

There is no possibility of having a null electromagnetic field. For a non singular electromagnetic field one finds:

$$\begin{cases} \beta^2 = \frac{1}{\alpha^2} = K + \frac{m}{x} + \frac{\sigma\sigma^*}{2x} \\ \overset{+}{F} = \frac{\sigma}{x^2} Z^3 . \end{cases} \tag{4.22}$$

There are as in (2.3a) two possible cases of having a 4-parametric isometry group, and one case of a 5-parametric isometry group. When there is a G_5 the metric is:

$$ds^2 = \frac{dx^2}{x^2} - x^{2B}\, dt^2 - x^2 d\zeta\, d\zeta^* . \tag{4.23}$$

The group structure is the one defined by (4.17).

d) Metric (2.3c)

$$ds^2 = 2p^2(x, y)\, dx\, dy - 2(1 + \lambda x)^2 \frac{d\zeta\, d\zeta^*}{\left(1 - \frac{K}{4} \zeta\zeta^*\right)} . \tag{4.24}$$

The admissible coordinate transformations are:

$$y \to \varphi(y) \tag{4.25}$$

and (4.2c). The components of the curvature tensor are:

$$\begin{cases} \gamma - \frac{R}{6} = 0 \\ E_{1\bar{1}} = 0 \\ E_{2\bar{2}} = \frac{-4\lambda p_x}{p^3(1+\lambda x)} \\ 4\gamma + \frac{R}{3} = -\frac{K}{(1+\lambda x)^2} + \frac{4}{p^2}\left(\frac{p_y}{p}\right)_x \\ E_{3\bar{3}} = \frac{K}{(1+\lambda x)^2} + \frac{4}{p^2}\left(\frac{p_y}{p}\right)_x . \end{cases} \tag{4.26}$$

There exists a 4-parametric isometry group which is the direct product of the "rotation" group by a one parametric group. The orbits of the group are the null surfaces $x = x_0$.

There are no empty spaces. If one adds a cosmological constant one finds the Bertotti-Robinson metric. There are no perfect fluid nor dust models. Non singular electromagnetic fields give again rise to BERTOTTI-ROBINSON. Singular electromagnetic fields do not exist.

e) Metric (2.6)

$$ds^2 = \frac{f^2(u)}{4}\left(dt + iC_1\frac{\zeta d\zeta^* - \zeta^* d\zeta}{1 - \frac{K}{4}\zeta\zeta^*}\right)^2 - \frac{du^2}{f^2(u)} - q^2(u)\frac{d\zeta\, d\zeta^*}{\left(1 - \frac{K}{4}\zeta\zeta^*\right)^2} \tag{4.27}$$

The allowable transformations are:

$$\begin{aligned} \zeta &\to \frac{\zeta + 4K^{-1}\sigma e^{-i\beta}}{\bar{\sigma}\zeta + e^{-i\beta}} \\ t &\to t + t_0 + \frac{4iC_1}{K}\ln\frac{\bar{\sigma}\zeta + e^{-i\beta}}{\sigma\bar{\zeta} + e^{i\beta}} . \end{aligned} \tag{4.28}$$

The components of the curvature tensor are:

$$\begin{cases} E_{1\bar{1}} = E_{2\bar{2}} = \frac{f^2}{4}\left(\frac{q''}{q} - \frac{C_1^2}{q^4}\right) \\ 2E_{3\bar{3}} = \frac{K}{q^2} - ff'' - \frac{3f^2C_1^2}{q^4} + \frac{q'^2}{q^2}f^2 - f'^2 \\ 12\gamma = -\frac{K}{q^2} - ff'' + \frac{4f^2C_1^2}{q^4} - \frac{q'^2}{q^2} - f'^2 + \frac{f^2q''}{q} + \frac{2q'ff'}{q} \\ \qquad + \frac{6iC_1ff'}{q^2} - \frac{6iC_1q'f^2}{q^2} \\ R = -\frac{K}{q^2} - ff'' + \frac{f^2C_1^2}{q^4} - \frac{q'^2}{q^2}f^2 - f'^2 - \frac{2q''f^2}{q} - \frac{4q'ff'}{q} . \end{cases} \tag{4.29}$$

The components of the Killing vectors are:

$$\begin{cases} X_1 = [0, i\zeta, -i\zeta^*, 0] \\ X_2 = \left[0, \frac{i}{2}\left(1 + \frac{K}{4}\zeta^2\right), \frac{-i}{2}\left(1 + \frac{K}{4}\zeta^{*2}\right), \frac{C_1}{2}(\zeta + \zeta^*)\right] \\ X_3 = \left[0, \frac{1}{2}\left(1 - \frac{K}{4}\zeta^2\right), \frac{1}{2}\left(1 - \frac{K}{4}\zeta^{*2}\right), \frac{iC_1}{2}(\zeta - \zeta^*)\right] \\ X_4 = [0, 0, 0, 1]. \end{cases} \tag{4.30}$$

The orbits of the group are the time like surfaces $u = u_0$. Their second fundamental form is:

$$\Omega = -\frac{1}{2} f' \left[dt + iC_1 \frac{\zeta d\zeta^* - \zeta^* d\zeta}{1 - \frac{K}{4}\zeta\zeta^*}\right]^2 + \frac{2qq'}{f} \frac{d\zeta\, d\zeta^*}{\left(1 - \frac{K}{4}\zeta\zeta^*\right)^2}. \tag{4.31}$$

The eigen directions of Ω lie in the 2-plane (ζ, ζ^*) and there is one other eigen direction orthogonal to this plane.

The empty spaces are the Taub-Nut spaces [15]:

$$\begin{cases} q^2(u) = u^2 + C_1^2 \\ f^2(u) = -K + \dfrac{2KC_1^2}{u^2 + C_1^2} + \dfrac{mu}{u^2 + C_1^2}. \end{cases} \tag{4.32}$$

In the presence of a cosmological constant one has:

$$\begin{cases} q^2(u) = u^2 + C_1^2 \\ f^2(u) = -K + \dfrac{2KC_1^2}{u^2 + C_1^2} + \dfrac{mu}{u^2 + C_1^2} + \dfrac{\Lambda u}{u^2 + C_1^2}\left(2C_1^2 u + \dfrac{u^2}{3} - \dfrac{C_1^4}{u}\right). \end{cases} \tag{4.33}$$

The dust models have been found by ELLIS, G.F.R. [4]. There are three cases corresponding respectively to $\Lambda < 0$, $\Lambda > 0$, and $\Lambda = 0$. One has:

$$\begin{aligned} &\Lambda > 0 \quad 1)\ f^2 = k^2;\ q^2 = \frac{2C_1^2 k^2}{K};\ \Lambda = \frac{K^2}{2C_1^2 k^2} \\ &\qquad\quad 2)\ f^2 = k^2;\ q^2 = \frac{2K}{\mu^2 k^2} + a e^{\mu u} + b e^{-\mu u};\ \Lambda = \frac{\mu^2 k^2}{2} \\ &\qquad\qquad a b k^2 \mu^2 = \frac{K^2}{\mu^2 k^2} - C_1^2 k^2 \\ &\Lambda = 0 \qquad f^2 = k^2;\ q^2 = -\frac{K}{k^2} u^2 + 2C_1 u \\ &\Lambda < 0 \qquad f^2 = k^2;\ q^2 = -\frac{2K}{\mu^2 k^2} + a \cos \mu u + b \sin \mu u;\ \Lambda = -\frac{\mu^2 k^2}{2} \\ &\qquad \frac{k^2 \mu^2}{2}(a^2 + b^2) = 2C_1^2 k^2 + \frac{2K^2}{k^2 \mu^2}. \end{aligned} \tag{4.34}$$

In the case $\Lambda = 0$, it is not difficult to show that the only solutions are the ones given by (4.34). A. H. TAUB pointed to us that, in the limit of a vanishing density, the space becomes flat; one does not recover the metrics (4.32).

In the case of a perfect fluid one ends up with one differential equation $[v = f^2]$:

$$-\frac{1}{2} v'' + \left(-\frac{2C_1^2}{q^4} + \frac{q'^2}{q} - \frac{q''}{q}\right) v + \frac{K}{q^2} = 0 \,. \tag{4.35}$$

There is no possibility of having a null electromagnetic field. In the non singular case however one has:

$$\begin{cases} q^2(u) = u^2 + C_1^2 \\ f^2(u) = -K + \dfrac{2KC_1^2}{u^2 + C_1^2} + \dfrac{mu}{u^2 + C_1^2} - \dfrac{\sigma\sigma^*}{2(u^2 + C_1^2)} \,. \end{cases} \tag{4.36}$$

This metric is equivalent to one of the Carter's metrics [8]. There are two metrics of the type (2.6) which admit a 5-parametric group of isometries which is the direct product of the group (2.7) by a one parametric group

$$ds^2 = \frac{1}{4}\left(dt + i\,\frac{\zeta d\zeta^* - \zeta^* d\zeta}{1 - \frac{K}{4}\zeta\zeta^*}\right)^2 - du^2 - \frac{d\zeta\, d\zeta^*}{\alpha^2\left(1 - \frac{K}{4}\zeta\zeta^*\right)^2} \tag{4.37}$$

and:

$$ds^2 = \frac{u^2}{4}(dt + i(\zeta d\zeta^* - \zeta^* d\zeta))^2 - \frac{du^2}{\alpha^2 u^2} - \frac{2u}{\beta}\, d\zeta\, d\zeta^* \,. \tag{4.38}$$

The first of these metrics has been found by I. OZSVATH [16]. If:

$$2\alpha^2 > K \qquad K\alpha^2 > 2\Lambda \qquad -K\alpha^2 + 2\Lambda + 4\alpha^2 > 0$$

this metric contains a perfect fluid, a non singular electromagnetic field and the cosmological constant.

f) Metric (2.8)

$$ds^2 = \frac{du^2}{f^2(u)} - \frac{1}{4} f^2(u)\left(dt + i\,\frac{\zeta d\zeta^* - \zeta^* d\zeta}{1 - \frac{K}{4}\zeta\zeta^*}\right)^2 - q^2(u)\,\frac{d\zeta\, d\zeta^*}{\left(1 - \frac{K}{4}\zeta\zeta^*\right)^2} \,. \tag{4.39}$$

The admissible change of variables are the translations along t and the transformations of the ζ variable given by (4.28). The components of the curvature are:

$$\begin{cases} 12\gamma = ff'' + f'^2 - \dfrac{4f^2C_1^2}{q^4} + f^2\dfrac{q'^2}{q^2} - \dfrac{K}{q^2} - \dfrac{q''f^2}{q} - \dfrac{2q'ff'}{q} \\ \qquad + 6iC_1\dfrac{f^2}{q^2}\left(\dfrac{f'}{f} - \dfrac{q'}{q}\right) \\ R = ff'' + f'^2 - \dfrac{f^2C_1^2}{q^4} + f^2\dfrac{q'^2}{q^2} - \dfrac{K}{q^2} + \dfrac{2q''f^2}{q} + \dfrac{4q'ff'}{q} \\ 2E_{1\bar{1}} = 2E_{2\bar{2}} = \dfrac{f^2}{2}\left(\dfrac{q''}{q} - \dfrac{C_1^2}{q^4}\right) \\ 2E_{3\bar{3}} = ff'' + f'^2 + \dfrac{3f^2C_1^2}{q^4} - f^2\dfrac{q'^2}{q^2} + \dfrac{K}{q^2} \,. \end{cases} \tag{4.40}$$

The 4 Killing vectors have components given by (4.30). The orbits of the group are the surfaces $u = u_0$. Their second fundamental form is given by:

$$\Omega = \frac{1}{2} f_u \left[dt + i C_1 \frac{\zeta d\zeta^* - \zeta^* d\zeta}{\left(1 - \frac{K}{4} \zeta\zeta^*\right)} \right]^2 + 2 \frac{q q_u}{f} \frac{d\zeta\, d\zeta^*}{\left(1 - \frac{K}{4} \zeta\zeta^*\right)^2} . \quad (4.41)$$

The eigen directions are the t direction and the directions of the $\zeta\zeta^*$ plane. Various solutions analoguous to the ones corresponding to the metric (2.6) can be exhibited:

a) Empty space:

$$\begin{cases} q^2(u) = u^2 + C_1^2 \\ f^2(u) = K - \dfrac{2KC_1^2}{u^2 + C_1^2} + \dfrac{mu}{u^2 + C_1^2} . \end{cases} \quad (4.42)$$

This anti-Nut space was given in [17].

b) With Λ term:

$$\begin{cases} q^2(u) = u^2 + C_1^2 \\ f^2(u) = K - \dfrac{2KC_1^2}{u^2 + C_1^2} + \dfrac{mu}{u^2 + C_1^2} - \dfrac{\Lambda u}{u^2 + C_1^2} \left[\dfrac{u^3}{3} + 2C_1^2 u - \dfrac{C_1^4}{u} \right] . \end{cases} \quad (4.43)$$

c) There does not exist a model with singular electromagnetic field.

d) Non singular electromagnetic field

$$\begin{cases} q^2 = u^2 + C_1^2 \\ f^2 = K - \dfrac{2KC_1^2}{u^2 + C_1^2} + \dfrac{mu}{u^2 + C_1^2} + \dfrac{k^2}{2(u^2 + C_1^2)} \\ \overset{+}{F} = \sigma (u^2 + C_1^2)^{-1} e^{-2i \operatorname{arctg} u/C_1} Z^3; \quad \sigma\sigma^* = k^2 . \end{cases} \quad (4.44)$$

e) There are 2 models which admit a 5-dimensional group of motions:

$$ds^2 = du^2 - \frac{1}{4} \left(dt + i \frac{\zeta d\zeta^* - \zeta^* d\zeta}{1 - \frac{K}{4} \zeta\zeta^*} \right)^2 - \frac{d\zeta\, d\zeta^*}{\alpha^2 \left(1 - \frac{K}{4} \zeta\zeta^*\right)^2} . \quad (4.45)$$

This metric was determined by OZSVATH [16]. Also:

$$ds^2 = \frac{du^2}{\alpha^2 u^2} - \frac{u^2}{4} (dt + i(\zeta d\zeta^* - \zeta^* d\zeta))^2 - \frac{2u}{\beta} d\zeta\, d\zeta^* . \quad (4.46)$$

The group structure is the same as the one mentioned earlier.

g) Metric (2.9)

$$ds^2 = q^2(x) \frac{du\, dv}{\left(1 - \frac{K}{4} uv\right)^2} - \alpha^2(t, x)\, dt^2 - \beta^2(t, x)\, dx^2 . \quad (4.47)$$

The admissible coordinate transformations are:

$$t \to \varphi(t) \qquad x \to \psi(x)$$

and the homographic transformations on (u, v).

The components of the curvature tensor are:

$$
\begin{aligned}
8\gamma + \frac{R}{3} &= \frac{K}{q^2} - \frac{1}{\alpha}\left(\frac{\beta_t}{\alpha\beta}\right)_t - \frac{1}{\beta}\left(\frac{\alpha_x}{\alpha\beta}\right)_x - \left(\frac{\beta_t}{\alpha\beta}\right)^2 - \left(\frac{\alpha_x}{\alpha\beta}\right)^2 - \left(\frac{q_x}{q\beta}\right)^2 \\
2E_{3\bar{3}} &= \frac{K}{q^2} + \frac{1}{\alpha}\left(\frac{\beta_t}{\alpha\beta}\right) + \frac{1}{\beta}\left(\frac{\alpha_x}{\alpha\beta}\right)_x + \left(\frac{\beta_t}{\alpha\beta}\right)^2 + \left(\frac{\alpha_x}{\alpha\beta}\right)^2 - \left(\frac{q_x}{q\beta}\right)^2 \\
2E_{1\bar{2}} &= -\frac{i}{\alpha}\frac{q_x}{q}\frac{\beta_t}{\beta^2} - \frac{1}{2\beta}\left(\frac{q_x}{q\beta}\right)_x - \frac{1}{2}\left(\frac{q_x}{q\beta}\right)^2 + \frac{1}{2}\frac{q_x}{q}\frac{\alpha_t}{\alpha\beta^2} \\
2\gamma - \frac{R}{6} &= \frac{1}{2\beta}\left(\frac{q_x}{q\beta}\right)_x + \frac{1}{2}\left(\frac{q_x}{q\beta}\right)^2 + \frac{1}{2}\frac{q_x}{q\beta}\frac{\alpha_x}{\alpha\beta}\,.
\end{aligned}
\tag{4.48}
$$

There exists a 3-parametric group of isometries. The components of the Killing vectors are:

$$
\begin{cases}
X_1 = [0, 0, u, -v] \\
X_2 = \left[0, 0, \frac{1}{2}\left(1 + \frac{K}{4}u^2\right), -\frac{1}{2}\left(1 + \frac{K}{4}v^2\right)\right] \\
X_3 = \left[0, 0, \frac{1}{2}\left(-1 + \frac{K}{4}u^2\right), -\frac{1}{2}\left(1 - \frac{K}{4}v^2\right)\right].
\end{cases}
\tag{4.49}
$$

The orbits of the group are the surfaces $x = \text{Cte}$, $t = \text{Cte}$. Their 2 second fundamental forms are $[q \to 1 + \lambda x]$

$$
\begin{cases}
\Omega^{(1)} = \frac{\lambda}{\beta}(\delta^3_a\delta^4_b + \delta^3_b\delta^4_a) \\
\Omega^{(2)} = 0\,.
\end{cases}
\tag{4.50}
$$

Certain particular metrics of this family are worth mentioning:

a) Empty spaces: $q = x$

$$\alpha^2 = \frac{1}{\beta^2} = K + \frac{m}{x}\,. \tag{4.51}$$

b) With Λ term: $q = x$

$$\alpha^2 = \frac{1}{\beta^2} = K + \frac{m}{x} + \frac{\Lambda}{x^2}\,. \tag{4.52}$$

c) There are no dust model, no model with perfect fluid, no model with a null electromagnetic field.

d) Non singular electromagnetic field: $q = x$

$$
\begin{cases}
\alpha^2 = \frac{1}{\beta^2} = K + \frac{m}{x} + \frac{\sigma\sigma^*}{2x^2} \\
\overset{+}{F} = \frac{\sigma}{x^2} Z^3\,.
\end{cases}
\tag{4.53}
$$

e) There is a 4-parametric group of isometries if α and β are functions of x only or if q is a constant and α and β are functions of t only.

f) There is a metric admitting a 5 dimensional group of isometries:

$$ds^2 = x^2\,du\,dv - x^{2B}\,dt^2 - \frac{dx^2}{x^2}\,. \tag{4.54}$$

h) Metric (2.11)

$$ds^2 = q^2(x) \frac{du\,dv}{\left(1 - \frac{K}{4} uv\right)^2} - \frac{f^2(x)}{4} \left(dt + C_1 \frac{udv - vdu}{1 - \frac{K}{4} uv}\right)^2 - \frac{dx^2}{f^2(x)} . \quad (4.55)$$

The admissible coordinate transformations are given by:

$$\begin{aligned} u &\to \frac{u + 4K^{-1}\beta e^{\lambda}}{\alpha u + e^{\lambda}} \\ v &\to \frac{v + 4K^{-1}\alpha e^{-\lambda}}{\beta v + e^{-\lambda}} \\ t &\to t + t_0 + 4K^{-1} C_1 \ln \frac{\alpha u + 1}{\beta v + 1} . \end{aligned} \quad (4.56)$$

The components of the curvature tensor are:

$$\begin{aligned} 2E_{1\bar{2}} &= \frac{f^2}{2}\left(-\frac{q''}{q} + \frac{C_1^2}{q^4}\right) \\ 2E_{3\bar{3}} &= \frac{K}{q^2} + f''f + f'^2 - f^2 \frac{q'^2}{q^2} + \frac{3C_1^2 f^2}{q^4} \\ R &= \frac{K}{q^2} - f''f - f'^2 + \frac{C_1^2 f^2}{q^4} - f^2 \frac{q'^2}{q^2} - 4ff'\frac{q'}{q} - 2f^2 \frac{q''}{q} \\ 12\gamma &= \frac{K}{q^2} - f''f - f'^2 + \frac{4C_1^2 f^2}{q^4} - f^2 \frac{q'^2}{q^2} + 2ff'\frac{q'}{q} + f^2 \frac{q''}{q} \\ &\quad + 6iC_1 f^2 \left(\frac{f'}{f} - \frac{q'}{q}\right) . \end{aligned} \quad (4.57)$$

The 4 Killing vectors have components

$$\begin{cases} X_1 = [0, 0, u, -v] \\ X_2 = \left[0, -C_1 \frac{u+v}{2}, \frac{1}{2}\left(1 + \frac{K}{4}u^2\right), -\frac{1}{2}\left(1 + \frac{K}{4}v^2\right)\right] \\ X_3 = \left[0, C_1 \frac{-u+v}{2}, \frac{1}{2}\left(-1 + \frac{K}{4}u^2\right), -\frac{1}{2}\left(1 - \frac{K}{4}v^2\right)\right] \\ X_4 = [0, 1, 0, 0] . \end{cases} \quad (4.58)$$

The orbits are the surfaces $x = x_0$. Their second fundamental form is:

$$\Omega = \frac{qq_x}{f} \frac{du\,dv}{\left(1 - \frac{K}{4}uv\right)^2} - \frac{f'}{2}\left[dt + C_1 \frac{udv - vdu}{1 - \frac{K}{4}uv}\right]^2 . \quad (4.59)$$

We have considered the following particular cases [17]:

a) Empty spaces:

$$\begin{cases} q^2 = x^2 + C_1^2 \\ f^2 = K - \frac{2KC_1^2}{x^2 + C_1^2} + \frac{mx}{x^2 + C_1^2} . \end{cases} \quad (4.60)$$

b) With Λ term:

$$\begin{aligned} q^2 &= x^2 + C_1^2 \\ f^2 &= K - \frac{2KC_1^2}{x^2 + C_1^2} + \frac{mx}{x^2 + C_1^2} + \frac{\Lambda x}{x^2 + C_1^2}\left(\frac{x^3}{3} + 2C_1^2 x - \frac{C_1^4}{x}\right) . \end{aligned} \quad (4.61)$$

c) There are no perfect fluid model, no dust model, no model containing a null electromagnetic field.

d) Non singular electromagnetic field:

$$\begin{aligned} q^2 &= x^2 + C_1^2 \\ f^2 &= K - \frac{2KC_1^2}{x^2 + C_1^2} + \frac{mx}{x^2 + C_1^2} + \frac{\sigma\sigma^*}{2(x^2 + C_1^2)} \\ \overset{+}{F} &= \sigma(x^2 + C_1^2)^{-1}\, e^{2i \operatorname{arc\,tg} x/C_1} Z^3 . \end{aligned} \tag{4.62}$$

e) There are 2 models having a 5-parametric isometry group:

$$ds^2 = \frac{2y}{\alpha} du\, dv - \frac{y^2}{4}(dt - v du + u\, dv) - \frac{dy^2}{\beta^2 y^2} \tag{4.63}$$

and:

$$ds^2 = \frac{1}{\alpha^2} \frac{du\, dv}{\left(1 - \frac{K}{4} uv\right)^2} - \left(dt + \frac{udv - vdu}{2\left(1 - \frac{K}{4} uv\right)}\right)^2 - dx^2 . \tag{4.64}$$

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