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The N-Body Problem with Spin-Orbit or Coulomb Interactions

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Abstract. A study is made of the resolvent $R(\lambda)$ for a system of n particles with spin-orbit coupling, an interaction which necessarily has a long range in momentum space. For short-range interactions, it has been known for several years that $R(\lambda)$ satisfies a Fredholm equation whose kernel is in the Schmidt-class. The corresponding spin-orbit kernel is not in the Schmidt-class, but it is shown that it does belong to a certain class of compact operators which is larger than the Schmidt-class. A modified Fredholm theory is presented which applies to all operators in this larger class. This enables $R(\lambda)$ to be found for all values of λ in the complex plane cut along the continuous spectrum of the Hamiltonian. It is shown that the modified Fredholm theory also holds for the Coulomb interaction.

1. Introduction

In recent years our understanding of the n-body problem has considerably improved. There are now rigorous mathematical methods available to discuss non-relativistic systems consisting of any finite number of particles with two-body interactions. For three particles, a powerful approach is due to FADDEEV [1, 2]. Under the assumption that the two-body scattering amplitudes are known, this gives a set of coupled equations for the three-body amplitude. Faddeev's work became widely known through two papers by LOVELACE [3, 4] and was subsequently generalized to larger numbers of particles by several authors [5–9]. Alternative equations in terms of the two-body scattering amplitude were given by ROSENBERG [10] and NEWTON [11] and applied by NOBLE [12]. For further information see refs. [13–15].

A different method was proposed by WEINBERG [16] and further discussed by HUNZIKER [17]. This is closely related to a formalism which was developed independently by one of us [18]. In this formalism, it is assumed that the two-body interactions are known, and a sequence of equations is constructed which must be solved successively for the resolvents referring to $2, 3, \ldots, n$ particles. In the present paper, these equations are examined for a class of interactions for which the n-body problem has not been considered before. Our previous paper is henceforth referred to as paper I.

In all current n-body theories, there are integral equations of the Fredholm type which under favourable circumstances have kernels in the Schmidt-class. Specifically, in the case of two particles interacting through a local potential $v(\boldsymbol{x})$, one considers an equation of the form

$$h(\boldsymbol{k}) = h_0(\boldsymbol{k}) - (2\pi)^{-3/2} \int (k^2 - \lambda)^{-1} \hat{V}(\boldsymbol{k} - \boldsymbol{l})\, h(\boldsymbol{l})\, d^3 l\,, \qquad (1.1)$$

where $\hat{V}$ is the Fourier transform of v, the function $h_0(\boldsymbol{k})$ is known and $h(\boldsymbol{k})$ is to be found. If $\hat{V}(\boldsymbol{k})$ is square-integrable and λ is not on the positive real axis, the kernel of Eq. (1.1) is a square-integrable function of $\boldsymbol{k}$ and $\boldsymbol{l}$ and is thus in the Schmidt-class. This enables the equation to be solved by the Fredholm method. This approach to the problem has been generalized to larger numbers of particles, but the actual computational work then becomes extremely laborious. For practical applications many authors have therefore considered interactions of finite rank. This is usually referred to as the separable approximation. It results in considerable computational simplifications, but it has the disadvantage of not being invariant under translations. It is therefore not adopted in the present paper.

In the context of the Faddeev equations, the separable approximation was first advocated by Lovelace [3, 4]. It also arises in special three-particle models due to Amado [19] and Sugar and Blankenbecler [20]. It has been used in many papers by Mitra and coworkers [21, 22]. A general exposition is given in refs. [14] and [22]. For various applications, see refs. [12] and [23–34].

In the separable approximation, one can introduce spin-dependent interactions without any difficulty. As a matter of fact, this was done in most of the papers cited above, and the question has never received any special attention. Consider, however, a translationally invariant spin-orbit interaction of the form

$$\boldsymbol{s} \cdot (\boldsymbol{x} \times \boldsymbol{k})\, v(\boldsymbol{x})\,, \qquad (1.2)$$

where $\boldsymbol{s}$ is a spin operator, $\boldsymbol{x}$ is a position and $\boldsymbol{k}$ a momentum operator. With the interaction (1.2), the analogue of Eq. (1.1) becomes

$$h(\boldsymbol{k}) = h_0(\boldsymbol{k}) - \boldsymbol{s} \cdot \int (k^2 - \lambda)^{-1} \boldsymbol{W}(\boldsymbol{k} - \boldsymbol{l}) \times \boldsymbol{k}\, h(\boldsymbol{l})\, d^3 l\,, \qquad (1.3)$$

$\boldsymbol{W}$ being proportional to the Fourier transform of $\boldsymbol{x}\, v(\boldsymbol{x})$. The kernel of this equation contains a factor $\boldsymbol{k}(k^2 - \lambda)^{-1}$, and so it cannot be in the Schmidt-class, no matter how $\boldsymbol{W}(\boldsymbol{k} - \boldsymbol{l})$ is chosen. There is thus a problem which has not been envisaged before. It is the main purpose of the present paper to show how this can be solved for any finite number of particles.

The problem is given a concise form in section 2. In order that there be a self-adjoint Hamiltonian H, a general condition is imposed on the interaction function $\boldsymbol{W}(\boldsymbol{k})$ in section 3. This guarantees that there exists a resolvent $R(\lambda) \equiv (H - \lambda)^{-1}$. Some general properties of $R(\lambda)$ are also discussed in section 3. Section 4 is devoted to the Fredholm equation for $R(\lambda)$ which we propose to solve. This has a kernel $K(\lambda)$ which is not in the Schmidt-class. However, in appendix A2 we introduce a class of integral operators which is denoted by (rc) and which is considerably larger than the Schmidt-class. For values of λ in the complex plane cut along the real axis from a certain point Λ_0 to ∞, it is shown in section 5 that $K(\lambda)$ does belong to the class (rc). It is explained in appendix A2 that (rc) is a subclass of a certain class of compact operators which some authors denote by $\mathfrak{K}_4$, but which we prefer to call (ϱc). Also, it is indicated in section 6 how one can generalize the Fredholm theory of integral equations so as to apply to kernels in (ϱc). This method is based on a forthcoming paper by one of us [35]. It makes it possible to construct $R(\lambda)$ for all values of λ in the complex plane cut from Λ_0 to ∞. In the cut plane, $R(\lambda)$ is found to be analytic, except for possible poles corresponding to bound states. To the left of Λ_0, there can thus be a discrete spectrum at most. It is shown in section 7 that the essential spectrum of H runs from Λ_0 to ∞.

The difficulty of Eq. (1.3) derives from the factor $\boldsymbol{k}$, and thus from the long range of the interaction in momentum space. This suggests that we also try our methods on interactions with a long range in position space. As an example of the latter category, we have chosen the Coulomb interaction, for which the separable approximation is known to be particularly unsuitable. In the context of the Faddeev equations, it has therefore not received much attention until recently [36, 37]. For three particles, there are now approximation schemes available due to SCHULMAN [36] and NOBLE [37], but it appears that the problem has never been investigated systematically. It is shown in section 8 that the Coulomb interaction gives rise to a kernel in the class (ϱc). For values of λ not in the continuous spectrum, the resolvent can thus be found with the help of section 6.

The main body of the paper ends with a general discussion. There is a mathematical appendix on compact operators, with special emphasis on the classes (ϱc) and (rc).

2. General Remarks

2.1. Positions and Momenta

The present paper is concerned with a system of n particles with positions $\boldsymbol{X}_i$, momenta $\boldsymbol{K}_i$, spins $\boldsymbol{s}_i$, and masses m_i. It is assumed that the Hamiltonian takes the form

$$H' = \sum_{i=1}^{n} \frac{K_i^2}{2m_i} + \sum_{i<j} V_{ij}\left(X_i - X_j; \frac{K_i}{2m_i} - \frac{K_j}{2m_j}; s_i, s_j\right). \qquad (2.1)$$

The interaction thus consists of $\frac{1}{2} n(n-1)$ two-body terms which are invariant under translations in position and velocity space.

Owing to the invariance properties of the interaction, the motion of the centre of mass can be separated off. This can, in fact, be done in many ways, a convenient set of new coordinates being [38]

$$\begin{aligned} x_i &= (2m_{i+1}/M_i M_{i+1})^{1/2} \sum_{j=1}^{i} m_j (X_{i+1} - X_j), \\ x_n &= (2/M_n)^{1/2} \sum_{j=1}^{n} m_j X_j, \qquad M_j = \sum_{i=1}^{j} m_i. \end{aligned} \qquad (2.2)$$

The transformation (2.2) is of the general form

$$x_i = \sum_{j=1}^{n} U_{ij} X_j (2m_j)^{1/2}, \qquad (2.3)$$

where $\{U_{ij}\}$ is an orthogonal matrix. In the position representation, the momentum K_j is represented by the operator $-i\nabla(X_j)$. If k_j stands for the quantity which in the position representation is represented by $-i\nabla(x_j)$, one has

$$k_i = \sum_{j=1}^{n} U_{ij} K_j (2m_j)^{-1/2}, \qquad K_i (2m_i)^{-1/2} = \sum_{j=1}^{n} U_{ji} k_j. \qquad (2.4)$$

The kinetic energy takes the form

$$H_0' = \sum_{i=1}^{n} \frac{K_i^2}{2m_i} = \sum_{i=1}^{n} k_i^2. \qquad (2.5)$$

Also,

$$X_i - X_j = \sum_{h=1}^{n-1} c_{ijh} x_h, \quad \frac{K_i}{2m_i} - \frac{K_j}{2m_j} = \sum_{h=1}^{n-1} c_{ijh} k_h, \qquad (2.6)$$

with some set of coefficients such that c_{ijn} vanishes.

According to Eqs. (2.2–4), $x_i (k_i)$ is proportional to the position (velocity) of particle $i+1$ with respect to the centre of mass of the subsystem consisting of the particles $1, 2, \ldots, i$ $(1 \leqq i \leqq n-1)$. The coordinate $x_n (k_n)$ is proportional to the position (velocity) of the centre of mass of the system as a whole. The relations (2.6) express the fact that the interaction does not depend on the motion of the centre of mass. The Hamiltonian for the internal motion is

$$H = \sum_{i=1}^{n-1} k_i^2 + \sum_{i<j} V_{ij}\left(\sum_{h=1}^{n-1} c_{ijh} x_h; \sum_{h=1}^{n-1} c_{ijh} k_h; s_i, s_j\right). \qquad (2.7)$$

To discuss a particular interaction term V_{ij}, it is convenient to use a set of coordinates which slightly differs from the set (2.2), and is chosen in such a way that

$$\boldsymbol{x}_1 = \boldsymbol{x}_{ij} \equiv (2 m_i m_j / m_i + m_j)^{1/2} (\boldsymbol{X}_i - \boldsymbol{X}_j) . \tag{2.8}$$

At times it is convenient first to split the particles into two groups, and to choose $\boldsymbol{x}_{n-1}$ proportional to the position of group 2 relative to group 1. One will then introduce internal coordinates in the two groups separately.

2.2. Wave Functions

The internal motion of a system of n particles may be described with the help of a wave function

$$f = f(\boldsymbol{k}) = f(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{n-1}) ,$$

where f is an element of the Hilbert space $L^2(R^{3n-3})$, that is, the space of all measurable functions $f(\boldsymbol{k})$ which satisfy

$$\|f\| = [\int |f(\boldsymbol{k})|^2 d^{3n-3} k]^{1/2} < \infty .$$

The space $L^2(R^{3n-3})$ is occasionally denoted by $L^2(\boldsymbol{k})$ or by L^2. If f and g are any two elements of L^2, their inner product is written as

$$(f, g) = \int f(\boldsymbol{k}) \bar{g}(\boldsymbol{k}) \, d^{3n-3} k . \tag{2.9}$$

In the case of particles with spins, it is appropriate to use wave functions of the form

$$f' = \sum_{a=1}^{M} f_a(\boldsymbol{k}) \, \chi_a , \tag{2.10}$$

where $\{\chi_a\}$ $(a = 1, \ldots, M)$ is a set of spinors and each $f_a(\boldsymbol{k})$ is in L^2. If all particles have spin 1/2, the index a runs through 2^n values, and similarly for other spins. With the inner product

$$(f', g')' = \sum_{a=1}^{M} (f_a, g_a) ,$$

the set of all functions of the form (2.10) is a Hilbert space. Let this be denoted by L'^2. A linear operator A' on L'^2 is associated with a matrix $\{A_{ab}\}$ of linear operators on L^2. If $\|A'\|'$ and $\|A_{ab}\|$ stand for the operator norms on L'^2 and L^2, respectively, one has

$$\|A'\|' \leqq \sum_{a,b} \|A_{ab}\| .$$

A relation of this form also applies to the Schmidt-norm, and, in fact, to all the norms discussed in appendix A1. If ν' denotes one of these norms on L'^2, and ν the corresponding norm on L^2, it is easily seen that

$$\begin{aligned} \nu'(A') &\leqq \sum_{a,b} \nu(A_{ab}) , \\ \nu(A_{ab}) &\leqq \nu'(A') \quad (a, b = 1, \ldots, M) . \end{aligned} \tag{2.11}$$

From this it follows that $\nu'(A')$ is finite if and only if each $\nu(A_{ab})$ is finite.

2.3. *Interactions*

For spin-independent interactions

$$V_{ij} = v_{ij}(\boldsymbol{x}_{ij})$$

such that

$$\int [v_{ij}(\boldsymbol{x}_{ij})]^2 \, d^3 x_{ij} < \infty \,, \tag{2.12}$$

it is known from paper I that the resolvent satisfies an inhomogeneous equation with a Hilbert-Schmidt-kernel. Owing to this, the resolvent can be found with the help of the Fredholm theory of functional equations.

Now consider spin-spin and tensor interactions of the form

$$V_{ij} = (\boldsymbol{s}_i \cdot \boldsymbol{s}_j)\, v_{ij}(\boldsymbol{x}_{ij}) \,, \qquad V_{ij} = (x_{ij})^{-2} (\boldsymbol{s}_i \cdot \boldsymbol{x}_{ij})\, (\boldsymbol{s}_j \cdot \boldsymbol{x}_{ij})\, v_{ij}(\boldsymbol{x}_{ij}) \,,$$

assume that $v_{ij}(\boldsymbol{x}_{ij})$ again satisfies Eq. (2.12), and examine the resolvent equations of paper I. These are now equations on L'^2. Owing to Eq. (2.11), their kernels are Hilbert-Schmidt on L'^2, and so the results of paper I can easily be generalized to spin-spin and tensor interactions.

For spin-orbit interactions

$$V_{ij} = (\boldsymbol{s}_i + \boldsymbol{s}_j) \cdot (\boldsymbol{x}_{ij} \times \boldsymbol{k}_{ij})\, v_{ij}(\boldsymbol{x}_{ij})$$

the methods of paper I do not suffice. This is not due to the spin, however, but to the velocity dependence of the interaction. For the sake of notational convenience we shall therefore discuss in detail the interaction

$$V_{ij} = \boldsymbol{s}_{ij} \cdot (\boldsymbol{x}_{ij} \times \boldsymbol{k}_{ij})\, v_{ij}(\boldsymbol{x}_{ij}) \,, \tag{2.13}$$

where $\boldsymbol{s}_{ij}$ is a vector whose components are real numbers, rather than Hermitian spin operators. The interaction (2.13) gives rise to a problem on L^2. Once this has been solved, the generalization to spin-dependent interactions is straightforward and may be left to the reader.

3. The Hamiltonian

3.1. *The Kinetic Energy*

On L^2, the kinetic energy H_0 is simply the multiplication by $k^2 = \sum_{i=1}^{n-1} k_i^2$. If $\mathfrak{D}(H_0)$ stands for the set of functions $f(\boldsymbol{k}) \in L^2$ such that $k^2 f(\boldsymbol{k})$ is in L^2, then the operator H_0 with domain $\mathfrak{D}(H_0)$ is self-adjoint (KATO [39], ChV, section 5.2). Because H_0 is a non-negative operator, its resolvent $(H_0 - \lambda)^{-1}$ is defined and is a bounded operator for any complex number λ other than $\lambda \geqq 0$. The resolvent is henceforth denoted by $R_0(\lambda)$. It is the operator of multiplication by $(k^2 - \lambda)^{-1}$, so its bound satisfies

$$\|R_0(\lambda)\| = \sup |k^2 - \lambda|^{-1} \leqq |\lambda|^{-1/2} (\operatorname{Im} \lambda^{1/2})^{-1} \,. \tag{3.1}$$

The range of $R_0(\lambda)$ is $\mathfrak{D}(H_0)$.

3.2. The Interaction

In order to define the interaction term V_{ij} on L^2, it is convenient to choose coordinates $\boldsymbol{x}$, such that $\boldsymbol{x}_{ij}$ is denoted by $\boldsymbol{x}_1$. The interaction (2.13) then formally yields

$$\begin{aligned}(V_{ij} f)(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{n-1}) \\ = \boldsymbol{s}_{ij} \cdot \int [\boldsymbol{W}_{ij}(\boldsymbol{k}_1 - \boldsymbol{l}_1) \times \boldsymbol{k}_1] f(\boldsymbol{l}_1, \boldsymbol{k}_2, \ldots, \boldsymbol{k}_{n-1}) \, d^3 l_1\end{aligned} \tag{3.2}$$

where

$$\boldsymbol{W}_{ij}(\boldsymbol{k}) = (2\pi)^{-3} \int e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \boldsymbol{x} v_{ij}(\boldsymbol{x}) \, d^3 x .$$

In order that V_{ij} be invariant under rotations, $v_{ij}(\boldsymbol{x})$ must be a function of x only. That gives

$$\boldsymbol{k} \times \boldsymbol{W}_{ij}(\boldsymbol{k}) = 0 . \tag{3.3}$$

In order that V_{ij} be a symmetric operator, $v_{ij}(x)$ must be real, and so

$$\boldsymbol{W}_{ij}(-\boldsymbol{k}) = \overline{\boldsymbol{W}}_{ij}(\boldsymbol{k}) . \tag{3.4}$$

In the following it is assumed, that the relations (3.4) and (3.5) hold true. In Eq. (3.2), the factor $\boldsymbol{W}_{ij}(\boldsymbol{k}_1 - \boldsymbol{l}_1) \times \boldsymbol{k}_1$ may then be replaced by $\boldsymbol{W}_{ij}(\boldsymbol{k}_1 - \boldsymbol{l}_1) \times \boldsymbol{l}_1$. It is further assumed that $\boldsymbol{W}_{ij}(\boldsymbol{k})$ is measurable, and that the function

$$W_{ij}(\boldsymbol{k}) = |\boldsymbol{W}_{ij}(\boldsymbol{k})|$$

satisfies

$$\int [W_{ij}(\boldsymbol{k})]^{4/3} \, d^3 k < \infty . \tag{3.5}$$

From this it follows that ([40], section 4.1)

$$\int [x v_{ij}(x)]^4 \, d^3 x < \infty .$$

At the origin, the function $v_{ij}(x)$ is thus less singular than $x^{-7/4}$ and at infinity it tends to zero faster than $x^{-7/4}$.

Given the relations (3.3–5), the interaction term V_{ij} is defined as the operator with domain $\mathfrak{D}(H_0)$ which acts according to Eq. (3.2). The total interaction V also has domain $\mathfrak{D}(H_0)$, and is equal to $\sum_{i<j} V_{ij}$. That this is a meaningful definition is ensured by the following lemma.

Lemma 3.1. *For any λ other than $\lambda \geqq 0$, the operator $VR_0(\lambda)$ is bounded. Its bound satisfies*

$$\|VR_0(\lambda)\| \leqq \text{const} \, |\lambda|^{1/4} (\operatorname{Im} \lambda^{1/2})^{-3/4} . \tag{3.6}$$

Proof. For two particles $VR_0(\lambda)$ is an integral operator with kernel

$$\boldsymbol{s}_{12} \cdot [\boldsymbol{W}_{12}(\boldsymbol{k}_1 - \boldsymbol{l}_1) \times \boldsymbol{l}_1] (l_1^2 - \lambda)^{-1} .$$

Its r-norm, which is defined in appendix A2, satisfies

$$\begin{aligned}[r(VR_0(\lambda))]^4 \leqq (s_{12})^4 \int W_{12}(\boldsymbol{k}_1 - \boldsymbol{l}_1) \, W_{12}(\boldsymbol{k}_1' - \boldsymbol{l}_1) \, l_1^2 \, |l_1^2 - \lambda|^{-2} \\ \cdot W_{12}(\boldsymbol{k}_1 - \boldsymbol{l}_1') \, W_{12}(\boldsymbol{k}_1' - \boldsymbol{l}_1') \, l_1'^2 |l_1'^2 - \lambda|^{-2} \, d^3 k_1 \, d^3 k_1' \, d^3 l_1 \, d^3 l_1' .\end{aligned}$$

The substitution

$$\boldsymbol{l} = \boldsymbol{l}_1,\ \boldsymbol{u} = \boldsymbol{k}_1 - \boldsymbol{l}_1, \quad \boldsymbol{v} = \boldsymbol{k}_1' - \boldsymbol{l}_1, \quad \boldsymbol{w} = \boldsymbol{l}_1 - \boldsymbol{l}_1'$$

and Schwarz's inequality give

$$\begin{aligned}[r(VR_0(\lambda))]^4 \leqq (s_{12})^4 \int l^4\, |l^2 - \lambda|^{-4}\, d^3 l \\ \cdot \int W_{12}(\boldsymbol{u})\, W_{12}(\boldsymbol{v})\, W_{12}(\boldsymbol{u}+\boldsymbol{w})\, W_{12}(\boldsymbol{v}+\boldsymbol{w})\, d^3u\, d^3v\, d^3w \\ \leqq \text{const}\, |\lambda|\, (\operatorname{Im}\lambda^{1/2})^{-3}\, \{\int [W_{12}(\boldsymbol{u})]^{4/3}\, d^3u\}^3 .\end{aligned}$$

Hence, by virtue of Eq. (3.5),

$$\|VR_0(\lambda)\| \leqq r(VR_0(\lambda)) \leqq \text{const}\, |\lambda|^{1/4}\, (\operatorname{Im}\lambda^{1/2})^{-3/4} . \tag{3.7}$$

This completes the proof for $n = 2$.

For larger numbers of particles, it is convenient to consider each term V_{ij} separately. If $\boldsymbol{k}'$ stands for $\boldsymbol{k}_2, \ldots, \boldsymbol{k}_{n-1}$,

$$\begin{aligned}(V_{ij}R_0(\lambda)\, f)\, (\boldsymbol{k}_1, \boldsymbol{k}') \\ = \boldsymbol{s}_{ij} \cdot \int [\boldsymbol{W}_{ij}(\boldsymbol{k}_1 - \boldsymbol{l}_1) \times \boldsymbol{l}_1]\, (l_1^2 + k'^2 - \lambda)^{-1}\, f(\boldsymbol{l}_1, \boldsymbol{k}')\, d^3 l_1 .\end{aligned}$$

If this expression is denoted by $g(\boldsymbol{k}, \boldsymbol{k}')$, the proof for $n = 2$ shows that

$$\begin{aligned}\int |g(\boldsymbol{k}_1, \boldsymbol{k}')|^2\, d^3k_1 \\ \leqq \text{const}\, |\lambda - k'^2|^{1/2}\, [\operatorname{Im}(\lambda - k'^2)^{1/2}]^{-3/2} \int |f(\boldsymbol{k}_1, \boldsymbol{k}')|^2\, d^3k_1,\end{aligned}$$

for almost every $\boldsymbol{k}'$. One can now use the inequality

$$|\lambda - k^2|^{1/2}\, [\operatorname{Im}(\lambda - k^2)^{1/2}]^{-3/2} \leqq |\lambda|^{1/2}\, (\operatorname{Im}\lambda^{1/2})^{-3/2} .$$

Integration over $\boldsymbol{k}'$ then shows that Eq. (3.6) holds true generally. This proves the lemma.

Since the range of $R_0(\lambda)$ is $\mathfrak{D}(H_0)$, it follows from lemma 3.1 that Vf is in L^2 whenever f is in $\mathfrak{D}(H_0)$.

Lemma 3.2. *The operator V is symmetric.*

Proof. Consider any particular term V_{ij} and choose functions f and g in $\mathfrak{D}(H_0)$. This yields

$$\begin{aligned}(V_{ij}f, g) = \int \bar{g}(\boldsymbol{k}_1, \boldsymbol{k}')\, d^{3n-6}k'\, d^3k_1 \int \boldsymbol{s}_{ij} \\ \cdot [\boldsymbol{W}_{ij}(\boldsymbol{k}_1 - \boldsymbol{l}_1) \times \boldsymbol{l}_1]\, f(\boldsymbol{l}_1, \boldsymbol{k}')\, d^3 l_1 .\end{aligned} \tag{3.8}$$

Now suppose that the integral on the right converges absolutely. The integrations may then be interchanged, and so it follows with Eq. (3.4) that either side is equal to

$$\begin{aligned}\int f(\boldsymbol{l}_1, \boldsymbol{k}')\, d^{3n-6}k'\, d^3 l_1 \int \boldsymbol{s}_{ij} \\ \cdot [\overline{\boldsymbol{W}}_{ij}(\boldsymbol{l}_1 - \boldsymbol{k}_1) \times \boldsymbol{l}_1]\, \bar{g}(\boldsymbol{k}_1, \boldsymbol{k}')\, d^3k_1 = (f, V_{ij}g) .\end{aligned}$$

This then shows that V_{ij} is symmetric.

In order to justify changing the order of integration, write K_{ij} for the integral operator with kernel

$$W_{ij}(\boldsymbol{k}_1 - \boldsymbol{l}_1)\, l_1 (l_1^2 + 1)^{-1} .$$

Obviously K_{ij} is an operator on $L^2(\boldsymbol{k}_1)$. By the proof of lemma 3.1, it has a finite r-norm on $L^2(\boldsymbol{k}_1)$; let this be denoted by $r_1(K_{ij})$. It then follows from Eq. (A2.11) of appendix A2 that

$$\begin{aligned}\int |g(\boldsymbol{k}_1, \boldsymbol{k}')\, W_{ij}(\boldsymbol{k}_1 - \boldsymbol{l}_1)\, l_1\, f(\boldsymbol{l}_1, \boldsymbol{k}')|\, d^{3n-6} k'\, d^3 k_1\, d^3 l_1 \\ \leqq \|g\|\, \|(H_0 + 1)\, f\|\, r_1(K_{ij})\,.\end{aligned}$$

This shows that the integral in Eq. (3.8) converges absolutely and thus completes the proof of the lemma.

3.3. The Hamiltonian

We now proceed to the major result of the present section, which reads as follows.

Theorem 3.3. *The operator $H = H_0 + V$ with domain $\mathfrak{D}(H_0)$ is self-adjoint.*

Proof. This theorem is closely related to a result due to Kato ([39], Ch.V, sections 5.3 and 4.2). It is proved in much the same way. Since H is symmetric by section 3.1 and lemma 3.2, it suffices to show that there is a real number μ such that the range of $H - \mu$ is as large as L^2 ([41], section 41).

It follows from Eq. (3.6) that

$$\|V R_0(\mu)\| \leqq \text{const.}\, |\mu|^{-1/8} < 1 \tag{3.9}$$

whenever μ is negative and $|\mu|$ is sufficiently large. If this relation is fulfilled, the operator $1 + V R_0(\mu)$ has a bounded inverse, which is given by

$$\sum_{n=0}^{\infty} [-\, V R_0(\mu)]^n\,.$$

This shows that the range of $1 + V R_0(\mu)$ is equal to L^2. Since $(H_0 - \mu)^{-1}$ exists, the range of $H_0 - \mu$ is also equal to L^2. Then it follows from

$$H - \mu = [1 + V R_0(\mu)]\, (H_0 - \mu) \tag{3.10}$$

that the operator $H - \mu$ with domain $\mathfrak{D}(H_0)$ has range L^2. This proves the theorem.

3.4. The Resolvent

Because H is self-adjoint, it has a resolvent $R(\lambda) = (H - \lambda)^{-1}$, which is a bounded operator for $\operatorname{Im}\lambda \neq 0$. In the upper and lower half-planes $(R(\lambda)\, f, g)$ is an analytic function of λ. If λ and μ are any two complex numbers such that $R(\lambda)$ and $R(\mu)$ are bounded, one has

$$\begin{aligned}[R(\lambda)]^* &= R(\bar{\lambda})\,, \\ R(\lambda) - R(\mu) &= (\lambda - \mu)\, R(\lambda)\, R(\mu) \\ dR(\lambda)/d\lambda &= [R(\lambda)]^2\,.\end{aligned} \tag{3.11}$$

Under the present assumptions on the interaction, there is a real number Λ such that the resolvent even exists for real values of λ such that $\lambda < \Lambda$. In other words, the spectrum of H is bounded below. To see this, choose a negative number μ such that Eq. (3.9) holds true. From Eq. (3.10) it then follows that

$$R(\mu) = R_0(\mu)\,[1 + VR_0(\mu)]^{-1}\,, \tag{3.12}$$

and so

$$\|R(\mu)\| \leqq \|R_0(\mu)\|\,[1 - \|VR_0(\mu)\|]^{-1}\,.$$

Hence $R(\mu)$ is a bounded operator. The third relation (3.11) now shows that $R(\lambda)$ is differentiable for $\lambda = \mu$. Since it is already known that $(R(\lambda)\,f, g)$ is analytic in the neighbourhood of the real axis, it follows that this quantity is an analytic function in the λ-plane cut from some point Λ to ∞.

For $\operatorname{Re}\lambda \leqq \Lambda$, it was shown in [18], Eq. (1.4.34) that

$$\|R(\lambda)\| \leqq |\lambda - \Lambda|^{-1}\,.$$

Together with the general relation

$$\|R(\lambda)\| \leqq |\operatorname{Im}\lambda|^{-1}$$

this yields

$$\|R(\lambda)\| \leqq |\lambda - \Lambda|^{-1/2}\,[\operatorname{Im}(\lambda - \Lambda)^{1/2}]^{-1}\,. \tag{3.13}$$

Now consider the operator $H_0R(\lambda)$. By virtue of Eq. (3.12), the operator $H_0R(\mu)$ is bounded if μ is near $-\infty$. For a suitable fixed value of μ, the relation

$$H_0R(\lambda) = H_0R(\mu) + (\lambda - \mu)\,H_0R(\mu)\,R(\lambda)$$

together with Eq. (3.13) thus yields

$$\|H_0R(\lambda)\| \leqq \text{const}\,(|\lambda - \Lambda|^{-1/2} + |\lambda - \Lambda|^{1/2})\,[\operatorname{Im}(\lambda - \Lambda)^{1/2}]^{-1}\,. \tag{3.14}$$

There is an inequality of the same form for $VR(\lambda)$.

Since H_0 is non-negative, there is a uniquely defined non-negative operator $(H_0)^{1/2}$. Because

$$\|(H_0)^{1/2}\,R(\lambda)\,f\|^2 = (R(\lambda)\,f, H_0R(\lambda)\,f)$$

for any $f \in L^2$,

$$\begin{aligned}\|(H_0)^{1/2}\,R(\lambda)\| &\leqq \|R(\lambda)\|^{1/2}\,\|H_0R(\lambda)\|^{1/2}\\ &\leqq \text{const}\,(|\lambda - \Lambda|^{-1/2} + 1)\,[\operatorname{Im}(\lambda - \Lambda)^{1/2}]^{-1}\,.\end{aligned} \tag{3.15}$$

In the following, there often occur operators of the form $R(\lambda)\,V$ or $R(\lambda)\,(H_0)^{1/2}$. These are certainly defined on the dense subset $\mathfrak{D}(H_0)$ of L^2. They can uniquely be extended to L^2 by the definitions

$$R(\lambda)\,V = [VR(\bar{\lambda})]^*\,, \qquad R(\lambda)\,(H_0)^{1/2} = [(H_0)^{1/2}\,R(\bar{\lambda})]^*\,. \tag{3.16}$$

It is henceforth assumed that this extension has been made. Because they are the adjoints of bounded operators, the operators under discussion are bounded themselves. Their bounds satisfy inequalities like Eqs. (3.14) and (3.15).

4. The Resolvent Equation

The resolvent satisfies the well-known equation

$$R(\lambda) = R_0(\lambda) - R_0(\lambda)\, V R(\lambda)\,. \tag{4.1}$$

For two particles, this is a useful equation, because the kernel $-R_0(\lambda)\, V$ is a compact operator. For larger numbers of particles, however, more powerful equations are required. These were derived in paper I, and, independently, by WEINBERG [16] and HUNZIKER [17]. The present section is devoted to a summary of some results that are needed in the following.

We first explain the notation, which is almost the same as in paper I. Let the system of n particles be divided into k groups ($1 \leqq k \leqq n$), and let $p(k)$ denote the particular division in question. Consider the case that all particles in any particular group interact, but that particles belonging to different groups do not. The total interaction which then remains is denoted by $V_{p(k)}$, the corresponding resolvent is

$$R^{(n)}_{p(k)}(\lambda) = (H_0 + V_{p(k)} - \lambda)^{-1}\,. \tag{4.2}$$

For the special cases $k = 1$ and $k = n$, we also write

$$R^{(n)} = R^{(n)}_{p(1)},\ R^{(n)}_0 = R^{(n)}_{p(n)}\,, \qquad V = V_{p(1)}\,. \tag{4.3}$$

The resolvents $R^{(n)}_{p(k)}$, with $k \geqq 2$, can be expressed in terms of the resolvents $R^{(m)}$, with $2 \leqq m \leqq n-1$, as follows. Let the groups of the division $p(k)$ consist of $n_1, \ldots, n_k$ particles, with $n_1, \ldots, n_j \geqq 2$ and $n_{j+1} = \cdots = n_k = 1$. Then

$$R^{(n)}_{p(k)} = R^{(n_1)} * R^{(n_2)} * \cdots * R^{(n_j)} * R^{(k)}_0\,. \tag{4.4}$$

Here the resolvent $R^{(n_i)}$ refers to the internal motion in group i, and $R^{(k)}_0$ refers to the internal motion of the centres of mass of the k groups with respect to each other. The operation $*$ is defined by

$$([R_a * R_b](\lambda)\, f, g) = (2\pi i)^{-1} \int_C (R_a(\sigma)\, R_b(\lambda - \sigma)\, f, g)\, d\sigma \tag{4.5}$$

for every f and g in L^2. Here C is a contour in the σ-plane such that the singularities of $R_a(\sigma)$ are on the right of C, and the singularities of $R_b(\lambda - \sigma)$ on the left of C.

In the case of the present paper, the resolvents $R_a(\sigma)$ and $R_b(\lambda - \sigma)$ are analytic for any λ, σ other than $\sigma \geqq \Lambda_a$ and $\lambda - \sigma \geqq \Lambda_b$. Owing to this, the integral is defined and is analytic for any λ other than $\lambda \geqq \Lambda_a + \Lambda_b$. The contour C is conveniently chosen as

$$\sigma = \frac{1}{2}(\lambda - \Lambda_b + \Lambda_a) + s \exp\left[\frac{1}{2} i \arg(\lambda - \Lambda_b - \Lambda_a)\right] \quad (-\infty < s < \infty)\,, \tag{4.6}$$

but it may be modified in various ways. It is easily seen that the convolution product $*$ is associative and commutative.

Now let the symbol $p(k) \subset p(k-j)$ express that the division $p(k)$ is obtained from $p(k-j)$ by further splitting some groups of the latter division. The resolvent equation for n particles then reads

$$\begin{aligned}\sum_{k=1}^{n} (-1)^{k-1} (k-1)! \sum_{p(k)} R^{(n)}_{p(k)} = (-1)^{n-1} &\sum_{\substack{p(n-1)\ldots p(2)\\ p(n-1)\subset\cdots\subset p(2)}} \cdots \sum \\ \cdot \{R^{(n)}_{p(n)} V_{p(n-1)} R^{(n)}_{p(n-1)} [V_{p(n-2)} - V_{p(n-1)}] & \\ \cdot R^{(n)}_{p(n-2)} \ldots R^{(n)}_{p(2)} [V - V_{p(2)}] R^{(n)}_{p(1)}\} \,. &\end{aligned} \tag{4.7}$$

In an obvious notation, Eq. (4.7) is of the form

$$R^{(n)} = Q^{(n)} + K^{(n)} R^{(n)} \,, \tag{4.8}$$

where

$$K^{(n)} = \sum_{p(2)} K^{(n)}_{p(2)} \,. \tag{4.9}$$

For $n = 2$, Eq. (4.7) is identical with Eq. (4.1). For further reference we define

$$F^{(n)} = K^{(n)} R^{(n)} = R^{(n)} - Q^{(n)} \,. \tag{4.10}$$

The relation (4.7) can be generalized as follows. Choose any particular division $p(l)$. A division $p(k) \subset p(l)$ gives rise to l numbers $k_1 \ldots k_l$, where k_j is the number of subgroups into which group j of $p(l)$ is divided by $p(k)$. For fixed $p(l)$, one has

$$\begin{aligned}&\sum_{k=l}^{n} \sum_{p(k)\subset p(l)} (-1)^{k-1} (k_1 - 1)! \ldots (k_l - 1)! \, R^{(n)}_{p(k)} \\ &= (-1)^{n-1} \sum_{\substack{p(n-1)\ldots p(l+1)\\ p(n-1)\subset\cdots\subset p(l+1)\subset p(l)}} \cdots \sum R^{(n)}_{p(n)} V_{p(n-1)} \\ &\quad \cdot R^{(n)}_{p(n-1)} \ldots R^{(n)}_{p(l+1)} [V_{p(l)} - V_{p(l+1)}] R^{(n)}_{p(l)} \,.\end{aligned} \tag{4.11}$$

For $n = l$ this equation is obvious and for $n = l + 1$ it is of the form (4.1). Once it has been proved for any particular pair n, l, it can be proved for $n + 1, l$ in the same way as Eq. (4.7) ([18], section 1.7.1). It thus holds true for every pair n, l such that $n \geqq l$.

Now choose $l = 2$, and suppose that the division $p(2)$ yields groups consisting of n_1 and n_2 particles such that $n_1 \geqq 2$ and $n_2 \geqq 2$. Compare the right-hand side of Eq. (4.11) with the kernel $K^{(n)}_{p(2)}$. Use Eq. (4.4) on the left of Eq. (4.11) and compare this with the equations for $R^{(n_1)}$ and $R^{(n_2)}$. With Eq. (4.10) this yields the important result

$$K^{(n)}_{p(2)} = - F^{(n_1)} * F^{(n_2)} * R^{(2)}_0 [V - V_{p(2)}] \,. \tag{4.12}$$

We wish to emphasize that this formula differs slightly from [18], Eq. (1.7.48), as may be seen by taking $n = 4$, $n_1 = n_2 = 2$. The proof in [18] is not correct, because use has been made of equations like [18], Eq. (1.7.39), which is false. The relation [18], (1.7.33) is correct.

For divisions $p(2)$ such that $n_1 = 1$ and $n_2 = n - 1$, the analogue of Eq. (4.12) reads

$$K^{(n)}_{p(2)} = - F^{(n-1)} * R_0^{(2)} [V - V_{p(2)}] \,. \tag{4.13}$$

The kernel $K^{(2)}(\lambda) = - R_0^{(2)}(\lambda)\, V$ and the inhomogeneous part $Q^{(2)}(\lambda) = R_0^{(2)}(\lambda)$ of the two-particle equation are defined for any λ other than $\lambda \geqq 0$. It is convenient to write $\Lambda_0^{(2)} = 0$. In solving the equation (section 6.2), one finds a number $\Lambda^{(2)} \leqq \Lambda_0^{(2)}$ such that the resolvent $R^{(2)}(\lambda)$ is analytic for any λ other than $\lambda \geqq \Lambda^{(2)}$.

Now suppose that the equations for $R^{(m)}(\lambda)$ $(m = 2, \ldots, n-1)$ have successively been solved, so that $R^{(m)}(\lambda)$ has been found for any λ other than $\lambda \geqq \Lambda^{(m)}$. With the help of Eqs. (4.4), (4.12) and (4.13), $K^{(n)}(\lambda)$ and $Q^{(n)}(\lambda)$ can be constructed for any λ other than $\lambda \geqq \Lambda_0^{(n)}$, where

$$\Lambda_0^{(n)} = \min_{n_1 + n_2 = n;\, n_1, n_2 \geqq 2} \{\Lambda^{(n-1)}, \Lambda^{(n_1)} + \Lambda^{(n_2)}\} \qquad (n \geqq 3)\,.$$

Again a number $\Lambda^{(n)} \leqq \Lambda_0^{(n)}$ is found, such that $R^{(n)}(\lambda)$ is analytic for any λ other than $\lambda \geqq \Lambda^{(n)}$. In section 6.2, Eq. (6.11), the numbers $\Lambda^{(n)}$ will be defined more precisely. In section 7 it will be proved that $\Lambda^{(n)}$, thus defined, is the lower bound of the spectrum of $H^{(n)}$.

The relations between the numbers Λ may be summarized as

$$\begin{aligned} &\Lambda^{(1)} = 0\,, \\ &\Lambda_0^{(n)} = \min_{n_1 + n_2 = n;\, n_1, n_2 \geqq 1} \{\Lambda^{(n_1)} + \Lambda^{(n_2)}\} \qquad (n \geqq 2)\,, \\ &\Lambda^{(n)} \leqq \Lambda_0^{(n)} \qquad (n \geqq 2)\,. \end{aligned} \tag{4.14}$$

5. The Kernel

5.1. The Relation Ω

In the present section it is shown that the kernel $K^{(n)}(\lambda)$ of Eq. (4.8) is sufficiently well-behaved for this equation to be soluble. In order that the argument may be presented as consisely as possible, we first define the relation Ω. If $z(\lambda)$ is any non-negative function of λ, the expression

$$z(\lambda) = \Omega(p, \Lambda) \tag{5.1}$$

means that $z(\lambda)$ satisfies an inequality of the form

$$0 \leqq z(\lambda) \leqq \sum_{q=1}^{Q} \gamma_q |\lambda - \Lambda|^{\alpha_q} [\mathrm{Im}(\lambda - \Lambda)^{1/2}]^{-\beta_q}\,,$$

where Q is some integer, $\gamma_q > 0$, $\alpha_q \geqq 0$, $\beta_q > 0$, and

$$\max_q \left(\alpha_q - \frac{1}{2}\beta_q\right) \leqq p < 0\,.$$

The relation Ω has the following properties.

Lemma 5.1.1. *If $z(\lambda)$ satisfies Eq.* (5.1), *it follows that*

$$z(\lambda) = \Omega(p_1, \Lambda_1)$$

for every $p_1 \geqq p$ and every $\Lambda_1 \leqq \Lambda$.

This lemma is a consequence of the inequalities

$$[\operatorname{Im}(\lambda - \Lambda)^{1/2}]^{-1} \leqq [\operatorname{Im}(\lambda - \Lambda_1)^{1/2}]^{-1} ,$$

$$|\lambda - \Lambda|^{1/2} [\operatorname{Im}(\lambda - \Lambda)^{1/2}]^{-1} \leqq |\lambda - \Lambda_1|^{1/2} [\operatorname{Im}(\lambda - \Lambda_1)^{1/2}]^{-1} .$$

Lemma 5.1.2. *From*

$$z_1(\lambda) = \Omega(p_1, \Lambda) , \qquad z_2(\lambda) = \Omega(p_2, \Lambda)$$

it follows that

$$z_1(\lambda)\, z_2(\lambda) = \Omega(p_1 + p_2, \Lambda) .$$

Lemma 5.1.3. *Suppose that*

$$z_1(\lambda) = \Omega(p_1, \Lambda_1) , \qquad z_2(\lambda) = \Omega(p_2, \Lambda_2) ,$$

where $p_1 + p_2 < -1$. Define

$$(z_1 * z_2)(\lambda) = (2\pi)^{-1} \int_{C_0} z_1(\sigma)\, z_2(\lambda - \sigma)\, |d\sigma| ,$$

where C_0 is given by

$$\sigma = \frac{1}{2}(\lambda - \Lambda_2 + \Lambda_1) + s \exp\left[\frac{1}{2} i \arg(\lambda - \Lambda_2 - \Lambda_1)\right] \qquad (-\infty < s < \infty) .$$

Then

$$(z_1 * z_2)(\lambda) = \Omega(p_1 + p_2 + 1, \Lambda_1 + \Lambda_2) .$$

This lemma is proved in appendix A4.

5.2. The Kernel

It can now be shown that the operator $K^{(n)}(\lambda)$ belongs to the class (rc) of $L^2(R^{3n-3})$. The proof is based on a number of lemmas.

Lemma 5.2. *The operator $K^{(2)}$ is an integral operator such that*

$$r(K^{(2)}(\lambda)) = \Omega\left(-\frac{1}{8}, 0\right) .$$

Proof. Since

$$K^{(2)}(\lambda) = - R_0^{(2)}(\lambda)\, V = - [V R_0^{(2)}(\bar{\lambda})]^* ,$$

it follows from Eq. (3.7) that

$$r(K^{(2)}(\lambda)) \leqq \text{const}\, |\lambda|^{1/4} (\operatorname{Im} \lambda^{1/2})^{-3/4} . \tag{5.2}$$

Lemma 5.3. *Let $F^{(m)}$ and $F^{(m)}(H_0)^{1/2}$ be integral operators such that*

$$r(F^{(m)}(\lambda)) = \Omega\left(-\frac{1}{8} m - \frac{7}{8}, \Lambda^{(m)}\right) \qquad (2 \leqq m \leqq n-1) , \tag{5.3}$$

$$r(F^{(m)}(\lambda)\,(H_0)^{1/2}) = \Omega\left(-\frac{1}{8} m - \frac{3}{8}, \Lambda^{(m)}\right) \qquad (2 \leqq m \leqq n-1) . \tag{5.4}$$

Then $K^{(n)}$ *is an integral operator such that*

$$r(K^{(n)}(\lambda)) = \Omega\left(-\frac{1}{8}n + \frac{1}{8}, \Lambda_0^{(n)}\right). \tag{5.5}$$

This relation is compatible with lemma 5.2.

Proof. According to Eq. (4.9), $K^{(n)}$ is a sum of terms $K_{p(2)}^{(n)}$. It is convenient to discuss these separately. To start with, consider a division $p(2)$ such that $n_1 = 1$ and $n_2 = n - 1$. Suppose that it is particle n which forms a group by itself. Choose internal coordinates $\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{n-2}$ among the particles $1, \ldots, n-1$, and introduce a coordinate $\boldsymbol{k}_{n-1}$ to describe the motion of particle n relative to the centre of mass of the large group. Now select a particle j in the large group, and choose coordinates $\boldsymbol{k}'_1, \ldots, \boldsymbol{k}'_{n-1}$ in such a way that

$$\boldsymbol{k}'_1 = (2m_j m_n / m_j + m_n)^{1/2}\left(\frac{\boldsymbol{K}_j}{2m_j} - \frac{\boldsymbol{K}_n}{2m_n}\right).$$

Make sure that the transformation from the unprimed to the primed coordinates is an orthogonal one. Write

$$\boldsymbol{k}'_1 = \sum_{i=1}^{n-1} c_i \boldsymbol{k}_i .$$

Then it is obvious that $c_{n-1} \neq 0$. If f is any function in L^2, define f' by

$$f'(\boldsymbol{k}'_1, \ldots, \boldsymbol{k}'_{n-1}) = f(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{n-1}).$$

Then

$$f'(\boldsymbol{k}'_1 - \boldsymbol{m}, \boldsymbol{k}'_2, \ldots, \boldsymbol{k}'_{n-1}) = f(\boldsymbol{k}_1 - c_1 \boldsymbol{m}, \ldots, \boldsymbol{k}_{n-1} - c_{n-1}\boldsymbol{m}).$$

According to Eq. (3.2), the interaction term V_{jn} is defined by

$$(V_{jn} f')(\boldsymbol{k}'_1, \ldots, \boldsymbol{k}'_{n-1}) \\ = - \boldsymbol{s}_{jn} \cdot \int [\boldsymbol{k}'_1 \times \boldsymbol{W}_{jn}(\boldsymbol{m})]\, f'(\boldsymbol{k}'_1 - \boldsymbol{m}, \boldsymbol{k}'_2, \ldots, \boldsymbol{k}'_{n-1})\, d^3 m,$$

and so

$$(V_{jn} f)(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{n-1}) \\ = - \boldsymbol{s}_{jn} \cdot \int \left[\left(\sum_{h=1}^{n-1} c_h \boldsymbol{k}_h\right) \times \boldsymbol{W}_{jn}(\boldsymbol{m})\right] \\ \cdot f(\boldsymbol{k}_1 - c_1 \boldsymbol{m}, \ldots, \boldsymbol{k}_{n-1} - c_{n-1}\boldsymbol{m})\, d^3 m. \tag{5.6}$$

If $\boldsymbol{k}^{(1)}$ stands for $\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{n-2}$, the operator $F^{(n-1)}(\lambda)$ is an integral operator with kernel $F^{(n-1)}(\boldsymbol{k}^{(1)}, \boldsymbol{l}^{(1)}; \lambda)$. The operator

$$F^{(n-1)}(\sigma)\, R_0^{(2)}(\lambda - \sigma)\, V_{jn}$$

is therefore an integral operator with kernel

$$- |c_{n-1}|^{-3} F^{(n-1)}\left(\boldsymbol{k}^{(1)}, \boldsymbol{l}^{(1)} + \frac{c^{(1)}}{c_{n-1}}(\boldsymbol{k}_{n-1} - \boldsymbol{l}_{n-1}); \sigma\right)(k_{n-1}^2 - \lambda + \sigma)^{-1} \boldsymbol{s}_{jn} \\ \cdot \left\{\sum_{h=1}^{n-2} c_h \left[\boldsymbol{l}_h + \frac{c_h}{c_{n-1}}(\boldsymbol{k}_{n-1} - \boldsymbol{l}_{n-1})\right] + c_{n-1}\boldsymbol{k}_{n-1}\right\} \times \boldsymbol{W}_{jn}\left(\frac{\boldsymbol{k}_{n-1} - \boldsymbol{l}_{n-1}}{c_{n-1}}\right). \tag{5.7}$$

Its r-norm is not larger than the sum of the r-norms of the two operators with kernels

$$|c_{n-1}|^{-3} F^{(n-1)}(\boldsymbol{k}^{(1)}, \boldsymbol{l}^{(1)}; \sigma)\,(k_{n-1}^2 - \lambda + \sigma)^{-1} \cdot s_{jn}\left(\sum_{h=1}^{n-2} |c_h|\, l_h\right) W_{jn}\left(\frac{\boldsymbol{k}_{n-1} - \boldsymbol{l}_{n-1}}{c_{n-1}}\right)$$

and (5.8)

$$|c_{n-1}|^{-3} F^{(n-1)}(\boldsymbol{k}^{(1)}, \boldsymbol{l}^{(1)}; \sigma)\,(k_{n-1}^2 - \lambda + \sigma)^{-1} \cdot s_{jn}\,|c_{n-1}|\, k_{n-1}\, W_{jn}\left(\frac{\boldsymbol{k}_{n-1} - \boldsymbol{l}_{n-1}}{c_{n-1}}\right).$$

This transition is a consequence of the definition of the r-norm. To justify it, one needs Schwarz's inequality and the relation

$$|\boldsymbol{s}\cdot(\boldsymbol{k}\times\boldsymbol{W})| \leqq s k W\,.$$

Now write $L_{jn}(\lambda - \sigma)$ for the operator with kernel

$$(k_{n-1}^2 - \lambda + \sigma)^{-1}\, W_{jn}\left(\frac{\boldsymbol{k}_{n-1} - \boldsymbol{l}_{n-1}}{c_{n-1}}\right).$$

The two operators (5.8) each correspond to an operator on $L^2(\boldsymbol{k}^{(1)})$ times one on $L^2(\boldsymbol{k}_{n-1})$. Write r_1 and r_3 for the r-norms on $L^2(\boldsymbol{k}^{(1)})$ and $L^2(\boldsymbol{k}_{n-1})$ respectively. This gives

$$\begin{aligned} &r(F^{(n-1)}(\sigma)\, R_0^{(2)}(\lambda - \sigma)\, V_{jn}) \\ &\leqq \text{const}\,[r_1(F^{(n-1)}(\sigma)\,(H_0)^{1/2})\, r_3(L_{jn}(\lambda-\sigma)) \\ &\quad + r_1(F^{(n-1)}(\sigma))\, r_3((H_0)^{1/2} L_{jn}(\lambda - \sigma))]\,. \end{aligned} \tag{5.9}$$

Here $(H_0)^{1/2}$ is an operator on $L^2(\boldsymbol{k}^{(1)})$ or on $L^2(\boldsymbol{k}_{n-1})$, as the case may be. If W_{jn} satisfies Eq. (3.5), it follows from the proof of lemma 3.1 that

$$\begin{aligned} r_3((H_0)^{1/2} L_{jn}(\lambda)) &\leqq \text{const}\,|\lambda|^{1/4}\ \ (\operatorname{Im}\lambda^{1/2})^{-3/4} = \Omega\left(-\frac{1}{8}, 0\right), \\ r_3(L_{jn}(\lambda)) &\leqq \text{const}\,|\lambda|^{-1/4}\,(\operatorname{Im}\lambda^{1/2})^{-3/4} = \Omega\left(-\frac{5}{8}, 0\right). \end{aligned} \tag{5.10}$$

With Eqs. (5.3) and (5.4), appendix A3 and lemma 5.1.3, the relation (5.9) thus yields

$$\begin{aligned} &r((F^{(n-1)} * R_0^{(2)}\, V_{jn})\,(\lambda)) \\ &\leqq (2\pi)^{-1} \int_{C_0} r(F^{(n-1)}(\sigma)\, R_0^{(2)}(\lambda-\sigma)\, V_{jn})\,|d\sigma| = \Omega\left(-\frac{1}{8} n + \frac{1}{8}, \Lambda^{(n-1)}\right). \end{aligned}$$

Since a result of this form applies to every term V_{jn} of the interaction $V - V_{p(2)}$, it follows with Eq. (4.13) that $K_{p(2)}^{(n)}$ is an integral operator such that

$$r(K_{p(2)}^{(n)}(\lambda)) = \Omega\left(-\frac{1}{8} n + \frac{1}{8}, \Lambda^{(n-1)}\right). \tag{5.11}$$

The terms $K^{(n)}_{p(2)}$ with $n_1 \geqq 2$, $n_2 \geqq 2$ can be discussed in the same way. The analogue of Eq. (5.9) is

$$\begin{aligned} &r(F^{(n_1)}(\tau)\, F^{(n_2)}(\sigma-\tau)\, R_0^{(2)}(\lambda-\sigma)\, V_{ij}) \\ &\leqq \text{const}\,[r_1(F^{(n_1)}(\tau)\,(H_0)^{1/2})\, r_2(F^{(n_2)}(\sigma-\tau))\, r_3(L_{ij}(\lambda-\sigma)) \\ &\quad + r_1(F^{(n_1)}(\tau))\, r_2(F^{(n_2)}(\sigma-\tau)\,(H_0)^{1/2})\, r_3(L_{ij}(\lambda-\sigma)) \\ &\quad + r_1(F^{(n_1)}(\tau))\, r_2(F^{(n_2)}(\sigma-\tau))\, r_3((H_0)^{1/2}\, L_{ij}(\lambda-\sigma))]\,. \end{aligned} \tag{5.12}$$

From this it follows that

$$r(K^{(n)}_{p(2)}(\lambda)) = \Omega\left(-\frac{1}{8}n + \frac{1}{8}, \Lambda^{(n_1)} + \Lambda^{(n_2)}\right). \tag{5.13}$$

If this result is combined with Eq. (5.11), lemma 5.1.1 gives the desired relation (5.5). This completes the proof of lemma 5.3.

Lemma 5.4. *Let $F^{(m)}$ and $K^{(n)}$ be integral operators such that*

$$r(F^{(m)}(\lambda)) = \Omega\left(-\frac{1}{8}m - \frac{7}{8}, \Lambda^{(m)}\right) \quad (2 \leqq m \leqq n-1)\,,$$

$$r(K^{(n)}(\lambda)) = \Omega\left(-\frac{1}{8}n + \frac{1}{8}, \Lambda_0^{(n)}\right).$$

Then $F^{(n)}$ is an integral operator such that

$$r(F^{(n)}(\lambda)) = \Omega\left(-\frac{1}{8}n - \frac{7}{8}, \Lambda^{(n)}\right). \tag{5.14}$$

The operator $F^{(2)}$ is an integral operator satisfying Eq. (5.14) *with $n = 2$.*

Proof. Suppose that $n \geqq 3$. By repeated application of resolvent Eq. (4.7) for small numbers of particles, the term $Q^{(n)}$ in the resolvent equation can be written in such a form that resolvents other than R_0 only occur in the combination $F^{(n_i)} = K^{(n_i)} R^{(n_i)}$. The term $Q^{(n)}$ can thus be written as

$$Q^{(n)} = R_0^{(n)} + \tilde{Q}^{(n)}\,,$$

where $\tilde{Q}^{(n)}$ consists of a finite number of terms $\tilde{Q}_p^{(n)}$ of the form

$$\begin{aligned} &\tilde{Q}_p^{(n)} = F^{(n_1)} * F^{(n_2)} * \cdots * F^{(n_j)} * R_0^{(p)}\,, \\ &\sum_{i=1}^{j} (n_i - 1) + (p-1) = n-1\,. \end{aligned} \tag{5.15}$$

This yields

$$F^{(n)} = K^{(n)} R^{(n)} = K^{(n)} R_0^{(n)} + K^{(n)} \tilde{Q}^{(n)} + [K^{(n)}]^2 R^{(n)}\,. \tag{5.16}$$

To prove the lemma, we now consider each term on the right separately.

Term 1. On $L^2(\boldsymbol{k})$, the resolvent $R_0^{(n)}(\lambda)$ is the operator of multiplication by $(k^2 - \lambda)^{-1}$. Hence, if $K^{(n)}$ is an integral operator, so is $K^{(n)} R_0^{(n)}$. By the definition of the r-norm

$$r(K^{(n)}(\lambda)\, R_0^{(n)}(\lambda)) \leqq r(K^{(n)}(\lambda)) \sup |k^2 - \lambda|^{-1}\,.$$

Because

$$\sup |k^2 - \lambda|^{-1} = \Omega(-1, 0)\,,$$

it follows with lemmas 5.1.1 and 5.1.2 that

$$r(K^{(n)}(\lambda)\,R_0^{(n)}(\lambda)) = \Omega\left(-\frac{1}{8}n - \frac{7}{8}, \Lambda_0^{(n)}\right). \tag{5.17}$$

Term 2. Owing to Eq. (5.15), the term $K^{(n)}\tilde{Q}^{(n)}$ is a sum of terms $K^{(n)}\tilde{Q}_p^{(n)}$ of the form

$$(2\pi i)^{-j}\int K^{(n)}(\lambda)\,F^{(n_1)}(\sigma_1)\,F^{(n_2)}(\sigma_2-\sigma_1)\ldots F^{(n_j)}(\sigma_j-\sigma_{j-1})$$
$$\cdot\, R_0^{(p)}(\lambda-\sigma_j)\,d\sigma_1\ldots d\sigma_j\,.$$

The integrand is an integral operator in the class (rc) of $L^2(R^{3n-3})$, by the data of the present lemma and appendix A2, lemma A2.

By repeated application of appendix A3 and lemma 5.1.3 it follows that $K^{(n)}\tilde{Q}_p^{(n)}$ is an integral operator such that

$$r(K^{(n)}(\lambda)\,\tilde{Q}_p^{(n)}(\lambda)) = \Omega\left(-\frac{1}{4}n + \frac{1}{8}p - \frac{7}{8}, \Lambda_0^{(n)}\right).$$

Since p does not exceed $n-1$, it follows with lemma 5.1.1 that

$$r(K^{(n)}(\lambda)\,\tilde{Q}^{(n)}(\lambda)) = \Omega\left(-\frac{1}{8}n - 1, \Lambda_0^{(n)}\right). \tag{5.18}$$

It may be remarked here, that the integral kernels of the operators $K^{(n)}(\lambda)\,\tilde{Q}_p^{(n)}(\lambda)$ can be found with the procedures given in appendix A2, lemma A2 and appendix A3.

Term 3. This term is most easily discussed in terms of the Schmidt-class (appendix A1 and A2). Since $K^{(n)}$ is in the class (rc) by assumption, and $R^{(n)}$ is a bounded operator, it follows that $[K^{(n)}]^2 R^{(n)}$ is in the Schmidt-class. This operator is therefore an integral operator. Owing to Eqs. (A1.18) and (A2.10)

$$\begin{aligned} r([K^{(n)}(\lambda)]^2 R^{(n)}(\lambda)) &\leqq \sigma([K^{(n)}(\lambda)]^2 R^{(n)}(\lambda)) \\ &\leqq \sigma([K^{(n)}(\lambda)]^2)\,\|R^{(n)}(\lambda)\| \\ &\leqq [r(K^{(n)}(\lambda))]^2\,\|R^{(n)}(\lambda)\|\,. \end{aligned}$$

Since

$$\|R^{(n)}(\lambda)\| = \Omega(-1, \Lambda^{(n)})$$

by Eq. (3.13), it follows that

$$r([K^{(n)}(\lambda)]^2 R^{(n)}(\lambda)) = \Omega\left(-\frac{1}{4}n - \frac{3}{4}, \Lambda^{(n)}\right). \tag{5.19}$$

The required result (5.14) now follows from Eqs. (5.17–19) and lemma 5.1.1. This proves the assertion for $n \geqq 3$.

If $n = 2$, the term $\tilde{Q}^{(n)}$ is absent. The assertion on $F^{(2)}$ can be justified by the arguments for term 1 and term 3. This completes the proof of lemma 5.4.

Lemma 5.5. *Let $F^{(m)}$, $F^{(m)}(H_0)^{1/2}$ and $K^{(n)}$ be integral operators such that*

$$r(F^{(m)}(\lambda)) = \Omega\left(-\frac{1}{8}m - \frac{7}{8}, \Lambda^{(m)}\right) \qquad (2 \leqq m \leqq n-1),$$

$$r(F^{(m)}(\lambda)(H_0)^{1/2}) = \Omega\left(-\frac{1}{8}m - \frac{3}{8}, \Lambda^{(m)}\right) \qquad (2 \leqq m \leqq n-1),$$

$$r(K^{(n)}(\lambda)) = \Omega\left(-\frac{1}{8}n + \frac{1}{8}, \Lambda_0^{(n)}\right).$$

Then $F^{(n)}(H_0)^{1/2}$ is an integral operator such that

$$r(F^{(n)}(\lambda)(H_0)^{1/2}) = \Omega\left(-\frac{1}{8}n - \frac{3}{8}, \Lambda^{(n)}\right). \tag{5.20}$$

The operator $F^{(2)}(H_0)^{1/2}$ is an integral operator satisfying Eq. (5.20) *with $n = 2$.*

Proof. This lemma can be proved in much the same way as lemma 5.4, the only difference being that one requires the relations

$$\sup |k(k^2 - \lambda)^{-1}| = \Omega\left(-\frac{1}{2}, 0\right),$$

$$\|R^{(n)}(\lambda)(H_0)^{1/2}\| = \Omega\left(-\frac{1}{2}, \Lambda^{(n)}\right).$$

The first of these is obvious, the second follows from Eq. (3.15).

Theorem 5.6. *For every $n \geqq 2$, the operators $K^{(n)}$, $F^{(n)}$ and $F^{(n)}(H_0)^{1/2}$ are integral operators satisfying Eqs.* (5.5), (5.14) *and* (5.20).

Proof. For $n = 2$, the assertion follows from lemmas 5.2, 5.4 and 5.5. Now suppose that it has been proved for $n = 2, \ldots, N-1$. Then for $n = N$ it follows from lemmas 5.3, 5.4 and 5.5. The theorem can thus be proved by complete induction.

Because the operator $K^{(n)}(\lambda)$ is in the class (rc), it is also in the class (ϱc) discussed in appendix A1. Given $Q^{(n)}(\lambda)$, this makes it possible to solve the resolvent equation explicitly. The method of solution is explained in section 6. In the cause of the argument, the following result is used.

Theorem 5.7. *For $n \geqq 2$, the ϱ-norm of $K^{(n)}(\lambda)$ satisfies*

$$\varrho(K^{(n)}(\lambda)) \leqq \text{const}\left(|\lambda - \Lambda_0^{(n)}|^{\frac{1}{4}n - \frac{1}{4}} + |\lambda - \Lambda_0^{(n)}|^{\frac{3}{4}n - \frac{5}{4}}\right) \cdot [\operatorname{Im}(\lambda - \Lambda_0^{(n)})^{1/2}]^{-\frac{7}{4}n + \frac{11}{4}}, \tag{5.21}$$

the constant on the right depending on n.

Proof. Because $\varrho(K) \leqq r(K)$, lemma 5.2 yields the inequality (5.21) for $n = 2$.

Now suppose that Eq. (5.21) has been proved for $n = 2, \ldots, N-1$. Consider the integrals by which one will evaluate the ϱ-norms of the operators $F^{(n-1)}(\sigma)\, R_0^{(2)}(\lambda - \sigma)\, V_{jn}$ and $F^{(n_1)}(\tau)\, F^{(n_2)}(\sigma - \tau)\, R_0^{(2)}(\lambda - \sigma)\, V_{ij}$ [appendix A2, Eq. (A2.9)].

According to theorem 5.6, the relations (5.3) and (5.4) are true. Hence all the statements in the proof of lemma 5.3 are correct. In particular, the two operators under discussion have finite r-norms, and so the integrals for their ϱ-norms converge absolutely. Owing to this, the order of integration may be changed. With Schwarz's inequality, this makes it possible to derive inequalities of the forms (5.9) and (5.12), with r everywhere replaced by ϱ.

Now the ϱ-norm, unlike the r-norm, satisfies the useful relations

$$\varrho(KR) \leqq \varrho(K)\,\|R\|\,, \qquad \varrho(KR(H_0)^{1/2}) \leqq \varrho(K)\,\|R(H_0)^{1/2}\|\,. \tag{5.22}$$

Given Eq. (5.21) for $n = 2, \ldots, N-1$ and Eqs. (3.13) and (3.15), the relations (5.22) enable bounds to be derived for $\varrho(F^{(n)})$ and $\varrho(F^{(n)}(H_0)^{1/2})$, with $n = 2, \ldots, N-1$. If these are combined with Eq. (5.10) and the integrations over σ and τ are performed using appendices A3 and A4, the desired result (5.21) is obtained for $n = N$. By complete induction, Eq. (5.21) thus holds true for every $n \geqq 2$.

6. The Solution of the Resolvent Equation

6.1. A Generalized Fredholm Formula

Choose any $f \in L^2$, write

$$h = R^{(n)}(\lambda)\, f\,, \qquad h_0 = Q^{(n)}(\lambda)\, f$$

and consider the equation

$$h = h_0 + K^{(n)}(\lambda)\, h \tag{6.1}$$

for any λ other than $\lambda \geqq \Lambda_0^{(n)}$. Because $K^{(n)}(\lambda)$ belongs to the class (ϱc), this equation can be solved explicitly by a generalization of the Fredholm theory of integral equations. Details of this formalism will be published in a separate paper [35]. In the present section we summarize the more important results.

For values of λ for which the homogeneous equation $h = K^{(n)}(\lambda)\, h$ does not possess a solution, Eq. (6.1) has precisely one solution, which is of the general form

$$h = h_0 + K^{(n)}(\lambda)\, h_0 + [\delta^{(n)}(\lambda)]^{-1}\, Z^{(n)}(\lambda)\, h_0\,. \tag{6.2}$$

Here $Z^{(n)}(\lambda)$ is an operator and $\delta^{(n)}(\lambda)$ is a number. The homogeneous equation has a non-vanishing solution if and only if $\delta^{(n)}(\lambda) = 0$.

The quantities $Z^{(n)}(\lambda)$ and $\delta^{(n)}(\lambda)$ are most easily constructed with the help of the series expansions

$$Z^{(n)}(\lambda) = \sum_{q=0}^{\infty} Z_q^{(n)}(\lambda)\,, \; \delta^{(n)}(\lambda) = \sum_{q=0}^{\infty} \delta_q^{(n)}(\lambda)\,. \tag{6.3}$$

Given $K^{(n)}(\lambda)$, the terms of the series can be found by the recurrence relations

$$\begin{aligned} Z_0^{(n)}(\lambda) &= [K^{(n)}(\lambda)]^2 , \\ Z_q^{(n)}(\lambda) &= \delta_q^{(n)}(\lambda)\,[K^{(n)}(\lambda)]^2 + Z_{q-1}^{(n)}(\lambda)\,K^{(n)}(\lambda) \qquad (q \geqq 1) , \\ \delta_0^{(n)}(\lambda) &= 1,\ \delta_1^{(n)}(\lambda) = \delta_2^{(n)}(\lambda) = \delta_3^{(n)}(\lambda) = 0 , \\ \delta_q^{(n)}(\lambda) &= -\,q^{-1}\,\mathrm{tr}\,Z_{q-4}^{(n)}(\lambda)\,[K^{(n)}(\lambda)]^2 \qquad (q \geqq 4) . \end{aligned} \tag{6.4}$$

Because $K^{(n)}(\lambda)$ is in (ϱc), the operator $[K^{(n)}(\lambda)]^2$ is in the Schmidt-class (σc) and $[K^{(n)}(\lambda)]^4$ is in the trace-class (τc). The quantities $Z_q^{(n)}(\lambda)$ and $\delta_q^{(n)}(\lambda)$ are therefore properly defined, and the $Z_q^{(n)}(\lambda)$ are in (σc). The series for $\delta^{(n)}(\lambda)$ converges absolutely, the series for $Z^{(n)}(\lambda)$ converges in the Schmidt-norm.

Some remarks on this solution and its connection with the classical Fredholm theory will be given in section 9.

Owing to Eq. (6.2), the solution of the resolvent equation takes the form

$$\begin{aligned} R^{(n)}(\lambda) = Q^{(n)}(\lambda) &+ K^{(n)}(\lambda)\,Q^{(n)}(\lambda) \\ &+ [\delta^{(n)}(\lambda)]^{-1}\,Z^{(n)}(\lambda)\,Q^{(n)}(\lambda) . \end{aligned} \tag{6.5}$$

Because $Z^{(n)}$ is in (σc), the quantities $Z^{(n)}$ and $Z^{(n)}\,Q^{(n)}$ are integral operators. By the proof of lemma 5.4, $K^{(n)}\,Q^{(n)}$ is also an integral operator.

6.2. The Fredholm Denominator

If f and g are two functions in L^2, the quantity $(R^{(n)}(\lambda)\,f, g)$ is analytic in the λ-plane cut from $\Lambda^{(n)}$ to ∞. Since $Q^{(n)}$ is a sum of resolvents, it is also analytic. Hence so is $F^{(n)}$, by Eq. (4.10). From Eqs. (4.12) and (4.13) and the properties of the convolution product it now follows that $K^{(n)}$ is analytic in the λ-plane cut from $\Lambda_0^{(n)}$ to ∞. This suggests the following lemma.

Lemma 6.1. *The Fredholm denominator $\delta^{(n)}(\lambda)$ is analytic in the λ-plane cut from $\Lambda_0^{(n)}$ to ∞.*

Proof. Because $K^{(n)}$ is analytic, it follows from the recurrence relations (6.4) that $\delta_q^{(n)}(\lambda)$ is an analytic function for every q. It is shown in [35] that

$$|\delta_q^{(n)}(\lambda)| \leqq (3\,e/q)^{\frac{1}{4}q}\,[\varrho(K^{(n)}(\lambda))]^q . \tag{6.6}$$

If ε and ε' are two positive numbers, the series $\sum \delta_q^{(n)}(\lambda)$ thus converges uniformly in the region

$$0 < \varepsilon < |\lambda - \Lambda_0^{(n)}| , \qquad 0 < \varepsilon' < \arg(\lambda - \Lambda_0^{(n)}) < 2\pi - \varepsilon' ,$$

by Eq. (5.21). The sum $\delta^{(n)}(\lambda)$ is therefore analytic in the λ-plane cut from $\Lambda_0^{(n)}$ to ∞. This proves the lemma.

According to Eqs. (6.4) and (6.6), $\delta^{(n)}(\lambda)$ tends to 1 as λ tends to $-\infty$. Hence $\delta^{(n)}(\lambda)$ does not vanish identically, and it can have only isolated zeros. It vanishes at $\lambda = \lambda_\alpha$ if and only if there is an element $\varphi_\alpha \in L^2$ such that $\varphi_\alpha \neq 0$ and

$$[1 - K^{(n)}(\lambda_\alpha)]\, \varphi_\alpha = Q^{(n)}(\lambda_\alpha)\,(H - \lambda_\alpha)\, \varphi_\alpha = 0\,. \tag{6.7}$$

Here $Q^{(n)}(\lambda_\alpha)\, H$ is the bounded operator which is the adjoint of $H\, Q^{(n)}(\bar{\lambda}_\alpha)$. The relation (6.7) is equivalent to

$$\left[1 + \sum_{k=2}^{n-1} (-1)^k\, (k-1)! \sum_{p(k)} R_{p(k)}^{(n)}(\lambda_\alpha)\, V_{p(k)}\right] [1 + R_0^{(n)}(\lambda_\alpha)\, V]\, \varphi_\alpha = 0\,. \tag{6.8}$$

Now suppose that there is a non-trivial element $\varphi_\alpha \in \mathfrak{D}(H_0)$ such that

$$(H - \lambda_\alpha)\, \varphi_\alpha = 0\,. \tag{6.9}$$

Then $\delta^{(n)}(\lambda_\alpha) = 0$, so λ_α is an isolated singularity of $R^{(n)}(\lambda)$. Owing to Eq. (3.11) it must be a simple pole. Because $K^{(n)}(\lambda_\alpha)$ is a compact operator, the multiplicity of its eigenvalue 1 is finite. Hence, by Eq. (6.7) the multiplicity of the eigenvalue λ_α of H is also finite.

Conversely, let $\delta^{(n)}(\lambda_\alpha) = 0$. Then there are two possibilities.

Case 1. There is an element $\varphi_\alpha \in L^2$ such that

$$[1 + R_0(\lambda_\alpha)\, V]\, \varphi_\alpha = R_0(\lambda_\alpha)\,(H - \lambda_\alpha)\, \varphi_\alpha = 0\,. \tag{6.10}$$

For $n = 2$, this is in fact the only possibility. It yields

$$R_0(\lambda_\alpha)\, H\, \varphi_\alpha = \lambda_\alpha\, R_0(\lambda_\alpha)\, \varphi_\alpha\,.$$

It is obvious that the right member of this relation is in $\mathfrak{D}(H_0)$. Hence $H\varphi_\alpha$ is in L^2 and φ_α is in $\mathfrak{D}(H_0)$. From this it follows that Eq. (6.9) is satisfied. The resolvent therefore has a simple pole. Because H is self-adjoint, λ_α must be real. The spectrum of H is bounded below, so λ_α is restricted to a finite interval of the negative real axis. Define

$$\begin{aligned} \Lambda^{(n)} &= \min_\alpha \lambda_\alpha\,, && \text{if there are zeros } \lambda_\alpha; \\ \Lambda^{(n)} &= \Lambda_0^{(n)}\,, && \text{otherwise}\,. \end{aligned} \tag{6.11}$$

Case 2. There is an element $\varphi_\alpha \in L^2$ such that

$$[1 + R_0(\lambda_\alpha)\, V]\, \varphi_\alpha = \chi_\alpha \neq 0\,,$$

$$\left[1 + \sum_{k=2}^{n-1} (-1)^k\, (k-1)! \sum_{p(k)} R_{p(k)}^{(n)}(\lambda_\alpha)\, V_{p(k)}\right] \chi_\alpha = 0\,.$$

In this case λ_α is not an eigenvalue of H. The quantities $Z^{(n)}(\lambda_\alpha)$ and $\delta^{(n)}(\lambda_\alpha)$ vanish simultaneously in such a way that the resolvent is analytic in the neighbourhood of $\lambda = \lambda_\alpha$, λ_α included. The number λ_α may be complex. For $n = 2$ this case does not apply. For $n = 3$ an example of these spurious zeros was discovered by FEDERBUSH [43] and discussed by NOBLE [44]. It is not known if the spurious zeros can be avoided by imposing suitable conditions on the interaction.

The FADDEEV equations do not give rise to such spurious zeros ([2], theorem 7.1), and it was pointed out recently by NOBLE [45] that they do not occur either in the equations proposed by ROSENBERG [10] and NEWTON [11].

7. The Spectrum of the Hamiltonian

First, some remarks will be made on the spectrum of a self-adjoint operator T. It consists of the numbers λ for which the operator $T - \lambda$ does not possess a bounded inverse. In defining different parts of the spectrum, we will follow KATO ([39], Ch.X, sections 1.1. and 1.2) and RIESZ and SZ.-NAGY ([46], sections 132 and 133).

The point spectrum consists of the eigenvalues of T. Let P be the orthogonal projection on the subspace, spanned by all the eigenspaces corresponding to the different eigenvalues. The continuous spectrum of T is then the spectrum of $T(1 - P)$. Finally, the essential spectrum is obtained by removing from the spectrum the isolated points which are eigenvalues with finite multiplicities. The essential spectrum thus consists of the continuous spectrum, the limit points of the point spectrum and the eigenvalues with infinite multiplicity.

The essential spectrum may be characterized as follows ([46], section 133).

Lemma 7.1. *The number λ belongs to the essential spectrum of T precisely if there exists a sequence $\{f_p\} \in \mathfrak{D}(T)$, such that*

$$\|f_p\| = 1 , \quad \|(T - \lambda) f_p\| \to 0 , \quad (g, f_p) \to 0$$

for any element g in the Hilbert space.

We will prove in this section, that the essential spectrum of the Hamiltonian ranges from $\Lambda_0^{(n)}$ to ∞. Hence, according to Eq. (6.11), $\Lambda^{(n)}$ is the lower bound of the spectrum of H. It is not known, whether or not the interval $[\Lambda_0^{(n)}, \infty)$ contains a part of the point spectrum.

Theorem 7.2. *The essential spectrum of the Hamiltonian $H^{(n)}$ runs from $\Lambda_0^{(n)}$ to ∞.*

Proof. We first show that the half-line $[0, \infty)$ belongs to the essential spectrum. Choose some set of coordinates of the type considered in section 2.1. Each term k_i^2 of the kinetic energy $H_0^{(n)}$ can be considered as an operator on $L^2(\boldsymbol{k}_i)$. It has a continuous spectrum from 0 to ∞. It follows from lemma 7.1 that for any $\mu \geqq 0$ there exists a sequence $\{d_p(\boldsymbol{k}_i)\}$ such that

$$\begin{aligned} \|d_p\|' &= [\int |d^p(\boldsymbol{k}_i)|^2 \, d^3 k_i]^{1/2} = 1 , \\ (d_p, g)' &= \int d_p(\boldsymbol{k}_i) \, \bar{g}(\boldsymbol{k}_i) \, d^3 k_i \to 0 \qquad (7.1) \\ \|[H_0 - (n-1)^{-1} \mu] \, d_p\|' &= \{\int [k_i^2 - (n-1)^{-1} \mu]^2 \, |d_p(\boldsymbol{k}_i)|^2 \, d^3 k_i\}^{1/2} \to 0 , \end{aligned}$$

the second relation being true for every $g(\boldsymbol{k}_i) \in L^2(\boldsymbol{k}_i)$. In the present case, the functions $d_p(\boldsymbol{k}_i)$ may be chosen non-negative.

Now define the sequence $\{f_p\}$ in $L^2(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{n-1})$ by

$$f_p(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{n-1}) = \prod_{i=1}^{n-1} d_p(\boldsymbol{k}_i) . \tag{7.2}$$

Then it is obvious that $f_p \in \mathfrak{D}(H_0)$. It follows from Eq. (7.1) that, for any $g \in L^2(R^{3n-3})$

$$\|f_p\| = 1 , \quad (f_p, g) \to 0 , \quad \|(H_0^{(n)} - \mu) f_p\| \to 0 . \tag{7.3}$$

In order to prove that μ belongs to the essential spectrum of $H^{(n)}$, it is sufficient to show that $\|V_{ij} f_p\|$ tends to 0 for any term V_{ij} of the potential energy V. Suppose that

$$\boldsymbol{k}_{ij} = \sum_{h=1}^{n-1} c_h \boldsymbol{k}_h .$$

Then at least one of the numbers c_h differs from 0. Let it be c_1. The relations (5.6) and (3.3) then yield

$$\begin{aligned}
&(V_{ij} f_p)(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{n-1}) \\
&= \boldsymbol{s}_{ij} \cdot \int \left[\boldsymbol{W}_{ij}(\boldsymbol{m}) \times \sum_{h=1}^{n-1} c_h (\boldsymbol{k}_h - c_h \boldsymbol{m})\right] \prod_{h=1}^{n-1} d_h(\boldsymbol{k}_h - c_h \boldsymbol{m})\, d^3 m \\
&= |c_1|^{-3}\, \boldsymbol{s}_{ij} \cdot \int \boldsymbol{W}_{ij}\left(\frac{\boldsymbol{k}_1 - \boldsymbol{l}_1}{c_1}\right) \times \left\{c_1 \boldsymbol{l}_1 + \sum_{h=2}^{n-1} c_h \left[\boldsymbol{k}_h - \frac{c_h}{c_1} (\boldsymbol{k}_1 - \boldsymbol{l}_1)\right]\right\} \\
&\quad \cdot d_p(\boldsymbol{l}_1) \prod_{h=2}^{n-1} d_p\left(\boldsymbol{k}_h - \frac{c_h}{c_1} (\boldsymbol{k}_1 - \boldsymbol{l}_1)\right) d^3 l_1 .
\end{aligned}$$

From this it follows with Schwarz's inequality that the norm of $V_{ij} f_p$ is not larger than the sum of the norms of the two functions

$$|c_1|^{-3}\, s_{ij} \int W_{ij}\left(\frac{\boldsymbol{k}_1 - \boldsymbol{l}_1}{c_1}\right) |c_1|\, l_1\, d_p(\boldsymbol{l}_1) \prod_{h=2}^{n-1} d_p(\boldsymbol{k}_h)\, d^3 l_1$$

and

$$|c_1|^{-3}\, s_{ij} \int W_{ij}\left(\frac{\boldsymbol{k}_1 - \boldsymbol{l}_1}{c_1}\right) \left(\sum_{h=2}^{n-1} |c_h|\, k_h\right) d_p(\boldsymbol{l}_1) \prod_{h=2}^{n-1} d_p(\boldsymbol{k}_h)\, d^3 l_1 ,$$

because the functions $d_p(\boldsymbol{k}_i)$ are non-negative.

Write W_{ij} for the integral operator on $L^2(\boldsymbol{k}_1)$ with kernel $W_{ij}((\boldsymbol{k}_1 - \boldsymbol{l}_1)/c_1)$. Then it is clear that

$$\|V_{ij} f_p\| \leqq \text{const} \left[\|W_{ij} (H_0)^{1/2} d_p\|' + \|W_{ij} d_p\|'\, \|(H_0)^{1/2} d_p\|'\right] .$$

Here the primed norms are those in $L^2(R^3)$. It has been used that $\|d_p\|' = 1$.

According to Eq. (7.1), $((H_0 + 1)\, d_p, g)'$ tends to 0 for any $g \in L^2(R^3)$, that is, the sequence $\{(H_0 + 1)\, d_p\}$ converges weakly to 0. By the proof of lemma 3.1, the operators $W_{ij}(H_0)^{1/2} R_0(-1)$ and $W_{ij} R_0(-1)$ belong to the class (rc) of $L^2(R^3)$, and so they are compact. As a consequence, they transform the sequence $\{(H_0 + 1)\, d_p\}$ into strongly convergent sequences. From this it follows that

$$\|W_{ij}(H_0)^{1/2}\, d_p\|' \to 0\,, \qquad \|W_{ij} d_p\|' \to 0\,,$$

and so

$$\|V_{ij} f_p\| \to 0\,. \tag{7.4}$$

With Eq. (7.3), this shows that the half-line $[0, \infty)$ belongs to the essential spectrum of $H^{(n)}$.

It follows from section 6.2 that the half-line $(-\infty, \Lambda_0^{(n)})$ does not contain points of the essential spectrum of $H^{(n)}$. So if $\Lambda_0^{(n)} = 0$, the assertion of the theorem has been proved. In particular, the assertion is true for $n = 2$.

On the other hand, let $\Lambda_0^{(n)}$ be negative. Then $\Lambda_0^{(n)}$ can be split into numbers $\Lambda^{(n_1)}$ and $\Lambda^{(n_2)}$ according to Eq. (4.14). At least one of the numbers must be negative. Let this be $\Lambda^{(n_1)}$. Then $\Lambda^{(n_1)}$ is either an eigenvalue of $H^{(n_1)}$, or it is equal to $\Lambda_0^{(n_1)}$. In the latter case it may be split further. Because $\Lambda^{(n_1)}$ can only be negative owing to one or more subgroups of the n-particle system having bound states, it follows, ultimately, that $\Lambda_0^{(n)}$ is the sum of energies of bound states of subgroups.

Now choose internal coordinates in these subgroups, and denote these by $\boldsymbol{k}^{(1)}$. Introduce coordinates $\boldsymbol{k}^{(2)}$ to describe the motion of the remaining particles and the centres of mass of the subgroups. Write $H(1)$ for the Hamiltonian which refers to the internal motion of the subgroups. Then there exists a function $\varphi(\boldsymbol{k}^{(1)})$ of norm 1 in $L^2(\boldsymbol{k}^{(1)})$ such that

$$[H(1) - \Lambda_0^{(n)}]\, \varphi(\boldsymbol{k}^{(1)}) = 0\,.$$

In $L^2(\boldsymbol{k}^{(2)})$ a sequence $\{f_p(\boldsymbol{k}^{(2)})\}$ can be constructed with the properties (7.2) and (7.3). If F_p is now defined by

$$F_p(\boldsymbol{k}) = \varphi(\boldsymbol{k}^{(1)})\, f_p(\boldsymbol{k}^{(2)})\,,$$

then

$$\|F_p\| = 1\,, \qquad (F_p, g) \to 0 \tag{7.5}$$

for any $g \in L^2(R^{3n-3})$. Also,

$$[H(1) - \Lambda_0^{(n)}]\, F_p = 0\,, \qquad \|[H_0(2) - \mu]\, F_p\| \to 0\,.$$

The argument for the half-line $[0, \infty)$ can be repeated to show that $\|V_{ij} F_p\|$ tends to 0 for every interaction V_{ij} which does not occur in $H(1)$. Hence

$$\|(H - \Lambda_0^{(n)} - \mu)\, F_p\| \to 0\,. \tag{7.6}$$

By Eqs. (7.5) and (7.6) and lemma 7.1, $\Lambda_0^{(n)} + \mu$ belongs to the essential spectrum of $H^{(n)}$ for every $\mu \geqq 0$. Since $(-\infty, \Lambda_0^{(n)})$ does not contain points of the essential spectrum, it follows that theorem 7.2 is true.

8. The Coulomb Interaction

In the context of the present paper, the difficulties associated with the spin-orbit coupling are due to the factor $\boldsymbol{k}_{ij}$ in the interaction (2.13).

This factor results in the interaction having a very long range in momentum space. In the case of a short-range interaction, one would expect the operator $K^{(n)}(\lambda)$ to be in the Schmidt-class (σc), but, as we have seen, it is merely in (ϱc). Now it is pointed out in appendix A1 that the class (ϱc) is invariant under unitary transformations. This suggests that it also refers to interactions with a long range in position space. The Coulomb interaction is now examined as an example of these.

In a system of n particles with two-body interactions of the form

$$V_{ij}(\boldsymbol{x}_{ij}) = e_{ij}/x_{ij}\,, \tag{8.1}$$

it is well known from Kato's work ([39], Ch.V, sections 4 and 5) that the Hamiltonian (2.7) is a self-adjoint operator with domain $\mathfrak{D}(H_0)$. Its spectrum is bounded below, and so the resolvent satisfies Eq. (3.13). If $\Lambda_0^{(n)}$ is defined by Eq. (4.14), it follows from papers by ZHISLIN [47] and DOLLARD [48] that $\Lambda_0^{(n)}$ is the lower bound of the essential spectrum of H, which is continuous in the strict sense.

Now consider two particles and examine the operator $-R_0(\lambda)\,V$ for non-positive λ. In the position representation this is an integral operator whose kernel is

$$K(\boldsymbol{x}_1, \boldsymbol{y}_1) = -\,e_{ij}(4\pi)^{-1}\exp[i\lambda^{1/2}\,|\boldsymbol{x}_1 - \boldsymbol{y}_1|]\,|\boldsymbol{x}_1 - \boldsymbol{y}_1|^{-1}\,y_1^{-1}\,. \tag{8.2}$$

Thus,

$$\begin{aligned}&\int |K(\boldsymbol{x}_1, \boldsymbol{y}_1)\,K(\boldsymbol{x}_1, \boldsymbol{y}_1')|\,d^3x_1\\ &= \text{const}\,(\operatorname{Im}\lambda^{1/2})^{-1}\exp[-\operatorname{Im}\lambda^{1/2}\,|\boldsymbol{y}_1 - \boldsymbol{y}_1'|]\,y_1^{-1}\,y_1'^{-1}\,.\end{aligned} \tag{8.3}$$

To obtain the r-norm $r(R_0(\lambda)\,V)$, one must square Eq. (8.3) and integrate over $\boldsymbol{y}_1$ and $\boldsymbol{y}_1'$. Choosing $\boldsymbol{u} = \boldsymbol{y}_1 - \boldsymbol{y}_1'$, $\boldsymbol{v} = \boldsymbol{y}_1$, we have

$$\begin{aligned}&\text{const}\,(\operatorname{Im}\lambda^{1/2})^{-2}\int \exp[-2\operatorname{Im}\lambda^{1/2}u]\,v^{-2}\,|\boldsymbol{v} - \boldsymbol{u}|^{-2}\,d^3u\,d^3v\\ &= \text{const}\,(\operatorname{Im}\lambda^{1/2})^{-2}\int \exp[-2\operatorname{Im}\lambda^{1/2}u]\,u^{-1}\,d^3u = \text{const}\,(\operatorname{Im}\lambda^{1/2})^{-4}\,.\end{aligned}$$

This shows that the operator $-R_0(\lambda)\,V$ is in the class (rc) and satisfies

$$r(R_0(\lambda)\,V) \leqq \text{const}\,(\operatorname{Im}\lambda^{1/2})^{-1}\,. \tag{8.4}$$

In the notation of section 5.1 we have for two particles

$$r(K^{(2)}(\lambda)) = \Omega\left(-\frac{1}{2}\,,\,0\right). \tag{8.5}$$

This result is to be compared with lemma 5.2. There is an analogue of lemma 5.3 which reads as follows.

Lemma 8.1. *Let the operators $F^{(m)}$ be integral operators such that*

$$r(F^{(m)}(\lambda)) = \Omega\left(-\frac{1}{2}\,m - \frac{1}{2}\,,\,\Lambda^{(m)}\right) \qquad (2 \leqq m \leqq n-1)\,. \tag{8.6}$$

Then $K^{(n)}(\lambda)$ is an integral operator such that

$$r(K^{(n)}(\lambda)) = \Omega\left(-\frac{1}{2}\,n + \frac{1}{2}\,,\,\Lambda_0^{(n)}\right). \tag{8.7}$$

Proof. First examine the terms $K^{(n)}_{p(2)}$ of the form (4.13). If $F^{(n-1)}(\lambda)$ is in the class (rc), then so is

$$F^{(n-1)}(\sigma)\, R_0^{(2)}(\lambda-\sigma)\, V_{jn}\,.$$

The analogue of Eq. (5.9) is

$$r(F^{(n-1)}(\sigma)\, R_0^{(2)}(\lambda-\sigma)\, V_{jn}) \leqq r_1(F^{(n-1)}(\sigma))\, r_3(K^{(2)}(\lambda-\sigma))\,.$$

This yields

$$r([F^{(n-1)} * R_0^{(2)}\, V_{jn}]\,(\lambda)) = \Omega\left(-\frac{1}{2}\,n+\frac{1}{2}\,,\,\Lambda^{(n-1)}\right).$$

with Eqs. (8.5) and (8.6), appendix A3 and lemma 5.1.3. That is, the terms $K^{(n)}_{p(2)}$ of the form (4.13) are compatible with Eq. (8.7). There is a similar argument for the operators $K^{(n)}_{p(2)}$ of the form (4.12), which proves Eq. (8.7) for the full kernel $K^{(n)}(\lambda)$.

We proceed to the analogue of lemma 5.4.

Lemma 8.2. *Let $F^{(m)}$ and $K^{(n)}$ be integral operators such that*

$$r(F^{(m)}(\lambda)) = \Omega\left(-\frac{1}{2}\,m-\frac{1}{2}\,,\,\Lambda^{(m)}\right) \qquad (2 \leqq m \leqq n-1)\,,$$

$$r(K^{(n)}(\lambda)) = \Omega\left(-\frac{1}{2}\,n+\frac{1}{2}\,,\,\Lambda_0^{(n)}\right).$$

Then $F^{(n)}$ is an integral operator such that

$$r(F^{(n)}(\lambda)) = \Omega\left(-\frac{1}{2}\,n-\frac{1}{2}\,,\,\Lambda^{(n)}\right). \tag{8.8}$$

Proof. As in the proof of lemma 5.4, it is convenient to decompose $F^{(n)}$ according to Eq. (5.16) and to consider three types of terms separately.

Term 1. This is a term $K^{(n)} R_0^{(n)}$. Because in the present section we are considering the position representation, $R_0^{(n)}$ is an integral operator whose kernel is of the form $G_0^{(n)}(\boldsymbol{x}-\boldsymbol{y};\lambda)$ ([18], Eq. (1.2.17)). From this it follows with Schwarz's inequality that

$$\begin{aligned} r(K^{(n)}(\lambda)\, R_0^{(n)}(\lambda)) &\leqq r(K^{(n)}(\lambda)) \int |G_0^{(n)}(\boldsymbol{x}-\boldsymbol{y};\lambda)|\, d^{3n-3}(\boldsymbol{x}-\boldsymbol{y}) \\ &\leqq \text{const}\; r(K^{(n)}(\lambda))\, |\lambda|^{\frac{3}{4}n-\frac{3}{2}}\, (\operatorname{Im}\lambda^{1/2})^{-\frac{3}{2}n+1} \end{aligned}$$

([18], Eq. (1.7.83)). Thus, with Eq. (8.7),

$$r(K^{(n)}(\lambda)\, R_0^{(n)}(\lambda)) = \Omega\left(-\frac{1}{2}\,n-\frac{1}{2}\,,\,\Lambda_0^{(n)}\right). \tag{8.9}$$

Term 2. This is a sum of operators of the form

$$K^{(n)}\, \tilde{Q}_p^{(n)} = K^{(n)}\, F^{(n_1)} * \cdots * F^{(n_j)} * R_0^{(p)}\,, \qquad 2 < p < n\,.$$

Here, $R_0^{(p)}$ can be handled as indicated under term 1. The further analysis follows the lines of term 2 in the proof of lemma 5.4. The result is

$$r(K^{(n)}(\lambda)\, \tilde{Q}_p^{(n)}(\lambda)) = \Omega\left(-n+\frac{1}{2}\,p-\frac{1}{2}\,,\,\Lambda_0^{(n)}\right), \tag{8.10}$$

a relation which is compatible with Eq. (8.8) because $p < n$.

Term 3. It remains to discuss the term $(K^{(n)})^2 R^{(n)}$. As in the proof of lemma 5.4, this satisfies

$$r([K^{(n)}(\lambda)]^2 R^{(n)}(\lambda)) \leqq [r(K^{(n)}(\lambda))]^2 \|R^{(n)}(\lambda)\| ,$$

and so

$$r([K^{(n)}(\lambda)]^2 R^{(n)}(\lambda)) = \Omega(-n, \Lambda^{(n)}) . \tag{8.11}$$

The results (8.9–11) together with lemma 5.1.1 now lead to Eq. (8.8), which concludes the proof of this lemma.

Again, consider the case $n = 2$. The relation (8.5) gives an estimate for $r(K^{(2)})$. From this an estimate for $F^{(2)}$ follows with lemma 8.2. Next, lemma 8.1 leads to an estimate for $r(K^{(3)})$. Then again lemma 8.2 gives an estimate for $r(F^{(3)})$, and so on. Summarizing, we have the following theorem.

Theorem 8.3. *For a system of n particles with Coulomb interactions, the operators* $K^{(n)}(\lambda)$ *and* $F^{(n)}(\lambda)$ *defined by Eqs.* (4.8) *and* (4.10) *are operators in the class* (rc) *of position space. Their r-norms satisfy Eqs.* (8.7) *and* (8.8).

This is the analogue of theorem 5.6. If we restrict ourselves to values of λ not in the essential spectrum of the Hamiltonian, the Fredholm theory of section 6 makes it possible to find the resolvent $R^{(n)}(\lambda)$ for n particles with Coulomb interactions.

We will conclude this section with some remarks on the momentum representation. The potential energy is then defined by

$$(V_{ij}\hat{f})(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{n-1}) = (2\pi)^{-2} e_{ij} \int |\boldsymbol{k}_1 - \boldsymbol{l}_1|^{-2} \hat{f}(\boldsymbol{l}_1, \boldsymbol{k}_2, \ldots, \boldsymbol{k}_{n-1})\, d^3 l_1 ,$$

where $\boldsymbol{k}_1 = \boldsymbol{k}_{ij}$ etc. Thus, V is a properly defined symmetric operator on $\mathfrak{D}(H_0)$, leading to a self-adjoint Hamiltonian with domain $\mathfrak{D}(H_0)$. The relations (8.4–8) remain valid for the r-norms in the momentum representation. This is not a trivial result, because the r-norm is not unitarily invariant. However, the relation (8.4) and the lemmas 8.1 and 8.2 can be proved without difficulty for the momentum representation.

9. Discussion

For certain classes of long-range interactions, we have formulated an equation for the n-particle resolvent $R(\lambda)$, and we have shown that this has a kernel $K(\lambda)$ which belongs to the class (ϱc). This enables the equation to be solved by the methods of section 6. It appears worth while to comment briefly on the significance of this procedure.

Consider the equation

$$h = h_0 + Kh , \tag{9.1}$$

where h_0 and K are known and h is to be found. The classical Fredholm theory gives a solution in the form

$$h = Dh_0/d , \tag{9.2}$$

where D is an operator and d is a number. Both D and d can be evaluated explicitly. These quantities involve K and all the powers K^p, as well as the traces $\operatorname{tr} K^p$ $(p = 1,2, \ldots)$. The solution can therefore only be expected to hold for kernels K in the trace-class.

If it is merely known that K is in the Schmidt-class, $\operatorname{tr} K$ does not necessarily exist. However, K^2 still is in the trace-class, and so the equation

$$h = h_0 + K h_0 + K^2 h \tag{9.3}$$

can be solved by the classical theory. In practice, however, this is never done. One prefers the modified Fredholm theory due to SMITHIES [42], in which D and d are replaced by quantities which no longer depend on $\operatorname{tr} K$. Now if K is merely in the class (ϱc), then $\operatorname{tr} K$, $\operatorname{tr} K^2$, and $\operatorname{tr} K^3$ need not exist. On the other hand, K^2 is in the Schmidt-class in this case, and so Eq. (9.3) can be solved by Smithies's formulas. We have chosen not to do this, however, and we have further modified the expression for D/d so as to eliminate also $\operatorname{tr} K^2$ and $\operatorname{tr} K^3$. This is the contents of section 6, which thus generalizes Smithies's ideas.

If instead of Eq. (9.1) one considers Eq. (9.3), then the corresponding homogeneous equation has an eigenfunction $h = K^2 h$ not only if there is a function h such that $h = Kh$, but also if $h = -Kh$. Thus, the Fredholm denominator may have spurious zeros which are cancelled exactly by zeros of the numerator. These spurious zeros are avoided if one adopts the modified solution which applies directly to Eq. (9.1). This thus results in the Fredholm numerator and denominator having a less complicated structure. One may hope that it makes the Fredholm series converge better. It is not difficult to see that this method for long-range interactions can also be used profitably in the context of the FADDEEV [1, 2] and ROSENBERG-NEWTON [10, 11] equations.

Our results on the resolvent $R(\lambda)$ apply only in the λ-plane cut from a certain point Λ_0 to ∞. For spin-orbit interactions, it is shown in section 7 that Λ_0 is the lower bound of the essential spectrum of H. The method of proof is different from, but related to, an argument for local interactions due to HUNZIKER [49]. Similar work was done by ZHISLIN [47]. The results of these authors apply to the Coulomb interaction as a special example. Our method can also easily be formulated so as to include this case.

In all this work on the essential spectrum, very few assumptions are made about the interaction, but also very little information is gained on the detailed structure of the spectrum. This question was discussed from a general point of view by GLAZMAN ([50] section 65). For short-range local interactions, one can use the time-dependent scattering theory due to JAUCH [51] and discussed by KATO ([39] Ch.X), and one can show that that there exist wave operators [52, 53]. From this it then follows that the Hamiltonian H contains parts which are unitarily equivalent to Hamiltonians that refer to the system of n particles being split into a number of subsystems which are allowed to move freely with respect to one another. For three particles with sufficiently smooth interactions, it was shown by Faddeev ([2] theorem 9.2) that the sum of all parts of H so obtained in fact accounts for all of the essential spectrum of H, which is thus absolutely continuous. Dollard [48] generalized the method of wave operators to Coulomb interactions, and the essential spectrum of the Coulomb Hamiltonian was also discussed recently by Weidmann [54]. For interactions with a long range in position space, there is at present very little known beyond this. As regards the spin-orbit coupling, which is non-local and has a long range in momentum space, a preliminary investigation indicates that wave operators do exist if the functions $W_{ij}(\boldsymbol{k})$ are somewhat more restricted than is indicated by Eq. (3.5). This would give some additional information on the structure of the spectrum. Eventually, however, one will want to construct $R(\lambda)$ explicitly for all values of λ on the real axis. This requires further investigation.

Appendix

A 1. Compact Operators on a Separable Hilbert Space

In the present paper, we are concerned with a Hilbert space $\mathfrak{H}$ whose dimension is finite or denumerably infinite. If f and g are any two elements of $\mathfrak{H}$, their inner product is denoted by (f, g), the notation being such that

$$(\alpha f, \beta g) = \alpha\bar{\beta}(f, g) \tag{A 1.1}$$

for every pair of complex numbers α and β.

By the term "operator" is meant "linear operator". If A is an operator on $\mathfrak{H}$, its bound is

$$\|A\| = \sup_{\|f\|=1} \|Af\| . \tag{A 1.2}$$

If $\|A\|$ is finite, the operator A is called *bounded*.

A bounded operator is called *compact* or *completely continuous* if it transforms every weakly convergent sequence into a strongly convergent one. In other words, let f be an element of $\mathfrak{H}$, and let $\{f_n\}$ $(n = 1, 2, \ldots)$ be any sequence such that

$$\lim_{n\to\infty} (f_n, g) = (f, g) \tag{A 1.3}$$

for every g in $\mathfrak{H}$. Then A is compact if and only if Eq. (A 1.3) implies that

$$\lim_{n\to\infty} \|Af_n - Af\| = 0 . \tag{A 1.4}$$

The compact operators are precisely the ones that admit a polar decomposition of the form

$$Af = \sum_i \lambda_i \varphi_i(f, \psi_i) . \tag{A 1.5}$$

Here $\{\varphi_i\}$ and $\{\psi_i\}$ $(i = 1, 2, \ldots)$ are orthonormal sets in $\mathfrak{H}$, and the quantities λ_i are positive numbers such that λ_i tends to 0 as i tends to ∞ (SCHATTEN [55] p. 18).

If A is compact and p is any positive number such that $p \geqq 1$, one may define

$$\|A\|_p = \left[\sum_i (\lambda_i)^p\right]^{1/p}. \tag{A 1.6}$$

This quantity has the properties of a norm, as is brought out by the following relations.

$$\begin{aligned} &\|A\|_p \geqq 0; \quad \|A\|_p = 0 \text{ implies } A = 0\,, \\ &\|\alpha A\|_p = |\alpha|\, \|A\|_p \text{ for any complex number } \alpha\,, \\ &\|A_1 + A_2\|_p \leqq \|A_1\|_p + \|A_2\|_p\,. \end{aligned} \tag{A 1.7}$$

The p-norm defined by Eq. (A 1.6) was investigated in great detail by SCHATTEN [55]. The relations (A 1.7) can be proved by combining various sections of Schatten's book.

If X and Y are any two bounded operators,

$$\|XAY\|_p \leqq \|X\|\, \|A\|_p\, \|Y\|\,. \tag{A 1.8}$$

In particular, for unitary operators U and V one has

$$\|UAV\|_p = \|A\|_p\,. \tag{A 1.9}$$

If $\mathfrak{K}_p$ denotes the class of compact operators A for which $\|A\|_p < \infty$, it is easy to see that

$$\mathfrak{K}_p \subseteqq \mathfrak{K}_q\,, \quad \|A\|_p \geqq \|A\|_q \quad (1 \leqq p \leqq q)\,. \tag{A 1.10}$$

Also, whenever A is compact,

$$\|A\| = \max_i \{\lambda_i\}\,, \tag{A 1.11}$$

and so

$$\|A\| \leqq \|A\|_p \quad (p \geqq 1)\,. \tag{A 1.12}$$

With the norm $\|A\|_p$, the class $\mathfrak{K}_p$ is a normed linear space. It follows from Schatten's work ([56] Ch.V, section 11, see also [55] Ch.V, section 7) that this is complete, hence a Banach space.

In the present paper, the cases $p = 1, 2$, and 4 are of particular importance. The corresponding classes and their norms are denoted by

$$\begin{aligned} &(\tau c) = \mathfrak{K}_1\,, \quad \tau(A) = \|A\|_1 \quad \text{(trace-class)}\,, \\ &(\sigma c) = \mathfrak{K}_2\,, \quad \sigma(A) = \|A\|_2 \quad \text{(Schmidt-class)}\,, \\ &(\varrho c) = \mathfrak{K}_4\,, \quad \varrho(A) = \|A\|_4\,. \end{aligned} \tag{A 1.13}$$

If A is in the Schmidt-class and $\{\chi_i\}$, $\{\omega_i\}$ are two complete orthonormal sets in $\mathfrak{H}$, the Schmidt-norm $\sigma(A)$ satisfies

$$[\sigma(A)]^2 = \sum_i \|A\chi_i\|^2 = \sum_{i,j} |(A\chi_i, \omega_j)|^2 = \sum_j \|A^*\omega_j\|^2 \tag{A 1.14}$$

([55] p. 29). For operators in the trace-class, the trace is defined by

$$\operatorname{tr} A = \sum_i (A\chi_i, \chi_i) \tag{A 1.15}$$

([55] p. 37). It satisfies

$$|\mathrm{tr} A| \leqq \tau(A) . \tag{A 1.16}$$

If A and B are in the Schmidt-class, then $A B$ is in the trace-class and the norms τ and σ are related according to

$$\begin{aligned} \tau(A^* A) &= \tau(A A^*) = [\sigma(A)]^2 , \\ \tau(A B) &\leqq \sigma(A)\, \sigma(B) . \end{aligned} \tag{A 1.17}$$

Likewise, if A and B are in the class (ϱc), then $A B$ is in the Schmidt-class, with

$$\begin{aligned} \sigma(A^* A) &= \sigma(A A^*) = [\varrho(A)]^2 , \\ \sigma(A B) &\leqq \varrho(A)\, \varrho(B) . \end{aligned} \tag{A 1.18}$$

A 2. Integral Operators

As our Hilbert space, we now choose a space consisting of square-integrable functions. Specifically, let R^n be the set of all systems of n real numbers $(x_1, \ldots, x_n)$, where each x_i may vary continuously over the interval $-\infty < x_i < \infty$. Let Z_m be the set of all systems of m real numbers $(s_1, \ldots, s_m)$, where each s_j takes certain discrete values only. If X stands for some finite or infinite interval in $R^n \times Z^m$, the set $L^2(X)$ consisting of all functions which are square-integrable over X is a Hilbert space. In this space, the inner product takes the form

$$(f,g) = \sum_{s_i} \int f(x_1,\ldots,x_n,s_1,\ldots,s_m)\, \bar{g}(x_1,\ldots,x_n,s_1,\ldots,s_m)\, dx_1\ldots dx_n . \tag{A2.1}$$

It is convenient to take the variables together and to denote them by x. This yields

$$(f, g) = \int f(x)\, \bar{g}(x)\, dx . \tag{A 2.2}$$

An *integral operator* on $L^2(X)$ is an operator which acts according to

$$(A f)(x) = \int A(x, y)\, f(y)\, dy , \tag{A 2.3}$$

the function $A(x, y)$ on $X \times X$ being the *integral kernel* of A.

On $L^2(X)$, the Schmidt-class is precisely the set of all integral operators whose kernels belong to $L^2(X \times X)$. The Schmidt-norm is given by

$$\sigma(A) = [\int\int |A(x, y)|^2\, dx\, dy]^{1/2} . \tag{A 2.4}$$

If A_1 and A_2 are any two operators in the Schmidt-class of $L^2(X)$, their product is an integral operator with kernel

$$(A_1 A_2)(x, y) = \int A_1(x, z)\, A_2(z, y)\, dz . \tag{A 2.5}$$

The adjoint A^* is an integral operator with kernel

$$A^*(x, y) = \bar{A}(y, x) . \tag{A 2.6}$$

For the special case of the space $L^2(0, 1)$, these statements are proved in Schatten's book ([55] p. 35). The proof also applies to more general spaces $L^2(X)$.

Now suppose that the operator A belongs to the trace-class. Then it is in the Schmidt-class, and so it is an integral operator. Its trace is given by

$$\operatorname{tr} A = \int B(x, y)\, C(y, x)\, dx\, dy = \int A(x, x)\, dx\,, \tag{A 2.7}$$

where B and C are two Schmidt-operators whose product is A.

If, on the other hand, it is merely known that A is in the class (ϱc), then the situation becomes much more difficult. It is not obvious that A is still an integral operator, and if it is, its kernel may be very unmanageable. It is in view of this complication that we now define a subset $(r c) \subsetneqq (\varrho c)$ which consists entirely of integral operators with useful properties analogous to Eqs. (A 2.5) and (A 2.6). This set is investigated in detail in a separate paper [35]. For easy reference we only quote some major results here.

An integral operator is said to belong to the class $(r c)$ if its kernel $A(x, y)$ is a measurable function on $X \times X$ satisfying

$$\begin{aligned} [r(A)]^4 &= \int |A(x, y)\, A(x', y)\, A(x, y')\, A(x', y')|\, dx\, dx'\, dy\, dy' \\ &= \int [\int |A(x, y)\, A(x', y)|\, dy]^2\, dx\, dx' \\ &= \int [\int |A(x, y)\, A(x, y')|\, dx]^2\, dy\, dy' < \infty\,. \end{aligned} \tag{A 2.8}$$

If A is in $(r c)$, it is bounded. In fact, it is in (ϱc). The norm $\varrho(A)$ can be evaluated according to

$$[\varrho(A)]^4 = \int \bar{A}(x, y)\, A(x, y')\, A(x', y)\, \bar{A}(x', y')\, dx\, dx'\, dy\, dy'\,, \tag{A 2.9}$$

the integral converging absolutely owing to Eq. (A 2.8). This relation shows that $\varrho(A)$ does not exceed $r(A)$. More generally, one has

$$\begin{aligned} (\sigma c) \subsetneqq (r c) \subsetneqq (\varrho c)\,, \\ \sigma(A) \geqq r(A) \geqq \varrho(A)\,. \end{aligned} \tag{A 2.10}$$

If $A(x, y)$ is the kernel of an operator A in $(r c)$, the kernel of A^* is given by Eq. (A 2.6). The product of two operators in $(r c)$ is an integral operator whose kernel satisfies Eq. (A 2.5). For any two functions f and g in $L^2(X)$, and any operator A in $(r c)$, one has

$$\int |g(x)\, A(x, y)\, f(y)|\, dx\, dy \leqq \|f\|\, \|g\|\, r(A)\,. \tag{A 2.11}$$

In defining the class $(r c)$, use is made of the special realization of the Hilbert space. The class $(r c)$ can therefore not be expected to have the same general significance as the classes $\mathfrak{R}_p$. To be specific, the quantity $r(A)$ has the properties of a norm, thus defining $(r c)$ as a Banach algebra. Unlike the ϱ-norm, however, the r-norm does not have the properties (A 1.8) and (A 1.9). In [35] an example is given of an operator in (ϱc) which is not in $(r c)$. The present paper is essentially devoted to studying operators in $(r c)$ which are not in (σc).

In dealing with products of operators, the following lemma is useful.

Lemma A 2. *Let A be an operator in the class (rc) of the space $L^2(X_1)$, let its kernel be $A(x_1, y_1)$, and let its r-norm in $L^2(X_1)$ be denoted by $r_1(A)$. Let B be an operator in the class (rc) of the space $L^2(X_1 \times X_2)$, let its kernel be $B(x_1, x_2; y_1, y_2)$, and let its r-norm be denoted by $r(B)$. Then the products AB and BA are integral operators in the class (rc) of the space $L^2(X_1 \times X_2)$. Their kernels are given by*

$$(AB)(x_1, x_2; y_1, y_2) = \int A(x_1, z_1)\, B(z_1, x_2; y_1, y_2)\, dz_1\,, \tag{A 2.12}$$

$$(BA)(x_1, x_2; y_1, y_2) = \int B(x_1, x_2; z_1, y_2)\, A(z_1, y_1)\, dz_1\,. \tag{A 2.13}$$

The r-norms satisfy

$$r(AB) \leqq r_1(A)\, r(B)\,, \tag{A 2.14}$$

$$r(BA) \leqq r_1(A)\, r(B)\,. \tag{A 2.15}$$

Proof. Consider the operator C whose kernel is given by

$$C(x_1, x_2; y_1\, y_2) = \int |A(x_1, z_1)\, B(z_1, x_2; y_1, y_2)|\, dz_1\,. \tag{A 2.16}$$

It is not difficult to show that C belongs to (rc). In fact, straight forward application of Schwarz's inequality yields

$$r(C) \leqq r_1(A)\, r(B)\,. \tag{A 2.17}$$

Owing to Eq. (A 2.11),

$$\begin{aligned}\int |g(x_1, x_2)\, A(x_1, z_1)\, B(z_1, x_2; y_1, y_2)\, f(y_1, y_2)|\, dx_1\, dx_2\, dy_1\, dy_2\, dz_1 \\ \leqq \|f\|\, \|g\|\, r_1(A)\, r(B)\,.\end{aligned} \tag{A 2.18}$$

The theorems of FUBINI and TONELLI on repeated and multiple integrals now immediately yield Eq. (A 2.12). Next, Eq. (A. 2.14) follows with Eq. (A. 2.17). The relations (A 2.13) and (A 2.15) can be proved by regarding BA as the adjoint of $A^* B^*$. This completes the proof of the lemma.

A 3. Integrals of Operators

Let t take values in a measurable set T in R^n. Suppose that, for every t in T, there is a bounded operator $A(t)$ acting on some Hilbert space $\mathfrak{H}$. Define A by

$$(Af, g) = \int_T (A(t)\, f, g)\, dt\,, \tag{A 3.1}$$

for every f and g in $\mathfrak{H}$.

Lemma A 3. *Let $A(t)$ be an operator on a space $L^2(X)$ for every t in T, let it be in the class (rc), and suppose that*

$$\int_T r(A(t))\, dt < \infty\,. \tag{A 3.2}$$

Then the operator A defined by Eq. (A 3.1) *is in* (rc). *If $A(x, y; t)$ stands for the kernel of $A(t)$, then the kernel of A is given by*

$$A(x, y) = \int_T A(x, y; t)\, dt\,, \tag{A 3.3}$$

the integral converging absolutely for almost every x, y. Also,

$$r(A) \leqq \int_T r(A(t))\, dt\,, \tag{A 3.4}$$

$$\varrho(A) \leqq \int_T \varrho(A(t))\, dt\,. \tag{A 3.5}$$

Proof. Choose f and g in $L^2(X)$. By virtue of Eq. (A 2.11)

$$\int_T dt \int |g(x)\, A(x, y; t)\, f(y)|\, dx\, dy \leqq \|f\|\, \|g\| \int_T r(A(t))\, dt < \infty\,. \tag{A 3.6}$$

Hence

$$\begin{aligned}(Af, g) &\equiv \int_T dt \int \bar{g}(x)\, A(x, y; t)\, f(y)\, dx\, dy \\ &= \int \bar{g}(x) \left[\int_T A(x, y; t)\, dt\right] f(y)\, dx\, dy\,.\end{aligned} \tag{A 3.7}$$

This yields Eq. (A 3.3). The relation (A 3.4) follows from this with the help of Eq. (A 2.8) and Schwarz's inequality. Similarly, Eq. (A 3.5) follows with Eq. (A 2.9). The argument requires certain integrations to be interchanged; this is permitted because all integrals involved converge absolutely.

A 4. Some Integrals

Let a and b be complex numbers such that

$$b - a = 2l \exp(i\varphi)\,, \tag{A 4.1}$$

where l is positive and $0 < \varphi < 2\pi$. Let C be the path in the complex z-plane given by

$$z = \frac{1}{2}(a + b) + t \exp\left(\frac{1}{2} i \varphi\right) \quad (-\infty < t < \infty)\,. \tag{A 4.2}$$

Consider the integral

$$\begin{aligned}J_1 &= \int_C |z - a|^\alpha [\mathrm{Im}(z - a)^{1/2}]^{-\beta}\, |dz| \\ &= \int_C |b - z|^\alpha [\mathrm{Im}(b - z)^{1/2}]^{-\beta}\, |dz|\,,\end{aligned} \tag{A 4.3}$$

where

$$\alpha \geqq -1\,, \quad \frac{1}{2}\beta - \alpha > 1\,. \tag{A 4.4}$$

In the second member of Eq. (A 4.3), z may be replaced by the variable ω defined according to

$$t + l \cos\frac{1}{2}\varphi = l \cos\frac{1}{2}\varphi \cosh\omega + \sinh\omega\,. \tag{A 4.5}$$

This yields

$$\begin{aligned}J_1 = {} & 2^{\frac{1}{2}\beta}\, l^{1+\alpha-\frac{1}{2}\beta} \left(\sin\frac{1}{2}\varphi\right)^{-\beta} \\ & \cdot \int_{-\infty}^{\infty} (1 + \cosh\omega)^{-\frac{1}{2}\beta} \left(\cosh\omega + \cos\frac{1}{2}\varphi \sinh\omega\right)^{1+\alpha} d\omega\,.\end{aligned} \tag{A 4.6}$$

Owing to Eq. (A 4.4) the integral is bounded uniformly in φ, and so

$$J_1 \leqq \text{const}\, |b-a|^{1+\alpha}\, [\text{Im}\,(b-a)^{1/2}]^{-\beta}\,. \tag{A 4.7}$$

Now consider the integral

$$J_2 = \int_c |z-a|^{\alpha_1} [\text{Im}\,(z-a)^{1/2}]^{-\beta_1} |b-z|^{\alpha_2} [\text{Im}\,(b-z)^{1/2}]^{-\beta_2} |dz|\,, \tag{A 4.8}$$

where

$$\begin{aligned} &\alpha_1+\alpha_2 \geqq -1\,, \tfrac{1}{2}\beta_1-\alpha_1+\tfrac{1}{2}\beta_2-\alpha_2>1\,,\\ &\alpha_1>-1\,,\quad \beta_1>0\,,\quad \tfrac{1}{2}\beta_1-\alpha_1>0\,,\\ &\alpha_2>-1\,,\quad \beta_2>0\,,\quad \tfrac{1}{2}\beta_2-\alpha_2>0\,. \end{aligned} \tag{A 4.9}$$

The conditions (A 4.9) are necessary and sufficient in order that there be numbers p, q such that

$$\begin{aligned} &p>1\,,\quad q>1\,,\quad (1/p)+(1/q)=1\,,\quad \alpha_1 p \geqq -1\,,\\ &\left(\tfrac{1}{2}\beta_1-\alpha_1\right)p>1\,,\quad \alpha_2 q \geqq -1\,,\quad \left(\tfrac{1}{2}\beta_2-\alpha_2\right)q>1\,. \end{aligned} \tag{A 4.10}$$

Hölder's inequality and Eq. (A 4.7) thus yield

$$J_2 \leqq \text{const}\, |b-a|^{\alpha_1+\alpha_2+1}\, [\text{Im}\,(b-a)^{1/2}]^{-\beta_1-\beta_2}\,. \tag{A 4.11}$$

This result leads directly to lemma 5.1.3.

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