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A Rigorous Formulation of LSZ Field Theory

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Abstract. An exact formulation of LSZ field theory is given. It is based on the Wightman axioms, asymptotic completeness, and a technical assumption stating the existence of retarded products of field operators. Reduction formulae are derived directly from the strong asymptotic condition. The GLZ-theorem, which states the conditions under which a given set of retarded functions defines a field theory, is formulated and proved in a rigorous way.

1. Introduction

The LSZ formulation of relativistic quantum field theory [1–3] has proved to be well adapted to the needs of elementary particle physics. In comparison with the Wightman formulation [4, 5] it suffers, however, from a lower degree of mathematical rigour. Existence problems are not always treated with due care, and limits are sometimes exchanged without full justification. Even though this lack of rigour has, up to now, not been prejudicial to the applications, it would obviously be desirable to have a rigorous version, preferably based on the Wightman axioms. Important steps in this direction have already been made. HAAG [6] and RUELLE [7] proved, starting from Wightman's axioms, the so-called strong asymptotic condition. This condition provides us with an asymptotic particle interpretation of the theory and makes it possible to add the axiom of asymptotic completeness to the familiar axioms of field theory. An exact definition of the S-matrix can then be given. HEPP [8] succeeded in deriving the LSZ reduction formulae rigorously for S-matrix elements between states with non-overlapping wave functions. These non-overlapping states form a total set in the Hilbert space of states. Nevertheless, Hepp's result is not completely satisfactory, because overlapping states are physically of considerable interest. They occur, for instance, always when in the course of a calculation the well-known summation over intermediate states has to be performed.

In this paper we propose to make some further steps towards a rigorization of LSZ. Besides the Wightman axioms and asymptotic completeness we shall have to assume the existence of retarded operators (see Chapter 3 for an exact formulation of this new postulate). On this basis we shall derive reduction formulae for the matrix elements of the field operator and the S-matrix, in the latter case also for overlapping states, directly from the strong asymptotic condition. The LSZ asymptotic condition will not be needed. Furthermore, we shall give a rigorous derivation of the unitarity equations of GLASER, LEHMANN and ZIMMERMANN, and a rigorous version of the GLZ theorem [3] which states the conditions under which a given set of retarded functions defines a field theory.

Existence of sharp retarded products seems to be necessary for the development of the formalism. We shall briefly discuss in Chapter 7 why the various known ways of defining smooth retarded products do not work satisfactorily. At present it is not known whether the existence of retarded products really has to be assumed or whether it is a consequence of the Wightman axioms. As long as this question remains unsettled the formalism as presented in this paper cannot be considered to be satisfactory.

As may have become clear by now, we shall treat LSZ as a field theoretical formalism, i.e. we consider the fields as the primary objects of the theory, the S-matrix as a derived quantity. We will therefore not treat the problem of interpolating a given S-matrix with an interacting field, local or otherwise [2, 9].

As usual we shall restrict ourselves to the simplest possible case of a single, hermitian, scalar field $A(x)$, associated with a single kind of spinless, uncharged particles with non vanishing rest mass m.

So as not to hide the essential aspects of the formalism in a jungle of computation we have abbreviated, or even omitted, some proofs which are straightforward in principle but involve lengthy algebraic or combinatorial considerations. This applies in particular to Chapter 3, dealing with the definition of retarded and time-ordered products, where proofs are almost completely suppressed.

Conventions as to signs and (2π)-factors, etc., have been chosen in accordance with the accepted usage in the theory of free fields.

2. The Basic Postulates

We start from the theory of a scalar, hermitian, Wightman field with mass $m > 0$. The axioms of such a theory are, in brief [4, 5]:

Postulate 1 (Quantum mechanics). The possible states of a physical system can be represented by vectors in a separable Hilbert space $\mathscr{H}$, observables and other physical quantities by operators in $\mathscr{H}$.

Postulate 2 (Relativistic invariance). A strongly continuous unitary representation $U(\Lambda, a)$ of the connected Poincaré group is defined on $\mathscr{H}$.

Postulate 3 (Spectral properties). Let $P_\mu = \int p_\mu dE(p)$ be the infinitesimal generators of the representation $U(1, a)$ of the translation group. Then the support of the spectral measure $dE(p)$ consists of the isolated point $p = 0$, the one particle hyperboloid $p^2 = m^2$, $p_0 > 0$, and a continuum in $p^2 \geqq 4m^2$, $p_0 > 0$. Let $\mathscr{H}_0$, $\mathscr{H}_1$ be the eigenspaces of the mass operator $M^2 = P_\mu P^\mu$ belonging to the eigenvalues 0, m^2, respectively. $\mathscr{H}_0$ is one-dimensional. It is spanned by a normalized vector $|0\rangle$, called the vacuum. On $\mathscr{H}_1$ the operators $U(\Lambda, a)$ define an irreducible representation of the Poincaré group to mass m and spin 0.

Postulate 4 (Field theory). In $\mathscr{H}$ there exists an operator valued distribution $A(x)$ with the following properties.

a) To each strongly decreasing test function $\varphi(x_1, \ldots, x_n) \in \mathscr{S}$, n arbitrary, there corresponds an (in general unbounded) closed operator

$$A^n(\varphi) = \int A(x_1) \ldots A(x_n)\, \varphi(x_1, \ldots, x_n)\, dx_1 \ldots dx_n\,, \tag{1}$$

called a *field monomial*. All field monomials A^n (note: $A^1 = A$) are defined on $|0\rangle$, and the vector $A^n(\varphi)\,|0\rangle$ depends on φ strongly continuously. The linear hull D of the set of all vectors $A^n(\varphi)\,|0\rangle$ is dense in $\mathscr{H}$.

It follows from these assumptions that $A^n(\varphi)$ is defined on all of D.

b) $A(\varphi)$ is hermitian for real φ.

c) A transforms under the Poincaré group as a scalar:

$$A(\Lambda x + a) = U(\Lambda, a)\, A(x)\, U^*(\Lambda, a)\,. \tag{2}$$

d) A is local:

$$[A(x), A(y)] = 0 \quad \text{for} \quad (x - y)^2 < 0\,. \tag{3}$$

e) $A(x)$ has non-vanishing matrix elements between $|0\rangle$ and $\mathscr{H}_1$. By multiplication with a constant we can then obtain that $\langle 1|\, A(x)\, |0\rangle$ is equal to the corresponding free field matrix element for all $|1\rangle \in \mathscr{H}_1$. This normalization will be assumed.

Under these assumptions the Haag-Ruelle asymptotic condition can be proved [5, 7], which we shall use in HEPP's formulation [8].

Let $\mathfrak{G} \subset \mathscr{S}$ be the space of test functions $\tilde{f}(p)$ with support in the set

$$G = \{p\colon 0 \leqq p^2 \leqq 4m^2, p_0 > 0\}\,. \tag{4}$$

To each $\tilde{f} \in \mathfrak{G}$ we associate an operator

$$A_f(t) = \int d^4q\; e^{-itq^-}\, \tilde{f}^*(q)\, \tilde{A}(q)\,. \tag{5}$$

with

$$q^\pm = q_0 \pm \omega(\boldsymbol{q})\,, \qquad \omega(\boldsymbol{q}) = +\,(\boldsymbol{q}^2 + m^2)^{1/2}\,, \tag{6}$$

$$A(x) = (2\pi)^{-3/2} \int d^4p\, \tilde{A}(p)\, e^{-ipx}\,. \tag{7}$$

Define

$$\Phi(t) = A_{f_1}^{(*)}(t) \ldots A_{f_n}^{(*)}(t)\, |0\rangle\,, \tag{8}$$

$\tilde{f}_i \in \mathfrak{G}$, n arbitrary. The superscript $^{(*)}$ means that a star may or may not be present. $\Phi(t)$ is a well defined vector in $\mathscr{H}$ for all finite values of t, depending continuously on t. The strong asymptotic condition states then the existence of a free, scalar, hermitian, field $A^{\text{in}}(x)$ with mass m, such that

$$\lim_{t \to -\infty} \Phi(t) = A_{f_1}^{\text{in}(*)} \ldots A_{f_n}^{\text{in}(*)}\, |0\rangle\,. \tag{9}$$

Here the limit is to be taken in the strong topology of $\mathscr{H}$, and

$$\hat{f}(\boldsymbol{p}) = \tilde{f}(\omega(\boldsymbol{p}), \boldsymbol{p}) \in \mathscr{S} \quad \text{in} \quad \boldsymbol{p}\,. \tag{10}$$

$A_{\hat{f}}^{\text{in}}$ is the destruction operator

$$A_{\hat{f}}^{\text{in}} = \int \frac{d^3 q}{2\omega(\boldsymbol{q})} \hat{f}^*(\boldsymbol{q})\, \hat{A}^{\text{in}}(\boldsymbol{q})\,, \tag{11}$$

if

$$\tilde{A}^{\text{in}}(q) = \delta_+(q)\, \hat{A}^{\text{in}}(\boldsymbol{q}) + \delta_-(q)\, \hat{A}^{\text{in}*}(-\boldsymbol{q})\,, \tag{12}$$

$$\delta_\pm(q) = \theta(\pm q_0)\, \delta(q^2 - m^2)\,. \tag{13}$$

Stars appear on the r.h.s. of (9) in the same places as in (8).

Similarly there exists another free field A^{out}, such that (9) holds in the limit $t \to +\infty$ with A^{in} replaced by A^{out}.

Let $\mathscr{Z}$ be the set of all vectors of the form (8), $\mathscr{Z}^{\text{in}}$ the set of all vectors (9), $\mathscr{L}$ and $\mathscr{L}^{\text{in}}$ the linear hulls of $\mathscr{Z}$ and $\mathscr{Z}^{\text{in}}$ respectively. The closure of $\mathscr{L}^{\text{in}}$ will be called $\mathscr{H}^{\text{in}}$. Analogously we define $\mathscr{Z}^{\text{out}}$, $\mathscr{L}^{\text{out}}$, $\mathscr{H}^{\text{out}}$. The asymptotic condition (9) can of course be extended to $\mathscr{L}$ by linearity[1].

We can now formulate the fifth postulate of the theory.

Postulate 5 (Asymptotic completeness). We have

$$\mathscr{H}^{\text{in}} = \mathscr{H}^{\text{out}} = \mathscr{H}\,. \tag{14}$$

A sixth and last postulate of a more technical nature will be formulated in the next chapter.

Finally, let us introduce some additional notations which will be useful in the sequel. The functions $\hat{f}$, $\hat{g}$, are called non-overlapping if $\operatorname{supp}\hat{f} \cap \operatorname{supp}\hat{g} = \emptyset$. The functions $\tilde{f}, \tilde{g} \in \mathfrak{S}$ are called non-overlapping if the corresponding $\hat{f}$, $\hat{g}$ defined by (10) are non-overlapping. $\mathscr{Z}_0$, $\mathscr{Z}_0^{\text{in}}$ are the subsets of $\mathscr{Z}$, $\mathscr{Z}^{\text{in}}$ obtained by taking only the vectors (8), (9) in which the wave functions $\hat{f}_i$ are pairwise non-overlapping. $\mathscr{L}_0$, $\mathscr{L}_0^{\text{in}}$ are again the respective linear hulls.

3. The Retarded and Time Ordered Products

The retarded [2] and time ordered [1] products of fields play an important part in the formalism. They are *formally* defined by

$$R(x_1, \ldots, x_n) = (-i)^{n-1} \sum \theta(x_1, x_{i_2}, \ldots, x_{i_n}) [\ldots [A(x_1), A(x_{i_2})], \ldots, A(x_{i_n})]\,,$$

$$T(x_1, \ldots, x_n) = \sum \theta(x_{i_1}, \ldots, x_{i_n})\, A(x_{i_1}) \ldots A(x_{i_n})\,. \tag{15}$$

[1] It is actually possible to prove the strong asymptotic condition for states of the form

$$\Phi(t) = \int d^4 q_j \exp(-\,\text{it} \sum \pm q_j^\pm)\, \tilde{f}(\pm q_1, \ldots, \pm q_n)\, \tilde{A}(q_1) \ldots \tilde{A}(q_n)\, |0\rangle$$

with $\tilde{f} \in \mathscr{S}$, $\operatorname{supp}\tilde{f} \subset G \otimes \cdots \otimes G$, i.e. for non-factorizing multiparticle wave functions. This fact is, however, not essential for our purposes. We will therefore not give the proof, which is rather lengthy.

Here $\theta(x_1, \ldots, x_n)$ is a function which is equal to one if $x_1^0 > x_2^0 > \cdots > x_n^0$, zero otherwise. The summation extends in the first case over all permutations of the arguments $(x_2, \ldots, x_n)$, in the second case over all permutations of $(x_1, \ldots, x_n)$.

Unfortunately the expressions (15) have no exact meaning because distributions can in general not be multiplied with discontinuous functions. We shall therefore use a different definition, namely we shall define R, T as objects having a certain number of properties, which will be used later on, and which are formally satisfied by the expressions (15). It is not known at present whether objects having these properties do always exist in a Wightman field theory. Their existence must therefore be assumed.

Postulate 6 (Existence of retarded products). There exist operator valued distributions $R(x_1, \ldots, x_n)$ for all positive integers n, with the following properties.

i) $R(\varphi) = \int dx_j\, R(x_1, \ldots, x_n)\, \varphi(x_1, \ldots, x_n)$, $\varphi \in \mathscr{S}$, has the properties of a field monomial as specified in Postulate 4, i.e. $R(\varphi)$ is a closed operator which is defined on D and maps D into itself.

ii) $R(\varphi)$ is hermitian for real φ.

iii) $R(x_1) = A(x_1)$.

iv) $R(x_1, x_2, \ldots, x_n)$ is symmetrical in the variables $x_2, \ldots, x_n$.

v) The R satisfy the identities [3]

$$\begin{aligned} &R(x, y, x_1, \ldots, x_n) - R(y, x, x_1, \ldots, x_n) \\ &= -i \sum [R(x, x_{i_1}, \ldots, x_{i_\alpha}), R(y, x_{i_{\alpha+1}}, \ldots, x_{i_n})]\,, \end{aligned} \tag{16}$$

where the summation extends over all partitions of $\{x_1, \ldots, x_n\}$ into two subsets, one of which may be empty.

vi) The support of $R(x_1, \ldots, x_n)$ is contained in the set

$$T_n = \{(x_1, \ldots, x_n) : (x_1 - x_i) \in \overline{V}_+, i = 2, \ldots, n\}\,, \tag{17}$$

$\overline{V}_+$ the closed forward cone.

vii) The R are covariant ,i.e.

$$R(\Lambda x_1 + a, \ldots, \Lambda x_n + a) = U(\Lambda, a)\, R(x_1, \ldots, x_n)\, U^*(\Lambda, a) \tag{18}$$

for all Poincaré transformations (Λ, a).

The support condition vi) could be relaxed without seriously impeding the usefulness of the formalism. It would suffice to demand that $R(x_1, \ldots, x_n)$ vanish outside a ε-neighbourhood of T_n, or even that it decrease strongly at infinity outside T_n. With this latter formulation the formalism becomes applicable to quasilocal fields. Such a weakening of the support condition would of course necessitate the abandonment of the covariance condition (18) for Lorentz transformations. The formalism would then lose its explicit invariance, but this is not necessarily more than a minor nuisance. Covariance under translations, however, is essential. In this paper we shall always work with the sharp support (17).

Once we have the retarded products R, we can define the chronological and antichronological products $T(x_1, \ldots, x_n)$ and $\overline{T}(x_1, \ldots, x_n)$ with the help of the relations [10]

$$\begin{aligned} &R(x_1, x_2, \ldots, x_n) \\ &= (-i)^{n-1} \sum (-1)^{k-1}\, \overline{T}(x_{i_2}, \ldots, x_{i_k})\, T(x_1, x_{i_{k+1}}, \ldots, x_{i_n}) \\ &= (-i)^{n-1} \sum (-1)^{k-1}\, \overline{T}(x_1, x_{i_2}, \ldots, x_{i_k})\, T(x_{i_{k+1}}, \ldots, x_{i_n})\,. \end{aligned} \tag{19}$$

The summations extend as in (16) over all partitions of the set $\{x_2, \ldots, x_n\}$ into two subsets. The relations (19), together with the requirement of total symmetry of T and $\overline{T}$, allow the recursive construction of T, $\overline{T}$ in terms of R-operators.

The following important properties of R, T, and $\overline{T}$ can be derived from these definitions[2]

a) $$R(x_1, \ldots, x_n) = -i\,[R(x_1, \ldots, x_{n-1}), A(x_n)] \tag{20}$$

if $x_n^0 < x_i^0$ for $i = 1, \ldots, n-1$.

b) $$\begin{aligned} T(x_1, \ldots, x_n) &= T(x_1, \ldots, x_k)\, T(x_{k+1}, \ldots, x_n)\,, \\ \overline{T}(x_1, \ldots, x_n) &= \overline{T}(x_{k+1}, \ldots, x_n)\, \overline{T}(x_1, \ldots, x_k)\,, \end{aligned} \tag{21}$$

if $x_i^0 > x_j^0$ for all $i = 1, \ldots, k$, $j = k+1, \ldots, n$.

c) $$T(x_1, \ldots, x_n)^* = T(x_1, \ldots, x_n) \text{ on } D\,. \tag{22}$$

Besides the retarded products R we shall have occasion to consider also the generalized retarded products (g.r.p.), if only in an auxiliary role. For their formal definition and a comprehensive discussion of their properties we refer to ref. [11] and the original papers quoted there. Here we shall use the following rigorous definition, based on postulate 6.

Let S be the index set $(1, \ldots, n)$. An *n-cell* is a set of signs $\sigma_I = +$ or $-$, attached to the proper subsets I of S, such that $\sigma_I \neq \sigma_{\complement I}$, and $\sigma_{I' \cup I''} = \sigma_{I'}$, if $I' \cap I'' = \emptyset$ and $\sigma_{I'} = \sigma_{I''}$. Two n-cells are called *adjacent* if all σ_I except two are the same in the two cells. The two differing signs belong then to two complementary subsets I_0, $\complement I_0$ of S. I_0 is called the border between the two cells.

With each n-cell C_μ we associate a g.r.p. $G_\mu(x_1, \ldots, x_n)$, such that the following two conditions are satisfied.

a) Let C_μ, C_ν be adjacent, with border $I_0 = \{i_1, \ldots, i_k\}$. Let σ_{I_0} be negative in C_μ, positive in C_ν. We define a k-cell C_α in I_0 by attributing to each proper subset of I_0 the sign that the same subset has in C_μ, and

[2] As has already been remarked in the introduction we shall not give any proofs in this chapter, since they are not central to our purpose and would lengthen the paper in an inadmissible way. The reader can easily convince himself, that everything stated in the remainder of this chapter is formally true for (15).

a $(n-k)$-cell C_β in $\complement I_0$ in the same way. Then

$$\begin{aligned} & G_\mu(x_1, \ldots, x_n) - G_\nu(x_1, \ldots, x_n) \\ & = -i\,[G_\alpha(x_{i_1}, \ldots, x_{i_k}), G_\beta(x_{i_{k+1}}, \ldots, x_{i_n})]\,. \end{aligned} \tag{23}$$

b) A particular n-cell is specified through the condition that $\sigma_i = +$ for $i = 2, \ldots, n$, if σ_i is the sign attached to the subset consisting of the single element i. The g.r.p. corresponding to this cell is $R(x_1, \ldots, x_n)$.

It can be shown that operator distributions satisfying these requirements exist and are uniquely determined (if Postulate 6 holds). They can be written as $R(x_1, \ldots, x_n)$ plus a sum of multiple commutators of R-operators of lower order. The G_μ have the general properties of field monomials and are hermitian. The commutator of two arbitrary g.r.p. G_α, G_β, depending on non-overlapping sets of variables, can always be written in the form (23) as difference of two g.r.p.

From $G_\mu(x_1, \ldots, x_n)$ we derive another g.r.p. $G_\mu^+(x_1, \ldots, x_n; x_{n+1})$ by fixing its cell as follows: the signs of proper subsets of $(1, \ldots, n)$ shall be the same in C_μ^+ as in C_μ, and σ_{n+1} is positive. Then:

$$\begin{aligned} G_\mu^+(x_1, \ldots, x_n; x_{n+1}) &= 0 \quad \text{if} \quad x_{n+1}^0 > x_i^0 \quad \text{for} \quad i = 1, \ldots, n\,, \\ G_\mu^+(x_1, \ldots; x_{n+1}) &= -i\,[G(x_1, \ldots, x_n), A(x_{n+1})] \\ & \quad \text{if} \quad x_{n+1}^0 < x_i^0 \quad \text{for} \quad i = 1, \ldots, n\,. \end{aligned} \tag{24}$$

The g.r.p. obtained through m-fold iteration of this process will be denoted $G_\mu^+(x_1, \ldots, x_n; x_{n+1}, \ldots, x_{n+m})$. They are symmetrical in the variables behind the semicolon. For ordinary retarded products we have $R^+(x_1, \ldots, x_n; x_{n+1}) = R(x_1, \ldots, x_{n+1})$.

The vacuum expectation values of R, T, $\overline{T}$, G_μ are denoted r, τ, $\overline{\tau}$, g_μ, respectively. The time ordered function $\tau(x_1, \ldots, x_n)$ has the familiar cluster expansion

$$\tau(x_1, \ldots, x_n) = \sum \tau^T(x_{i_1}, \ldots) \ldots \tau^T(\ldots, x_{i_n})\,, \tag{25}$$

where the truncated functions τ^T have the correct space like asymptotic decay, i.e.

$$\int \prod d^4 x_i\, \tau^T(x_1, \ldots, x_n)\, \varphi(x_1, x_2 + \boldsymbol{a}_2, \ldots, x_n + \boldsymbol{a}_n)\,, \qquad \varphi \in \mathscr{S}$$

is as a function of $\boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$, in the Schwartz space $\mathscr{S}$ of tempered test functions.

4. Reduction Formulae

In this chapter we wish to derive reduction formulae, i.e. relations expressing matrix elements of the field operator on the one hand, of the S-matrix on the other hand, in terms of retarded or time ordered functions. For this we prove first a variant of the strong asymptotic condition.

Let $\tilde{f}(p) \in \mathfrak{S}$. Following HEPP [8] we introduce the function

$$\bar{f}(x, t) = (2\pi)^{-5/2} \int d^4 p\, e^{-ipx}\, e^{itp^-}\, \tilde{f}(p)\,. \tag{26}$$

From the equivalent representation

$$\bar{f}(x, t) = (2\pi)^{-5/2} \int d^4q\, \delta_+(q)\, [2\omega(\boldsymbol{q})]\, e^{-iqx} \int dp_0\, \tilde{f}(p_0, \boldsymbol{q})\, e^{ip_0(t-x_0)}$$

we obtain with the help of RUELLE's work on smooth solutions of the Klein-Gordon equation [7, 12] the estimates

$$\begin{aligned} |\bar{f}(x, t)| &\leqq c_N |t|^{-3/2} (1 + |x_0 - t|)^{-N}, \\ \int d^3x\, |\bar{f}(x_0, \boldsymbol{x}, t)| &\leqq c'_N |t|^{3/2} (1 + |x_0 - t|)^{-N} \end{aligned} \tag{27}$$

for all integers $N > 0$. Estimates of the same form hold for all derivatives of $\bar{f}$.

We can now prove

Theorem 1. *Let*

$$\begin{aligned} \Phi(t) &= \int \prod_1^n \{d^4p_i\, \tilde{f}_i(-p_i)\, e^{-itp_i^+}\} \prod_1^m \{d^4q_j\, \tilde{g}^*(q_j)\, e^{-itq_j^-}\} \\ &\quad \cdot \tilde{T}(p_1, \ldots, p_n, q_1, \ldots, q_m)\, |0\rangle \\ &= \int \prod_1^n \{d^4x_i\, \bar{f}_i(x_i, t)\} \prod_1^m \{d^4y_j\, \bar{g}_j^*(y_j, t)\}\, T(x_1, \ldots, x_n, y_1, \ldots, y_m)\, |0\rangle \end{aligned} \tag{28}$$

with $\tilde{f}_i, \tilde{g}_j \in \mathfrak{S}$.

Then

$$\underset{t\to-\infty}{\text{st.}\lim}\, \Phi(t) = \sum_{\{i_j\}} \prod_{j=1}^m \tau(g_j, f_{i_j}) \prod_{k\notin\{i_j\}} A_k^{\text{in}*}\, |0\rangle\,. \tag{29}$$

Here the summation extends over all ordered subsets $\{i_1, \ldots, i_m\}$ *with* m *elements of the index set* $\{1, \ldots, n\}$. $A_k^{\text{in}*}$ *is the creation operator for a particle with wave function* $\hat{f}_k$, *and*

$$\tau(g, f) = \int dp\, dq\, \tilde{f}(-p)\, \tilde{g}^*(q)\, \tilde{\tau}(p, q)\,. \tag{30}$$

The limit (29) *vanishes if* $n < m$.

The same result, with A^{in} replaced by A^{out}, holds of course in the limit $t \to +\infty$. In both cases the chronological product T can be replaced by the antichronological product $\bar{T}$. The τ in (30) has then of course also to be replaced by $\bar{\tau}$.

The proof of Theorem 1 follows narrowly the familiar proof of the asymptotic condition (9). First we show that the strong limit of $\Phi(t)$ exists, by showing that $\left|\frac{d}{dt}\Phi(t)\right| \leqq c|t|^{-3/2}$. In order to do this we consider

$$\begin{aligned} &\langle\dot{\Phi}(t)|\dot{\Phi}(t)\rangle \\ &= \int \prod_1^n \{d^4p_i\, \tilde{f}_i^*(-p_i)\, e^{itp_i^+}\} \prod_1^m \{d^4q_j\, \tilde{g}_j(q_j)\, e^{itq_j^-}\} \left[\sum_1^n p_i^+ + \sum_1^m q_j^-\right] \\ &\quad \cdot \prod \{d^4p'_i\, \tilde{f}_i(-p'_i)\, e^{-itp_i'^+}\} \prod \{d^4q'_j\, \tilde{g}_j^*(q'_j)\, e^{-itq_j'^-}\} \left[\sum p_i'^+ + \sum q_j'^-\right] \\ &\quad \cdot \langle 0|\, \tilde{\bar{T}}(-p_1, \ldots, -q_m)\, \tilde{T}(p'_1, \ldots, q'_m)\, |0\rangle\,. \end{aligned} \tag{31}$$

The vacuum expectation value occurring here can be developped into its cluster expansion. The corresponding truncated functions can be shown, as was the case in (25), to have the correct decrease in space like directions. Terms in this cluster expansion containing truncated functions of three or more points give contributions vanishing in the limit $|t| \to \infty$ at least as $|t|^{-3}$, because

$$\left| \int \prod_1^l dx_i \bar{f}_i^{(*)}(x_i, t) \prod_1^k dy_j \bar{g}_j^{(*)}(y_j, t) \langle 0 | \bar{T}(x_1, \ldots, x_l) \, T(y_1, \ldots, y_k) | 0 \rangle^T \right| \leqq c |t|^{-\frac{3}{2}(k+l-2)} .$$

This follows as usual from the estimates (27) and the space like decrease of truncated vacuum expectation values. ($\langle 0 | \bar{T} \, T | 0 \rangle^T$ can be written as a finite sum of derivatives of continuous functions with strong decrease in space like directions.)

The contribution to (31) from the 2-point function terms in the cluster expansion vanishes for the following reason. Consider any variable p_i. Because of the support properties of $\tilde{f}_i, \tilde{g}_j$, this variable can only be associated in a non-vanishing 2-point function with either a q_j, or a p'_k. In the former case we have

$$(p_i^+ + q_j^-) \, \tilde{\bar{\tau}}(-p_i, -q_j) = 0$$

because of the factor $\delta^4(p_i + q_j)$ contained in $\tilde{\bar{\tau}}$. The corresponding terms p_i^+, q_i^- can thus be dropped in the first square bracket in (31). In the second case we obtain

$$\langle 0 | \tilde{A}(-p_i) \, \tilde{A}(p'_k) | 0 \rangle \, p_i^+ = 0$$

because in the support of $\tilde{f}_i(-p_i)$ we have

$$\langle 0 | \tilde{A}(-p_i) \, \tilde{A}(p'_k) | 0 \rangle = \delta^4(p'_k - p_i) \, \delta_-(p_i) \, .$$

Thus p_i^+ can again be dropped in the first square bracket. For the same reason we can drop q_j^-, if q_j is associated with a q_k. This takes care of all the terms in this square bracket, i.e. after dropping all of them there is nothing left.

Hence the convergence of $\Phi(t)$ is proved. The limit Φ_{in} can be determined by calculating its scalar products $\langle \Psi_{\text{in}} | \Phi_{\text{in}} \rangle$ with a dense set of states Ψ_{in} in $\mathscr{H}^{\text{in}}$. This is easily done for Ψ_{in} of the form (9):

$$\langle \Psi_{\text{in}} \mid \Phi_{\text{in}} \rangle = \lim_{t \to -\infty} \langle \Psi(t) | \Phi(t) \rangle \, ,$$

$\Psi(t)$ as in (8), $\Phi(t)$ given by (28). This limit can be determined as in the convergence proof. One has to study an expression of a structure similar to (31), where now, however, the two square brackets are missing. The 2-point function terms survive alone in the limit and can easily be calculated to give the result demanded by (29).

In the special case $m = 0$ Eq. (29) becomes

$$\lim_{t \to -\infty} \Phi(t) = \prod_i A_i^{\text{in}*} | 0 \rangle \, , \tag{32}$$

hence all vectors in $\mathscr{L}^{\text{in}}$ can be obtained as limits of vectors of the form (28).

We are now in a position to derive reduction formulae for the matrix elements

$$M = \langle \Phi_{\text{in}} | \, G_\mu(\varphi) \, | \Psi_{\text{in}} \rangle \tag{33}$$

with

$$\begin{aligned} G_\mu(\varphi) &= \int \prod_1^l d^4x_i \, G_\mu(x_1, \ldots, x_l) \, \varphi(x_1, \ldots, x_l) \, , \\ &= \int \prod_1^l d^4k_i \, \tilde{G}_\mu(k_1, \ldots, k_l) \, \tilde{\varphi}(k_1, \ldots, k_l) \, , \qquad \varphi \in \mathscr{S} \end{aligned} \tag{34}$$

$$\Phi_{\text{in}} = A_{\hat{f}_1}^{\text{in}*} \ldots A_{\hat{f}_n}^{\text{in}*} |0\rangle \, , \qquad \Psi_{\text{in}} = A_{\hat{g}_1}^{\text{in}*} \ldots A_{\hat{g}_m}^{\text{in}*} |0\rangle \, , \tag{35}$$

$\hat{f}_i, \hat{g}_j \in \mathscr{S}$. In order to be sure that M exists we shall assume that either $\Phi_{\text{in}} \in \mathscr{H}_0^{\text{in}}$ or $\Psi_{\text{in}} \in \mathscr{H}_0^{\text{in}}$, i.e. at least one of the two states is non-overlapping. In this case M exists because $G_\mu(\varphi)$ is defined on $\mathscr{H}_0^{\text{in}}$, as can be proved with the methods of Hepp's paper [8]. This non-overlap assumption is not used in the derivation of the reduction formula: our result (Theorem 2) is valid for Φ_{in}, Ψ_{in} both overlapping, if $G_\mu(\varphi)$ is defined on all of $\mathscr{H}^{\text{in}}$. Whether this is generally true as a consequence of our assumptions is not known.

We shall avoid a lot of irrelevant combinatorics by giving the explicit derivation of the reduction formula only for a typical special case, namely for the case that G_μ is the field operator A, and that all $\hat{f}_i$ are orthogonal to all $\hat{g}_j$, except possibly for a single pair:

$$(\hat{f}_i, \hat{g}_j) = \int \frac{d^3p}{2\omega(\boldsymbol{p})} \hat{f}_i^*(\boldsymbol{p}) \, \hat{g}_j(\boldsymbol{p}) = 0 \text{ for all } i, j, \text{ except } i = j = 1 \, . \tag{36}$$

Let K be the index set $\{1, \ldots, n\}$, K' the set $\{2, \ldots, n\}$, K_1 and K_2 two complementary subsets of K, and K_1', K_2' two complementary subsets of K'. Analogously we define $L = \{1, \ldots, m\}$, $L' = \{2, \ldots, m\}$, $L_{1,2}$, $L_{1,2}'$. n_1, m_1, etc., are the numbers of elements in K_1, L_1, etc.

Define

$$\begin{aligned} F(K_1, L_1, t) = \int \prod_1^n dx_i \bar{f}_i^*(x_i, t) \prod_1^m dy_j \bar{g}_j(y_j, t) \, dz \, \varphi(z) \\ \cdot \langle 0 | \, \bar{T}(X_1, Y_1) \, A(z) \, T(X_2, Y_2) \, | 0 \rangle \, . \end{aligned} \tag{37}$$

Here X_α, Y_α stand for the sets of variables $\{x_i\}$, $i \in K_\alpha$, and $\{y_j\}$, $j \in L_\alpha$. $F'(K_1', L_1', t)$ is defined analogously, with integration over the variables $x_2, \ldots, x_n, y_2, \ldots, y_m$.

From Theorem 1 we obtain

$$\lim_{t \to -\infty} F(K_1, L_1, t) = \begin{cases} M \quad \text{if} \quad K_2 = L_1 = \emptyset \\ \tau(g_1, f_1) \, \langle \Phi_{\text{in}}' | \, A(\varphi) \, | \Psi_{\text{in}}' \rangle \text{ if } K_2 = \{x_1\}, L_1 = \emptyset \\ \bar{\tau}(g_1, f_1) \, \langle \Phi_{\text{in}}' | \, A(\varphi) \, | \Psi_{\text{in}}' \rangle \text{ if } K_2 = \emptyset, L_1 = \{y_1\} \\ 0 \quad \text{in all other cases} \, . \end{cases} \tag{38}$$

Φ_{in}', Ψ_{in}' are obtained fron Φ_{in}, Ψ_{in} by dropping the first creation operator $A_{\hat{f}_1}^{\text{in}*}$ or $A_{\hat{g}_1}^{\text{in}*}$ respectively.

Because of

$$\tau(x, y) + \bar{\tau}(x, y) = \langle 0|\, A(x)\, A(y)\, |0\rangle + \langle 0|\, A(y)\, A(x)\, |0\rangle$$

we have $\tau(g_1, f_1) + \bar{\tau}(g_1, f_1) = (\hat{f}_1, \hat{g}_1)$. With this we obtain from (38):

$$\begin{aligned} M = (-1)^n \lim_{t \to -\infty} \sum_{K_1, L_1} (-1)^{n_1 + m_1} F(K_1, L_1, t) \\ - (-1)^n (\hat{f}_1, \hat{g}_1) \sum_{K_1', L_1'} (-1)^{n_1' + m_1'} F'(K_1', L_1', t)\,. \end{aligned} \tag{39}$$

For evaluating this expression we need the result of the following auxiliary consideration. Let $G(t_1, \ldots, t_n)$ be an expression of the form

$$G(t_1, \ldots, t_n) = \int \prod_1^n \{dx_i \bar{f}_i(x_i, t_i)\}\, D(x_1, \ldots, x_n)\,. \tag{40}$$

with D a tempered distribution, and $\bar{f}_i$ functions of the form (26), where the $\tilde{f}_i$ may now be arbitrary functions in $\mathscr{S}$, without restrictions on their support. $\bar{f}(x, t)$ is, for fixed t, a test function in x. With the familiar notations

$$\boldsymbol{x}^\alpha = \prod_{i=1}^3 x_i^{\alpha_i}, \quad \partial_\alpha = \prod_{i=1}^3 \frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}\,,$$

$$\alpha = \{\alpha_1, \alpha_2, \alpha_3\}, \quad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3$$

we obtain

$$\begin{aligned} &(x_0 - t)^N \boldsymbol{x}^\alpha \bar{f}(x, t) \\ &= (2\pi)^{-5/2} (-i)^{N + |\alpha|} \sum_{\beta=0}^{|\alpha|} t^\beta \int d^4p \exp[-i p_0 (x_0 - t)]\, e^{-i\boldsymbol{p}\boldsymbol{x}} e^{-it\,\omega(\boldsymbol{p})} \\ &\cdot \frac{\partial^N}{\partial p_0{}^N} \tilde{f}_{\alpha\beta}(p)\,, \end{aligned} \tag{41}$$

where the $\tilde{f}_{\alpha\beta}$ are defined by

$$\partial_\alpha [\tilde{f}(p)\, e^{-it\,\omega(\boldsymbol{p})}] = e^{-it\,\omega(\boldsymbol{p})} \sum_\beta t^\beta \tilde{f}_{\alpha\beta}(p)\,.$$

From (41) we obtain the estimates, for any integers N, $M \geqq 0$:

$$|\bar{f}(x, t)| \leqq \frac{c_{MN}\,(1 + |t|)^M}{(1 + |\boldsymbol{x}|)^M\, (1 + |x_0 - t|)^N}\,. \tag{42}$$

c_{MN} a positive constant. Similar estimates hold for all x-derivatives of $\bar{f}$.

Let $\sigma(x_2, \ldots, x_n, t_2, \ldots, t_n)$ be a C^∞-function, all of whose x-derivatives are for fixed t absolutely bounded with bounds increasing at most polynomially with increasing t_i. Then

$$\begin{aligned} G_1(x_1, t_1, \ldots, t_n) = \int \prod_{i=2}^n \{dx_i \bar{f}_i(x_i, t_i)\}\, \sigma(x_2, \ldots, x_n, t_2, \ldots, t_n) \\ \cdot D(x_1, \ldots, x_n) \end{aligned} \tag{43}$$

is a tempered distribution in x_1, which is polynomially bounded in the parameters t_i. G_1 can be written as a derivative of a continuous function

$G_2(x_1, t_i)$ [13]:

$$G_1(x, t_i) = D_x G_2(x, t_i) \tag{44}$$

with

$$|G_2(x, t_i)| \leqq c(1 + |\boldsymbol{x}|)^A (1 + |x_0|)^B (1 + t)^C \tag{45}$$

for suitable integers A, B, C, and a positive constant c. t stands for Max $|t_i|$.

Let $\varrho(u) \in \mathfrak{D}$, with

$$\begin{aligned} \varrho &= 1 \quad \text{in} \quad |u| \leqq 1\,, \\ \varrho &= 0 \quad \text{in} \quad |u| \geqq 2\,. \end{aligned} \tag{46}$$

Let

$$\varrho_t(u) = \varrho\left(u/\sqrt{t}\right), \tag{47}$$

$$\varrho_t'(u) = 1 - \varrho_t(u)\,. \tag{48}$$

Finally, let

$$G'(t_i) = \int dx_1 \bar{f}_1(x_1, t_1)\, \varrho_t'(x_1^0 - t_1)\, D_{x_1} G_2(x_1, t_i)\,,$$

or, through integration by parts:

$$G'(t_i) = \int dx\, G_2(x, t_i)\, \check{D}_x[\bar{f}_1(x, t_1)\, \varrho_t'(x^0 - t_1)]\,. \tag{49}$$

The derivation $\check{D}_x$ does not alter the relevant properties of $\bar{f}_1$ and ϱ_t' and will therefore be dropped for the sake of simplicity. Then

$$|G'(t_i)| \leqq c c_{MN}(1 + t)^{C+M} \int d^3x (1 + |\boldsymbol{x}|)^{A-M} \cdot \int dx^0\, |\varrho_t'(x^0 - t_1)|\, \frac{(1 + |x^0|)^B}{(1 + |x^0 - t_1|)^N}\,.$$

The $\boldsymbol{x}$-integral exists if M is chosen sufficiently large. The x^0-integral can be estimated as follows:

$$\int dx^0 |\varrho_t'(x^0 - t)|\, \frac{(1 + |x^0|)^B}{(1 + |x^0 - t_1|)^N} \leqq \operatorname*{Max}_u |\varrho_t'(u)| \cdot \int\limits_{|u - t_1| \geqq \sqrt{|t_1|}} du\, \frac{(1 + |u|)^B}{(1 + |u - t_1|)^N}$$

and, with $N > B + 2$:

$$\int\limits_{|u-t_1| \geqq \sqrt{|t_1|}} du\, \frac{(1 + |u|)^B}{(1 + |u - t_1|)^N} \leqq (1 + \sqrt{|t_1|})^{N-B-2} \int du\, \frac{(1 + |u + t_1|)^B}{(1 + |u|)^{B+2}} \leqq K_N\left(1 + \sqrt{t}\right)^{N-2}$$

for some positive constant K_N. Hence

$$|G'(t_i)| \leqq C_{MN}(1 + t)^{C+M-\frac{N}{2}+1} \tag{50}$$

for a fixed $M > A + 3$ and arbitrary N, i.e. G' decreases for $t \to \infty$ stronger than any inverse power of t.

From this result we derive easily

$$\lim_{t \to \infty} t^N |G(t_i) - G^0(t_i)| = 0 \quad \text{for all } N > 0\,, \tag{51}$$

if

$$G^0(t_i) = \int \prod_1^n \{dx_i\, \bar{f}_i(x_i, t_i)\, \varrho_t(x_i^0 - t_i)\}\, D(x_1, \ldots, x_n)\,. \tag{52}$$

The function F defined in (37) is of the form (40). We can therefore apply (51) and obtain

$$\lim_{t\to-\infty} F(K_1, L_1, t)$$
$$= \lim_{t\to-\infty} \int \prod_1^n \{dx_i\, \bar{f}_i^*(x_i, t)\, \varrho_{|t|}(x_i^0 - t)\} \prod_1^m \{dy_j\, \bar{g}_j(y_j, t)\, \varrho_{|t|}(y_j^0 - t)\}$$
$$\cdot\, dz\, \varphi(z)\, \varrho_{|t|}(z^0)\, \langle 0|\, \bar{T}(X_1, Y_1)\, A(z)\, T(X_2, Y_2)\, |0\rangle\,. \tag{53}$$

The supports of $\varrho_{|t|}(u)$ and $\varrho_{|t|}(u - t)$ do not overlap for sufficiently large $|t|$. Because of (21) we can then replace the expectation value occurring in (53) by $\langle 0|\, \bar{T}(X_1, Y_1)\, T(z, X_2, Y_2)\, |0\rangle$ without changing the limit. The $\varrho_{|t|}$-factors can then again be dropped due to (51). From (39) and (19) we conclude

$$M = r(\Phi_{\text{in}}, \Psi_{\text{in}}, \varphi) + (\hat{f}_1, \hat{g}_1)\, r(\Phi'_{\text{in}}, \Psi'_{\text{in}}, \varphi) \tag{54}$$

with

$$r(\Phi_{\text{in}}, \Psi_{\text{in}}, \varphi)$$
$$= (-1)^n\, i^{m+n} \lim_{t\to-\infty} \int \prod [dx_i\, \bar{f}_i^*(x_i, t)] \prod [dy_j\, \bar{g}_j(y_j, t)]\, dz\, \varphi(z)$$
$$\cdot\, r(z, x_1, \ldots, x_n, y_1, \ldots, y_m) \tag{55}$$
$$= (-1)^n\, i^{m+n} \lim_{t\to-\infty} \int \prod [dp_i\, \tilde{f}_i^*(p_i)\, e^{-itp_i^-}] \prod [dq_j\, \tilde{g}_j(-q_j)\, e^{-itq_j^+}]$$
$$\cdot\, dk\, \tilde{\varphi}(k)\, \tilde{r}(k, p_1, \ldots, p_n, q_1, \ldots, q_m)\,.$$

Consider what happens to the x-space form of (55) if in one or more of the functions $\bar{f}_i, \bar{g}_j$ the argument t is replaced by $-t$. The limit in this case can again be computed by introducing the factors $\varrho_{|t|}$. It is then seen at once that the limit vanishes because of the support condition (17). We can therefore replace the factors $\exp(-itp_i^-)$, $\exp(-itq_j^+)$ in the p-space form of (55) by $[\exp(-itp_i^-) - \exp(itp_i^-)]$ and $[\exp(-itq_j^+) - \exp(itq_j^+)]$ respectively, without changing the result.

Let

$$\delta_t(u) = i\,\frac{e^{-itu} - e^{itu}}{u}\,. \tag{56}$$

The functions $\delta_t(u)$ converge for $t \to -\infty$ to $2\pi\,\delta(u)$ in the topology of tempered distributions.

Define

$$\tilde{r}^{\text{amp}}(k; p_1, \ldots, p_n) = \prod_{i=1}^n (p_i^2 - m^2)\, \tilde{r}(k, p_1, \ldots, p_n)\,. \tag{57}$$

Then

$$r(\Phi_{\text{in}}, \Psi_{\text{in}}, \varphi) = (-1)^n \lim_{t\to-\infty} \int \prod \left[dp_i\, \tilde{f}_i^*(p_i)\, \frac{\delta_t(p_i^-)}{p_i^+}\right]$$
$$\prod \left[dq_j\, \tilde{g}_j(-q_j)\, \frac{\delta_t(q_j^+)}{q_j^-}\right] dk\, \tilde{\varphi}(k)\, \tilde{r}^{\text{amp}}(k; p_1, \ldots, p_n, q_1, \ldots, q_m)\,. \tag{58}$$

$r(\Phi_{\text{in}}, \Psi_{\text{in}}, \varphi)$ and therefore also the r.h.s. of (58), depends only on the values of $\tilde{f}_i, \tilde{g}_j$, on the mass shell: for any $\chi(p) \in C^\infty$, $\operatorname{supp}\chi \subset G$,

$\chi(\omega(\boldsymbol{p}), \boldsymbol{p}) = 1$, the limit

$$\tilde{r}_{MS}^{\mathrm{amp}}(k; \boldsymbol{p}_1, \ldots, \boldsymbol{p}_n, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_m) = (2\pi)^{-n-m} \lim_{t\to\infty} \int \Pi \left[dp_{i0}\, \chi(p_1)\, \delta_t(p_i^-)\right] \cdot \int \Pi \left[dq_{j0}\, \chi(-q_j)\, \delta_t(q_j^+)\right] \tilde{r}^{\mathrm{amp}}(k; p_1, \ldots, q_m) \tag{59}$$

exists after integration over the test function $\tilde{\varphi}(k)\, \Pi\, \hat{f}_i^*(\boldsymbol{p}_i)\, \Pi\, \hat{g}_j(-\boldsymbol{q}_j)$, and is independent of the special choice of χ. We can therefore consider $\tilde{r}_{MS}^{\mathrm{amp}}$ as the restriction of $\tilde{r}^{\mathrm{amp}}$ to the mass shell in the variables p_i, q_j, i.e. we can use (59) to *define* this restriction. If $\tilde{r}^{\mathrm{amp}}$ is continuous in the variables p_i^-, q_j^+ in a neighborhood of the mass shell, then this definition coincides with the familiar one. This is the case, in particular, if all the wave functions $\hat{f}_i$, $\hat{g}_j$ are mutually non-overlapping [8].

With this definition (58) becomes

$$r(\Phi_{\mathrm{in}}, \Psi_{\mathrm{in}}, \varphi) = (-2\pi)^{n+m} \int \Pi \left[\frac{d^3 p_i}{2\,\omega(\boldsymbol{p}_i)}\, \hat{f}_i^*(\boldsymbol{p}_i)\right] \Pi \left[\frac{d^3 q_j}{2\,\omega(\boldsymbol{q}_j)}\, \hat{g}_j(-\boldsymbol{q}_j)\right] \cdot dk\, \tilde{\varphi}(k)\, \tilde{r}_{MS}^{\mathrm{amp}}(k; \boldsymbol{p}_1, \ldots, \boldsymbol{p}_n, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_m)\,. \tag{60}$$

Formally this can be written

$$\begin{aligned} &r(\Phi_{\mathrm{in}}, \Psi_{\mathrm{in}}, \varphi) \\ &= (-2\pi)^{n+m} \int \Pi \left[d^4 p_i\, \delta_+(p_i)\, \hat{f}_i^*(\boldsymbol{p}_i)\right] \Pi \left[d^4 q_j\, \delta_+(q_m)\, \hat{g}_j(\boldsymbol{q}_j)\right] \\ &\quad \cdot dk\, \tilde{\varphi}(k)\, \tilde{r}^{\mathrm{amp}}(k; p_1, \ldots, p_n, -q_1, \ldots, -q_m) \\ &= (-1)^{n+m} \int \Pi \left[d^4 x_i\, f_i^*(x_i)\right] \Pi \left[d^4 y_j\, g_j(y_j)\right] \\ &\quad \cdot dz\, \varphi(z)\, r^{\mathrm{amp}}(z; x_1, \ldots, x_n, y_1, \ldots, y_m)\,. \end{aligned} \tag{61}$$

Here

$$f_i(x) = (2\pi)^{-3/2} \int dp\, e^{-ipx}\, \delta_+(p)\, \hat{f}(\boldsymbol{p}) \tag{62}$$

is a smooth solution of the Klein-Gordon equation. Note, that f_i is not the Fourier transform of $\hat{f}_i$!

In order to give a rigorous meaning to (61) the right-hand sides have to be defined by a limiting procedure of the type (58). This will be understood in the future.

Substitution of (61) into (54) yields, finally, the desired reduction formula. In the general case, i.e. without the restriction (36), one proves in the same way:

Theorem 2. *Let* $G_\mu(\varphi)$, Φ_{in}, Ψ_{in} *be defined by* (34), (35). *Assume* $\Phi_{\mathrm{in}} \in \mathscr{D}_0^{\mathrm{in}}$ *or* $\Psi_{\mathrm{in}} \in \mathscr{D}_0^{\mathrm{in}}$.

Then

$$\langle \Phi_{\mathrm{in}} |\, G_\mu(\varphi)\, | \Psi_{\mathrm{in}} \rangle = \Sigma \prod_{i=1}^{r} (\hat{f}_{\alpha_i}, \hat{g}_{\beta_i})\, g_\mu(\Phi_{\mathrm{in}}(\alpha_i), \Psi_{\mathrm{in}}(\beta_i), \varphi) \tag{63}$$

with

$$g_\mu(\Phi_{\mathrm{in}}, \Psi_{\mathrm{in}}, \varphi) = (-2\pi)^{n+m} \int \prod_i \left[d^4 p_i\, \delta_+(p_i)\, \hat{f}_i^*(\boldsymbol{p}_i)\right] \cdot \prod_j \left[d^4 q_j\, \delta_+(q_j)\, \hat{g}_j(\boldsymbol{q}_j)\right] \prod_h dk_h\, \tilde{\varphi}(k_1, \ldots, k_l) \cdot \tilde{g}_{\mu MS}^{+\,\mathrm{amp}}(k_1, \ldots, k_l; \boldsymbol{p}_1, \ldots, \boldsymbol{p}_n, -\boldsymbol{q}_1, \ldots, -\boldsymbol{q}_m)\,. \tag{64}$$

The superscripts $^+$ *and* $^{\mathrm{amp}}$ *refer asusual to the variables behind the semicolon. The mass-shell restriction of* $\tilde{g}_\mu^{+\,\mathrm{amp}}$ *is defined as in* (59), *by*

$$\begin{aligned}&\tilde{g}_{\mu MS}^{+\,\mathrm{amp}}(k_1, \ldots; p_1, \ldots, -q_m)\\ &= (2\pi)^{-n-m} \lim_{t\to\infty} \int \prod [dp_{i0}\,\chi(p_i)\,\delta_t(p_i^-)]\\ &\cdot \prod [dq_{j0}\,\chi(q_j)\,\delta_t(q_j^-)]\,\tilde{g}_\mu^{+\,\mathrm{amp}}(k_1, \ldots; p_1, \ldots, p_n, -q_1, \ldots, -q_m)\,.\end{aligned} \tag{65}$$

The scalar product $(\hat{f}, \hat{g})$ *is defined as in* (36). *The sum in* (63) *extends over all possible ways of pairing an arbitrary number* $r (0 \leqq r \leqq \mathrm{Min}(n, m))$ *of* α_i*'s with the same number of* β_i*'s.* $\Phi_{\mathrm{in}}(\alpha_i)$, $\Psi_{\mathrm{in}}(\beta_i)$ *are obtained from* Φ_{in}, Ψ_{in}, *by omitting those creation operators whose wave functions appear in one of the* factors $(\hat{f}_{\alpha_i}, \hat{g}_{\beta_i})$.

The following important remark has been made by POHLMEYER [14]. Assume that $\Psi_{\mathrm{in}} \in \mathscr{L}_0^{\mathrm{in}}$. The vector $G_\mu(\varphi)|\Psi_{\mathrm{in}}\rangle$ is then defined, and therefore also the matrix element $\langle\Phi|\, G_\mu(\varphi)\, |\Psi_{\mathrm{in}}\rangle$ for arbitrary $\Phi \in \mathscr{H}^{\mathrm{in}}$, in particular for all vectors of the form

$$\Phi = \int \prod_1^n \frac{d^3 p_i}{2\omega(\boldsymbol{p}_i)} \hat{f}(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n) \prod_1^n \hat{A}^{\mathrm{in}*}(\boldsymbol{p}_i)\,|0\rangle\,. \tag{66}$$

where $\hat{f}$ is square integrable with respect to the measure $\prod [2\omega(\boldsymbol{p}_i)]^{-1} d^3p_i$. Let L_2^n be the space of these functions. From the fact, that the products $\prod \hat{f}_i(\boldsymbol{p}_i)$, $\hat{f}_i \in \mathscr{S}$, are total in L_2^n we obtain easily

Theorem 3 *(Pohlmeyer's Theorem). Eq.* (65)' *defines a linear form* $\tilde{g}_{\mu MS}^{+\,\mathrm{amp}}(k_h; \boldsymbol{p}_i, -\boldsymbol{q}_j)$ *on the space* $\mathscr{S}(k_1, \ldots, k_l) \otimes L_2^n(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n) \otimes N(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_m)$, *continuous in the first two factors. Here* $N(\boldsymbol{q}_j)$ *is the linear space spanned by the products* $\prod \hat{g}_j(\boldsymbol{q}_j)$ *of non-overlapping functions* $\hat{g}_j \in \mathscr{S}$.

In other words: the expression

$$\int \prod dk_h\, \tilde{\varphi}(k_h) \prod [d^3q_j\, \hat{g}(\boldsymbol{q}_j)]\, \tilde{g}_{\mu MS}^{+\,\mathrm{amp}}(k_h; \boldsymbol{p}_i, -\boldsymbol{q}_j)$$

can be identified with a function $F(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n) \in L_2^n$.

An analogous result with $\{\boldsymbol{p}_i\}$, $\{\boldsymbol{q}_j\}$ interchanged is of course obtained if $\Phi_{\mathrm{in}} \in \mathscr{L}_0^{\mathrm{in}}$.

Reduction formulae for S-matrix elements can be derived in a similar way. Let

$$\begin{aligned}\Phi_{\mathrm{out}} &= A_{\hat{f}_1}^{\mathrm{out}*} \ldots A_{\hat{f}_n}^{\mathrm{out}*}\,|0\rangle\\ \Psi_{\mathrm{in}} &= A_{\hat{g}_1}^{\mathrm{in}*} \ldots A_{\hat{g}_m}^{\mathrm{in}*}\,|0\rangle\,.\end{aligned} \tag{67}$$

Both states may be overlapping.

Consider

$$\begin{aligned}F(t) &= \int \prod_1^n dx_i\, \bar{f}_i^*(x_i, \pm t)\\ &\cdot \prod_1^m dy_j\, \bar{g}_j(y_j, \pm t)\, \tau(x_1, \ldots, x_n, y_1, \ldots, y_m)\,.\end{aligned} \tag{68}$$

The limit for $t \to \infty$ of these expressions for all possible choices of the signs of t can be calculated by again introducing the auxiliary function ϱ_t, and making use of (21) and Theorem 1. In the special case $n = m = 1$ we obtain

$$(\hat{f}, \hat{g}) = \langle 0 | A_{\hat{f}}^{\text{out}} A_{\hat{g}}^{\text{in}*} | 0 \rangle$$

$$= \lim_{t \to \infty} \int dp\, dq\, \tilde{f}^*(p)\, \tilde{g}(-q)\, \tilde{\tau}^{\text{amp}}(p, q) \frac{\delta_t(p^-)}{p^+} \frac{\delta_t(q^+)}{q^-} + 2\,\tau(f, g)\,. \quad (69)$$

Define the expression $\tilde{\sigma}(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n, -\boldsymbol{q}_1, \ldots, -\boldsymbol{q}_m)$ as follows. Take

$$\tilde{\tau}^{\text{amp}}(p_1, \ldots, -q_m) = \Pi\,(p_i^2 - m^2)\, \Pi\,(q_j^2 - m^2)\, \tilde{\tau}(p_1, \ldots, -q_m)\,,$$

and expand it into a cluster sum according to (25). Replace all 2-point functions $\tilde{\tau}^{\text{amp}\,T}(p, q)$ by $2\,\omega(\boldsymbol{q})\,\delta^3(\boldsymbol{p} + \boldsymbol{q})$. For the $\tilde{\tau}^{\text{amp}\,T}$ with 3 or more variables substitute their mass shell restrictions defined as in (59).

With the help of (69) we obtain then the general result:

Theorem 4. *Let Φ_{out}, Ψ_{in}, be defined by* (67). *Then*

$$\langle \Phi_{\text{out}} | \Psi_{\text{in}} \rangle$$

$$= (2\pi i)^{n+m} \int \Pi \left\{ \frac{d^3 p_i}{2\,\omega(\boldsymbol{p}_i)} \hat{f}_i^*(\boldsymbol{p}_i) \right\} \Pi \left\{ \frac{d^3 q_j}{2\,\omega(\boldsymbol{q}_j)} \hat{g}_j(\boldsymbol{q}_j) \right\} \quad (70)$$

$$\cdot\, \tilde{\sigma}(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n, -\boldsymbol{q}_1, \ldots, -\boldsymbol{q}_m)\,.$$

In analogy to Theorem 3 we see that $\tilde{\sigma}$ is a separately continuous, bilinear form on $L_2^n \otimes L_2^m$, i.e. it is the kernel of a continuous linear mapping of L_2^m into L_2^n, or vice versa.

5. Completeness Equations

In this chapter we shall derive what is usually called, somewhat inappropriately, unitarity equations. We prefer to speak of completeness equations, since these equations essentially express the consequences of asymptotic completeness for the g.r.p.

Let $\{\hat{f}_1, \hat{f}_2, \ldots\}$ be a complete orthonormal basis of L_2^1, with $\hat{f}_i \in \mathscr{S}$:

$$(\hat{f}_\alpha, \hat{f}_\beta) = \int \frac{d^3 p}{2\,\omega(\boldsymbol{p})} \hat{f}_\alpha^*(\boldsymbol{p})\, \hat{f}_\beta(\boldsymbol{p}) = \delta_{\alpha\beta}\,, \quad (71)$$

$$\sum_\alpha \hat{f}_\alpha^*(\boldsymbol{p})\, \hat{f}_\alpha(\boldsymbol{q}) = 2\,\omega(\boldsymbol{p})\, \delta^3(\boldsymbol{p} - \boldsymbol{q})\,. \quad (72)$$

Let

$$A_\alpha^{\text{in}} = \int \frac{d^3 p}{2\,\omega(\boldsymbol{p})} \hat{f}_\alpha^*(\boldsymbol{p})\, \hat{A}^{\text{in}}(\boldsymbol{p})\,. \quad (73)$$

Consider the set of sequences

$$N_i = \{n_1^i, n_2^i, \ldots\} \quad (74)$$

of non-negative integers, with

$$|N_i| = \sum_\alpha n_\alpha^i < \infty\,. \quad (75)$$

Define

$$|N_i\rangle = \prod_\alpha (n^i_\alpha!)^{-1/2} \prod_\alpha [A^{\mathrm{in}*}_\alpha]^{n^i_\alpha} |0\rangle\,. \tag{76}$$

The vectors $|N_i\rangle$ form a complete orthonormal basis of $\mathscr{H}$.

We take the vacuum expectation value of the identity (23):

$$\begin{aligned}\tilde{g}_\mu(p_1, \ldots, p_n) - \tilde{g}_\nu(p_1, \ldots, p_n)\\ = -i\{\langle 0|\, \tilde{G}_\alpha(p_1, \ldots, p_k)\, \tilde{G}_\beta(p_{k+1}, \ldots, p_n)\, |0\rangle\\ - \langle 0|\, \tilde{G}_\beta(p_{k+1}, \ldots, p_n)\, \tilde{G}_\alpha(p_1, \ldots, p_k)\, |0\rangle\}\,.\end{aligned} \tag{77}$$

(For the sake of simplicity we assume that $(i_1, \ldots, i_n) = (1, \ldots, n)$.) The two terms on the right can be evaluated by summing over the system $|N_i\rangle$ as intermediate states. The vacuum contributions cancel, so that we have to compute

$$\langle \tilde{G}_\alpha \tilde{G}_\beta\rangle' = \sum_{l=1}^{\infty} \sum_{|N_i|=l} \langle 0|\, \tilde{G}_\alpha |N_i\rangle\, \langle N_i|\, \tilde{G}_\beta |0\rangle\,. \tag{78}$$

The reduction formula (63) gives, if we make use of the symmetry properties of $g^{+\,\mathrm{amp}}_\mu$:

$$\begin{aligned}\langle \tilde{G}_\alpha \tilde{G}_\beta\rangle' = \sum_{l=1}^{\infty} \frac{(2\pi)^{2l}}{l!}\\ \cdot \sum_{\alpha_1, \ldots \alpha_l} \int \left[\prod_1^l \left\{\frac{d^3 q_j}{2\,\omega(\boldsymbol{q}_j)}\, \hat{f}^*_{\alpha_j}(\boldsymbol{q}_j)\right\} \tilde{g}^{+\,\mathrm{amp}}_{\alpha MS}\, (p_1, \ldots, p_n;\, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_l)\right]\\ \cdot \int \left[\prod \left\{\frac{d^3 q_j}{2\,\omega(\boldsymbol{q}_j)}\, \hat{f}_{\alpha_j}(\boldsymbol{q}_j)\right\} \tilde{g}^{+\,\mathrm{amp}}_{\beta MS}\, (p_{k+1}, \ldots, p_n;\, -\boldsymbol{q}_1, \ldots, -\boldsymbol{q}_l)\,.\right.\end{aligned} \tag{79}$$

The functions $\prod \hat{f}_{\alpha_j}(\boldsymbol{q}_j)$ form a basis of L^l_2. Hence, because of Theorem 3:

$$\begin{aligned}\langle \tilde{G}_\alpha \tilde{G}_\beta\rangle'\\ = \sum_{l=1}^{\infty} \frac{(2\pi)^{2l}}{l!} = \int \prod \{dq_j\, \delta_+(q_j)\}\, \tilde{g}^{+\,\mathrm{amp}}_\alpha\, (p_1, \ldots;\, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_l)\\ \cdot\, \tilde{g}^{+\,\mathrm{amp}}_\beta (\ldots, p_n;\, -\boldsymbol{q}_1, \ldots, -\boldsymbol{q}_l)\,.\end{aligned}$$

This can be substituted into (77) to give the result

$$\begin{aligned}\tilde{g}_\mu(p_1, \ldots, p_n) - \tilde{g}_\nu(p_1, \ldots, p_n)\\ = -i \sum_{l=1}^{\infty} \frac{(2\pi)^{2l}}{l!} \int \prod_1^l dq_j \left\{\prod_1^l \delta_+(q_j) - \prod_1^l \delta_-(q_j)\right\}\\ \cdot\, \tilde{g}^{+\,\mathrm{amp}}_\alpha (p_1, \ldots, p_k;\, q_1, \ldots, q_l)\, \tilde{g}^{+\,\mathrm{amp}}_\beta (p_{k+1}, \ldots, p_n;\, -q_1, \ldots, -q_l)\,.\end{aligned} \tag{80}$$

This is the desired general completeness equation.

The well-known GLZ equations (Ref. [3], Eq. (15)) can be obtained in the same way from (16), or directly as a linear combination of suitable equations of the set (80).

With the help of the completeness equations the following expansion theorem can be proved.

Theorem 5. *The series*

$$\sum_{l=0}^{\infty} \frac{(-2\pi)^l}{l!} \int dp_i\, \tilde{\varphi}(p_1, \ldots, p_n) \int dq_j\, \tilde{g}_\mu^{+\,\mathrm{amp}}(p_1, \ldots, p_n; -q_1, \ldots, -q_l) \cdot :A^{\mathrm{in}}(q_1) \ldots A^{\mathrm{in}}(q_l): \tag{81}$$

converges on $\mathscr{D}_0^{\mathrm{in}}$ *strongly to* $\tilde{G}_\mu(\tilde{\varphi})$ *for all* $\tilde{\varphi} \in \mathscr{S}$.

The double colon denotes of course Wick ordering.

A straightforward calculation shows that (81) describes the matrix elements $\langle \Phi_{\mathrm{in}} | \tilde{G}_\mu(\tilde{\varphi}) | \Psi_{\mathrm{in}} \rangle$, $\Phi_{\mathrm{in}} \in \mathscr{D}^{\mathrm{in}}$, $\Psi_{\mathrm{in}} \in \mathscr{D}_0^{\mathrm{in}}$, correctly. Only a finite number of terms contribute to such a matrix element, so that no convergence problems arise. Theorem 5 is therefore proved if we can demonstrate that the series (81) converges strongly on $\mathscr{D}_0^{\mathrm{in}}$. We shall demonstrate this only for the special case of a one-particle state $|\hat{f}\rangle = A_f^{\mathrm{in}*}\,|0\rangle$. The proof of the general case proceeds in complete analogy, but involves a good deal of uninteresting combinatorics.

The integration over the test function $\tilde{\varphi}$ is not relevant for our purpose and will be dropped.

Let G_l be the l^{th} term in (81). Consider

$$\langle \hat{f}| (\sum G_l)(\sum G_{l'}) |\hat{f}\rangle = \sum_l \sum_{l'} \frac{(-2\pi)^{l+l'}}{l!\,l'!} \int dq_i\, dq_j'\, \tilde{g}_\mu^{+\,\mathrm{amp}}(-p_h; -q_1, \ldots, -q_l) \tag{82}$$
$$\cdot\, \tilde{g}_\mu^{+\,\mathrm{amp}}(p_h'; -q_1', \ldots, -q_{l'}') \langle \hat{f}| :A^{\mathrm{in}}(q_1) \ldots A^{\mathrm{in}}(q_l): :A^{\mathrm{in}}(q_1') \ldots A^{\mathrm{in}}(q_{l'}'): |\hat{f}\rangle\,.$$

The integral occurring here can be written as a sum of integrals, each of which extends in each variable q_i, q_j' over one of the half spaces q_{i0}, $q_{j0}' \gtreqless 0$. The expectation value in (82) is different from zero only in the four following cases.

1st case. The integration goes over $q_{i0} \geqq 0$, $q_{j0}' \leqq 0$, all i, j, and $l = l'$. Then

$$\langle \hat{f}| : \ldots : : \ldots : |\hat{f}\rangle \cong l!\, \langle \hat{f}|\hat{f}\rangle \prod_{i=1}^{l} \{2\omega(\boldsymbol{q}_i)\, \delta^3(\boldsymbol{q}_i + \boldsymbol{q}_i')\, \delta_+(q_i)\, \delta_-(q_i')\}$$
$$+ l^2(l-1)! \prod_{i=1}^{l-1} \{2\omega(\boldsymbol{q}_i)\, \delta^3(\boldsymbol{q}_i + \boldsymbol{q}_i')\, \delta_+(q_i)\, \delta_-(q_i')\} \tag{83}$$
$$\cdot\, \delta_+(q_l)\, \delta_-(q_l')\, \hat{f}(\boldsymbol{q}_l)\, \hat{f}^*(-\boldsymbol{q}_l')\}\,.$$

The sign $\cong$ means that the two sides give the same result if substituted in (82), when we take due regard to the symmetry of $\tilde{g}_\mu^{+\,\mathrm{amp}}$ in the variables q_i.

The contribution to (82) of the first term in (83) is

$$\langle \hat{f} | \hat{f} \rangle \sum \frac{(2\pi)^{2l}}{l!} \int dq_i \, \tilde{g}_\mu^{+\,\mathrm{amp}}(-p_h; q_1, \ldots, q_l) \, \tilde{g}_\mu^{+\,\mathrm{amp}}(p'_h; -q_1 \ldots, -q_l) \cdot \prod \delta_-(q_i) \,. \tag{84}$$

Consider the completeness Eq. (80) associated with the commutator $[\tilde{G}_\mu(-p_1, \ldots, -p_n), \tilde{G}_\mu(p'_1, \ldots, p'_n)]$. The r.h.s. is, up to a factor, equal to (84) minus the corresponding sum in which the δ_- have been replaced by δ_+. These two terms have non overlapping support in the variable $(p_1 + \ldots + p_n)$, hence they exist separately if their difference exists, which is the case because of (80). The existence of (84) is thus proved.

The contribution of the second term in (83) is

$$(2\pi)^2 \sum \frac{(2\pi)^{2(l-1)}}{(l-1)!} \int \prod_1^{l-1} \{dq_i \, \delta_-(q_i)\} \, dq_l \, dq'_l \, \delta_+(q_l) \, \hat{f}(q_l) \, \delta_-(q'_l) \, \hat{f}^*(-q'_l) \cdot \tilde{g}_\mu^{+\,\mathrm{amp}}(-p_h; -q_1, \ldots, -q_l) \, \tilde{g}_\mu^{+\,\mathrm{amp}}(p'_h; q'_l, q_1, \ldots, q_{l-1}) \,.$$

This is again one of the two terms in the r.h.s. of the completeness equation associated with $[\tilde{G}_\mu^+(-p_1, \ldots, -p_n; q_l), \tilde{G}_\mu^+(p'_1, \ldots, p'_n; -q'_l)]$, where the variables q_l, q'_l are to be amputated and restricted to the mass shell. This does not destroy the existence of our expression because of Theorem 3.

2 nd case. Integration over $q_{i0} > 0$, all i, $q'_{j0} < 0$, all j except one, and $l + 2 = l'$. For symmetry reasons we get the total contribution of all these terms by taking the case where $q'_{l'}$ is the exceptional variable, and multiplying the result with l'. As in the first case it can be shown that the resulting contribution to (82) is, up to a factor, equal to one term in the r. h. s. of the completeness equation for $[\tilde{G}_\mu(-p_1, \ldots, -p_n), \tilde{G}_\mu^{+\,\mathrm{amp}}(p'_1, \ldots, p'_n; q_{l'}, -q_{l'-1})]$ and therefore exists.

3 rd case. Integration over $q_{i0} > 0$ for all i except one, $q'_{j0} < 0$ for all j, and $l = l' + 2$. This is exactly analogous to the second case.

4 th case. Integration over $q_{i0} > 0$ for all i except one, $q'_{j0} < 0$ for all j except one, and $l = l'$. Again we can restrict ourselves to the special case $q_{l0} < 0$, $q'_{l0} > 0$, and multiply the corresponding contribution with l^2. We obtain as contribution to (82) one term of the r.h.s. of the completeness equation for $[\tilde{G}_\mu^{+\,\mathrm{amp}}(-p_1, \ldots, -p_n; -q'_l), \tilde{G}_\mu^{+\,\mathrm{amp}}(p'_1, \ldots, p'_n; q_l)]$.

This completes the proof of Theorem 5 in the special case of a one-particle state.

Theorem 5 contains as a special case the Haag expansion of the field operator [3, 15]:

$$\tilde{A}(p) = \tilde{A}^{\mathrm{in}}(p) \tag{85}$$

$$+ \sum_{l=2}^{\infty} \frac{(-2\pi)^l}{l} \int dq_j \, \tilde{r}^{\mathrm{amp}}(p; -q_1, \ldots, -q_l) :\tilde{A}^{\mathrm{in}}(q_1) \ldots \tilde{A}^{\mathrm{in}}(q_l):$$

We assume that knowledge of the field $A(x)$ completely fixes the physical content of the theory. The functions r are therefore physically relevant only insofar as they intervene in the expansion (85), i.e. two sets of retarded functions $r(x_1, \ldots, x_n)$ are *physically equivalent* if the $\tilde{r}^{\mathrm{amp}}(p; q_1, \ldots, q_l)$ of the two sets coincide on the mass shell of the q_i.

6. The GLZ-Theorem

In this chapter we want to give a rigorous version of Theorem II of the paper by GLASER, LEHMANN and ZIMMERMANN [3]. This theorem states the conditions under which a given set of retarded functions defines a field theory. It is thus a counterpart to Wightman's reconstruction theorem.

Theorem 6. *Let* $r(x_1, \ldots, x_n)$, $g_\mu(x_1, \ldots, x_n)$, $n = 2, \ldots,$ *be tempered distributions with the following properties:*

a) $r(x_1, \ldots, x_n)$ *is real, invariant under the connected Poincaré group, and symmetrical in the variables* $x_2, \ldots, x_n$.

b) *The support of* r *is contained in* T_n, *defined in* (17).

c) *The mass shell restrictions* $\tilde{g}^{+\,\mathrm{amp}}_{\mu M S}(k_1, \ldots, k_l; \boldsymbol{p}_1, \ldots, \boldsymbol{p}_m, -\boldsymbol{q}_1, \ldots, -\boldsymbol{q}_n)$ *defined by* (65) *exist if integrated over test functions of the form* $\tilde{\varphi}(k_1, \ldots, k_l) \prod_1^m \hat{f}_i(\boldsymbol{p}_i)\, \hat{g}(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n)$, $\tilde{\varphi} \in \mathscr{S}$, $\hat{f}_i \in \mathscr{S}$ *non-overlapping*, $\hat{g} \in L_2^n$.

d) *The* $\tilde{g}_\mu$ *satisfy the completeness equations* (80).

Then the series (85) *defines a Wightman field* $A(x)$ *in the Fock space* $\mathscr{H}^{\mathrm{in}}$, *which satisfies Postulates 1–6 of chapters 2 and 3. The retarded functions calculated from this field are physically equivalent to the given* r.

Before proving this theorem we wish to make two comments.

1) If the support conditions for R are weakened in the way mentioned after Eq. (18), then condition b) of the theorem has of course to be similarily weakened. Invariance of r can then be demanded for translations, while Lorentz invariance has to be demanded only for $\tilde{r}^{\mathrm{amp}}(p_1; p_2, \ldots, p_n)$ with $p_2, \ldots, p_n$ on the mass shell.

2) The conditions given here are apparently much stronger than the ones in the original formulation of GLASER, LEHMANN, and ZIMMERMANN, where no mention of generalized retarded products was made. To this we wish to say the following. The GLZ equations containing r exclusively are a consequence of the completeness equations postulated in d). If distributions r satisfying a), b), c), and the GLZ equations, are given, then the generalized g_μ can be formally calculated from them with the help of the completeness equations. The assumptions on g_μ demand then that these formally defined g_μ exist in a rigorous sense, and satisfy condition c). It is thus in principle possible to formulate all the conditions in terms of the ordinary retarded functions exclusively. However, such a formulation would look extremely complicated.

Proof of Theorem 6. That the series (85) converges strongly on $\mathscr{L}_0^{\text{in}}$ has already been demonstrated in the proof of Theorem 5. In order to show that A is a Wightman field we have to prove the existence of the Wightman functions. To this end let us introduce the more general series

$$\tilde{G}_\mu(p_1, \ldots, p_n) \tag{86}$$
$$= \sum_{l=0}^{\infty} \frac{(-2\pi)^l}{l!} \int dq_j\, \tilde{g}_\mu^{+\,\text{amp}}(p_1, \ldots, p_n; -q_1, \ldots, -q_l) :\tilde{A}^{\text{in}}(q_1) \ldots \tilde{A}^{\text{in}}(q_l):$$

which converges strongly on $\mathscr{L}_0^{\text{in}}$. We intend to prove the existence of $\langle 0|\, \tilde{G}_{\mu_1}(P_1) \ldots \tilde{G}_{\mu_k}(P_k)\, |0\rangle$ by induction with respect to k. Here the P_i are non-overlapping sets of variables $\{p_1^i, \ldots, p_{\alpha_i}^i\}$, with $\cup P_i = \{p_1, \ldots, p_n\}$. The n-point Wightman function obtains as the special case $k = n$.

For $k = 1$ we have $\langle 0|\, \tilde{G}_\mu\, |0\rangle = \tilde{g}_\mu$, i.e. existence. We assume that the vacuum expectation values of at most $(k-1)$ factors $\tilde{G}_\mu$ exist and wish to show their existence for k factors.

Let $\theta(P)$ be a C^∞-function with $\theta(P) = 1$ for $\sum_P p_{j0} \geqq 0$. $\theta(P) = 0$ for $\sum p_{j0} \leqq -m$, if $P = \{p_j\}$. Then

$$\theta(P)\, \tilde{G}_\mu(P)\, |0\rangle = |0\rangle \langle 0|\, \tilde{G}_\mu |0\rangle\,,$$
$$\theta(P) \langle 0|\, \tilde{G}_\mu(P) = \langle 0|\, \tilde{G}_\mu(P)\,,$$

and

$$\langle 0|\, \tilde{G}_{\mu_1} \ldots \tilde{G}_{\mu_k}\, |0\rangle$$
$$= \theta(P_1) \langle 0|\, [\tilde{G}_{\mu_1}, \tilde{G}_{\mu_2} \ldots \tilde{G}_{\mu_k}]\, |0\rangle + \tilde{g}_{\mu_1}(P_1) \langle 0|\, \tilde{G}_{\mu_2} \ldots \tilde{G}_{\mu_k} |0\rangle\,. \tag{87}$$

The second term in this expression exists according to assumption. For the first term we have

$$\langle 0|\, [\tilde{G}_{\mu_1}, \tilde{G}_{\mu_2} \ldots \tilde{G}_{\mu_k}]\, |0\rangle = \sum_{i=2}^{k} \langle 0|\, \tilde{G}_{\mu_2} \ldots [\tilde{G}_{\mu_1}, \tilde{G}_{\mu_i}] \ldots |0\rangle\,. \tag{88}$$

Let $|N_i\rangle$ be the basis of $\mathscr{H}^{\text{in}}$ defined in (76). Let $|N_i'\rangle$ be another orthogonal basis of $\mathscr{H}^{\text{in}}$, all of whose elements are in $\mathscr{L}_0^{\text{in}}$. Such a basis exists because $\mathscr{L}_0^{\text{in}}$ is dense in $\mathscr{H}^{\text{in}}$. The individual terms in (88) can then be defined as

$$\langle 0| \ldots [\tilde{G}_{\mu_1}, \tilde{G}_{\mu_i}] \ldots |0\rangle$$
$$= \lim_{m_2, \ldots, m_k \to \infty}\ \lim_{m_1 \to \infty}\ \sum_{j_\alpha = 1}^{m_\alpha} \langle 0|\, \tilde{G}_{\mu_2}\, |N_{j_2}'\rangle \langle N_{j_2}'| \ldots |N_{j_{i-1}}'\rangle \tag{89}$$
$$\cdot \{\langle N_{j_{i-1}}'|\, \tilde{G}_{\mu_1}\, |N_{j_1}\rangle \langle N_{j_1}|\, \tilde{G}_{\mu_i}\, |N_{j_i}'\rangle - \langle N_{j_{i-1}}'|\, \tilde{G}_{\mu_i}\, |N_{j_1}\rangle \langle N_{j_1}|\, \tilde{G}_{\mu_1}\, |N_{j_i}'\rangle\}$$
$$\cdot \langle N_{j_i}'| \ldots |N_{j_{k-1}}'\rangle \langle N_{j_{k-1}}'|\, \tilde{G}_{\mu_k}\, |0\rangle\,.$$

All the matrix elements in this expression exist due to assumption c) of the theorem. The matrix elements inside the curly bracket can be calculated from (86), which calculation yields of course the values (63) given in Theorem 2. If we keep $N'_{j_{i-1}}, N'_{j_i}$ fixed and sum over N_{j_1}, we find, as in the proof of Theorem 5, that the resulting expression can be written as a finite sum over terms, each of which is the r.h.s. of a suitable completeness equation, and therefore exists. In short, we obtain

$$\sum_{j_1=1}^{\infty} \{\ldots\} = i \langle N'_{j_{i-1}} | \tilde{G}_\mu(P_1, P_i) - \tilde{G}_\nu(P_1, P_i) | N'_{j_i} \rangle .$$

where $[G_{\mu_1}, G_{\mu_i}] = i(G_\mu - G_\nu)$ is one of the relations (23).

Hence

$$\begin{aligned} &\langle 0| \ldots [\tilde{G}_{\mu_1}, \tilde{G}_{\mu_i}] \ldots |0\rangle \\ &= i\{\langle 0| \tilde{G}_{\mu_2} \ldots \tilde{G}_{\mu_{i-1}} \tilde{G}_\mu \tilde{G}_{\mu_{i+1}} \ldots |0\rangle \\ &\quad - \langle 0| \tilde{G}_{\mu_2} \ldots \tilde{G}_{\mu_{i-1}} \tilde{G}_\nu \tilde{G}_{\mu_{i+1}} \ldots |0\rangle\} , \end{aligned}$$

and this exists according to the inductive assumption. This completes the existence proof of $\langle 0| \tilde{G}_{\mu_1} \ldots \tilde{G}_{\mu_k} |0\rangle$, and thus of the Wightman functions.

It is easy to see that the Wightman functions have the correct invariance, reality, and spectral properties. Locality we obtain from

$$\begin{aligned} &\langle 0| A(x_1) \ldots [A(x_i), A(x_{i+1})] \ldots A(x_n) |0\rangle \\ &= i\langle 0| \ldots R(x_i, x_{i+1}) \ldots |0\rangle - i\langle 0| \ldots R(x_{i+1}, x_i) \ldots |0\rangle \end{aligned}$$

and the support properties of $R(x, y)$ which follow from (86) and condition b). It is also easily demonstrated that the $R(x_1, \ldots, x_n)$ defined by (86) satisfy the conditions of Postulate 6. Any other solution R' of these conditions will give retarded functions which are physically equivalent to r. This is so because Postulate 6 fixes $r(x_1, \ldots, x_n)$ uniquely up to terms with support on the manifolds $x_i = x_j$ for some pair (i, j) of indices. These terms, transformed into p-space, vanish if all variables but one are amputated and restricted to the mass shell.

7. Smooth Retarded Products

It is tempting to try to change Postulate 6 into a theorem by giving the formal definition (15) a rigorous meaning with the help of a suitable smoothing procedure. Unfortunately, the known methods of smoothing do not work satisfactorily for various reasons. A common feature of all these methods is that they require a relaxation of the support condition (17) and the covariance condition (18). We have already remarked that this would not necessarily be fatal to the formalism. More serious difficulties do, however, exist.

1 st possibility. Instead of working directly with the field $A(x)$ one can introduce the regularized quasilocal fields

$$B(x) = \int dy\, \varphi(x - y)\, A(y) , \qquad \varphi \in \mathfrak{D}$$

as auxiliary quantities and build up the formalism from this [6]. $B(x)$ is a continuous function of x, and the expressions (15) are therefore well defined. The drawback of this method is that the strong asymptotic condition, formulated for B, is not true in the form (9): the wave functions $\hat{f}_i$ on the right have to be multiplied by $\hat{\varphi}$. This leads to serious difficulties in the derivation of reduction formulae, and makes it impossible to prove completeness equations of the form given here.

2nd possibility. Hepp [8] made (15) meaningful by regularizing the functions θ with a function $\chi(s_1 - s_2, \ldots, s_{n-1} - s_n) \in \mathfrak{S}$, whose properties we need not put down here explicitly. It is important that χ is not a product of functions of the individual arguments. One bad consequence of this fact is that the cluster expansion (25) does not hold. Theorem 1 is then wrong, and reduction formulae cannot be deduced from the strong asymptotic condition. They can, however, be deduced from the LSZ asymptotic condition, provided that the latter is true. This has as yet only been proved on non-overlapping states. The second bad feature of this approach is that the identities (16) are not satisfied, i.e. the completeness equations are not true.

3rd possibility. The Eq. (15) have an exact meaning if we define

$$\theta(x_1, \ldots, x_n) = \chi(x_1^0 - x_2^0) \ldots \chi(x_{n-1}^0 - x_n^0),$$

with χ a C^∞ function which is equal to 1 for sufficiently large arguments, and vanishes for sufficiently small arguments. The same objections as in the 2nd possibility apply, even though for different reasons: the cluster expansion (25) is not possible, and the identities (16) do not hold. Again it *is* possible to derive reduction formulae in the cases where the LSZ asymptotic condition holds.

Instead of using (15) we can define

$$R(x_1, \ldots, x_n) = (-i)^{n-1} \sum_{i_1, \ldots, i_n} (-1)^\alpha \chi(x_{i_2}^0 - x_{i_1}^0) \ldots \chi(x_1^0 - x_{i\alpha}^0)$$
$$\cdot \chi(x_1^0 - x_{i\alpha+1}^0) \ldots \chi(x_{i_{n-1}}^0 - x_{i_n}^0) A(x_{i_1}) \ldots A(x_1) \ldots A(x_{i_n}).$$

This is (if χ is replaced by θ) formally equivalent to (15). The identities (16) are true with this definition. Completeness equations can be derived if the LSZ asymptotic condition holds on all of $\mathscr{H}^{\text{in}}$. If turns out, however, that in this approach the retarded functions are not truncated, i.e. they do not tend to zero if the variables separate in space-like directions. This is too large a price to pay. It would for instance make it impossible to derive any useful analyticity properties for $\tilde{r}$.

8. Conclusions

We have tried to find out what the assumptions are that go into the LSZ formalism, and what exactly can be derived from them. Besides the generally accepted axioms (Wightman axioms plus asymptotic completeness) we were forced to introduce the postulate of the existence of retarded products. This necessity is rather disagreeable because the said postulate is very technical in nature and has no intuitive appeal whatever. It would be very satisfying if a way could be found to either circumvent or prove this assumption. Chances for the latter seem to be slim, judging from our experience with the related problem of defining retarded functions within the linear program (see Ref. [16] for the present state of this problem).

Another unpleasant aspect of the formalism is the cumbersome definition (65) of the mass shell restriction of $\tilde{g}_\mu^{+\,\mathrm{amp}}$. Here the question remains whether Hepp's continuity proof can or cannot be extended to the overlapping case.

For these two reasons our formulation of LSZ cannot be regarded to be final.

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