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Number Operators for Representations of the Canonical Commutation Relations

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Abstract. A *number operator* for a representation of the canonical commutation relations is defined as a self-adjoint operator satisfying an exponentiated form of the equation $Na^* = a^*(N + I)$, where a^* is an arbitrary creation operator. When N exists it may be chosen to have spectrum $\{0, 1, 2, \ldots\}$ (in a direct sum of Fock representations) or $\{0, \pm 1, \pm 2, \ldots\}$ (otherwise). Examples are given of representations having number operators, and a necessary and sufficient condition is given for a direct-product representation to have a number operator.

Introduction

The Fock representation of the canonical commutation relations has a total occupation number operator N. One way of completely describing N is to say

(i) it is self-adjoint

(ii) its spectrum is $\{0, 1, 2, \ldots\}$

and

(iii) it satisfies the commutation relation,

$$N a^*(\varphi) = a^*(\varphi)(N + I) \tag{0.1}$$

in a suitably rigorous form. Here $a^*(\varphi)$ is the creation operator for a wavefunction φ, and (0.1) is to hold for all φ.

In fact, the only representations of the canonical commutation relations which have a number operator N satisfying (i)—(iii) are direct sums of Fock representations [2, 4, 5].

If we relax the requirements on N by eliminating the assumption (ii) about the spectrum, then there exist other representations of the canonical commutation relations possessing such number operators. We call them *particle representations*.

In Section 1 we discuss general properties of particle representations. For a *strange* particle representation (other than a direct sum of Fock

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representations) the number operator is always unbounded below (Theorem 1.3). Furthermore, given a strange particle representation, one can always select a number operator which has every integer (negative as well as positive) as an eigenvalue (Theorem 1.3).

In Sections 3 and 4 we consider direct-product representations of the canonical commutation relations as described by KLAUDER, MCKENNA, and WOODS [11] and by STREIT [20]. We determine precisely which ones are particle representations; they are the ones having in the representation space a vector $\varphi_1 \otimes \varphi_2 \otimes \cdots$ where each φ_j is a multiple of a Hermite function (Theorem 3.3). We discuss the problem of extending these representations so that they are defined over a Hilbert space, showing that this can be done in a smooth way if and only if the indices on the Hermite functions are bounded (Theorem 4.7). We understand that M. REED [14] has considered similar and related questions about direct-product representations, but his work was not yet available at the time of this writing.

In Section 5 we discuss a class of particle representations which includes the extreme universally invariant representations as described by SHALE and SEGAL [18] and the representations corresponding to a non-relativistic infinite free Bose gas as described by ARAKI and WOODS [1]. They have generating functionals of the form

$$\mu(z) = \exp[-1/4 \,\|Tz\|^2]\,,$$

where $T \geqq I$. The corresponding representation is a direct sum of Fock representations if and only if $T^2 - I$ is trace class (Theorem 5.1).

1. Number Operators

We consider representations of the canonical commutation relations over a space $\mathfrak{H}$ of test functions. $\mathfrak{H}$ is assumed to be a complex inner product space, with the imaginary part of the inner product serving as the commutator bracket. This means that the commutation relations, in the Weyl form, are

$$W(z)\, W(z') = \exp\left[\frac{1}{2} i \operatorname{Im}(z, z')\right] W(z + z')\,, \tag{1.1}$$

where z and z' are arbitrary elements of $\mathfrak{H}$, and (z, z') is their inner product (linear on the left).

By a *representation of the Weyl relations (or a Weyl system) over* $\mathfrak{H}$ we mean a map W from $\mathfrak{H}$ into the unitary operators on some complex Hilbert space $\mathfrak{K}$ such that the *Weyl relation* (1.1) is satisfied, and, in addition, for each fixed $z \in \mathfrak{H}$, the function $W(tz)$ of the real variable t is weakly continuous at 0. For a description of the motivation for this definition and its connection with other formulations of the commutation relations, see [2] or [19].

If W is a representation of the Weyl relations over $\mathfrak{H}$ and $z \in \mathfrak{H}$, one can define the associated creation operator $a^*(z)$ to be the closure of $2^{-1/2}[R(z) - iR(iz)]$, where $R(z)$ is the self-adjoint generator of the group $t \to W(tz)$. In case W_F is the Fock-Cook representation [9, 3] and N is the total occupation number operator, a suitable exponentiated form of the commutation relation

$$N a_F^*(z) = a_F^*(z)\,(N + I) \tag{1.2}$$

is satisfied. To be precise, for each $z \in \mathfrak{H}$ the relation

$$e^{itN}\, W_F(z)\, e^{-itN} = W_F(e^{it} z) \tag{1.3}$$

is satisfied for all $t \in \mathbb{R}$ [3, 2].

As has been suggested by Segal [16, 19 p. 64], if one has a Weyl system W and an operator N satisfying the indicated commutation relations, then that N should have a physical interpretation as an occupation number operator. Accordingly, we take the commutation relation as a definition of a number operator, and then we investigate the properties of such operators.

1.1 Definition. Let W be a Weyl system over $\mathfrak{H}$ on $\mathfrak{K}$. A self-adjoint operator N on $\mathfrak{K}$ is a *number operator for* W if

$$e^{itN}\, W(z)\, e^{-itN} = W(e^{it} z)\,, \tag{1.4}$$

for all $z \in \mathfrak{H}$, $t \in \mathbb{R}$.

This definition differs in two respects from the definition

$$N = \sum_{k=1}^{\infty} a^*(e_k)\, a(e_k)\,,$$

where $\{e_k\}$ is an orthonormal basis of $\mathfrak{H}$. First, the infinite sum can converge in certain strange representations where Eq. (1.4) fails to hold [2]. These representations agree with the Fock-Cook representation on a dense subspace of $\mathfrak{H}$. Second, we shall see that number operators (in the sense of Definition 1.1) exist in many physically interesting representations where the infinite sum fails to converge. In fact the sum $\sum a^*(e_k)\, a(e_k)$ exists only in representations which are direct sums of those indicated above (i.e. those which agree with the Fock-Cook representation on a fixed dense subspace) [2, 4]. The distinction between the two definitions arises from the fact that when the sum exists, it is a non-negative operator, whereas number operators are not generally bounded below.

One difficulty with Definition 1.1 is that a number operator need not have integer eigenvalues. To see this, suppose W_F is the Fock-Cook representation of the Weyl relations on $\mathfrak{K}_F$, and N is the usual number operator. Let $\mathfrak{K}$ be an infinite dimensional Hilbert space and let A be a self-adjoint operator on $\mathfrak{K}$ having continuous spectrum. Then the self-adjoint generator of the group $t \to e^{itA} \otimes e^{itN}$ is a number operator for the Weyl system $I \otimes W_F$ acting on $\mathfrak{K} \otimes \mathfrak{K}_F$. ($I$ is the identity operator.) This number operator for $I \otimes W_F$ is easily seen to have no eigenvectors.

In this example one sees a feature which occurs in general; namely the representation $I \otimes W_F$ also has another number operator $I \otimes N$ which does have integer eigenvalues.

1.2 Lemma. *If a Weyl system W has a number operator N, then it has another number operator N' whose spectrum is a subset of the integers.*

Proof. Since, for every $z \in \mathfrak{H}$, $e^{2\pi iN} W(z)\, e^{-2\pi iN} = W(e^{2\pi i}z) = W(z)$, the unitary $U = e^{2\pi iN}$ commutes with all the $W(z)$'s. Now U has a spectral resolution $U = \int_0^1 e^{2\pi i\theta} dF(\theta)$ where the spectral projections $F(\theta)$ commute with every bounded operator which commutes with U (see [22], p. 307). Thus each $F(\theta)$ commutes with all the $W(z)$'s and also with e^{itN}, $t \in \mathbb{R}$. Let $A = -\int_0^1 \theta\, dF(\theta)$, and

$$V(t) = e^{itN} e^{itA} .$$

Then V is a continuous one-parameter unitary group, and

$$\begin{aligned} V(t)\, W(z)\, V(-t) &= e^{itN} [e^{itA}\, W(z)\, e^{-itA}]\, e^{-itN} \\ &= e^{itN}\, W(z)\, e^{-itN} , \end{aligned}$$

so the self adjoint generator N' of V is a number operator for W. The spectrum of N' is a subset of the integers since

$$e^{2\pi iN'} = e^{2\pi iN} e^{2\pi iA} = e^{2\pi iN} U^{-1} = I . \quad \blacksquare$$

Actually the spectrum of N' looks like $\{n_0, n_0 + 1, n_0 + 2, \ldots\}$ or $\{\cdots -2, -1, 0, 1, 2, \ldots\}$. It appears easy enough to prove this: Take an eigenvector φ of N', with eigenvalue n. Then $a^*(z)\, \varphi$ should be, according to (1.2), an eigenvector of N' with eigenvalue $n + 1$. However it is important to realize that (1.2) is symbolic, not rigorous, so to make this argument correct we would have to check that φ is in the domain of $a^*(z)$ and that $a^*(z)\, \varphi$ is in the domain of N. Instead of this, we shall prove the desired result, and more, using only bounded operators.

1.3 Theorem. *Suppose W is a Weyl system with a number operator N. If the spectrum of N is bounded below, then W is a direct sum of Fock-Cook representations. Otherwise W has a number operator N' whose spectrum is the set of all integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.*

Proof. Define A and N' as in the proof of Lemma 1.2. Since $N = N' + A$ in the sense of strong sum of commuting operators, and the spectrum of A is a subset of $[-1, 0]$, the spectrum of N' is bounded below if and only if that of N is. If the spectrum of N' is bounded below by the integer n, then $N' + nI$ is a number operator for W whose spectrum consists of non-negative integers. By Theorem 1 of [2], p. 64, W is a direct sum of Fock-Cook representations.

The possibility that N' is not bounded below is handled by the next lemma.

1.4 Lemma. *If N' is a number operator whose spectrum is unbounded below and consists of integers, then its spectrum is $\{0, \pm 1, \pm 2, \ldots\}$.*

Proof. Fix a unit vector $z_0 \in \mathfrak{H}$. Then by the Stone-von Neumann Theorem [12] (or see Ref. [2], p. 27), the representation of the Weyl relations over $\mathbb{C}$ given by $\alpha \to W(\alpha z_0)$ is unitarily equivalent to a direct sum of copies of the Schrödinger representation W_s. So we may assume that $\mathfrak{R} = \mathfrak{R}_1 \otimes \mathfrak{R}_s$, where $\mathfrak{R}_s$ is the representation space $L^2(\mathbb{R})$ for the Schrödinger representation, and

$$W(\alpha z_0) = I \otimes W_s(\alpha) .$$

Let N_s be the usual number operator $\frac{1}{2}(P^2 + Q^2 - 1)$ for the Schrödinger representation. Then, writing $U(t) = (I \otimes e^{itN_s})\, e^{-itN'}$, we have $U(t)\, W(\alpha z_0)\, U(-t) = W(\alpha z_0)$ for all $\alpha \in \mathbb{C}$, so $U(t)$ commutes with all the operators $I \otimes W_s(\alpha)$, $\alpha \in \mathbb{C}$. Thus $U(t)$ must lie in the commutator of the algebra $\{I \otimes W_s(\alpha) : \alpha \in \mathbb{C}\}''$. Since the Schrödinger representation is irreducible this commutator consists of all operators of the form $A_1 \otimes I$, so we have $U(t) = U_1(t) \otimes I$ or $e^{itN'} = U_1(t) \otimes e^{itN_s}$. Thus $U_1(t)$ is a continuous one-parameter unitary group; call its self-adjoint generator A. The spectrum of A is a subset of the integers since $e^{2\pi i A} \otimes I = I$. Furthermore A cannot be bounded below, because N_s is non-negative, and we are assuming N' is not bounded below.

Now we can prove that any integer m is in the spectrum of N'. Select an integer $m_0 \leqq m$ belonging to the spectrum of A. Since the spectrum of N_s is $\{0, 1, 2, \ldots\}$, $(m - m_0)$ is in the spectrum of N_s. But the spectrum of N' is the sum of that of A and that of N_s, so $m = m_0 + (m - m_0)$ is in the spectrum of N'. ∎

The existence of a number operator imparts a particle interpretation to the vectors in the representation space. To see this consider a Weyl system W acting on $\mathfrak{R}$, and suppose no subrepresentation of W is unitarily equivalent to the Fock-Cook representation. Then if W has a number operator N, we may suppose, according to Theorem 1.3, that N has spectrum $\{0, \pm 1, \pm 2, \ldots\}$. We may think of any eigenvector v of N with eigenvalue 0 as a "ground" state, even though it has an infinite number of "bare" particles with probability one [2]. Then an eigenvector of N with eigenvalue $n > 0$ has n more particles than v has, and an eigenvector with eigenvalue $-n < 0$ has n fewer particles than v. In fact, using the spectral representation $N = \sum_{n=-\infty}^{\infty} n P_n$, we may associate with any unit vector $x \in \mathfrak{K}$ the probability $\langle P_n x, x \rangle$ that the number of particles in x differs from the number in v by n. For this reason we adopt the following terminology.

1.5 Definition. A representation W of the Weyl relations which has a number operator is called a *particle representation*. A number operator for W is called *normalized* if its spectrum is either $\{0, \pm 1, \pm 2, \ldots\}$ or $\{0, 1, 2, \ldots\}$.

2. Generating Functionals for Particle Representations

We review briefly the definition of *generating functional*. Terms not defined here are explained in Ref. [2], p. 44—45.

Let W be a Weyl system over $\mathfrak{H}$. Define for each finite-dimensional subspace $\mathscr{M}$ of H the weakly-closed algebra $\mathfrak{A}_{\mathscr{M}}(W) = \{W(z) : z \in \mathscr{M}\}''$. Then the *Weyl algebra* $\mathfrak{A}(W)$ is the C^*-algebra generated by all the $\mathfrak{A}_{\mathscr{M}}(W)$'s as $\mathscr{M}$ varies over the finite-dimensional subspaces of $\mathfrak{H}$. As a C^*-algebra, $\mathfrak{A}$ is independent of W [16].

Given any state E of $\mathfrak{A}$, the Gelfand-Segal construction [10, 15], [6] yields a cyclic representation π_E of $\mathfrak{A}$ on a Hilbert space $\mathfrak{K}_E$ with a normalized cyclic vector v_E such that

$$E(A) = \langle \pi_E(A)\, v_E, v_E \rangle$$

for all $A \in \mathfrak{A}$.

Assuming E is *regular*, which means that E is strongly continuous on the unit ball of each $\mathfrak{A}_{\mathscr{M}}(W)$, $\mathscr{M}$ finite-dimensional, then the operators $W_E(z) = \pi_E(W(z))$ form a representation of the Weyl relations whose Weyl algebra is $\pi_E(\mathfrak{A})$. Furthermore the complex-valued function μ on $\mathfrak{H}$ defined by

$$\mu(z) = \langle W_E(z)\, v_E, v_E \rangle = E(W(z))$$

completely determines E and is called the *generating functional* of E [17].

Suppose W is a *particle* representation of the Weyl relations (Defn. 1.5). If N is a normalized number operator, and v is any eigenvector of N, then the generating functional

$$\mu(z) = \langle W(z)\, v, v \rangle$$

is *invariant under changes of phase*:

$$\mu(e^{it} z) = \mu(z) \quad \text{for all} \quad z \in \mathfrak{H}, \quad t \in \mathbb{R}. \tag{2.1}$$

In fact,

$$\begin{aligned} \mu(e^{it} z) &= \langle e^{itN}\, W(z)\, e^{-itN}\, v, v \rangle \\ &= \langle W(z)\, e^{-itN}\, v, e^{-itN}\, v \rangle \\ &= \mu(z)\,. \end{aligned}$$

It is easy to see that every particle representation is a direct sum of cyclic representations whose cyclic vectors are eigenvectors of N, and hence whose generating functionals satisfy (2.1). Conversely, as we prove below, any generating functional μ which has the property (2.1) corresponds to a particle representation. Since it is quite easy to exhibit generating functionals which are invariant under changes of phase, and it is also easy to determine whether a given functional has that property, this observation is quite helpful in studying particle representations.

We now proceed to a proof of the statement made above about generating functionals which satisfy (2.1).

2.1 Proposition. *Let μ be a generating functional which is invariant under changes of phase* (2.1). *Then the Weyl system W_E determined by μ via the Gelfand-Segal construction has a number operator which annihilates the cyclic vector v_E.*

Proof. A theorem of SEGAL [16] shows that the map $W(z) \to W(e^{it}z)$ induces an automorphism γ_t of the Weyl algebra $\mathfrak{A}$. The condition (2.1) implies that the regular state E determined by μ is invariant under γ_t:

$$E(\gamma_t(A)) = E(A) \quad \text{for all} \quad A \in \mathfrak{A}\,.$$

Now it is an easily checked property of the Gelfand-Segal construction that the invariance of E under γ_t implies the existence of a unitary $U(t)$ on the representation space $\mathfrak{K}_E$ which leaves the cyclic vector v_E invariant and which implements the automorphism:

$$U(t)\,A\,U(t)^{-1} = \gamma_t(A)\,.$$

In fact $U(t)$ is defined by

$$U(t)\,A v_E = \gamma_t(A)\,v_E, \qquad \text{for all} \quad A \in \mathfrak{A}\,.$$

Clearly $U(t+t') = U(t)\,U(t')$, so U is a one-parameter group of unitary operators. To prove the existence of a number operator N, we just have to show that $U(t)$ is a strongly continuous function of t at $t = 0$. Then its self-adjoint generator N will be a number operator since

$$e^{itN}\,W(z)\,e^{-itN} = \gamma_t(W(z)) = W(e^{it}z)\,.$$

To do this it suffices to prove that for all $z \in \mathfrak{H}$

$$\lim_{t\to 0} \|[U(t) - I]\,W(z)\,v_E\| = 0\,.$$

But

$$\begin{aligned}\|(U(t) - I)\,W(z)\,v_E\|^2 &= 2 - 2\,\mathrm{Re}\,\langle U(t)\,W(z)\,v_E,\, W(z)\,v_E\rangle \\ &= 2 - 2\,\mathrm{Re}\,\langle W(-z)\,W(e^{it}z)\,v_E,\, v_E\rangle \\ &= 2 - 2\,\mathrm{Re}\,\mu((e^{it}-1)\,z)\exp\left[\frac{1}{2} i \sin t\,\|z\|^2\right].\end{aligned}$$

This $\to 0$ as $t \to 0$ because μ is continuous on finite-dimensional subspaces and its value at 0 is 1. ∎

If a generating functional is not invariant under change of phase, then the corresponding representation may or may not have a number operator. In future work I hope to present a criterion by which one can tell directly from μ whether or not a number operator exists.

3. Number Operators for Direct-Product Representations

Suppose $\mathfrak{B} = \{e_1, e_2, \dots\}$ is an orthonormal basis of $\mathfrak{H}$, and $\mathscr{V}$ is the set of all finite linear combinations of vectors in $\mathfrak{B}$. We consider direct-product representations of the Weyl relations over $\mathscr{V}$ following KLAUDER, MCKENNA, and WOODS [11]. Our goal will be to determine which of them are particle representations.

For each $n = 1, 2, \dots$ let $\mathfrak{K}_n$ be the representation space $L^2(\mathbb{R})$ for the Schrödinger Weyl system W_s over $\mathbb{C}$. Denote by $\mathfrak{K}$ the complete infinite tensor product space $\mathfrak{K}_1 \otimes \mathfrak{K}_2 \otimes \cdots$ [13].

If $z \in \mathscr{V}$, say $z = \sum_{i=1}^{n} z_i e_i$, then defining

$$W(z) = W_s(z_1) \otimes \cdots \otimes W_s(z_n) \otimes I \otimes I \otimes \cdots,$$

we get a representation of the Weyl relations over $\mathscr{V}$ acting on $\mathfrak{K}$. This representation leaves invariant each incomplete infinite tensor product space. In fact, if $\psi = \psi_1 \otimes \psi_2 \otimes \cdots$ is a decomposable vector in $\mathfrak{K}$ such that $\|\psi_n\| = 1$ for all n, then the incomplete infinite tensor product space $\mathfrak{K}_\psi$ whose distinguished vector is ψ is defined as the closed subspace spanned by vectors $\varphi = \varphi_1 \otimes \varphi_2 \otimes \cdots$ such that $\sum_n |1 - \langle \varphi_n, \varphi \rangle|$ converges. Since, for $z \in \mathscr{V}$, $W(z)$ changes only a finite number of factors in φ, such a $W(z)$ maps $\mathfrak{K}_\psi$ into $\mathfrak{K}_\psi$.

So for each ψ, by restricting W to $\mathfrak{K}_\psi$ we get a Weyl system W_ψ on $\mathfrak{K}_\psi$, which we shall call a *direct-product* representation. It is known [11] that each W_ψ is irreducible, and that W_ψ is unitarily equivalent to W_φ if and only if ψ is weakly equivalent to φ (i.e. $\sum |1 - |\langle \varphi_n, \psi_n \rangle||$ converges).

One might guess that the self-adjoint generator N of the unitary group

$$U(t) = e^{itN_s} \otimes e^{itN_s} \otimes e^{itN_s} \otimes \cdots \tag{3.1}$$

is a number operator for the whole representation W because $U(t)\, W(z)\, U(-t) = W(e^{it} z)$, for $z \in \mathscr{V}$. (Here N_s is the usual number operator for the Schrödinger representation.) However it is easy to see that $t \to U(t)$ is not weakly continuous at zero, so it has no self-adjoint generator. In fact if $\varphi \in L^2(\mathbb{R})$ is a normalized eigenfunction of N_s with eigenvalue 1, then $\langle U(t)\,[\varphi \otimes \varphi \otimes \cdots], \varphi \otimes \varphi \otimes \cdots \rangle$ is *one* when t is an integer multiple of 2π, *zero* otherwise.

To find the representations W_ψ for which a number operator does exist, we first look for those ψ such that the generating functional

$$\mu_\psi(z) = \langle W_\psi(z)\, \psi, \psi\rangle \tag{3.2}$$

is invariant under change of phase.

3.1 Proposition. *Suppose $\psi = \psi_1 \otimes \psi_2 \otimes \cdots$, where each $\|\psi_k\| = 1$. The generating functional μ_ψ given by (3.2) is invariant under change of phase if and only if each ψ_k is an eigenfunction of N_s (i.e. each ψ_k is a multiple of some Hermite function).*

Proof. If each ψ_k is an eigenfunction of N_s then the generating functional

$$\mu_k(z) = \langle W_s(z)\, \psi_k, \psi_k\rangle, \quad z \in \mathbb{C} \tag{3.3}$$

is invariant under change of phase for each k. Then if $z = \sum_{k=1}^{n} z_k e_k$, we have

$$\begin{aligned}\mu_\psi(e^{it} z) &= \prod_{k=1}^{n} \mu_k(e^{it} z_k)\\ &= \prod_{k=1}^{n} \mu_k(z_k)\\ &= \mu_\psi(z)\,.\end{aligned}$$

On the other hand, if μ_ψ is invariant under change of phase, then each μ_k, as given in (3.3), will have the same property. By Prop. 2.1, this implies there exists a number operator for the Schrödinger representation which annihilates ψ_k. Because W_s is irreducible, any number operator for it differs from N_s by an additive constant. So ψ_k is an eigenfunction of N_s. ∎

3.2 Corollary. *If $\psi = \psi_1 \otimes \psi_2 \otimes \cdots$ and each ψ_k is an eigenfunction of N_s, then the direct-product representation W_ψ has a number operator.*

Proof. This follows immediately from the proposition, using Prop. 2.1 and the fact that W_ψ is irreducible (so that ψ is a cyclic vector). ∎

It is easy to exhibit explicitly the number operator N for W_ψ which annihilates ψ. In fact, if n_k is the eigenvalue of N_s corresponding to $\psi_k (N_s \psi_k = n_k \psi_k)$, then

$$e^{itN} = \exp it(N_s - n_1 I) \otimes \exp it(N_s - n_2 I) \otimes \cdots.$$

[This is proved by observing that the operator on the right leaves $\mathfrak{K}_\psi$ invariant, and (1.4) is satisfied.] So we see that N is obtained from $\sum a^*(e_k)\, a(e_k)$ by subtracting a constant multiple of the identity. The constant is infinite, except in the case where all but a finite number of the n_k's are zero, which is the case that W_ψ is unitarily equivalent to the Fock-Cook representation (Theorem 1.3; or this can be proved directly by calculating the generating functional).

The representations W_ψ singled out by Prop. 3.1 were among the first strange representations to be discussed; they are unitarily equivalent to the *discrete representations* of WIGHTMAN and SCHWEBER [21]. It may appear that Prop. 3.1 identifies these representations as the only direct-product representations having a number operator. However, there remains the possibility that some direct-product representation is a particle representation, but that no number operator for it annihilates any vector of the form $\psi_1 \otimes \psi_2 \otimes \cdots$. This is excluded by the next result.

3.3 Theorem. *The only direct-product representations of the Weyl relations which have number operators are the discrete representations, i.e. those with a vector $\varphi = \varphi_1 \otimes \varphi_2 \otimes \cdots$ in the representation space such that each φ_k is an eigenfunction of N_s (i.e. is a multiple of a Hermite function).*

Proof. Suppose $\psi = \psi_1 \otimes \psi_2 \otimes \cdots$, each $\|\psi_k\| = 1$, and W_ψ has a number operator N, assumed normalized.

Step 1. For every $t \in \mathbb{R}$, $U(t)\,\psi$ is weakly equivalent to ψ, where $U(t)$ is defined in (3.1).

Proof of Step 1: For each $t \in \mathbb{R}$, define $V(t) = U(t)\, e^{-itN}$. Considered as a map from $\mathfrak{K}_\psi$ to $\mathfrak{K}_{U(t)\psi}$, $V(t)$ is unitary. Moreover, for each $z \in \mathscr{V}$

$$\begin{aligned} V(t)^{-1}\, W_{U(t)\psi}(z)\, V(t) &= e^{itN}\,[U(-t)\, W_{U(t)\psi}(z)\, U(t)]\, e^{itN} \\ &= e^{itN}\, W_\psi(e^{-it}z)\, e^{itN} \\ &= W_\psi(z)\,. \end{aligned}$$

This shows that for each t $V(t)$ establishes a unitary equivalence between W_ψ and $W_{U(t)\psi}$. It follows [11] that for each t $U(t)\,\psi$ is weakly equivalent to ψ.

Step 2. There exist real constants $a_1, a_2, \ldots$ such that

$$e^{itN} = \exp it(N_s - a_1 I) \otimes \exp it(N_s - a_2 I) \otimes \cdots. \tag{3.4}$$

Proof of Step 2: By Step 1 and the definition of weak equivalence, we know that $\sum_{k=1}^{\infty} |1 - |\mu_k(t)||$ converges for each t, where

$$\mu_k(t) = \langle e^{itN_s}\psi_k, \psi_k\rangle\,.$$

Now each $\mu_k(t)$ is the characteristic function (Fourier transform) of a probability measure, as one sees by using the spectral resolution of N_s. It follows from a theorem in probability theory (e.g. Doob [8] Th. 2.7) that there exist real constants $a_1, a_2, \ldots$ such that

$$\sum_{k=1}^{\infty} |1 - e^{-ia_k t}\,\mu_k(t)| < \infty \tag{3.5}$$

for all t.

For each real s, let $Y(s) = e^{-ia_1 s} \otimes e^{-ia_2 s} \otimes \cdots$, a unitary operator on $\mathfrak{K}$ [13] which commutes with all the $U(t)$'s. Now (3.5) says $U(t)\, Y(t)\,\psi \in \mathfrak{K}_\psi$, so we may restrict $U(t)\, Y(t)$ to $\mathfrak{K}_\psi$ getting a unitary operator $Z(t)$.

Z is easily seen to be a one-parameter group, and $Z(t)$ is weakly measurable in t since

$$Z(t) = \operatorname*{st-lim}_{n\to\infty} \exp it(N_s - a_1 I) \otimes \cdots \otimes \exp it(N_s - a_n I) \otimes I \otimes I \otimes \cdots .$$

Since $\mathfrak{K}_\psi$ is separable this implies that $t \to Z(t)$ has a self-adjoint generator N'. Clearly N' is a number operator for W_ψ because

$$\begin{aligned} Z(t)\, W_\psi(z)\, Z(-t) &= U(t)\, W_\psi(z)\, U(-t) \\ &= W_\psi(e^{it} z) . \end{aligned}$$

Then, since $e^{itN'} e^{-itN}$ commutes with all the $W_\psi(z)$'s, the irreducibility of the representation implies that N' differs from N by a constant multiple of the identity. Hence by changing the real number a_1 selected above, we may suppose $N' = N$. This gives (3.4).

Step 3. The constants $a_1, a_2, \ldots$ in (3.4) may be selected to be integers $n_1, n_2, \ldots$

Proof of Step 3: Since each $\mu_k(2\pi) = 1$, we have from (3.5)

$$\sum |1 - e^{-2\pi i a_k}| < +\infty . \tag{3.6}$$

If we write $a_k = n_k + b_k$, where n_k is an integer and $-1/2 < b_k \leqq 1/2$, then (3.6) says $\sum |1 - e^{-2\pi i b_k}| < +\infty$, which implies [13] that $\sum |b_k|$ converges. So we have

$$e^{itN} = \exp\left(-it \sum_k b_k\right) \exp it(N_s - n_1 I) \otimes \exp it(N_s - n_2 I) \otimes \cdots .$$

Taking $t = 2\pi$, we see that $\sum b_k$ is an integer, which we may incorporate into n_1. We then have

$$e^{itN} = \exp it(N_s - n_1 I) \otimes \exp it(N_s - n_2 I) \otimes \cdots . \tag{3.7}$$

Step 4. If h_k is the kth Hermite function, then ψ is weakly equivalent to

$$h = h_{n_1} \otimes h_{n_2} \otimes h_{n_3} \otimes \cdots$$

where $n_1, n_2, \ldots$ are the integers in (3.7).

Proof of Step 4: Using (3.7) and the fact that $e^{itN} \psi \in \mathfrak{K}_\psi$, we have

$$\langle e^{itN} \psi, \psi \rangle = \prod_{k=1}^{\infty} \langle \exp it(N_s - n_k I)\, \psi_k, \psi_k \rangle . \tag{3.8}$$

Using the spectral theorem we see that the function $t \to \langle e^{itN} \psi, \psi \rangle$ is the characteristic function of a probability measure, and likewise $t \to \langle \exp it(N_s - n_k I)\, \psi_k, \psi_k \rangle$ is the characteristic function of a probability measure m_k. In fact, if P_n is the projection of $L^2(\mathbb{R})$ onto the one-dimensional subspace spanned by the Hermite function h_n, then $N_s = \sum_{n=0}^{\infty} n P_n$. Hence m_k assigns measure $\langle P_n \psi_k, \psi_k \rangle$ to the integer $n - n_k$, $n = 0, 1, 2, \ldots$

Now we use the Kolmogorov three series theorem (see [8], p. 111, or [20]) which tells us that since the infinite product of the characteristic functions of the dm_k's converges to a characteristic function, we have

$$\sum_k \int_{|x|>c} dm_k(x) < +\infty . \tag{3.9}$$

Here c is any positive number; for our purposes we take $c = 1/2$. Then

$$\begin{aligned}\int_{|x|<c} dm_k(x) &= m_k(\{0\}) \\ &= \langle P_{n_k}\psi_k, \psi_k\rangle \\ &= |\langle h_{n_k}, \psi_k\rangle|^2 .\end{aligned}$$

So

$$\int_{|x|>c} dm_k(x) = 1 - |\langle h_{n_k}, \psi_k\rangle|^2$$

and (3.9) says

$$\sum_{k=1}^{\infty} (1 - |\langle h_{n_k}, \psi_k\rangle|^2) < \infty .$$

Since each $|\langle h_{n_r}, \psi_k\rangle| \leqq 1$, this implies the convergence of

$$\sum (1 - |\langle h_{n_k}, \psi_k\rangle|) ,$$

which is the definition of weak equivalence of ψ with $h_{n_1} \otimes h_{n_2} \otimes \cdots$.

Step 5. The theorem is now proved, since if ψ is weakly equivalent to h, then [13] there exist constants $c_1, c_2, \ldots$ such that

$$\sum |1 - \langle c_k h_{n_k}, \psi_k\rangle|$$

converges. Then $\varphi = c_1 h_{n_1} \otimes c_2 h_{n_2} \otimes \cdots \in \mathfrak{K}_\psi$, and each $\varphi_k = c_k h_{n_k}$ is an eigenfunction of N_s. ▮▮

4. Continuity Properties of Discrete Representations

The direct-product representations, as described in Section 3, are defined only over the space $\mathscr{V}$, which is the algebraic span of a basis. But for physical applications such a space is too small; generally one needs a representation defined over a space of test functions or over a complete space. So it is of interest to inquire which of the discrete representations can be extended from $\mathscr{V}$ to $\mathfrak{H}$. And for our purposes it is not sufficient to prove abstractly that such an extension exists, since we would want the extended Weyl system over $\mathfrak{H}$ to have a number operator. Examples are known [2] of Weyl systems over $\mathfrak{H}$ which have no number operator, yet whose restrictions to $\mathscr{V}$ do have number operators.

The most natural idea is to extend the representation by continuity. For the case of a direct-product representation W, STREIT [20] has determined the precise set of $z = \sum_{j=1}^{\infty} z_j e_j$ in $\mathfrak{H}$ to which the representation can be extended via the formula

$$W(z) = \operatorname*{st-lim}_{n\to\infty} W\left(\sum_{j=1}^{n} z_j e_j\right). \tag{4.1}$$

However, to use his criterion to determine whether or not a particular discrete representation can be extended to every $z \in \mathfrak{H}$ using (4.1) is much more difficult than proceeding directly. So our method is independent of STREIT's Theorem. The result is that some of them can be extended to all of $\mathfrak{H}$ via (4.1) and some cannot be.

4.1 Definition. Let $\mathfrak{H}$ be an inner product space and W a Weyl system over $\mathfrak{H}$. W is *continuous (on all of $\mathfrak{H}$)* if the map $z \to W(z)$ is continuous from the metric topology of $\mathfrak{H}$ into the weak operator topology.

We recall that every Weyl system is continuous on finite-dimensional subspaces of $\mathfrak{H}$, but examples are known [2, 20] of Weyl systems which are not continuous on all of $\mathfrak{H}$. Our interest in continuous Weyl systems lies in the fact that they may be easily extended from dense subspaces to the whole space, and if the original representation had a number operator, so will the extended one. There is nothing difficult about these results, and the first is essentially proved elsewhere [1], but we give the proofs here for later reference.

4.2 Lemma. *Let $\mathfrak{H}$ be an inner product space and $\mathscr{V}$ a dense subspace of $\mathfrak{H}$. A continuous Weyl system over $\mathscr{V}$ has a unique extension to a continuous Weyl system over $\mathfrak{H}$.*

Proof. We just have to prove the existence of a continuous extension, since such an extension is clearly unique and satisfies the Weyl relation (1.1). Since every representation is a direct sum of cyclic representations it suffices to consider a cyclic continuous representation W over $\mathscr{V}$ acting on, say, $\mathfrak{K}$.

We must show that if $z_0 \in \mathfrak{H}$, and $\{z_n\}$ is any sequence in $\mathscr{V}$ converging to z_0, then the sequence $\{W(z_n)\,x\}$ is a Cauchy sequence in $\mathfrak{K}$ for every $x \in \mathfrak{K}$. Since the $W(z_n)$'s are unitary, it actually suffices to prove this only for those x lying in a total subset S of $\mathfrak{K}$. For S we choose $\{W(z)\,v : z \in \mathscr{V}\}$, where v is a unit cyclic vector. The Weyl relation (1.1) then gives directly

$$\begin{aligned} &\|[W(z_m) - W(z_n)]\, W(z)\, v\|^2 \qquad\qquad (4.2)\\ &\quad = 2 - 2\,\mathrm{Re}\left[\mu(z_n - z_m)\, \exp\frac{1}{2} i\, \mathrm{Im}\{(z_n, z_m) + 2(z_n - z_m, z)\}\right], \end{aligned}$$

where $\mu(z_n - z_m) = \langle W(z_n - z_m)\, v, v\rangle$. This $\to 0$ as $m, n \to \infty$ since $z_n - z_m \in \mathscr{V}$ and $\|z_n - z_m\| \to 0$, so that by the continuity of W at 0 in $\mathscr{V}$, $W(z_n - z_m) \to I$.

Hence we know there is an operator $W(z_0)$ such that for any sequence $\{z_n\}$ in $\mathscr{V}$ converging to z_0, st-lim $W(z_n) = W(z_0)$. Since $W(z_0)$ is the limit of unitaries, it is isometric. But since $W(-z_0)$ also exists, and the Weyl relation shows it is the inverse of $W(z_0)$, we know $W(z_0)$ is unitary. ∎

4.3 Lemma. *Let W be a continuous representation of the Weyl relations over $\mathfrak{H}$, and $\mathscr{V}$ a dense subspace of $\mathfrak{H}$. If the restriction of W to $\mathscr{V}$ has a number operator N, then N is a number operator for W over $\mathfrak{H}$.*

Proof. If $z \in \mathfrak{H}$, and $\{z_n\}$ is a sequence in $\mathscr{V}$ converging to z, then

$$\begin{aligned} e^{itN}\, W(z)\, e^{-itN} &= \operatorname*{st-lim}_{n\to\infty} e^{itN}\, W(z_n)\, e^{-itN} \\ &= \operatorname*{st-lim}_{n\to\infty} W(e^{it} z_n) \\ &= W(e^{it} z)\,. \end{aligned}$$ ∎

Now we need a practical criterion for deciding whether or not a representation is continuous, and this is given in the next result.

4.4 Proposition. *Let W be a cyclic Weyl system over $\mathscr{V}$ on $\mathfrak{K}$, let v be a unit cyclic vector and μ the generating functional*

$$\mu(z) = \langle W(z)\, v, v\rangle\,.$$

W is continuous on all of $\mathscr{V}$ if and only if μ is continuous at $0 \in \mathscr{V}$.

4.5 Corollary. *A Weyl system is continuous if and only if it is continuous at zero.*

Proof of Proposition (sufficiency). Suppose μ is continuous at 0. If $z_0 \in \mathscr{V}$, and $\{z_n\}$ is any sequence in $\mathscr{V}$ converging to z_0, we must prove that $\operatorname*{st-lim}_{n\to\infty} W(z_n) = W(z_0)$. This is done exactly as in the proof of Lemma 4.2, except that in (4.2) we replace z_m by z_0. ∎

Now we use these observations to analyse the discrete representations. In this case $\mathscr{V}$ is the algebraic span of the orthonormal basis $\{e_1, e_2, \ldots\}$. Each discrete representation is unitarily equivalent to a W_h, where h has the form $h = h_{n_1} \otimes h_{n_2} \otimes \cdots$, and h_n is the nth Hermite function. The generating functional

$$\mu(z) = \langle W_h(z)\, h, h\rangle \tag{4.3}$$

is entirely determined by the functions μ_n on $\mathbb{C}$ defined by

$$\mu_n(\alpha) = \langle W_s(\alpha)\, h_n, h_n\rangle,\ \alpha \in \mathbb{C}\,. \tag{4.4}$$

For if $z = \sum_{j=1}^{p} \alpha_j e_j \in \mathscr{V}$, then

$$\mu(z) = \prod_{j=1}^{p} \mu_{n_j}(\alpha_j)\,. \tag{4.5}$$

The functions μ_n are easily calculated using the fact that

$$h_n = (n!)^{-1/2}\, C^n h_0\,,$$

where $h_0(x) = \pi^{-1/4} e^{-(1/2)x^2}$, and C is the creation operator for the Schrödinger representation. We omit the details, and give the result:

$$\mu_n(\alpha) = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k!} \left(-\frac{|\alpha|^2}{2}\right)^k e^{-(1/4)|\alpha|^2}. \tag{4.6}$$

Here $\binom{n}{k}$ is the binomial coefficient.

Knowing the explicit form of the generating functionals, we can determine their continuity properties. The result is this:

4.6 Proposition. *Let $h = h_{n_1} \otimes h_{n_2} \otimes \cdots$, where h_n is the nth Hermite function. The generating functional μ defined on $\mathscr{V}$ by* (4.3) *is continuous at* 0 *if and only if the sequence $n_1, n_2, \ldots$ is bounded.*

As an immediate corollary of 4.2—4.4, 4.6 we have

4.7 Theorem. *Let $h = h_{n_1} \otimes h_{n_2} \otimes \cdots$, where h_n is the nth Hermite function, and let W_h be the direct-product representation of the Weyl relations over $\mathscr{V}$, which acts on the infinite tensor product space $L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \otimes \cdots$ with distinguished vector h. Then W_h has a continuous extension to a particle representation (Defin.* 1.5*) on $\mathfrak{H}$ (= completion of $\mathscr{V}$) if and only if the occupation numbers $n_1, n_2, n_3, \ldots$ are bounded.*

For the purpose of proving Proposition 4.6 we will need the following simple inequality.

4.8 Lemma. *Let μ_n be the generating functional* (4.3). *If $|\alpha| \leqq 2^{-n}$, then*

$$1 \geqq \mu_n(\alpha) \geqq \exp(-2^n |\alpha|^2). \tag{4.7}$$

Proof. Since μ_n is a generating functional, $|\mu_n(\alpha)| \leqq 1$ for all α, and since μ_n is real [cf. (4.6)], half of the inequality is proved. For the other half, write

$$\mu_n(\alpha) = (1 + b_n(\alpha))\, e^{-(1/4)|\alpha|^2}, \tag{4.8}$$

where

$$b_n(\alpha) = \sum_{k=1}^{n} \binom{n}{k} \frac{1}{k!} \left(-\frac{|\alpha|^2}{2}\right)^k.$$

Then for $|\alpha| < 2^{-n}$

$$\begin{aligned} |b_n(\alpha)| &\leqq \frac{|\alpha|^2}{2} \left(\sum_{k=1}^{n} \binom{n}{k}\right) = \frac{|\alpha|^2}{2} (2^n - 1) \\ &\leqq |\alpha|^2\, 2^{n-1} < 1/2. \end{aligned}$$

Hence

$$|\log(1 + b_n(\alpha))| \leqq 2\,|b_n(\alpha)| \leqq (2^n - 1)\, |\alpha|^2,$$

so

$$1 + b_n(\alpha) \geqq \exp[-(2^n - 1)\, |\alpha|^2].$$

In view of (4.8) this gives

$$\mu_n(\alpha) \geqq \exp(-2^n |\alpha|^2),$$

and the lemma is proved. ∎

Proof of Proposition 4.6 (sufficiency). Suppose the sequence $n_1, n_2, \ldots$ is bounded above by the integer M. Then if $z = \sum_{j=1}^{p} \alpha_j e_j \in \mathscr{V}$ is such that $\|z\|^2 < 2^{-2M}$, then each $|\alpha_j| < 2^{-M}$ so it follows from Lemma 4.8 that

$$1 \geqq \mu_{n_j}(\alpha_j) \geqq \exp(-2^M |\alpha_j|^2),$$

for $j = 1, \ldots, p$.

Hence, using (4.5), if $\|z\|^2 < 2^{-2M}$ we have

$$1 \geqq \mu(z) \geqq \prod_{j=1}^{p} \exp(-2^M |\alpha_j|^2) = \exp(-2^M \|z\|^2).$$

This shows μ is continuous at zero in $\mathscr{V}$.

(Necessity). Suppose the sequence $n_1, n_2, \ldots$ is unbounded. We will show that for any $\delta > 0$ there exists $z \in \mathscr{V}$ such that $\|z\| < \delta$ and yet

$$|1 - \mu(z)| > 1/4. \tag{4.9}$$

This proves μ is not continuous at 0.

So suppose $\delta > 0$ is given. Since the sequence $n_1, n_2, \ldots$ is unbounded we can find n_j such that $(2/n_j)^{1/2} < \delta$, and we suppose $n_j \geqq 3$. Select $z = (2/n_j)^{1/2} e_j$. Then $\|z\| < \delta$ and we see that (4.9) is true as follows (we drop the subscript j, writing n for n_j):

$$\begin{aligned} \mu(z) &= \mu_n((2/n)^{1/2}) \\ &= \left[1 - \binom{n}{1}\frac{1}{n} + r\right] \exp\left(-\frac{1}{2n}\right) \\ &= r \exp\left(-\frac{1}{2n}\right), \end{aligned}$$

where

$$r = \sum_{k=2}^{n} \binom{n}{k} \frac{1}{k!} \left(-\frac{1}{n}\right)^k.$$

Thus we have $|\mu(z)| < |r|$. But

$$\begin{aligned} |r| &\leqq \sum_{k=2}^{n} \frac{n!}{(n-k)!\,k!} \frac{1}{k!} \frac{1}{n^k} \\ &\leqq \sum_{k=2}^{n} \frac{n!}{n^k (n-k)!} \frac{1}{(k!)^2} \\ &\leqq \sum_{k=2}^{n} \frac{1}{(k!)^2} < e - 2 < \frac{3}{4}, \end{aligned}$$

so $|\mu(z)| < 3/4$ which implies $|1 - \mu(z)| > 1/4$. ▮▮

It is still thinkable that the discrete representations corresponding to unbounded occupation numbers $n_1, n_2, \ldots$ might be extendable to all of $\mathfrak{H}$ via (4.1), even though the Weyl system is not continuous. But an easy modification of the proof just completed shows that this does not happen. In general, such a representation can be extended by (4.1) to a Weyl system on a subspace strictly larger than $\mathscr{V}$, but not to all of $\mathfrak{H}$.

5. Some Other Particle Representations

If T is a self-adjoint operator on $\mathfrak{H}$ such that $T \geqq I$, and $\mathscr{D}$ is its domain, then a generating functional on $\mathscr{D}$ is given by

$$\mu(z) = \exp\left[-\frac{1}{4}\|Tz\|^2\right], \quad z \in \mathscr{D}. \tag{5.1}$$

Special examples of representations having generating functionals of this form are those given by the extreme universally invariant states found by SHALE, in which case T is a constant operator (see SEGAL [18]), and the states of an infinite free nonrelativistic Bose gas discussed by ARAKI and WOODS [1].

Since the μ given in (5.1) is invariant under change of phase, by Prop 2.1 there is a number operator for the cyclic representation of the Weyl relations determined by μ. Using a construction due essentially to ARAKI and WOODS, it is possible to exhibit explicitly the representation and its number operator. This construction also proves that (5.1) is actually a generating functional.

Let $A = \frac{1}{2}(T^2 - I)$, and let β be any conjugation of $\mathfrak{H}$ which commutes with A. (β is anti-linear, and $\beta^2 = I$; such a conjugation always exists.) Now let $\mathscr{M}$ be the closure of the range of $A^{1/2}$, a subspace of $\mathfrak{H}$, and denote by $\mathfrak{H}_F$ (resp. $\mathscr{M}_F$) the Fock-Cook space constructed over $\mathfrak{H}$ (resp. $\mathscr{M}$). For $z \in \mathscr{D}$, it is easy to see that $[I + A]^{1/2} z$ and $A^{1/2}\beta z$ are both defined, so we may define a unitary operator $W(z)$ on $\mathfrak{H}_F \otimes \mathscr{M}_F$ by

$$W(z) = W_F((I + A)^{1/2} z) \otimes W_F(A^{1/2}\beta z). \tag{5.2}$$

Here the first W_F is the Fock-Cook representation of the Weyl relations over $\mathfrak{H}$, and the second is the analogous representation over $\mathscr{M}$.

Direct calculations show that W is a representation of the Weyl relations over $\mathscr{D}$ and that the function μ given in (5.1) satisfies

$$\mu(z) = \langle W(z)\, v_0 \otimes v_0, v_0 \otimes v_0\rangle,$$

where v_0 is the zero-particle state in the Fock-Cook representation. Also simple modifications of the proofs in ARAKI-WOODS [1] show that $v_0 \otimes v_0$ is cyclic for $W(z)$, and W is a factor representation, reducible unless $A = 0$.

In case $A \neq 0$, an explicit normalized number operator for W is the closure of $N_F \otimes I - I \otimes N_F$, where the first N_F is the usual number operator for the Fock-Cook representation over $\mathfrak{H}$, and the second is the analogous operator for $\mathscr{M}$. In fact

$$\begin{aligned}
(e^{itN_F} \otimes e^{-itN_F})\, W(z)\, (e^{-itN_F} \otimes e^{itN_F}) \\
&= W_F(e^{it}[I + A]^{1/2} z) \otimes W_F(e^{-it} A^{1/2}\beta z) \\
&= W_F([I + A]^{1/2}(e^{it} z)) \otimes W_F(A^{1/2}\beta(e^{it} z)) \\
&= W(e^{it} z),
\end{aligned}$$

which shows that the self-adjoint generator of the group $t \to e^{itN_F} \otimes e^{-itN_F}$ is a number operator N for W, and this is the operator described above. Since e^{itN} leaves invariant the cyclic vector $v_0 \otimes v_0$ whose generating functional is μ, this is the operator one obtains using the construction described in the proof of Prop. 2.1.

The simplicity of constructing N disguises the fact that it is not really a natural number operator for W. First of all, N is not affiliated with the weakly-closed algebra $\mathfrak{B}$ generated by the $W(z)$'s, i.e. $e^{itN} \notin \mathfrak{B}$. Hence it is difficult to think of N as an observable or as a renormalization of the formal operator $\sum_{k=1}^{\infty} a_k^* a_k$, since the finite sums $\sum_{k=1}^{n} a_k^* a_k$ are all affiliated with $\mathfrak{B}$. To see that $e^{itN} \notin \mathfrak{B}$, observe that when $A \neq 0$ it is always possible to find non-zero $z_0 \in \mathscr{D}$ such that $[I + A]^{1/2} z_0 \in \mathscr{M}$. Then the Weyl relations show that the operator

$$W_F(A^{1/2} \beta z_0) \otimes W_F([I + A]^{1/2} z_0)$$

commutes with all the $W(z)$'s; but it does not commute with e^{itN} except when t is an integer multiple of 2π.

A second peculiarity of N, related to the first, is that its spectrum always consists of all integers (positive and negative). However for certain choices of T the cyclic representation (5.2) determined by μ is actually unitarily equivalent to a direct sum of Fock-Cook representations. So for these representations it is possible to find a *non-negative* number operator, in which case the operator N selected above is a particularly unnatural choice.

We conclude with a determination of which T's give rise to a direct sum of Fock-Cook representations. The proof uses the fact that such representations are the only finite-particle representations [2].

5.1 Theorem. *Suppose $T \geqq I$ is a self-adjoint operator with domain $\mathscr{D} \subset \mathfrak{H}$. The cyclic particle representation W of the Weyl relations determined by the generating functional*

$$\mu(z) = \exp\left[-\frac{1}{4}\|Tz\|^2\right], \quad z \in \mathscr{D}$$

is (unitarily equivalent to) a direct sum of Fock-Cook representations if and only if $T^2 - I$ has convergent trace.

Proof. First we show that if $A = \frac{1}{2}(T^2 - I)$ does not have pure point spectrum, then the representation (5.2) corresponding to μ is not a direct sum of Fock-Cook representations. Let $\mathfrak{H}_c$ be the continuous subspace for A, i.e. the orthogonal complement of the subspace of $\mathfrak{H}$ spanned by the eigenvectors of A. It suffices to show that the $W(z)$'s for $z \in \mathfrak{H}_c$ give a Weyl system which is not a direct sum of Fock-Cook representations over the same subspace $\mathfrak{H}_c$. So there is no loss of generality in assuming A has no point spectrum, i.e. $\mathfrak{H}_c = \mathfrak{H}$.

We shall use the following lemma which is proved, but not stated, by ARAKI and WOODS [1].

5.2 Lemma. *Suppose the weakly-closed algebra $\mathfrak{B}$ of operators on $\mathfrak{K}$ is a factor other than all bounded operators on $\mathfrak{K}$, and v is a cyclic vector for $\mathfrak{B}$. Suppose further that there exists a unitary operator U on $\mathfrak{K}$ such that $U\mathfrak{B}U^{-1} = \mathfrak{B}$ and v is the unique eigenvector of U. Then $\mathfrak{B}$ is not type I.*

We apply this lemma to the algebra $\mathfrak{B} = \{W(z): z \in \mathscr{D}\}''$, which is a reducible factor, as mentioned earlier. For v we take the cyclic vector $v_0 \otimes v_0$. For U we take the operator $V \otimes V^{-1}$ where

$$V = \bigoplus_{n=0}^{\infty} (e^{iA})^{\otimes n}$$

a unitary operator on the Fock-Cook space $\mathfrak{H}_F$. Since A has only continuous spectrum, the same is true of $(e^{iA})^{\otimes n}$, so that V has only one eigenvector, v_0. Hence U has only one eigenvector, namely $v = v_0 \otimes v_0$.

Furthermore, for $z \in \mathscr{D}$

$$\begin{aligned} U W(z) U^{-1} &= V W_F([I + A]^{1/2} z) V^{-1} \otimes V^{-1} W_F(A^{1/2} \beta z) V \\ &= W_F(e^{iA}[I + A]^{1/2} z) \otimes W_F(e^{-iA} A^{1/2} \beta z) \\ &= W(e^{iA} z) . \end{aligned}$$

This shows $U\mathfrak{B}U^{-1} = \mathfrak{B}$, and then the Lemma 5.2 says $\mathfrak{B}$ is not Type I. But the algebra generated by a direct sum of Fock-Cook representations *is* Type I, since the Fock-Cook representation is irreducible. So if A has continuous spectrum, the representation is not a direct sum of Fock-Cooks.

We are thus reduced to the case that A (or T) has pure point spectrum. So let $\{e_1, e_2, \ldots\}$ be an orthonormal basis of $\mathfrak{H}$ consisting of eigenvectors of T:

$$T e_j = t_j e_j .$$

For simplicity we shall first consider μ as defined only on the set $\mathscr{V}$ of finite linear combinations of the basis vectors $e_1, e_2, \ldots$

Let E be the regular state of the Weyl algebra over $\mathscr{V}$ whose generating functional is μ. According to Theorem 4, p. 77 [2], the cyclic representation determined by E is a direct sum of Fock representations if and only if the functions $\psi_{\mathscr{M}}(t) = E(e^{itN(\mathscr{M})})$ converge uniformly in t as $\mathscr{M} \to \mathscr{V}$ through the finite-dimensional subspace of $\mathscr{V}$. (Here $N(\mathscr{M})$ is the usual number operator over $\mathscr{M}$.) In the present case, since every finite-dimensional subspace of $\mathscr{V}$ is contained in some $\mathscr{M}_k = \operatorname{span}\{e_1, \ldots, e_k\}$, it can be shown that this is equivalent to the convergence of the sequence $\psi_k(t) = E(\exp itN(\mathscr{M}_k))$ to a characteristic function.

Now if $z = \sum_{j=1}^{n} z_j e_j \in \mathscr{V}$, then the generating functional μ factors as

$$\mu(z) = \prod_{j=1}^{n} \mu_j(z_j) \tag{5.3}$$

where

$$\mu_j(\alpha) = \exp\left[-\frac{1}{4} t_j^2 |\alpha|^2\right], \quad \alpha \in \mathbb{C}. \tag{5.4}$$

Let $\mathfrak{A}_j$ be the weakly-closed algebra generated by the Weyl operators corresponding to z's lying in the one-dimensional subspace $[e_j]$ spanned by e_j. And let E_j be the state of $\mathfrak{A}_j$ whose generating functional is μ_j (5.4). Then by (5.3) and the regularity of E it follows that if $A_j \in \mathfrak{A}_j$, $j = 1, \ldots, n$, we have

$$E(A_1 A_2 \ldots A_n) = \prod_{j=1}^{n} E_j(A_j).$$

But the operator $\exp(itN(\mathscr{M}_k))$ is just such a product. In fact (e.g. [2] p. 35, 37)

$$\exp(itN(\mathscr{M}_k)) = \exp(itN([e_1])) \ldots \exp(itN([e_k])),$$

so

$$\begin{aligned} \psi_k(t) &= E\big(\exp(itN(\mathscr{M}_k))\big) \\ &= \prod_{j=1}^{k} \varphi_j(t) \end{aligned} \tag{5.5}$$

where

$$\varphi_j(t) = E_j(\exp itN([e_j])). \tag{5.6}$$

Thus we are reduced to finding necessary and sufficient conditions for the infinite product $\prod \varphi_j$ to converge to a characteristic function.

These are given by the Kolmogorov Three-series Theorem if we know the measure whose characteristic function is φ_j. For this we need the explicit formula for E_j as given by SEGAL [18], Theorem 1. Namely

$$E_j(A) = (1 - c_j) \sum_{n=0}^{\infty} c_j^n \operatorname{trace}(A P_n(j)).$$

where c_j is selected between 0 and 1 so that $t_j^2 = \frac{(1 + c_j)}{(1 - c_j)}$, and $P_n(j)$ is the projection onto the n-particle subspace of $N([e_j])$. Then, using (5.6), we see that the measure whose characteristic function is φ_j has mass $(1 - c_j) c_j^n$ at n, $n = 0, 1, 2, \ldots$ The Three-series Theorem (e.g. [20]) then says that $\prod \varphi_j$ converges if and only if these three series converge:

$$\sum_j c_j, \quad \sum_j c_j/(1 - c_j), \quad \text{and} \quad \sum_j \left[\left(\frac{c_j}{1 - c_j}\right)^2 + \frac{c_j}{1 - c_j}\right].$$

The convergence of these three is equivalent to that of $\sum c_j/(1 - c_j)$ and since $t_j^2 - 1 = 2c_j/(1 - c_j)$, this is equivalent to the convergence of the trace of $T^2 - I$.

It follows that if $T^2 - I$ is *not* a trace class operator, then the representation W over $\mathscr{V}$ is not a direct sum of Fock-Cook representations over $\mathscr{V}$. In this case the representation over $\mathscr{D} \supset \mathscr{V}$ cannot be a direct sum of Fock-Cook representations over $\mathscr{D}$.

Conversely, if $T^2 - I$ is trace class, then the representation W over $\mathscr{V}$ is a direct sum of Fock-Cook representations over $\mathscr{V}$. Since T is bounded, $\mathscr{D} = \mathfrak{H}$ so the representation W is defined on all of $\mathfrak{H}$, and by Prop. 4.4 it is continuous on all of $\mathfrak{H}$. Since a direct sum of Fock-Cook representations is also continuous on all of $\mathfrak{H}$, the two agree everywhere. ∎

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