Commun. math. Phys. 7, 112—137 (1968)

Particle Localization in Field Theory

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Received August 1/September 20, 1967

Abstract. The localization properties of particles in field theory are studied with the help of a convenient mathematical description of counters. It is shown that field theory is capable of explaining the observed localization patterns. Apart from the usual axioms of field theory we have to assume some smoothness properties of the Green's functions in momentum space.

1. Introduction

Localizability is an essential ingredient of the intuitive notion of a particle. According to this notion a particle is an entity with some sort of stability, which is localized in a small region of space throughout its history, and which moves roughly according to the laws of classical mechanics, e.g., in a straight line if not subjected to external forces. This behaviour is exemplified by the well-known pictures of high energy events obtained in bubble chambers and similar instruments. These pictures show patterns of straight lines (in the absence of magnetic fields) joining in points, which can, most naturally, be explained as follows. The lines are tracks of particles. The particles proceed in straight lines until they meet another particle, whereupon they indulge in some reactions of a mysterious nature resulting in a certain number of other particles, which emerge from the region of interaction to proceed in straight lines, etc. This region of interaction is reasonably well localized in space (within the thickness of the tracks) and could presumably, by some more refined techniques, be localized in time to a similar degree of accuracy.

Field theory, on the other hand, is a theory of a continuum. Localized events of the type described above seem at first sight to be foreign to it. The definition of the term "particle" used in quantum field theory is indeed not based on this localization in x space, but instead, on the spectral properties of the energy momentum vector P_μ, i.e., a p space property. Roughly speaking, a particle is defined to be an object with a sharp value m of the mass, when the mass operator is $M^2 = P_0^2 - \mathbf{P}^2$. The question arises naturally whether this particle notion has anything to do with the more intuitive one discussed before. In other words:

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Can quantum field theory explain the qualitative features of a bubble chamber picture?

A result in this direction has been obtained by ARAKI and HAAG [1], [2]. Starting from the usual assumptions of field theory they were able to show that, asymptotically (for large negative or positive time), the points of Minkowski space, where a particle of momentum p can reasonably be expected to be found, lie in a direction (as seen from the origin) parallel to p. In the present paper we want to show that the particle behaviour at *finite* distances can be explained if some quite natural assumptions about the smoothness of the Green's functions are added to the familiar axioms of field theory. Namely, we shall assume that the time-ordered products of quasilocal fields are, in p-space, smooth functions of the momenta apart from the one-particle and threshold singularities that are known to exist as a consequence of the axioms. (For an exact formulation of these assumptions see the end of Chapter 2.) We will not discuss the problem in full generality but will only show in some simple examples that field theory yields the results to be expected from a particle behaviour. It is, however, clear that the methods used can be generalized to arbitrarily complex events.

In Chapter 2 the formalism used will be explained and the assumptions stated. In particular we shall introduce a convenient mathematical description of a counter. In Chapter 3 we shall derive the propagation in straight lines of individual particles. In Chapter 4 the scattering of two particles will be discussed. Chapter 5 deals with the description of unstable particles of sufficiently long life to produce tracks of measurable length. Some mathematical results, in particular estimates of certain integral expressions which will be used throughout the paper, are derived in an Appendix.

Our considerations hold only for $m > 0$. They cannot be applied to massless particles like the photon or the neutrino. This is hardly surprising since, at least, the photon is known to be a particle of a somewhat dubious character which it is hard to localize.

2. Description of Counters

We shall study our problem in the framework of the LSZ formalism [3]. For the sake of convenience we shall consider the simplest case of one Hermitian field $A(x)$ describing a single type of particles of mass $m > 0$. The generalization to more realistic cases is immediate. $A(x)$ is assumed to be local, i.e.,

$$[A(x), A(y)] = 0 \quad \text{for} \quad (x-y)^2 < 0 \,. \tag{1}$$

This assumption could actually be relaxed. It would suffice that the commutator decrease sufficiently rapidly for $x - y$ going to infinity in a spacelike direction.

The Hilbert space $\mathscr{H}$ of states is the familiar Hilbert space of field theory. In $\mathscr{H}$ there exists a unitary representation $U(a)$ of the translation group with

$$A(x+a) = U(a)\, A(x)\, U^*(a)\,. \tag{2}$$

Relativistic invariance will not be used. Therefore we do not need to make any assumptions about the transformation properties of A under the homogeneous Lorentz group.

$A(x)$ is assumed to satisfy the LSZ asymptotic condition

$$\lim_{t\to\pm\infty} \langle \Phi\, | A^f(t) |\, \Psi\rangle = \left\langle \Phi\, | A^f_{\substack{\text{aut}\\ \text{in}}} |\, \Psi\right\rangle \tag{3}$$

for Φ, Ψ in a dense set of $\mathscr{H}$, and all sufficiently smooth positive frequency solutions $f(x)$ of the Klein-Gordon equation for mass m. Here:

$$A^f(t) = i \int\limits_{x^0 = t} d^3x \left\{ A(x) \frac{\partial f(x)}{\partial x^0} - \frac{\partial A(x)}{\partial x^0} f(x) \right\}. \tag{4}$$

We assume completeness of the asymptotic states:

$$\mathscr{H} = \mathscr{H}_{\text{in}} = \mathscr{H}_{\text{aut}} \tag{5}$$

where $\mathscr{H}_{\text{ex}}$ is the Fock space of the free field A_{ex}.

In order to define the localization of a particle we shall try to translate the laboratory procedures used for that purpose into mathematical terms. The experimentalist employs counters to localize particles, where the word "counter" is used in a broad sense which includes, for instance, the bubbles of a bubble chamber (more exactly, the atoms whose ionization by the particle gives rise to the bubbles). A counter may roughly be described as an apparatus which is located in a certain region of space, switched on during a certain interval of time, and which may or may not be triggered, if subjected to the influence of the system under investigation. If the counter is triggered we say that a particle was in that region during that interval of time.

For describing this procedure in our formalism we will consider the triggering of a counter not as a measurement in the usual quantum mechanical sense, but as a pure filtering operation in the sense of HAAG and KASTLER [4]. This means the following:

A *state* Φ is, physically speaking, an ensemble of identical systems which has been prepared in a specified way, e.g., the particles of a beam extracted from an accelerator and guided through a system of slits, magnetic lenses, monitoring counters, etc. An *operation* is a procedure which is applied to all systems of the ensemble, thereby changing the properties of the ensemble. An operation is called a *filter* if it rejects some of the elements of the original ensemble Φ_i, so that the new ensemble Φ_f of the systems admitted by the operation (in our case: the systems that have triggered the counter) contains fewer elements than Φ_i. A *pure* operation is an operation that transforms pure states into pure states. In our case this means that the interaction of the system with the counter does not destroy more information than it creates.

We represent states, as usual, by vectors $\Phi, \ldots$ in the Hilbert space $\mathscr{H}$, but use the normalization

$$|\Phi|^2 = N \tag{6}$$

where N is the number of systems in the ensemble represented by Φ. The action of a counter on Φ can then be represented by a bounded linear operator C in $\mathscr{H}$, such that the triggering probability of a counter subjected to a system from Φ is given by

$$p_\Phi = |C\Phi|^2/|\Phi|^2 \, . \tag{7}$$

The operator C will be required to have the following properties:

i. *Boundedness.* In order that the expression (7) can be interpreted as a probability we have to demand that

$$\|C\| \leqq 1 \, , \tag{8}$$

where $\|C\|$ is the norm of C.

ii. *Annihilation of the vacuum.* We want the counter to register particles, but not to create or destroy them. In particular we assume

$$C|0> = C^*|0> = 0 \tag{9}$$

where $|0>$ is the vacuum state. Note that C is not assumed to be Hermitian.

iii. *Localization in x space.* We have somehow to formulate the fact that the counter represented by C is located in a bounded set B of space-time which we assume for the moment to contain the origin of our system of co-ordinates. At first sight it would seem natural to assume that C is an element of the von Neumann algebra $\mathfrak{A}(B)$ generated by the field operators in B [5]. This is, unfortunately, not possible because of the Reeh-Schlieder theorem [6] which states that there exist no local operators other than zero which annihilate the vacuum. We shall therefore assume that C is a quasi-local operator and shall achieve localization through the following considerations:

Any measurement can be performed only with a finite accuracy. For any set of counters in a given experimental arrangement there exists a small number $\eta > 0$ such that a triggering probability smaller than η is experimentally indistinguishable from an exactly vanishing probability. We shall therefore calculate probabilities only up to terms smaller than η. The counter C will be said to be *essentially localized* in B, if there exists a $C' \in \mathfrak{A}(B)$ with

$$\|C - C'\| < \alpha\eta \tag{10}$$

where $\alpha < 1$ is an appropriately chosen safety factor which insures that the totality of terms that will be neglected in our estimates do not add up to a quantity exceeding η. We assume that C is essentially localized in a bounded set B of a regular shape (spherical or cubical) of diameter d_1.

The region of essential localization of a counter is, of course, not uniquely determined by this definition. We will just choose a convenient region among the many possible ones.

iv. *Localization in p space.* Let $U(x)$ be the unitary representation of the translation group introduced in (2). The operator

$$C(x) = U(x)\, C\, U^*(x) \tag{11}$$

represents a counter identical to C which is essentially localized in a region B_x obtained from B by translation through the vector x.

We demand that the Fourier transform $\tilde{C}(p)$ of $C(x)$ be essentially localized in a neighbourhood $\tilde{B}$ of the origin in p space. More exactly: there shall exist a bounded operator C' such that the operator valued distribution $\tilde{C}'(p)$, defined in analogy to $\tilde{C}(p)$, has its support in $\tilde{B}$, and

$$\left\| \int d^4p\, [\tilde{C}(p) - \tilde{C}'(p)] \right\| < \alpha\eta \,. \tag{12}$$

This condition is satisfied, in particular, if $\tilde{C}(p)$ itself has its support in $\tilde{B}$. More general C's are, however, admitted by (12).

The diameter d_2 of $\tilde{B}$ will be assumed to be small with respect to m:

$$d_2 \ll m \,. \tag{13}$$

This condition means that the momentum transferred from the particle to the counter during their interaction is small.

The two diameter d_1, d_2 of course cannot be chosen to be arbitrarily small simultaneously, but they have to satisfy an inequality of the type of the uncertainty relations. The minimal possible value of their product depends on the choice of $\alpha\eta$.

This description of a counter is somewhat different from the one used by ARAKI and HAAG [1] since we do not treat C as an observable in the customary sense. The difference is merely a matter of convenience. The usual formulation could be recovered by making the additional assumption that C is a projection. This assumption is, however, neither necessary nor (in the present context) useful. It is important to note that, even if we made this assumption, the so defined projection operators would not satisfy the Newton-Wigner conditions [7].

The advantage of our interpretation lies in the fact that it allows us to describe arrangements of several counters in a simple way, even if the separations between the counters are not all space-like. Indeed, let $C(x_1)$, $C(x_2)$ be two counters whose regions of essential localization B_1 and B_2 satisfy the condition

$$y_2 - y_1 \notin \overline{V}_+ \quad \text{for all} \quad y_1 \in B_1,\ y_2 \in B_2 \,, \tag{14}$$

$\overline{V}_+$ the closed forward cone. Let Φ be defined as above. The ensemble of systems that have triggered both counters is then represented by the vector

$$\Phi_{12} = C(x_1)\, C(x_2)\, \Phi \tag{15}$$

and the probability that both counters are triggered by a system in the state Φ is

$$p_\Phi(x_1, x_2) = |C(x_1)\, C(x_2)\, \Phi|^2/|\Phi|^2 \,. \tag{16}$$

The condition (14) can only be satisfied if, either $x_1^0 > x_2^0$, or B_1 and B_2 are totally space-like to each other. In the latter case the order of factors in (15) is irrelevant since, then, $\|[C(x_1), C(x_2)]\| < \eta$. We can therefore replace (15) and (16) by

$$\Phi_{12} = T\, C(x_1)\, C(x_2)\, \Phi \tag{17}$$

$$p_\Phi(x_1, x_2) = |\Phi|^{-2}\, |T\, C(x_1)\, C(x_2)\, \Phi|^2 \tag{18}$$

where T denotes time-ordering of the C operators.

The generalization to more than two operators is immediate. If $C(x_1), \ldots, C(x_n)$ are n counters such that their localization regions B_i satisfy conditions of the form (14) pairwise, then the probability that all counters are triggered is given by

$$p_\Phi(x_1, \ldots, x_n) = |\Phi|^{-2}\, |T\, C(x_1) \ldots C(x_n)\, \Phi|^2 \,. \tag{19}$$

This is the expression which we are going to use. The normalization factor $|\Phi|^{-2}$ is irrelevant for our purposes and will be omitted in the sequel.

For the sake of convenience we shall only consider counter arrangements $C(x_1), \ldots, C(x_n)$ such that for any pair x_i, x_j of their locations one of the two conditions

$$x_i^0 = x_j^0 \quad \text{or} \quad |x_i^0 - x_j^0| > d_1 \tag{20}$$

is satisfied. The time ordering in (19) can then be defined with smooth θ functions:

$$T\, C(x_1) \ldots C(x_n) = \sum \theta(x_{i_1}^0 - x_{i_2}^0) \ldots \theta(x_{i_{n-1}}^0 - x_{i_n}^0)\, C(x_{i_1}) \ldots C(x_{i_n}) \,. \tag{21}$$

Here, the sum on the right extends over all permutations of the indices $(1, \ldots, n)$ and $\theta(x)$ is a C^∞ function with

$$\begin{aligned} \theta(x) &\equiv 1 \quad \text{for} \quad x \geqq \frac{d_1}{2} \\ \theta(0) &= \frac{1}{2} \\ \theta(x) &\equiv 0 \quad \text{for} \quad x \leqq -\frac{d_1}{2} \,. \end{aligned} \tag{22}$$

The Fourier transform $\tilde{\theta}(p)$ of $\theta(x)$ is of the form

$$\tilde{\theta}(p) = \frac{\varphi(p)}{p + i\varepsilon} \tag{23}$$

where we can choose $\varphi \in \mathscr{S}$ to be essentially localized in $\tilde{B}$, i.e., such that its values for $|p| > d_2$ will give negligible contributions to the p_Φ which we are going to discuss.

This definition of θ has the advantage that the formation of the T product conserves, to some extent, the localization in p space. Indeed, the Fourier transform of $T\, C(x_1), \ldots, C(x_n)$ is obtained by convolution of $\tilde{C}(p_1), \ldots, \tilde{C}(p_n)$ with $\tilde{\theta}$ functions of suitable arguments and is, under our conditions, still essentially localized in a neighbourhood of the origin of the same order of magnitude as $\tilde{B}$.

In order to get an analytic expression for p_Φ we write (19) (without the factor $|\Phi|^{-2}$) in the form

$$p_\Phi(x_1, \ldots, x_n) = \langle \Phi \,|(T C_1 \ldots C_n)^* \, T C_1 \ldots C_n| \, \Phi \rangle \,, \tag{24}$$

with $C_i = C(x_i)$.

Let $\{f_\alpha\}$ be a complete orthonormal set of positive frequency solutions of the Klein-Gordon equation. The vectors

$$|\alpha_1, \ldots, \alpha_l\rangle_{\text{out}} = c_{\{\alpha_i\}} \, A_{\text{out}}^{\alpha_1 *} \ldots A_{\text{out}}^{\alpha_l *} \, |0\rangle \tag{25}$$

($c_{\{\alpha_i\}}$ a normalization factor) form then a basis of $\mathscr{H}$. We insert these vectors as intermediate states in (24) and obtain

$$p_\Phi(x_1, \ldots, x_n) = \sum_{l=1}^{\infty} \sum_{\{\alpha_i\}} |c_{\{\alpha_i\}}|^2 \, |_{\text{out}}\langle \alpha_1, \ldots, \alpha_l| \, T C_1 \ldots C_n \, |\Phi\rangle|^2 \,. \tag{26}$$

We shall consider states Φ of the form

$$\Phi = A_{\text{in}}^{g_1 *} \ldots A_{\text{in}}^{g_m *} \, |0\rangle \equiv |g_1, \ldots, g_m\rangle_{\text{in}} \,, \tag{27}$$

where the g_i are positive frequency solutions of the Klein-Gordon equation which are not necessarily mutually orthogonal. Φ describes a system of m incoming particles with wave functions g_i.

A typical individual term in the sum (26) is, if we omit the irrelevant factor $|c_{\{\alpha_i\}}|^2$:

$$|T_\Phi^{\{\alpha_i\}}|^2 = |_{\text{out}}\langle \alpha_1, \ldots, \alpha_e| \, T C_1 \ldots C_n \,|g_1, \ldots, g_m\rangle_{\text{in}}|^2 \,. \tag{28}$$

This can be expressed with the help of the LSZ reduction formulae [3] as (again up to irrelevant constant factors):

$$\begin{aligned} & T_\Phi^{\{\alpha_i\}}(x_1, \ldots, x_n) \\ &= \int \prod_1^e du_i \prod_1^m dv_j \prod_1^e f_{\alpha_i}^*(u_i) \prod_1^m g_j(v_j) \, \tau(x_1, \ldots, x_n; u_1, \ldots, u_e, v_1, \ldots, v_m) \\ &= \int \prod_1^n dP_h \, e^{i P_h x_h} \int \prod dq_i \prod dp_j \prod \delta_+(q_i) \, \check{f}_{\alpha_i}(\boldsymbol{q}_i) \times \\ & \qquad \times \prod \delta_+(p_j) \, \check{g}_j(\boldsymbol{p}_j) \, \tilde{\tau}(P_h; -q_i, p_j) \,. \end{aligned} \tag{29}$$

The following notations have been used: $\tau(x_1, \ldots, x_n; y_1, \ldots, y_k)$ is the vacuum expectation value of the time ordered product of the operators $C_1, \ldots, C_n, A(y_1), \ldots, A(y_k)$, amputated (i.e., acted upon with Klein-Gordon operators) with respect to the field variables $y_1, \ldots, y_k$. We write the counter variables x_i in front of a semicolon, the field variables behind the semicolon. The T product is to be formed with the smooth

θ functions (22)[1]. $\tilde{\tau}(P_1, \ldots, P_n; Q_1, \ldots, Q_k)$ is the Fourier transform of $\tau(x_1, \ldots; y_1, \ldots)$. The wave functions f_α (and analogously g_j) are of the form

$$f_\alpha(x) = (2\pi)^{-2} \int d^4p\, e^{-ipx}\, \delta_+(p)\, \tilde{f}(\boldsymbol{p}), \tag{30}$$

where

$$\delta_+(p) = \theta(p_0)\, \delta(p^2 - m^2). \tag{31}$$

(Here θ stands of course for the ordinary sharp step function.)

Equation (29) is, strictly speaking, only correct if none of the incoming particle proceeds without, at any time, interacting with either another particle or a counter. We will not consider such processes. Our considerations can, of course, be easily extended to them.

We are now able to formulate the assumptions we need for drawing physical conclusions from (29).

$\tilde{\tau}(P_i; Q_j)$ is of the form

$$\tilde{\tau}(P_i; Q_j) = \delta^4(\Sigma P_i + \Sigma Q_j)\, \hat{\tau}(P_i; Q_j), \tag{32}$$

where $\hat{\tau}(P_i; Q_j)$ is only defined on the manifold $\Sigma P_i + \Sigma Q_j = 0$, and depends therefore on one vector variable less than $\tilde{\tau}$. We shall nevertheless exhibit all variables explicitly, including one redundant one.

We assume that the functions $\hat{\tau}(P_i; Q_j)$ contain the singularities that they can be expected to have from physical reasons [8], especially the one-particle poles as discussed by ZIMMERMANN [9]. Outside these singularities $\hat{\tau}$ shall be infinitely differentiable in all variables, and shall not exhibit any strong oscillations. More exactly, the following inequalities shall be satisfied sufficiently far away from the physical singularities:

$$\begin{aligned} \left|\frac{\partial \hat{\tau}}{\partial Q_j}\right| d_2 &\leqslant 1, \\ \left|\frac{\partial \hat{\tau}}{\partial P_i}\right| d_2 &< c\, \mathrm{Max}\,|\hat{\tau}|, \end{aligned} \tag{33}$$

where c is a constant of order of magnitude 1 and the maximum is over a d_2 neighbourhood of $(P_i; Q_j)$. Similar inequalities shall hold for the higher derivatives.

The existence of one-particle poles and of threshold singularities are consequences of the usual axioms of field theory. It is not known whether the C^∞ character of $\hat{\tau}$ outside these singularities follows from the axioms. Estimates of the type (33) do certainly *not* follow from the axioms, but constitute an additional assumption. Indeed, in Chapter 5 we shall discuss a case in which they are not satisfied.

[1] The θ functions originating in the reduction procedure, i.e., those containing at least one field variable y_j, could actually be chosen such that the function φ in (23) has strict support in $\tilde{B}$. This choice would be of some advantage in the sequel. For reasons of simplicity we shall, however, stick to (22).

The physical contents of (29) will be discussed with the help of the following procedure: we shall consider the expression $T_{\Phi}^{\{\alpha_i\}}(\lambda x_1, \ldots, \lambda x_n)$ for fixed initial and final states Φ and $|\alpha_i\rangle_{\text{out}}$, and for fixed geometrical arrangements of n counters. We shall determine the asymptotically leading term in this expression if λ tends to ∞ while everything else remains fixed (i.e., the x_i and thus the geometrical configuration of the counters remain fixed, while the distances between the counters tend to ∞). This asymptotically leading term is, under our smoothness assumptions, a sufficient approximation for the exact $T_{\Phi}^{\{\alpha_i\}}$ (i.e., within the η accuracy) already for relatively small λ, let us say for distances between the counters which are small multiples of d_1.

3. One-particle States

In this chapter we will consider the case $m = 1$, i.e.,

$$\Phi = A_{\text{in}}^{g*}|0\rangle \tag{34}$$

with

$$g(x) = (2\pi)^{-2} \int d^4q \, e^{-iqx} \, \delta_+(q) \, \check{g}(\mathbf{q}) \, . \tag{35}$$

As a first example we consider an incoming particle with a sharp momentum, i.e., we put

$$\check{g}(\mathbf{q}) = \delta^3(\mathbf{q} - \mathbf{p}) \tag{36}$$

for a fixed $\mathbf{p}$. In order to be completely rigorous we should of course consider a slightly smeared out $\check{g}$, e.g., an L_2 function with support in a small neighbourhood of $\mathbf{p}$. The integrals in which we are interested do, however, exist for the choice (36). This idealization is therefore harmless.

We place two counters in the space-time points $\lambda x_1, \lambda x_2, x_1^0 - x_2^0 = d_1$, and want to show that they can be triggered simultaneously only if their separation $\xi = x_1 - x_2$ is approximately parallel to the four-momentum $p = (\sqrt{\mathbf{p}^2 + m^2}, \mathbf{p})$ of the incoming particle.

The function $\hat{\tau}(P_1, P_2; -q_1, \ldots, -q_l, p)$ is in the variables P_1, P_2 essentially concentrated in a small neighbourhood of the origin. Therefore, only the terms with $l = 1$ contribute essentially to the sum (26). Let us consider one of these terms, corresponding to a wave function $\check{f}$, which is also assumed to be of the form (36). We have then to discuss the following expression:

$$T_p^q(\lambda x_1, \lambda x_2) = \frac{1}{q_0 p_0} \int d^4P_1 d^4P_2 \exp[i\lambda(P_1x_1 + P_2x_2)] \, \tilde{\tau}(P_1, P_2; -q, p) \tag{37}$$

with $p^2 = q^2 = m^2$, $p_0 > 0$, $q_0 > 0$. We fix our frame of reference such that $x_1 = 0$ and introduce the new variable $\xi = x_1 - x_2$. The P_1 integration can be carried out with the help of (32). We obtain

$$T_p^q(\lambda\xi) = \frac{1}{q_0 p_0} \int d^4P \, e^{-i\lambda P\xi} \, \hat{\tau}(-P + q - p, P; -q, p). \tag{38}$$

The integrand is essentially different from 0 only for $P \sim 0$, $P - q + p \sim 0$ and, therefore, $q \sim p$. In this region $\hat{\tau}$ has the pole structure [9]:

$$\hat{\tau}(-P+q-p, P; -q, p) = \frac{\hat{\tau}(P; -q, -P+q)\,\hat{\tau}(-P-p+q; p, P-q)}{(q-P)^2 - m^2 + i\varepsilon} + \frac{\hat{\tau}(P; p, -P-p)\,\hat{\tau}(-P-p+q; -q, P+p)}{(p+P)^2 - m^2 + i\varepsilon} + \sigma(P; q, p)\,, \qquad (39)$$

where σ contains no more singularities and is assumed to satisfy (33). The contribution of σ to (38) is then a strongly decreasing function of λ which becomes negligibly small already for quite small values of λ. It can therefore be disregarded.

Let us consider the first pole term. The pole manifold $(q-P)^2 = m^2$ consists of two sheets. Only the sheet in $(q-P) \in V_+$ is relevant because the residue is negligibly small on the other one.

Let

$$r(x; y, z) = K_y K_z\{\theta(x-y)\,\theta(y-z)\,\langle 0|\,[[C(x), A(y)], A(z)]\,|0\rangle + (y \leftrightarrow z)\}$$
$$(40)$$
$$a(x; y, z) = K_y K_z\{\theta(y-x)\,\theta(z-y)\,\langle 0|\,[[C(x), A(y)], A(z)]\,|0\rangle + (y \leftrightarrow z)\}$$

be the partially amputated retarded and advanced functions. Their Fourier transforms are

$$\begin{aligned} \tilde{r}(P; p, q) &= \delta^4(P+p+q)\,\hat{r}(-p-q; p, q) \\ \tilde{a}(P; p, q) &= \delta^4(P+p+q)\,\hat{a}(-p-q; p, q)\,. \end{aligned} \qquad (41)$$

In the region $q^2 = m^2$, $q_0 > 0$, $P \sim 0$, we have[2]

$$\begin{aligned} \hat{\tau}(P; -q, -P+q) &= \hat{r}(P; -q, -P+q) \\ \hat{\tau}(-P-p+q; p, P-q) &= \hat{a}(-P-p+q; p, P-q) \end{aligned} \qquad (42)$$

and

$$(q-P)^2 - m^2 + i\varepsilon = (q-P)^2 - m^2 + i\varepsilon(q_0 - P_0)\,. \qquad (43)$$

For large P, but fixed q, p, the difference between the two sides of (42) is negligible.

The functions on the right-hand side of (42) are in P_0 boundary values of regular functions, analytic in $\operatorname{Im} P_0 < 0$, which increase for increasing P_0 at most like $\exp(d_1 |\operatorname{Im} P_0|)$ multiplied with a polynomial in P_0. This exponential increase is a consequence of our use of smooth θ functions.

The contribution of the first pole term to (38) is thus essentially:

$$T_1(\lambda\xi) = \frac{1}{q_0 p_0}\int dP\, e^{-i\lambda P\xi}\, \frac{\hat{r}(P; -q, -P+q)\,\hat{a}(-P-p+q; p, P-q)}{(q-P)^2 - m^2 + i\varepsilon(q_0 - P_0)}\,. \qquad (44)$$

[2] For a comprehensive discussion of the connections between time ordered and retarded functions and their analyticity properties, see Ref. [10].

The integrand is analytic in P_0 for $\operatorname{Im} P_0 < 0$ and bounded (apart from a polynomial) by exp $(2d_1|\operatorname{Im} P_0|)$. $T_1(\lambda\xi)$ has therefore its support in $\lambda\xi^0 \leqq 2d_1$, i.e., this term does not contribute to T_p^q for $\lambda\xi^0 > 2d_1$.

The only remaining term is the second pole term in (39). As above it can be shown that its contribution to (38) is in a sufficient approximation

$$T_2(\lambda\xi) = \frac{1}{q_0 p_0}\int dP\, e^{-i\lambda P\xi}\frac{\hat{r}(P;p,-P-p)\,\hat{a}\,(-P-p+q;-q,P+q)}{(p+P)^2-m^2+i\varepsilon}. \quad (45)$$

It can also be shown that

$$T'(\lambda\xi) = \frac{1}{q_0 p_0}\int dP\, e^{-i\lambda P\xi}\frac{\hat{r}(P;p,-P-p)\,\hat{a}(-P-p+q;-q,P+q)}{(p+P)^2-m^2-i\varepsilon(p_0+P_0)}$$

vanishes for $\lambda\xi^0 > 2d_1$. For these values of λ we can therefore subtract T' from T_2 without changing the result:

$$\begin{aligned} T_2(\lambda\xi) = \frac{1}{q_0 p_0}\int dP\, e^{-i\lambda P\xi}\,\hat{r}(P;p,-P-p) \\ \hat{a}(-P-p+q;-q,P+q)\,\delta_+(p+P)\,. \end{aligned} \quad (46)$$

This is the simplest special case of the expressions discussed in the Appendix. From there we obtain the following result: $T_2(\lambda\xi)$ decreases for increasing λ like $\lambda^{-3/2}$. The coefficient of this power, as well as the coefficients multiplying the higher terms in the asymptotic development of T_2, are essentially different from zero only if ξ is parallel to $p+P$ for a value of P in which the functions $\hat{r}$, $\hat{a}$ appearing in (46) are essentially different from zero, i.e., for a P in a small neighbourhood of the origin. $T_p^q(\lambda\xi)$ is thus negligibly small already for reasonably small λ, unless ξ is almost parallel to p and therefore also to q. The probability that both counters are triggered by the particle is appreciable only if their separation is approximately parallel to the momentum of the particle. This result corresponds with what one would expect from a simple particle picture. The decrease of this probability like the inverse third power of the four-dimensional distance between the counters corresponds also to expectations.

As a second example we study an arrangement of three counters in λx_1, λx_2, λx_3, with $x_1^0 > x_2^0 > x_3^0$. monitoring a one-particle state (34). This time $\check{g}$ is supposed to be a function in $\mathfrak{D}$ with an arbitrarily large support. Equation (37) of the first example has to be replaced by

$$T_g^f(\lambda x_1,\lambda x_2,\lambda x_3) = \int dp\,dq\,\delta_+(p)\,\check{g}(\mathbf{p})\,\delta_+(q)\check{f}(\mathbf{q})\,G(\lambda x_1,\lambda x_2,\lambda x_3,p,q) \quad (47)$$

with

$$G(\lambda x_1,\lambda x_2,\lambda x_3,p,q) = \int dP_1\,dP_2\,dP_3\,e^{i\lambda\Sigma P_k x_k}\,\tilde{\tau}(P_1,P_2,P_3;-q,p). \quad (48)$$

We put again $x_1 = 0$ and write G as a function of $\xi = x_1 - x_2$, $\eta = x_2 - x_3$. In analogy to (38) we obtain

$$\begin{aligned} &G(\lambda\xi,\lambda\eta,p,q) \\ &= \int dP_2\,dP_3\,e^{-i\lambda(P_2\xi+P_3\eta)}\,\hat{\tau}(-P_2-P_3+q-p,P_2,P_3;-q,p)\,. \end{aligned} \quad (49)$$

We assume $\xi^0 > d_1$, $\eta^0 > d_1$ and study the asymptotic behaviour of G for large positive λ. As in the first example one can show that the only asymptotically relevant contribution to G comes from the following pole term in $\hat{\tau}$:

$$\frac{\hat{\tau}(P_3; p, -p-P_3)\,\hat{\tau}(P_2; p+P_3, -p-P_2-P_3) \times \times\, \hat{\tau}(-p+q-P_2-P_3; -q, p+P_2+P_3)}{[(p+P_3)^2-m^2+i\varepsilon]\,[(p+P_2+P_3)^2-m^2+i\varepsilon]} \,. \tag{50}$$

The numerator of this expression is only essentially different from zero if P_2, P_3 and $-p+q-P_2-P_3$ are sufficiently close to 0. This yields in particular the condition that p and q be almost equal.

The asymptotic behaviour of the contribution of (50) to G can be found with the methods described in the Appendix, with the following result: $G(\lambda\xi, \lambda\eta, p, q)$ is, for sufficiently large λ, essentially different from zero only if the vectors ξ, η, p, q, are all approximately parallel (within an accuracy determined by d_2). In this case G decreases like λ^{-3}. The p, q integration in (47) gives the result: $T_g^f(\lambda x_i)$ is essentially different from zero only if ξ and η are approximately parallel, in which case it decreases like λ^{-3} for increasing λ. Again this is already true for reasonably small λ.

This result means that all three counters can be triggered only if they lie roughly in a straight line. This is what we expect from a particle picture. The triggering of the first two counters selects obviously a subensemble of Φ consisting solely of particles with a momentum approximately parallel to the separation between these counters. The λ^{-6} decrease of the probability that all three counters are triggered is also what we expect.

4. Two-particle Scattering

We take Φ to be a two-particle state with particles of sharp momenta p_1, p_2:

$$\Phi = A^*_{\text{in}}(p_1)\, A^*_{\text{in}}(p_2)\, |0\rangle \,, \tag{51}$$

with $p_{1,2}{}^2 = m^2$, $p_{1,2} \in V_+$. This mathematically somewhat dubious choice of Φ will turn out to be harmless, as already happened in the first example of Chapter 3. (Note that we are mainly interested in relative probabilities.) For the sake of simplicity we assume that the energy of Φ is in the elastic region

$$4m^2 < (p_1+p_2)^2 < 9m^2 \,. \tag{52}$$

For the moment we assume also that $(p_1+p_2)^2$ is not close to either of the limiting points of this interval. The influence of the threshold singularity at $4m^2$ will be discussed later in this chapter.

The geometry of the process will be fixed with the help of three counters placed in the points λx, λy, λz, with $x^0 - y^0 = x^0 - z^0 = d_1$, $y^0 = z^0$, $|\mathbf{y} - \mathbf{z}| > d_1$.

Under condition (52) we have to take into account only the $l = 2$ terms in (26). We consider an individual term of this type and use again the simplification of choosing plane waves for the wave functions f_α.

We have then to study the quantity

$$T^{q_1 q_2}_{p_1 p_2}(\lambda x, \lambda y, \lambda z) = \frac{1}{q_{10} q_{20} p_{10} p_{20}} \int dP\, dQ\, dR\, e^{i\lambda(Px + Qy + Rz)} \times \\ \times \tilde{\tau}(P, Q, R; -q_1, -q_2, p_1, p_2) . \tag{53}$$

The factor in front of the integral is irrelevant and will be omitted in the sequel. Also the indices p_i, q_i on the left-hand side will be omitted.

Because of the essential support of $\tilde{\tau}$ the integral (53) is only non-negligible if $q_1 + q_2 \sim p_1 + p_2$. This will be assumed to be the case. We put $x = 0$ and introduce the relative separations $\xi = x - y$, $\eta = x - z$, as new variables. Integration over P yields

$$T(\lambda \xi, \lambda \eta) = \int dQ\, dR\, e^{-i\lambda(Q\xi + R\eta)}\, \hat{\tau}(P, Q, R; -q_1, -q_2, p_1, p_2) , \tag{54}$$

where P stands for $P = -Q - R + q_1 + q_2 - p_1 - p_2$.

We will only consider the contribution of the truncated part of $\hat{\tau}$ to (54). The rest describes free propagation of the two particles without scattering and can be reduced to the one-particle situation of Chapter 3.

The asymptotics of T is again governed by the one-particle poles of $\hat{\tau}$. Among the various pole combinations which contribute to the asymptotically leading term (all of which have a simple physical interpretation) we select the following typical one for further discussion:

$$I(Q, R; p_i, q_i) = \frac{\hat{\tau}(P; -q_1, q_1 - P)}{(P - q_1)^2 - m^2 + i\varepsilon} \frac{\hat{\tau}(Q; p_1, -p_1 - Q)}{(Q + p_1)^2 - m^2 + i\varepsilon} \times \\ \times \frac{\hat{\tau}(R; p_2, -p_2 - R)}{(R + p_2)^2 - m^2 + i\varepsilon}\, \hat{\tau}(; Q + p_1, R + p_2, P - q_1, -q_2) . \tag{55}$$

The contribution of (55) to (54) is

$$F(\lambda \xi, \lambda \eta) = \int dQ\, dR\, e^{-i\lambda[(Q,\xi) + (R,\eta)]}\, I(Q, R; p_1, \ldots, q_2) . \tag{56}$$

The function $\hat{\tau}(; k_1, \ldots, k_4)$ contains exclusively field operators (no counters) and is assumed to be slowly varying, i.e., to satisfy the first inequality (33). The corresponding factor in I can then be drawn in front of the integral because the other factors of I are concentrated in $P \sim Q \sim R \sim 0$. Thus

$$F(\lambda \xi, \lambda \eta) \cong \hat{\tau}(; p_1, p_2, -q_1, -q_2)\, G(\lambda \xi, \lambda \eta) , \tag{57}$$

$$G(\lambda \xi, \lambda \eta) \\ = \int dQ\, dR\, e^{-i\lambda(Q\xi + R\eta)} \times \\ \times \frac{\hat{\tau}(P; -q_1, -P + q_1)}{(P - q_1)^2 - m^2 + i\varepsilon} \frac{\hat{\tau}(Q; p_1, -Q - p_1)}{(Q + p_1)^2 - m^2 + i\varepsilon} \frac{\hat{\tau}(R; p_2, -R - p_2)}{(R + p_2)^2 - m^2 + i\varepsilon} . \tag{58}$$

This expression can be evaluated for large λ with the methods developed in the Appendix, with the following result:

$G(\lambda\xi, \lambda\eta)$ is not negligible only if there exist constants $\mu, \nu, \varrho > 0$, such that

$$\begin{aligned} \xi &\cong \mu p_1 + \varrho \bar{q}_1 \\ \eta &\cong \nu p_2 + \varrho \bar{q}_1 \,. \end{aligned} \tag{59}$$

Here $\bar{q}_1$ stands for $\bar{q}_1 = -q_2 + p_1 + p_2 \cong q_1$, and $\cong$ means equality up to terms of an order of magnitude given by $\mathrm{Max}(\mu d_2, \nu d_2, \varrho d_2)$. For these values of ξ, η, G decreases like $\lambda^{-5/2}$.

The probability that all three counters are triggered is therefore not negligible only if (59) holds and decreases like λ^{-5} for increasing λ. The condition (59) describes the following geometrical situation: the two rays drawn from the counters in λy, λz in the directions p_1 and p_2 respectively, meet approximately in a point. A ray drawn from this point in direction $\bar{q}_1$ (or q_1) meets approximately the third counter in λx. This has of course a simple interpretation in terms of particles. The first two counters register the two incoming particles. From the counters the particles proceed in direction of their momenta, then hit each other and are scattered. One of the scattered particles then triggers the third counter. The λ^{-5} decrease of the triggering probability is in accordance with this description.

The dependence of the probability on the values of the momenta p_i, q_i, is essentially determined by the factor $\hat{\tau}(;p_1, p_2, -q_1, -q_2)$ of Eq. (57). This is exactly the familiar S matrix element for two particle scattering.

It remains to discuss the influence of the threshold singularities. We shall only consider the lower threshold at $(p_1 + p_2)^2 = 4m^2$. In a neighbourhood of this point the 4 point function is of the form [11]:

$$\begin{aligned} &\hat{\tau}(;p_1, p_2, -q_1, -q_2) = \\ &= a(p_1, p_2, q_1, q_2) + [(p_1 + p_2)^2 - 4m^2]^{1/2}\, b(p_1, p_2, q_1, q_2)\,, \end{aligned} \tag{60}$$

where the functions a and b are assumed to be infinitely differentiable and slowly varying. The square root $(z - 4m^2)^{1/2}$ is defined in a plane cut along $z \geqq 4m^2$ and is positive imaginary for negative real z. It is in (60) to be taken as boundary value form the upper half plane. The root factor is of course *not* slowly varying in the region $(p_1 + p_2)^2 \sim 4m^2$.

Consider the function $F(\lambda\xi, \lambda\eta)$ defined in Eq. (56). For the moment we still assume that the threshold $(Q + R + p_1 + p_2)^2 = 4m^2$ is outside the essential support of I, but possibly close to it. $\hat{\tau}$ can then be developed in a series of the form

$$\begin{aligned} &\hat{\tau}(;Q + p_1, R + p_2, P - q_1, -q_2) = \\ &= \hat{\tau}(;p_1, p_2, -q_1, -q_2) + \sum_{i=1}^{N} \varrho_i(Q, R)\, \sigma_i(p_1, p_2, q_1, q_2) + \text{higher terms.} \end{aligned} \tag{61}$$

Here N is an arbitrary integer, the $\varrho_i(Q, R)$ are polynomials of degree i, the σ_i are derivatives of $\hat{\tau}$. They are small far away from the threshold, but increase over all bounds when approaching threshold. The contribution of the first term in (61) to F is the expression (57) which we have already studied. The contributions of the higher terms can be evaluated in the same way. The term of i^{th} order gives a contribution which is not negligible only if (59) is satisfied and which decreases like $\lambda^{-(5+i)/2}$. The first term is thus still dominant for sufficiently large λ. The higher terms are, however, multiplied with coefficients which diverge at threshold and become therefore more and more important for fixed finite values of λ, if threshold is approached.

In order to see what happens on actually reaching the threshold (i.e., for zero energy scattering), let us consider the special case

$$p_1 = p_2 = \overline{q}_1 \equiv p = (m, \mathbf{0}) \,, \tag{62}$$

$$\xi = \eta = \alpha p \,. \tag{63}$$

The factor α has to be introduced for dimensional reasons and will be assumed to have the numerical value 1.

The a term in (60) is uninteresting and we shall only discuss the root term. B is slowly varying and can be drawn in front of the integral. The relevant integral is then:

$$F(\lambda) = \int dQ\, dR\, e^{-i\lambda(p, Q+R)} [(p+Q)^2 - m^2 + i\varepsilon]^{-1} [(p+R)^2 - m^2 + i\varepsilon]^{-1} \times$$
$$\times [(p+Q+R)^2 - m^2 + i\varepsilon]^{-1} [(2p+Q+R)^2 - 4m^2)^{-1/2} g(Q, R) \,, \tag{64}$$

where g is a C^∞ function with essential support in a small neighbourhood of the origin.

This becomes, after introduction of the new variables $u = Q + R$ and $v = R$:

$$F(\lambda) = \int du\, dv\, e^{-i\lambda\, up} [(p+u-v)^2 - m^2 + i\varepsilon]^{-1} [(p+v)^2 - m^2 + i\varepsilon]^{-1} \times$$
$$\times [(p+u)^2 - m^2 + i\varepsilon]^{-1} [(2p+u)^2 - 4m^2]^{-1/2} g(u, v) \,. \tag{65}$$

The function

$$g(u) = \int dv [(p+u-v)^2 - m^2 + i\varepsilon]^{-1} [(p+v)^2 - m^2 + i\varepsilon]^{-1} g(u, v)$$

is continuous everywhere, C^∞ outside the manifold $(2p+u)^2 = 4m^2$ and, in general $g(0) \neq 0$.

We use now the form $p = (m, \mathbf{0})$ and introduce u_0 and $t = \mathbf{u}^2$ as new variables of integration. The integral of $g(u)$ over the angular variables in $\mathbf{u}$ will be denoted by $\bar{g}(u_0, t)$. Then

$$F(\lambda) = \int du_0\, e^{-i\lambda m u_0} \int_0^\infty dt \sqrt{t}\, \frac{(4mu_0 + u_0^2 - t + i\varepsilon)^{1/2}}{2mu_0 + u_0^2 - t + i\varepsilon} \bar{g}(u_0, t) \,. \tag{66}$$

By standard arguments it can be shown that the function $F'(\lambda)$, obtained from (66) by reversing the sign of ε, is negligible for large λ. Thus

$$F(\lambda) \sim F(\lambda) - F'(\lambda) = \int du_0\, e^{-i\lambda m u_0} J(u_0)\,, \tag{67}$$

$$J(u_0) = \int_0^\infty dt \sqrt{t}\; \bar{g}(u_0, t) \left\{ \frac{(4mu_0 + u_0^2 - t + i\varepsilon)^{1/2}}{2mu_0 + u_0^2 - t + i\varepsilon} - (\varepsilon \to -\varepsilon) \right\}. \tag{68}$$

Because of the support of $\bar{g}$ we are only interested in the behaviour of J in a small neighbourhood of $u_0 = 0$.

For $-2m < u_0 < 0$ we have $J(u_0) = 0$.

For $u_0 > 0$ we obtain

$$J(u_0) = 2 \int_0^{4mu_0 + u_0^2} dt \sqrt{t}\; (4mu_0 + u_0^2 - t)^{1/2} \frac{\mathcal{P}}{2mu_0 + u_0^2 - t} \bar{g}(u_0, t)$$

$$= (4mu_0 + u_0^2) \int_0^1 ds \sqrt{s} \sqrt{1-s}\; \frac{\mathcal{P}}{1 - 2s + \dfrac{u_0}{4m + u_0}} \bar{g}(u_0, s(4mu_0 + u_0^2))\,,$$

which tends to zero like u_0 for $u_0 \to 0$.

From this result we obtain, with the methods of the Appendix, that $F(\lambda) \sim \lambda^{-2}$. This is a slower decrease than the $\lambda^{-5/2}$ decrease that we obtained away from threshold. This result in itself is physically irrelevant. It takes into account only the contribution of the truncated part of $\hat{\tau}$ to (56). In the situation (62), which is a special case of forward scattering, the dominant contribution will come from the free propagation terms in $\hat{\tau}$, which decrease only like $\lambda^{-3/2}$. Our result shows, however, that the terms of higher order $i \geqq 1$ in (61) do indeed become important in the vicinity of the threshold. This means that, the closer we are to threshold (i.e., the lower the scattering energy), the longer we have to wait with the observation of the outgoing particles so as to be in the asymptotic region where the S matrix describes the observations with sufficient accuracy.

5. Unstable Particles

We consider the same case as in the beginning of Chapter 4, i.e., the state Φ of Eqs. (51) and (52), away from threshold, monitored by the same three counters in λx, λy, λz. But this time the factor $\hat{\tau}(\,; Q + p_1, \ldots)$ of (55) will not be assumed to be slowly varying. Instead it shall exhibit a resonance peak of the Breit-Wigner form:

$$\hat{\tau}(\,; k_1, \ldots, k_4) = \frac{a(k_1, \ldots, k_4)}{(k_1 + k_2)^2 - M^2 + iM\Gamma}\,, \qquad 4m^2 < M^2 < 9m^2, \quad \Gamma > 0\,. \tag{69}$$

The numerator a is assumed to be slowly varying. We assume that the resonance is very narrow:

$$\Gamma \ll d_2 \ll m \,, \tag{70}$$

and that the energy of Φ is exactly the resonance energy:

$$(p_1 + p_2)^2 = M^2 \,. \tag{71}$$

This latter assumption is not essential. It is made simply for convenience of notation.

The factor a can be drawn in front of the integral and we obtain instead of (57), (58) the expressions

$$F(\lambda\xi, \lambda\eta) = a(p_1, p_2, -q_1, -q_2)\, G(\lambda\xi, \lambda\eta) \,, \tag{72}$$

$$G(\lambda\xi, \lambda\eta) = \int dQ\, dR\, e^{-i\lambda(Q\xi + R\eta)} \frac{\hat{\tau}(P; -q_1, -P + q_1)}{(P - q_1)^2 - m^2 + i\varepsilon} \times$$
$$\times \frac{\hat{\tau}(Q; p_1, -Q - p_1)}{(Q + p_1)^2 - m^2 + i\varepsilon} \frac{\hat{\tau}(R; p_2, -R - p_2)}{(R + p_2)^2 - m^2 + i\varepsilon} \frac{1}{(p_1 + p_2 + Q + R)^2 - M^2 + iM\Gamma} \,. \tag{73}$$

The behaviour of G for large λ can again be discussed with the methods of the Appendix, with the following result:

$G(\lambda\xi, \lambda\eta)$ is, for sufficiently large λ, essentially different from zero only, if there exist positive constants μ, ν, ϱ, σ, with

$$\begin{aligned} \xi &\cong \mu p_1 + \varrho q_1 + \sigma(p_1 + p_2) \\ \eta &\cong \nu p_2 + \varrho q_1 + \sigma(p_1 + p_2) \,. \end{aligned} \tag{74}$$

The sign $\cong$ has the same meaning as in (59).

G decreases under these conditions like $\lambda^{-2} e^{-1/2 \lambda\sigma\Gamma M}$. The probability that all three counters are triggered is therefore only observably different from 0 if (74) is satisfied, and decreases with increasing λ like $\lambda^{-4} e^{-\lambda\sigma\Gamma M}$. The exponential factor $e^{-\lambda\sigma\Gamma M}$ is of course in the strict asymptotic sense (for really large λ) strongly decreasing. Its decrease is, however, very slow in the beginning, so that the factor is not small for the finite values of λ in which we are interested.

The condition (74) allows the following geometrical interpretation: the two straight lines drawn from the counter emplacements λy, λz, in the directions p_1, p_2, meet approximately in a point. From this point can be drawn a segment of length $\lambda\sigma M$ in direction $p_1 + p_2$, and a line drawn from the end of this segment in direction q_1 will approximately meet λx. This interpretation is not uniquely dictated by (74) but it could be made unique by using more than three counters for monitoring the process. It is the interpretation that agrees with the particle description of the process: the two original particles trigger the first two counters, then continue in direction of their momenta until they meet. On meeting they form an unstable bound state, which proceeds for a certain finite time roughly in direction of its momentum $p_1 + p_2$, then decays into two particles with momenta q_1 and q_2, one of which goes on to trigger the third counter.

The length τ of the segment in direction $p_1 + p_2$, i.e., the lifetime of the unstable particle in its rest system is $\tau = \lambda\sigma M$. The exponential factor in the probability is thus $e^{-\Gamma\tau}$, leading to the familiar decay law for unstable particles with decay constant Γ. The factor λ^{-4} agrees also with expectations.

We obtain thus a strictly exponential decay law for unstable particles. The so-called "non-eponential terms" in the decay [12] are simply the threshold terms discussed in the preceding chapter. They exist independently of whether resonances are present or not[3]. They have therefore nothing to do with the decay of unstable particles, but are a direct scattering effect, i.e., they are connected with scattering without formation of an intermediate unstable particle. If an unstable particle is defined as a particle with a measurable lifetime, or as a particle which leaves a trace of a measurable length in a track chamber, then the observation of its decay law will yield an exactly exponential decay. For resonances with a shorter half-time, i.e., larger Γ, our estimates are not sufficiently accurate. In this case, however, the decay law is not accessible to experiments, since a separation of scattering via formation of a bound state and direct scattering is not possible.

6. Final Remarks

It was the purpose of this paper to show that field theory is able to explain the observed localization properties of particles. Actually we have shown this only for some special examples dealing with low numbers of counters and simple processes (propagation of free particles, two-particle scattering). It is clear, however, that the same procedures can be applied to more complicated processes like the creation of particles, monitored by an arbitrary number of counters and will also, in these complex cases, yield results in accordance with a particle interpretation. It is also possible to treat with our methods composite processes, i.e., processes involving multiple scatterings. We will not dwell upon this point here, since we only want to establish the principle.

We mentioned already in the Introduction that the "counters" treated here may very well be the bubbles of a bubble chamber. Now we considered the counters to be localized in a small region of space-time, whilst in a bubble chamber the temporal development of a process is not observed. The exact time when the formation of a bubble is initiated is not measured, at least not with an accuracy that would be comparable to the accuracy of spatial localization (which is given by the diameter of the bubbles). This, however, is a purely technical default which is open to improvement. Our idealization is certainly permitted. It is clear that the

[3] This has, of course, already been noted by other authors, see e.g., Ref. [13].

three-dimensional bubble chamber picture is obtained from our four-dimensional track pattern simply by projection.

It is not clear to what extent our procedure can be reversed, i.e., whether it is possible to draw conclusions on the smoothness of Green's functions of the "correct" theory from the observed particle behaviour. In order to discuss this question we should need much more quantitative estimates of triggering probabilities than the ones given here. Even then it is doubtful whether any useful information could be derived from present experimental data, since experimentalists have up to now not been notably interested in the problem of establishing the spatio-temporal evolution of particle processes, their attention being focused on p space.

Appendix

In this Appendix we want to derive the asymptotic behaviour of certain integral expressions occurring in the context of this paper. The results are generalizations of RUELLE's work on the asymptotics of solutions of the Klein-Gordon equation [14].

First, we want to study the behaviour for large λ of the following expression:

$$G(\lambda, \xi_1, \ldots, \xi_n, p_1, \ldots, p_e, \tau_1, \ldots, \tau_e) = \tag{A.1}$$
$$= \int \prod_{i=1}^{n} d^4 q_i \exp\left[-i\lambda \sum_1^n (\xi_i, q_i)\right] \prod_{k=1}^{l} \delta_+(p_k + \sum_i \alpha_{ki} q_i, \tau_k)\, g(q_1, \ldots, q_n)\,.$$

Here

$$\delta_+(p, \tau) = \theta(p_0)\, \delta(p^2 - \tau)\,. \tag{A.2}$$

The p_k are vectors in the forward cone, with $p_k^2 = m_k^2 > 0$. g is a C^∞ function with support in a compact set S of diameter $d \ll m_k$, all k, and which has no strong oscillatory behaviour, i.e., satisfies conditions of the type (33). The α_{ki} are constant coefficients. For each i, $1 \leqq i < l$, there shall exist a k with $\alpha_{ki} \neq 0$.

Let

$$t_k = (p_k + \sum_{i=1}^{n} \alpha_{ki} q_i)^2 \tag{A.3}$$

be the arguments of the δ functions appearing in the integrand. t_k is for fixed p_k a function of the n four-vectors q_i. Let γ_k be the gradient of t_k in q space, with the components

$$\gamma_k^{i\nu} = \frac{\partial t_k}{\partial q_i^\nu} = 2\alpha_{ki}(p_k^\nu + \sum_j \alpha_{kj} q_j^\nu)\,. \tag{A.4}$$

We assume that the p_k are so chosen that the l gradients $\gamma_1, \ldots, \gamma_e$ are linearly independent everywhere in S. This implies in particular that $l \leqq 4n$.

The arguments ξ_i, p_k, τ_k of G will not be exhibited explicitly in the sequel.

Let $\mathfrak{M}$ be the $(4n - l)$ dimensional manifold in q space, defined by the conditions $t_k = 0$, all k. Let us assume for the moment that the vector $\Xi = (\xi_1, \ldots, \xi_n)$, considered as a vector in the same R_{4n}, is nowhere in S orthogonal to $\mathfrak{M}$. We can then introduce $t_1, \ldots, t_l$ and the exponent $\alpha = \Sigma \xi_i q_i$, together with sufficiently many appropriately chosen functions u_j of q, as new variables of integration such that the Jacobian

$$J = \frac{\partial (q_i^\nu)}{\partial (t_k, \alpha, u_j)}$$

is, in S, regular (analytic and non-vanishing). Thus:

$$G(\lambda) = \int d\alpha\, e^{-i\lambda\alpha} \int \prod du_j\, g(\tau_k, \alpha, u_j)\, J(\tau_k, \alpha, u_j) \tag{A.5}$$

in obvious notation. The function

$$f(\alpha) = \int \prod du_j\, g(\tau_k, \alpha, u_j)\, J(\tau_k, \alpha, u_j)$$

is in Schwartz space $\mathfrak{S}$ and has no strong oscillations. Its Fourier transform (A.5) is therefore strongly decreasing and may be negligibly small already for low values of λ, depending on the bounds for the derivatives of g.

Let us then consider the case that Ξ is orthogonal to $\mathfrak{M}$ in a point $P \in S$ which we choose to be the point $q_i = 0$, all i. This can always be achieved through a suitable redefinition of the p_k. This condition means that there exist constants σ_k such that

$$\xi_i = \sum_k \sigma_k \alpha_{k\,i} p_k\,. \tag{A.6}$$

We can again introduce $t_1, \ldots, t_l$ as new independent variables, together with other variables $u_1, \ldots, u_{4n-l}$ which parametrize $\mathfrak{M}$, such that the Jacobian

$$J(u_j, t_k) = \frac{\partial (q_1, \ldots, q_n)}{\partial (u_j, t_k)} \tag{A.7}$$

is regular in S. The u_j are chosen such that P is the origin of the u_j system and that the line element on $\mathfrak{M}$ in P is given by

$$ds^2 = \sum_j c_j\, du_j^2\,, \quad c_j \text{ positive constants} \tag{A.8}$$

The t_k integration in (A.1) can be carried out with the help of the δ functions, with the result:

$$G(\lambda) = \int \prod du_j \exp\left[-i\lambda \sum_h \xi_h \cdot q_h(u_j, \tau_k)\right] g(u_j, \tau_k)\, J(u_j, \tau_k)\,, \tag{A.9}$$

where $g(u_j, \tau_k) = g[q_i(u_j, \tau_k)]$. The use of the same letter g for two different functions will not lead to confusion since the functions can be distinguished by their arguments. Similar liberties will be taken tacitly in the future. The new g is again in $\mathfrak{S}$.

The functions $q_h(u_j, \tau_k)$ can, for τ_k fixed, be expanded into a Taylor series in u_j:

$$q_h = {}^0q_h(\tau_k) + {}^1q_h(u_j, \tau_k) + {}^2q_h(u_j, \tau_k) + \text{higher terms}, \tag{A.10}$$

where ${}^\nu q_h(u_j, \tau_k)$ stands for the terms of order ν in u_j.

The argument of the exponential in (A.9) is, if we take (A.6) into account:

$$\sum_h (\xi_h, q_h) = \sum_h \sum_k \sigma_h \alpha_{kh} (p_k, q_h) = \sum_k \sigma_k (p_k, Q_k) \tag{A.11}$$

with $Q_k = \sum_h \alpha_{kh} q_h$. (A.11) can be expanded in powers of u_j. The linear term of this expansion vanishes due to our choice of the u origin. Thus:

$$\Sigma(\xi_h, q_h) = A_0(\tau_k) + A_2(\tau_k, u_j) + R(\tau_k, u_j)\,, \tag{A.12}$$

where $A_2(\tau_k, u_j)$ is quadratic in u_j, and R contains the terms of third and higher degree.

The constant term is

$$A_0(\tau_k) = \sum_k \sigma_k (p_k, {}^0Q_k)\,. \tag{A.13}$$

${}^0Q(\tau_k)$ vanishes in $\tau_k = m_k^2$ (all k) and is analytic in a neighbourhood of this point. For $\tau_k \sim m_k$ we obtain

$$\begin{aligned} A_0(\tau_k) &= \Sigma\,\sigma_k\{p_k^0(-p_k^0 + [(\mathbf{p}_k + {}^0\mathbf{Q}_k)^2 + \tau_k]^{1/2}) - (\mathbf{p}_k, {}^0\mathbf{Q}_k)\} \\ &= \tfrac{1}{2}\,\Sigma\,\sigma_k(\tau_k - m_k^2) + \text{higher terms in } (\tau_k - m_k^2)\,. \end{aligned} \tag{A.14}$$

For A_2 we obtain

$$\begin{aligned} A_2(\tau_k, u_j) = \sum_k \sigma_k \Bigg\{ &\frac{p_k^0}{[(\mathbf{p}_k + {}^0\mathbf{Q}_k)^2 + \tau_k]^{1/2}} \left[({}^1\mathbf{Q}_k)^2 - \frac{1}{2}\,\frac{(\mathbf{p}_k, {}^1\mathbf{Q}_k)^2}{(\mathbf{p}_k + {}^0\mathbf{Q}_k)^2 + \tau_k}\right] + \\ &+ (\mathbf{p}_k, {}^2\mathbf{Q}_k)\left[\frac{p_k^0}{[(\mathbf{p}_k + {}^0\mathbf{Q}_k)^2 + \tau_k]^{1/2}} - 1\right]\Bigg\}\,. \end{aligned} \tag{A.15}$$

This is for $\tau_k = m_k^2$:

$$A_2(m_k^2, u_j) = \sum_k \sigma_k \left\{({}^1\mathbf{Q}_k)^2 - \frac{1}{2}\,\frac{(\mathbf{p}_k, {}^1\mathbf{Q}_k)^2}{(p_k^0)^2}\right\}. \tag{A.16}$$

Here the curly brackets are positive definite quadratic forms in ${}^1\mathbf{Q}_k$. Due to continuity the same is true for the corresponding brackets in (A.15) for τ_k sufficiently close to m_k^2. We will only consider the case that $A_2(m_k^2, u_j)$ is a non-degenerate quadratic form. This is true in particular if all σ_k are different from zero. The u_j can then be chosen such that

$$A_2(\tau_k, u_j) = \sum_{i=1}^{\mu} u_j^2 - \sum_{j=\mu+1}^{4n-l} u_j^2\,.$$

The u_j appearing in the second sum will be renamed $v_1, \ldots, v_{4n-l-\mu}$. Thus

$$A_2(\tau_k, u_j, v_h) = \sum_{j=1}^{\mu} u_j^2 - \sum_{h=1}^{\nu} v_h^2\,, \quad \mu + \nu = 4n - l\,. \tag{A.17}$$

(A.9) becomes:

$$\begin{aligned} G(\lambda) = e^{-i\lambda A_0} \int du_j\, dv_h\, g(u_j, v_h, \tau_k)\, J(u_j, v_h, \tau_k) \times \\ \times \exp\{-i\lambda[A_2(u_j, v_h, \tau_k) + R(u_j, v_h, \tau_k)]\}\,. \end{aligned} \tag{A.18}$$

We write this in the form

$$G(\lambda) = e^{-i\lambda A_0} \int dx\, e^{-i\lambda x} F(x) \tag{A.19}$$

with

$$F(x) = \int du_j\, dv_h\, g'\, \delta(x - A_2 - R), \quad g' = gJ\,. \tag{A.20}$$

The asymptotic behaviour of G is directly connected with the second order zero of the exponent appearing in (A.18), in $u_j = v_h = 0$. It suffices therefore to consider an arbitrarily small neighbourhood of the origin. We restrict ourselves to the set

$$U_r : \Sigma\, u_j^2 + \Sigma v_h^2 \leqq r^2, \quad |x| \leqq r\,, \tag{A.21}$$

where r is chosen such that we have in U_r:

$$|R(u_j, v_h, \tau_k)| \ll \Sigma\, u_j^2 + \Sigma\, v_h^2\,.$$

We introduce spherical co-ordinates in both u and v space by

$$u^2 = \sum_1^{\mu} u_j^2, \quad v^2 = \sum_1^{\nu} v_h^2\,, \tag{A.22}$$

the corresponding angular variables (suitably normalized) being denoted Ω_u, Ω_v, respectively.

Let us consider the character of $F(x)$ in U_r:

1st case: $\nu = 0$, $\mu = 4n - l$ (or $\mu = 0$, $\nu = 4n - l$).

In this case:

$$F(x) = \int du^2\, d\Omega_u\, u^{\mu-2} g(u^2, \Omega_u)\, \delta(x - u^2 - R)\,.$$

Because of the smallness of R we have

$$F(x) = 0 \quad \text{for} \quad -r < x < 0\,. \tag{A.23}$$

For $r > x > 0$ we can solve the equation

$$x - u_0^2 - R(u_0, \Omega_u) = 0 \tag{A.24}$$

for u_0^2, with the result

$$u_0^2 = x + x^{3/2} h(\sqrt{x}, \Omega_u)\,,$$

where h is infinitely differentiable in $x^{1/2}$. The derivative, with respect to u_0^2, of the left-hand side of (A.24) is

$$D = -1 + u_0 h'(u_0, \Omega_u) \tag{A.25}$$

where h' is again infinitely differentiable in its first argument. This yields:

$$F(x) = \int d\Omega_u\, u_0^{\mu-2} \frac{g'(u_0)}{1 - u_0 h'(u_0, \Omega_u)}\,.$$

This is infinitely differentiable in x for $x > 0$, and behaves for $x \to 0$ like

$$F(x) \sim g'(0)\, x^{\frac{\mu-2}{2}}\,,$$

so that we obtain for $x \geqq 0$:

$$F(x) = x^{\frac{\mu-2}{2}} f(\sqrt{x}) \tag{A.26}$$

$f(t)$ is C^∞ in $t < r^{1/2}$, and $f(0) = g'(0)$.

2nd case: $\mu \neq 0,\ \nu \neq 0,\ \mu + \nu = 4n - l$.

In this case we have

$$\begin{aligned} F(x) = \int du^2\, d\Omega_u\, dv^2\, d\Omega_v\, u^{\mu-2} v^{\nu-2} \times \\ \times g'(u, v, \Omega_u, \Omega_v)\, \delta(x - u^2 + v^2 - R)\,. \end{aligned} \tag{A.27}$$

In $r > x > 0$ the equation

$$x - u_0^2 + v^2 - R(u_0, v, \Omega_u, \Omega_v) = 0 \tag{A.28}$$

can be solved for u_0^2:

$$u_0^2 = x + v^2 + h(x, v, \Omega_u, \Omega_v) \tag{A.29}$$

with $h = 0((x + v^2)^{3/2})$.

The derivative of $x - u^2 + v^2 - R$ with respect to u^2 is, on the manifold defined by (A.28):

$$D = -1 + 0((x + v^2)^{1/2})\,. \tag{A.30}$$

The u^2 integration in (A.27) can be carried out:

$$F(x) = \int d\Omega_u\, d\Omega_v \int_0^\infty dv^2\, v^{\nu-2}\, u_0^{\mu-2} \frac{g'(u_0, v, \Omega_u, \Omega_v)}{1 + O((x + v^2)^{1/2})}\,. \tag{A.31}$$

In the same way we obtain for $-r < x < 0$:

$$F(x) = \int d\Omega_u\, d\Omega_v \int_0^\infty du^2\, u^{\mu-2}\, u_0^{\nu-2} \frac{g'(u, v_0, \Omega_u, \Omega_v)}{1 + O(v_0)} \tag{A.32}$$

with

$$v_0^2 = u^2 - x + 0((u^2 - x)^{3/2})\,.$$

For $N < \frac{\mu + \nu}{2} - 1$ exists $\lim\limits_{x \to 0} \frac{d^N F(x)}{dx^N}$ and is the same for approach from positive or negative x. The N^{th} derivative of F is thus continuous in $|x| < r$.

If $\mu + \nu$ is even (i.e., l even), then the derivative of order $N_0 = \frac{4n - l}{2} - 1$ has in $x = 0$ a singularity of the type $g'(0) \log|x|$ and we obtain, apart from an irrelevant C^∞ function:

$$F(x) = |x|^{N_0} \{\log|x| \cdot f_\pm(|x|^{1/2}) + h_\pm(|x|^{1/2})\} \quad \text{for} \quad x \gtreqless 0\,. \tag{A.33}$$

Here the functions $f_\pm, h_\pm$ are C^∞, and $f_+(0) = f_-(0) = cg'(0)$, c a numerical constant depending on ν.

For l odd we obtain in the same way, again up to a C^∞ function:

$$F(x) = |x|^{N_0} f_\pm(|x|^{1/2}) \quad \text{for} \quad x \gtreqless 0 \tag{A.34}$$

with the same $N_0 = \frac{4n - l}{2} - 1$. f_+ and f_- are C^∞, and $f_\pm(0) = cg'(0)$.

With this information on $F(x)$ we can evaluate $G(\lambda)$ as given by (A.19). The asymptotic behaviour of G is determined by the singularity of F at $x = 0$. We split the integral of (A.19) into two parts:

$$G(\lambda) = e^{-i\lambda A_0}\left\{\int_{-\infty}^{0} dx\, e^{-i\lambda x} F(x) + \int_{0}^{\infty} dx\, e^{-i\lambda x} F(x)\right\}$$

and insert the forms (A.23), (A.26), (A.33), (A.34) of $F(x)$. We shall only discuss the second integral $\int_0^\infty$ explicitly. The first integral has an asymptotic expansion of the same type. We will not study possible cancellations of terms in the two contributions. Cancellations cannot occur in the case which is most interesting to us, namely in the case $\nu = 0$, corresponding to $\sigma_k > 0$ for all k. In this case the first integral vanishes identically.

For $x \geqq 0$ we have

$$F(x) = x^{\frac{4n-l-2}{2}} e^{-x}(a_0 + a_1 x^{1/2} + \cdots + a_N x^{\frac{N}{2}}) + x^{\frac{4n-l+N-1}{2}} R_N(x^{1/2}), \tag{A.35}$$

or the same expression multiplied with $\log x$. R_N is infinitely differentiable and of strong decrease at infinity. a_0 is different from zero if $g'(0) \neq 0$, i.e., $g(u_j, \tau_k)|_{u_j=0} \neq 0$.

The remainder term $x^{\frac{4n-l+N-1}{2}} R_N(x)^{1/2}$ vanishes in $x = 0$ with its derivatives up to order $\delta = \frac{4n - l + N - 2}{2}$, or $\delta = \frac{4n - l + N - 3}{2}$ (for $l + N$ even or odd). Its contribution to $G(\lambda)$ is thus bounded by $c\lambda^{-\delta}$, c a constant. This is also true if an additional factor $\log x$ is present. N, and therefore δ, can be chosen arbitrarily large.

The other terms in (A.35) yield contributions of the form

$$Z_\beta(\lambda) = \int_0^\infty dx\, x^\beta e^{-i\lambda x} e^{-x},$$

or

$$Z'_\beta(\lambda) = \int_0^\infty dx\, x^\beta \log x\, e^{-i\lambda x} e^{-x}, \quad \beta > -1.$$

They can be calculated with the help of a suitable rotation of the integration path in the complex x plane. We obtain:

$$\begin{aligned} Z_\beta(\lambda) &= (1 + i\lambda)^{-\beta-1} \int_0^\infty dt\, t^\beta e^{-t} \sim \\ &\sim i^{-\beta-1} \lambda^{-\beta-1} \int_0^\infty dt\, t^\beta e^{-t}, \\ Z'_\beta(\lambda) &= (1 + i\lambda)^{-\beta-1} \int_0^\infty dt\, t^\beta \{\log t - \log(1 + i\lambda)\} e^{-t} \sim \\ &\sim -i^{-\beta-1} \lambda^{-\beta-1} \log\lambda \int_0^\infty dt\, t^\beta e^{-t}. \end{aligned} \tag{A.36}$$

The asymptotically leading term in $G(\lambda)$ is then of the form

$$G(\lambda) \sim c(\tau_k)\, \lambda^{-\frac{4n-l}{2}} \exp\{-i\lambda A_0(\tau_k)\}\,, \tag{A.37}$$

possibly multiplied with $\log\lambda$. $c(\tau_k)$ is a slowly varying function of the τ_k.

If $g(0, \tau_k) = 0$, then $c(\tau_k) = 0$, and $G(\lambda)$ decreases like $\lambda^{-\frac{4n-l-1}{2}}$ with a coefficient depending on the first derivative of g in $u_j = 0$, and so on. No logarithmic factors appear in the case $\nu = 0$.

We will now use these results for $G(\lambda)$ to study the asymptotics of

$$H(\lambda, \xi_i, p_k) = \int \prod_k \frac{d\tau_k}{\tau_k - m_k^2 + i\varepsilon_k}\, G(\lambda, \xi_i, p, \tau_k)\,. \tag{A.38}$$

Here ε_k may be either a fixed constant $0 < \varepsilon_k \ll d$, or it may be understood in the sense that the limit $\varepsilon_k \to 0$ has to be taken in the final result. The latter is usually the case in our applications, the only exception being Eq. (73), where one of the ε_k is finite.

If Ξ is nowhere in S orthogonal to any of the manifolds $\mathfrak{M}(\tau_k)$ we have:

$$|G(\lambda)| \leqq c_m(\tau_k)\, \lambda^{-m}$$

for all m, where the c_m have compact support. Thus:

$$|H(\lambda)| \leqq C_m \lambda^{-m}\,,$$

$$\begin{aligned} C_m = \frac{1}{2} \int \frac{d\tau_k}{(\tau_k - m_k^2)^2 + \varepsilon_k^2}\, |\tau_k - m_k^2|\, |c_m(\tau_k) - c_m(2m_k^2 - \tau_k)| \,+ \\ + \frac{\varepsilon_k}{2} \int \frac{d\tau_k}{(\tau_k - m_k^2)^2 + \varepsilon_k^2}\, [c_m(\tau_k) + c_m(2m_k^2 - \tau_k)]. \end{aligned} \tag{A.39}$$

C_m is small if c_m is small. In these directions H decreases thus rapidly.

Let us then consider the case (A.6). G has compact support in τ_k. The asymptotic behaviour of H can therefore be obtained by inserting (A.37) into (A.38):

$$H(\lambda) \sim \lambda^{-\frac{4n-l}{2}} \int \prod_k \frac{d\tau_k}{\tau_k - m_k^2 + i\varepsilon_k}\, c(\tau_k) \exp[-i\lambda A_0(\tau_k)]\,, \tag{A.40}$$

or the same multiplied by $\log\lambda$.

If the resonance peaks in $\tau_k \sim m_k^2$ are cut off smoothly so that what remains is a function in $\mathscr{S}$ with small derivatives, then we obtain rapid decrease of $H(\lambda)$ in view of (A.14). The asymptotically important terms in $H(\lambda)$ are therefore due to the contribution from a small neighbourhood of $\tau_k \sim m_k^2$ with diameter $\ll d$. $c(\tau_k)$ is slowly varying over distances of that order of magnitude and can be drawn in front of the integral:

$$H(\lambda) \sim \lambda^{-\frac{4n-l}{2}}\, c(m_k^2) \int \prod_k \frac{d\tau_k}{\tau_k - m_k^2 + i\varepsilon_k} \exp[-i\lambda A_0(\tau_k)]\,.$$

This becomes after insertion of (A.14):

$$\begin{aligned} H(\lambda) \sim \lambda^{-\frac{4n-l}{2}}\, c(m_k^2) \prod_k \int \frac{d\tau_k}{\tau_k - m_k^2 + i\varepsilon_k} \,\times \\ \times \exp\{-i\lambda\sigma_k[p_k^0(-p_k^0 + [(\mathbf{p}_k + {}^0\mathbf{Q}_k)^2 + \tau_k]^{1/2}) - (\mathbf{p}_k, {}^0\mathbf{Q}_k)]\}. \end{aligned} \tag{A.41}$$

The square roots $z_k^{1/2}$ in this expression are defined in planes cut along the negative real axis, such that $x_k^{1/2} > 0$ for $x_k > 0$. For $x_k < 0$ we define $x_k^{1/2}$ to be in the upper half plane if $\sigma_k < 0$, in the lower half plane if $\sigma_k > 0$. This yields an exponential decrease of the integrand for $\tau_k \to -\infty$.

Let us consider the k^{th} factor F_k in the product Π_k. For $\sigma_k < 0$ the integrand is the boundary value of a function of τ_k which is analytic in the upper half plane and decreases exponentially for $\tau_k \to \infty$ in the upper half plane. The path of integration can be closed by a half circle at infinity so that the integral vanishes. If $\sigma_k > 0$, then the path of integration can be closed by an infinite semicircle in the lower half plane and the method of residues gives

$$F_k(\lambda) = \exp\{-i\lambda\sigma_k p_k^0[-p_k^0 + (\mathbf{p}_k^2 + m_k^2 - i\varepsilon_k)^{1/2}]\},$$

since ${}^0\mathbf{Q}_k$ vanishes in $\tau_k = m_k^2$. With the assumption $\varepsilon_k \ll m_k^2$ this becomes

$$F_k(\lambda) \cong e^{-1/2\,\lambda\sigma_k\varepsilon_k}. \tag{A.42}$$

We arrive thus at the following result: $H(\lambda)$ is in our approximation only different from 0 if all $\sigma_k > 0$, in which case we have

$$H(\lambda) \sim c(m_k^2)\,\lambda^{-\frac{4n-l}{2}}\,e^{-1/2\,\lambda\Sigma\sigma_k\varepsilon_k}. \tag{A.43}$$

In the case of an infinitesimal ε_k the factor $e^{-\frac{\lambda}{2}\sigma_k\varepsilon_k}$ has, of course, to be replaced by 1. For strictly positive ε_k we obtain a factor $e^{-\frac{\lambda}{2}\sigma_k\varepsilon_k}$ which is for large λ strongly decreasing, which behaves, however, for small and medium values of λ sensibly like a constant.

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