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On the Simplicity of the even CAR Algebra and Free Field Models

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Abstract. An example of a local rings system where the quasilocal algebra is a simple countably generated C^*-algebra with unit is provided by the "local observables" for the free Fermi field.

1. Introduction

In this note we prove that the C^*-algebra generated by the even operators in the algebra of Canonical Anticommutation Relations (CAR) on an infinite dimensional Hilbert space is *simple* (i.e. does not contain any two sided ideal different from 0 and the whole algebra).

This result is of interest in field theory because it shows that the algebra of all quasilocal observables for a free Fermi Dirac field is a *simple separable* C^*-algebra *with unit*. In this connection we call local observables the self adjoint (uniform limits of) *even* polynomials in the field operators with test functions vanishing outside a given bounded set. The local system they define satisfies evidently the usual axioms of local field theory[1], and the C^*-algebra they generate (the quasilocal algebra) is the even part of the CAR algebra $\mathfrak{A}$ on the Hilbert space K which is the direct sum of the one electron and one positron Hilbert spaces; it is therefore simple (§ 2) in contrast with the subalgebra of all gauge invariant quantities in $\mathfrak{A}$[2].

This shows that the technical assumptions of $\mathfrak{A}$ being simple with unit and separable, sometimes used in the C^*-algebra formulation of field theory and statistical mechanics, do not conflict with the postulates of local theories.

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[1] See ref. [1]; the "additivity property" (see ibid. section IV) is clearly not satisfied in the present example.

[2] See ref. [1], Appendix II.

2. The Theorem

Let K be a (complex or real) separable infinite dimensional Hilbert space and $\mathfrak{A}$ the C^*-algebra of CAR on K[3]; by that we mean that there exists a linear map ψ from K into $\mathfrak{A}$, whose range generates $\mathfrak{A}$, such that, if $f, g \in K$,

$$\psi(f)\,\psi(g)^* + \psi(g)^*\,\psi(f) = \langle g, f\rangle I \tag{1}$$
$$\psi(f)\,\psi(g) + \psi(g)\,\psi(f) = 0$$

(where $\langle \cdot, \cdot \rangle$ is the inner product in K and I the identity operator in $\mathfrak{A}$); it follows that ψ is isometric (so that $\mathfrak{A}$ is separable) and, as f spans the unit sphere K_1 of K, the operators $U_f = \psi(f) + \psi(f)^*$ span a family of unitary self-adjoint elements which generate $\mathfrak{A}$. Moreover $\mathfrak{A}$ is simple.

The relation

$$\gamma(\psi(f)) = -\psi(f)$$

for all $f \in K$, defines an involuntary * automorphism γ of $\mathfrak{A}$; we denote by $\mathfrak{A}_+$ the set of all $A \in \mathfrak{A}$ such that $\gamma(A) = A$; $\mathfrak{A}_+$ is a C^*-subalgebra of $\mathfrak{A}$ containing I.

Remark. Each $A \in \mathfrak{A}_+$ is in the norm closure of the set $\{U_f A U_f; f \in K_1\}$; the even polynomials in ψ and ψ^* are in fact dense in $\mathfrak{A}_+$, and if $P(\psi(f_1), \ldots, \psi(f_n))$ is such a polynomial, it commutes with U_f if f is orthogonal to $f_1, \ldots, f_n$. Note that this fact is false if K is finite dimensional.

Lemma. *Let π be an irreducible representation of $\mathfrak{A}$ and suppose there is a projection E in $\pi(\mathfrak{A}_+)'$ with $0 < E < I_\pi$ (identity operator in the representation Hilbert space $\mathfrak{H}_\pi$), then the self adjoint unitary operator*

$$V = I_\pi - 2E$$

implements γ in the representation π and so $\pi(\mathfrak{A}_+)'$ is generated by I_π and E.

Proof. It is enough to show

$$V\pi(U_f)\,V = -\pi(U_f), \quad \text{if} \quad f \in K_1 . \tag{2}$$

The relation

$$\pi(U_{f_1} U_{f_2})\,V = V\pi(U_{f_1} U_{f_2})$$

if $f_1, f_2 \in K_1$, implies

$\pi(U_{f_1})\,V\pi(U_{f_1}) = \pi(U_{f_2})\,V\pi(U_{f_2}) = S$, independent from $f_1, f_2 \in K_1$. (3)

Then

$$\pi(U_f)\,(S + V)\,\pi(U_f) = S + V, \quad \text{all} \quad f \in K_1;$$

it follows $S + V = \lambda I_\pi$ (λ a complex number); introducing the operators $E = \frac{1}{2}(I_\pi - V)$ and $F = \frac{1}{2}(I_\pi - S)$, it follows that $E + F = \left(1 - \frac{\lambda}{2}\right) I_\pi$, which is only possible if $\lambda = 0$, since E, F are non zero self adjoint projections; then $S = -V$ and (2) follows from (3).

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[3] For a complete account of the basic facts on CAR quoted here, see e.g. ref. [2].

Theorem. $\mathfrak{A}_+$ *is simple.*

Proof. Let π_1 be an irreducible representation of $\mathfrak{A}_+$; it is enough to show that π_1 is faithful. Let π be an irreducible extension of π_1 to $\mathfrak{A}$, then

$$\pi_1 = E\pi \mid \mathfrak{A}_+$$

with E a self adjoint projection in $\pi(\mathfrak{A}_+)'$. If $E = I_\pi$, $\pi_1 = \pi \mid \mathfrak{A}_+$ is faithful since π is faithful. Suppose $E < I_\pi$ and let $A \in \mathfrak{A}_+$ be such that

$$\pi(A)E = 0\,;$$

then

$$\pi(A)\,(I_\pi - E) = 0 \quad \text{is equivalent to} \quad A = 0\,.$$

However the Lemma above implies $\pi(U_f)\,(I_\pi - E)\,\mathfrak{H}_\pi \subseteqq E\,\mathfrak{H}_\pi$ for all $f \in K_1$, so that

$$\pi(U_f A\, U_f)\,(I_\pi - E) = 0$$

and by the Remark this implies

$$\pi(A)\,(I_\pi - E) = 0$$

whence $A = 0$ and π_1 is faithful.

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