Commun. math. Phys. 6, 248—261 (1967)

Noether Equations and Conservation Laws*

ANDRZEJ TRAUTMAN**
Department of Physics, Syracuse University, Syracuse, N. Y.

Received June 18, 1967

Abstract. The purpose of the paper is to present a rigorous derivation of the relation between conservation laws and transformations leaving invariant the action integral. The (space-)time development of a physical system is represented by a cross section of a product bundle M. A Lagrange function is defined as a mapping $L : \overline{M} \to \mathbf{R}$, where $\overline{M}$ is the bundle space of the first jet extension of M. A symmetry transformation is defined as a bundle automorphism of M, carrying solutions of the Euler-Lagrange equation into solutions of the same equation. An important class of symmetry transformations is that of generalized invariant transformations: they are defined by specifying their action on the Euler-Lagrange equation. The generators of generalized invariant transformations are solutions of a system of linear, homogeneous partial differential equations (Noether equations). The set of all solutions of these equations has a natural structure of Lie algebra. In a simple manner, the Noether equations give rise to differential conservation laws.

1. Introduction

The nature of the connection between symmetries and the existence of conserved quantities is an intriguing physical problem. The theory of this connection, as it appears in classical physics, constitutes one of the most beautiful chapters of mathematical physics. The fundamental work on this problem was done by EMMY NOETHER in 1918 [1]. Since then, a rather large number of papers have appeared on this subject. They contain either generalizations of Noether's results [2] or their application to particular physical theories. Little work has been done on a precise statement and proof of the basic theorems relating properties of invariance to conservation laws.

The formulation that these theorems have been given until recently can be summarized as follows. To alleviate the exposition, we make here a number of simplifying assumptions. Let n and N be positive integers and consider a physical system whose history (space-time development)

* Supported by Air Force Office of Scientific Research and Aeronautical Research Laboratories.

** On leave of absence from the Institute of Theoretical Physics, Warsaw University, Warsaw, Poland.

may be represented as a mapping[1] of $\mathbf{R}^n$ into $\mathbf{R}^N$. The equation of motion,

$$L^A = 0, \quad A = 1, \ldots, N,$$

of the system is assumed to coincide with the Euler-Lagrange equation corresponding to a Lagrange function $L : \mathbf{R}^n \times \mathbf{R}^N \times \mathbf{R}^{nN} \to \mathbf{R}$. A transformation of $\mathbf{R}^n \times \mathbf{R}^N$, which may be written as

$$x' = \xi(x, y), \quad y' = \eta(x, y), \tag{1}$$

is said to be an invariant transformation for this system if it preserves the value of the action integral $\int L\, dx$. Given a continuous group of invariant transformations, one considers the 'infinitesimal' mapping

$$x' = x + \delta x, \quad y' = y + \delta y.$$

The invariance of the action integral, evaluated for $f : \mathbf{R}^n \to \mathbf{R}^N$, gives

$$L^A\, \bar{\delta} f_A + \frac{\partial T^i}{\partial x^i} = 0, \tag{2}$$

where

$$\bar{\delta} f_A = \delta y_A - \frac{\partial f_A}{\partial x^i}\, \delta x^i,$$

$$T^i = L\, \delta x^i + \frac{\partial L}{\partial y_{Ai}}\, \bar{\delta} f_A, \quad i = 1, \ldots, n$$

and everything is computed at $y = f(x)$. For any f satisfying the equation of motion, there is a differential conservation law $\partial T^i / \partial x^i = 0$.

Eq. (2) may be looked upon as the equation from which invariant transformations can be determined by solving it with respect to $(\delta x, \delta y)$ and then reconstructing the group. However, one is faced here with a somewhat unusual problem: Eq. (2) is supposed to hold for arbitrary mappings $f : \mathbf{R}^n \to \mathbf{R}^N$ and as such does not lend itself to treatment by the methods used for partial differential equations. Even if a solution $(\delta x, \delta y)$ of Eq. (2) is found, it is not obvious what is the behavior of the fields, of the action and of the equations of motion under finite transformations generated by $(\delta x, \delta y)$. In any case, it is clear that transformations of the form (1) are too general to be applicable to mappings (fields). For example, a rotation around the origin in $\mathbf{R}^2$ by an arbitrary angle $\neq k\pi$ maps the graph of the function $f(x) = x^2$ into another parabola which is not a graph of any (single-valued) function on $\mathbf{R}$.

[1] In this article, we adopt the following notation and terminology: A *map* is understood in its set-theoretic sense; if $f : E \to F$ and $g : F \to G$ are maps, then $g \circ f : E \to G$ denotes the composite map; a *bijection* is a one-to-one map onto. By a *manifold* we mean a differentiable manifold of class C^∞ and the adjective *differentiable* always refers to that class. A *mapping* is a differentiable map from one manifold into another. A *transformation* is a bijective mapping whose inverse is also a mapping. The 'number spaces' $\mathbf{R}^n$, $n = 1, 2, \ldots$, are assumed to have their natural manifold structure. If M is a manifold, then a mapping from M into $\mathbf{R}$ is called a *function*.

It is the purpose of this paper to give a precise characterization of invariant transformations and of their relation to conservation laws. In order to avoid the ambiguities of the infinitesimal language, we use exclusively the method of one-parameter groups of transformations. For simplicity, we restrict ourselves to the number spaces $\mathbf{R}^n$ and their mappings. Everything under consideration being of a local character, this restriction does not lead to any loss of generality. In Section 2 are summarized the essential facts about one-parameter groups of transformations of $\mathbf{R}^n$. Section 3 contains some elementary information on bundles, their homomorphisms and extensions. A history of a physical system may be thought of as a cross section of an appropriate bundle, space-time (or time) serving as the base. If the set Γ of cross sections under consideration is large enough, then the only maps that transform elements of Γ into cross sections are automorphisms of the bundle space (Lemma 1). In Section 4, it is shown how a differentiable automorphism of the bundle $\mathbf{R}^n \times \mathbf{R}^N$ over $\mathbf{R}^n$ may be extended to an automorphism of the bundle $\mathbf{R}^n \times \mathbf{R}^N \times \mathbf{R}^{nN}$ over $\mathbf{R}^n$. In other words, the transformation properties of derivatives of mappings are obtained from the transformation properties of the mappings themselves. Section 5 contains the definition of symmetry, invariant and generalized invariant transformations and also the derivation of some of their properties (Lemma 3). In Section 6, we present a system of partial differential equations whose solutions generate groups of generalized invariant transformations (Theorem 1) and write down the conservation law corresponding to any such solution (Theorem 2). The last section contains a few examples which are intended to show the connection of the formalism developed here with other approaches to the same problem.

2. One-parameter Groups of Transformations

A family $(\xi_t)_{t\in\mathbf{R}}$ of maps $\xi_t : \mathbf{R}^n \to \mathbf{R}^n$ is called a *one-parameter group of transformations* of $\mathbf{R}^n$ if $\chi : \mathbf{R}^{n+1} \to \mathbf{R}^n$, defined by $\chi(t, x) = \xi_t(x)$, is a mapping and

$$\xi_0 = \mathrm{id}\,, \tag{3}$$

$$\xi_t \circ \xi_s = \xi_{t+s} \quad \text{for any} \quad t, s \in \mathbf{R}\,. \tag{4}$$

It follows from (3) and (4) that $\xi_{-t} = \xi_t^{-1}$; therefore, for any t, ξ_t is a transformation of $\mathbf{R}^n$. The curves $t \to \chi(t, x)$, $x \in \mathbf{R}^n$, are called *trajectories* of the group. Through any point $x \in \mathbf{R}^n$ there passes exactly one trajectory of the group. Denoting $\frac{\partial \chi}{\partial t}(t, x)$ by $\frac{d\xi_t}{dt}(x)$, one can write the vector field tangent to the trajectories as

$$X = \left.\frac{d\xi_t}{dt}\right|_{t=0}.$$

The vector field X is said to be induced by the group (ξ_t). Differentiating both sides of Eq. (4) with respect to t and putting $s = -t$, one obtains the differential equation of the trajectories

$$\frac{d\xi_t}{dt} = X \circ \xi_t . \tag{5}$$

Conversely, if X is a vector field on $\mathbf{R}^n$, i.e., a mapping of $\mathbf{R}^n$ into $\mathbf{R}^n$, then the set of solutions of (5), each solution being defined in some neighborhood of an $x \in \mathbf{R}^n$ and for sufficiently small $t \in \mathbf{R}$, defines a *local, one-parameter group of local transformations of* $\mathbf{R}^n$ [3]. One says that this local group is generated by X, or that X is its *generator*. Subsequently, to simplify the language, we shall often say that X generates a one-parameter group of transformations although the group is defined only locally, in general.

A function a on $\mathbf{R}^n$ is called *invariant* with respect to the group (ξ_t) if it is constant along the trajectories of the group, i.e., if $a \circ \xi_t = a$ for any $t \in \mathbf{R}$. Let $X(a)$ denote the value of the vector field $X = (X^i)$ on the function a,

$$X(a) = X^i \frac{\partial a}{\partial x^i}, \quad i = 1, \ldots, n,$$

then, from Eq. (5) one derives

$$\frac{d}{dt}(a \circ \xi_t) = X(a) \circ \xi_t , \tag{6}$$

so that

$$a \circ \xi_t = a \Leftrightarrow X(a) = 0 .$$

3. Bundles, Homomorphisms and Extensions

A *bundle* is a triple (M, E, π) consisting of two sets M and E, and a map π of M onto E. The set M is called the bundle space, E — the base space and π — the projection. For any $x \in E$, the set $\pi^{-1}(x)$ is called the *fibre* over x. A subset $\Phi \subset M$ such that π restricted to Φ is a bijection onto E is called a *cross section* of the bundle (M, E, π). For reasons of linguistic simplicity, we shall often say that Φ is a cross section of M to mean that it is a cross section of the bundle (M, E, π). With any cross section Φ of M there is canonically associated a map $\varphi : E \to M$ such that $\pi \circ \varphi = \mathrm{id}$ and $\varphi(E) = \Phi$. Conversely, to any map $\varphi : E \to M$ such that $\pi \circ \varphi = \mathrm{id}$ there corresponds the cross section $\Phi = \varphi(E)$.

As an example, one can consider the *product bundle* $(E \times F, E, \pi)$, where $\pi(x, y) = x$, $(x, y) \in E \times F$. If f is a map of E into F and φ_f is the corresponding *graph map*, $\varphi_f(x) = (x, f(x))$, then $\varphi_f(E)$ is a cross section, called the *graph* of f. In the case of a product bundle, in addition to the bundle projection π, one has a second projection $\varrho : E \times F \to F$, defined by $\varrho(x, y) = y$.

Let (M_1, E_1, π_1) and (M_2, E_2, π_2) be bundles; a map $\omega : M_1 \to M_2$ is called a *bundle homomorphism* of M_1 into M_2 if there exists a map $\xi : E_1 \to E_2$ such that $\pi_2 \circ \omega = \xi \circ \pi_1$. (More precisely, one should say that the pair (ω, ξ) of maps $\omega: M_1 \to M_2$ and $\xi: E_1 \to E_2$ is a homomorphism of the bundle (M_1, E_1, π_1) in the bundle (M_2, E_2, π_2) if $\pi_2 \circ \omega = \xi \circ \pi_1$, cf. [4]). For any ω, the map ξ, if it exists, is unique. A bundle homomorphism may be characterized by the property that it maps fibres into fibres. Let ω be a bundle homomorphism of M_1 into M_2 such that the corresponding map $\xi: E_1 \to E_2$ is bijective; if $\varphi: E_1 \to M_1$ corresponds to the cross section Φ of M_1, then $\omega \circ \varphi \circ \xi^{-1}$ corresponds to the cross section $\omega(\Phi)$ of M_2. If there are given three bundles, (M_i, E_i, π_i), $i = 1, 2, 3$, and two bundle homomorphisms, (ω_j, ξ_j), $j = 1, 2$, such that $\omega_j: M_j \to M_{j+1}$ and $\xi_j: E_j \to E_{j+1}$ then the pair $(\omega_2 \circ \omega_1, \xi_2 \circ \xi_1)$ is a bundle homomorphism of (M_1, E_1, π_1) in (M_3, E_3, π_3). A bundle homomorphism ω of M_1 into M_2 such that both ω and ξ are bijective is called a *bundle isomorphism*. In particular, if the bundles (M_1, E_1, π_1) and (M_2, E_2, π_2) coincide, then a bundle isomorphism of M_1 onto M_2 is called a *bundle automorphism*.

A set Γ of cross sections of M is said to *separate the points* of the base space E if, for any $z_1, z_2 \in M$ not belonging to the same fibre, there exists a cross section $\Phi \in \Gamma$ containing both z_1 and z_2. Let ω be a bijection of M onto M and $\Phi \in \Gamma$, then $\omega(\Phi)$, in general, is not a cross section (cf. the example given in the Introduction). The question as to what bijections carry cross sections into cross sections is clarified by the following

Lemma 1. *If Γ separates the points of E, then a necessary and sufficient condition for both $\omega(\Phi)$ and $\omega^{-1}(\Phi)$ to be cross sections for all $\Phi \in \Gamma$ is that ω be a bundle automorphism.*

The sufficiency of this condition is obvious: if φ is the map associated with $\Phi \in \Gamma$ and ω is a bundle automorphism with $\pi \circ \omega = \xi \circ \pi$, then $\omega(\Phi)$ is the cross section corresponding to the map $\omega \circ \varphi \circ \xi^{-1}$. To prove that the condition is necessary, we note first that a bijection $\omega: M \to M$ is a bundle automorphism if and only if it permutes the set of all fibres. Suppose now that a bijection ω is not a permutation of the set of all fibres. This means that there exist $z_1, z_2 \in M$ such that either $\pi(z_1) \neq \pi(z_2)$ and $\pi \circ \omega(z_1) = \pi \circ \omega(z_2)$ or $\pi(z_1) = \pi(z_2)$ and $\pi \circ \omega(z_1) \neq \pi \circ \omega(z_2)$. In the first case, we take a cross section $\Phi \in \Gamma$ containing both z_1 and z_2 and observe that $\omega(\Phi)$ is not a cross section. In the second case, the same argument can be applied to ω^{-1}.

Let (M, E, π) and $(\overline{M}, E, \bar{\pi})$ be two bundles with the same base space. We say that the second bundle is a *j-extension* of the first, if there are given two sets Γ and $\bar{\Gamma}$, consisting of cross sections of M and $\overline{M}$, respectively, and a map $j: \Gamma \to \bar{\Gamma}$, called the extension map, such that

$$\text{for any} \quad \bar{z} \in \overline{M} \quad \text{there exists} \quad \Phi \in \Gamma \quad \text{such that} \quad \bar{z} \in j(\Phi)\,, \tag{7}$$

and

$$\text{if} \quad \Phi, \Psi \in \Gamma, \quad \text{then} \quad \bar{\pi}(j(\Phi) \cap j(\Psi)) \subset \pi(\Phi \cap \Psi)\,. \tag{8}$$

For the sake of simplicity, we shall also say that $\overline{M}$ is an extension of M; the cross section $j(\Phi)$ will be written as $\bar{\Phi}$.

If $\overline{M}$ is an extension of M defined by $j : \Gamma \to \bar{\Gamma}$, then there exists a unique bundle homomorphism τ of $\overline{M}$ onto M, such that

$$\pi \circ \tau = \bar{\pi} \quad \text{and} \quad \tau \circ j = \text{id} .$$

The map τ is called the projection of $\overline{M}$ onto M, defined by j. Let $(\overline{M}_i, E_i, \bar{\pi}_i)$ be an extension of (M_i, E_i, π_i), defined by $j_i : \Gamma_i \to \bar{\Gamma}_i$, $i = 1, 2$. A bundle homomorphism (ω, ξ), where $\omega : M_1 \to M_2$ and $\xi : E_1 \to E_2$, is said to be *compatible* with these extensions if

$$\omega(\Gamma_1) \subset \Gamma_2$$

and

$$(\xi \circ \bar{\pi}_1)(\bar{\Phi} \cap \bar{\Psi}) \subset \bar{\pi}_2(\overline{\omega(\Phi)} \cap \overline{\omega(\Psi)})$$

for any $\Phi, \Psi \in \Gamma_1$. If ω is compatible with the extensions then it can be extended to a bundle homomorphism $\bar{\omega} : \overline{M}_1 \to \overline{M}_2$ in such a way that

$$\tau_2 \circ \bar{\omega} = \omega \circ \tau_1 \quad \text{and} \quad \bar{\omega} \circ j_1 = j_2 \circ \omega ,$$

where τ_i is the projection of $\overline{M}_i$ on M_i, defined by j_i. If, in addition, $\xi : E_1 \to E_2$ is bijective, $\varphi : E_1 \to M_1$, corresponds to $\Phi \in \Gamma_1$, and $\bar{\varphi} : E_1 \to \overline{M}_1$ corresponds to $\bar{\Phi} = j_1(\Phi) \in \bar{\Gamma}_1$, then the map

$$\bar{\omega} \circ \bar{\varphi} \circ \xi^{-1} = \overline{\omega \circ \varphi \circ \xi^{-1}} \tag{9}$$

corresponds to $\bar{\omega}(\bar{\Phi}) = \overline{\omega(\Phi)} \in \bar{\Gamma}_2$.

If there are given three bundles (M_i, E_i, π_i), their extensions $(\overline{M}_i, E_i, \bar{\pi}_i)$, $i = 1, 2, 3$ and two bundle homomorphisms (ω_j, ξ_j), $\omega_j : M_j \to M_{j+1}$, $\xi_j : E_j \to E_{j+1}$, $j = 1, 2$, compatible with these extensions, then the bundle homomorphism $(\omega_2 \circ \omega_1, \xi_2 \circ \xi_1)$ is also compatible and

$$\overline{\omega_2 \circ \omega_1} = \bar{\omega}_2 \circ \bar{\omega}_1 . \tag{10}$$

As a corollary, one infers that if $\omega : M_1 \to M_2$ is a bundle isomorphism and both ω and ω^{-1} are compatible with the extensions, then $\bar{\omega} : \overline{M}_1 \to \overline{M}_2$ is also a bundle isomorphism and $\bar{\omega}^{-1} = \overline{\omega^{-1}}$.

4. Jet Extensions

Let n and N be positive integers and consider the manifolds $E = \mathbf{R}^n$, $F = \mathbf{R}^N$, $\dot{F} = E^* \otimes F \approx \mathbf{R}^{nN}$, $\ddot{F} = E^* \otimes E^* \otimes F \approx \mathbf{R}^{n^2N}$, $M = E \times F$, $\overline{M} = M \times \dot{F}$ and $\overline{\overline{M}} = \overline{M} \times \ddot{F}$. Here E^* is the dual of E, considered as a vector space and $\approx$ denotes a natural identification. Typical elements of these manifolds will be denoted in the following manner: $x = (x^i) \in E$, $y = (y_A) \in F$, $\dot{y} = (y_{A\,i}) \in \dot{F}$, $\ddot{y} = (y_{A\,ij}) \in \ddot{F}$, $z = (x, y) \in M$, $\bar{z} = (x, y, \dot{y}) \in \overline{M}$, etc., where $i, j = 1, \ldots, n$ and $A = 1, \ldots, N$. As explained in Section 3, the set of all mappings of E into F may be identified with the set Γ of all differentiable cross sections of the product bundle (M, E, π), where $\pi(x, y) = x$. Clearly, this set of cross sections of (M, E, π) separates the points of E. Let $\bar{\Gamma}$ denote the set of all differentiable cross sections of the bundle $(\overline{M}, E, \bar{\pi})$, where $\bar{\pi}(x, y, \dot{y}) = x$. We can now define a map $j: \Gamma \to \bar{\Gamma}$, allowing us to consider $\overline{M}$ as an extension of M. Given a mapping $f: E \to F$, the graph mapping $\varphi_f: E \to M$,

$$\varphi_f(x) = (x, f(x)), \tag{11}$$

associated with it, and the cross section $\Phi = \varphi_f(E)$, we define $j(\Phi)$ to be the cross section of $\overline{M}$ corresponding to $\bar{\varphi}_f: E \to \overline{M}$, where

$$\bar{\varphi}_f(x) = (x, f(x), f'(x))$$

and $f' = \operatorname{grad} f = (\partial f_A / \partial x^i)$. Clearly, $j: \Gamma \to \bar{\Gamma}$ is an extension map, i.e. conditions (7) and (8) are satisfied. One says that j defines $(\overline{M}, E, \bar{\pi})$ as the (first) *jet extension* of (M, E, π) [5]. The projection $\tau: \overline{M} \to M$ is now a mapping given by $\tau(x, y, \dot{y}) = (x, y)$. In addition, one can define two other projections, $\varrho: M \to F$, $\varrho(x, y) = y$ and $\dot{\varrho}: \overline{M} \to \dot{F}$, $\dot{\varrho}(x, y, \dot{y}) = \dot{y}$. Jet extensions of higher order may be defined in a similar way. Here we need only the second jet extension $(\overline{\overline{M}}, E, \bar{\bar{\pi}})$ of (M, E, π) defined by associating with the graph mapping (11) the mapping $\bar{\bar{\varphi}}_f: E \to \overline{\overline{M}}$ such that

$$\bar{\bar{\varphi}}_f(x) = (x, f(x), f'(x), f''(x)),$$

where $f'' = (\partial^2 f_A / \partial x^i \partial x^j)$ is the set of all second partial derivatives of f.

Any differentiable homomorphism (ω, ξ) of the bundle (M_1, E_1, π_1) in the bundle (M_2, E_2, π_2) is compatible with their jet extensions (here and subsequently, $E_i = \mathbf{R}^{n_i}$, $F_i = \mathbf{R}^{N_i}$, $M_i = E_i \times F_i$, etc., n_i and N_i are positive integers, $i = 1, 2$). For any $(x, y) \in M_1$,

$$\omega(x, y) = (\xi(x), \eta(x, y))$$

where $\eta: M_1 \to F_2$ is a mapping. In particular, let $E_1 = E_2 = E$, $\chi: M_1 \to F_2$ be a mapping and φ_χ denote the bundle homomorphism of M_1 into M_2 such that $\pi_2 \circ \varphi_\chi = \pi_1$ and $\varrho_2 \circ \varphi_\chi = \chi$, that is

$$\varphi_\chi(x, y) = (x, \chi(x, y)).$$

Given χ, one defines a mapping $\operatorname{Grad} \chi = (\operatorname{Grad}_i \chi): \overline{M}_1 \to \dot{F}_2$, $i = 1, \ldots, n$, by writing the extension $\bar{\varphi}_\chi$ of φ_χ as

$$\bar{\varphi}_\chi(x, y, \dot{y}) = (x, \chi(x, y), (\operatorname{Grad} \chi)(x, y, \dot{y})). \tag{12}$$

Note that $\operatorname{Grad} \varrho = \dot{\varrho}$. If $\chi: M_1 \to F_2 = E \otimes F_1$, then $\operatorname{Div} \chi$ denotes the obvious contraction of $\operatorname{Grad} \chi$:

$$(\operatorname{Div} \chi)_A = \operatorname{Grad}_i \chi^i_A; \quad i = 1, \ldots, n; \quad A = 1, \ldots, N_1.$$

With any transformation $\xi : E \to E$ one can associate the bundle homomorphism $\psi_\xi : M \to M$ given by

$$\psi_\xi(x, y) = (\xi(x), y) .$$

To evaluate the extended homomorphism $\bar{\psi}_\xi : \overline{M} \to \overline{M}$, one may use formula (9). The result is

$$\psi_\xi(x, y, \dot{y}) = (\xi(x), y, \dot{y} \cdot \xi^{-1\prime}(x)) \tag{13}$$

where the significance of the symbols is the obvious one:

$$(\dot{y} \cdot \xi^{-1\prime}(x))_{A\,i} = y_{A\,j} \frac{\partial \xi^{-1j}}{\partial x^i} \circ \xi(x) .$$

Any differentiable bundle automorphism $\omega : M \to M$, $\pi \circ \omega = \xi \circ \pi$ may be represented as

$$\omega = \psi_\xi \circ \varphi_\eta ,$$

where $\eta = \varrho \circ \omega$. Using Eqs. (10), (12) and (13) one obtains the following general formula for the extended automorphism $\bar{\omega}$,

$$\bar{\omega}(x, y, \dot{y}) = (\xi(x), \eta(x, y), (\operatorname{Grad}\eta)(x, y, \dot{y}) \cdot \xi^{-1\prime}(x)) . \tag{14}$$

For an arbitrary mapping $\chi : M_1 \to F_2$ and bundle automorphism $\omega : M_1 \to M_1$, equation (9) applied to $\varphi_\chi \circ \omega$ gives

$$(\operatorname{Grad}\chi) \circ \bar{\omega} = \operatorname{Grad}(\chi \circ \omega) \cdot \xi^{-1\prime} . \tag{15}$$

Similarly, if $\varphi_f : E \to M_1$ is a graph mapping, then

$$(\operatorname{Grad}\chi) \circ \bar{\varphi}_f = \operatorname{grad}(\chi \circ \varphi_f) . \tag{16}$$

Since the bundle $(\overline{\overline{M}}, E, \bar{\bar{\pi}})$ may be considered, in an obvious and natural manner, as an extension of $(\overline{M}, E, \bar{\pi})$, it is clear how the Grad map may be generalized to be applicable to mappings like $\dot{\chi} : \overline{M}_1 \to \dot{F}_2$. For the mapping $\operatorname{Grad}\chi : \overline{\overline{M}}_1 \to \ddot{F}_2$ the analogues of formulae (15) and (16) are

$$(\operatorname{Grad}\dot{\chi}) \circ \bar{\bar{\omega}} = \operatorname{Grad}(\dot{\chi} \circ \bar{\omega}) \cdot \xi^{-1\prime} \tag{17}$$

and

$$(\operatorname{Grad}\dot{\chi}) \circ \bar{\bar{\varphi}}_f = \operatorname{grad}(\dot{\chi} \circ \bar{\varphi}_f) , \tag{18}$$

where $\bar{\bar{\omega}}$ is the extension of ω to $\overline{\overline{M}}_1$.

Let us now consider a one-parameter group $(\omega_t)_{t \in \boldsymbol{R}}$ of automorphisms of M, $\pi \circ \omega_t = \xi_t \circ \pi$. Clearly, $(\xi_t)_{t \in \boldsymbol{R}}$ is a one-parameter group of transformations of E and

$$\omega_t(x, y) = (\xi_t(x), \eta_t(x, y)) .$$

The vector fields induced by the groups (ξ_t) and (ω_t) are, respectively,

$$X = \left.\frac{d\xi_t}{dt}\right|_{t=0} \quad \text{and} \quad Z = (X, Y), \quad \text{where} \quad Y = \left.\frac{d\eta_t}{dt}\right|_{t=0} . \tag{19}$$

The families $(\bar{\omega}_t)_{t\in\mathbf{R}}$ and $(\bar{\bar{\omega}}_t)_{t\in\mathbf{R}}$ of extensions of the automorphisms ω_t also constitute one-parameter groups of transformations, respectively of $\bar{M}$ and of $\bar{\bar{M}}$. The vector field $\bar{Z}$ induced on $\bar{M}$ by $(\bar{\omega}_t)$ may be evaluated from Eq. (14),

$$\bar{Z} = (X,\ Y,\ \mathrm{Grad}\, Y - \dot{\varrho} \cdot X')\ . \tag{20}$$

The map $Z \to \bar{Z}$ is a Lie algebra homomorphism: it is linear and

$$\overline{[Z_1, Z_2]} = [\bar{Z}_1, \bar{Z}_2]\ ,$$

where the brackets denote the usual commutators of vector fields.

5. Euler-Lagrange Equation and Symmetry Transformations

We wish now to consider physical systems whose evolution is described by differential equations derivable from a variational principle with a Lagrange function depending on field variables, their first derivatives and possibly also on the independent variables. More precisely, we assume that the *histories* of a physical system may be identified with, or described by, cross sections of a bundle (M, E, π). By a Lagrange function we understand a function L defined on the bundle space $\bar{M}$ of the first jet extension of (M, E, π), the significance of the symbols being the same as in the previous section.

Given the Lagrange function,

$$L : \bar{M} \to \mathbf{R}\ ,$$

one can form the *Euler-Lagrange mapping*

$$[L] : \bar{\bar{M}} \to F^*$$

defined by

$$[L] = \frac{\partial L}{\partial y} - \mathrm{Div}\, \frac{\partial L}{\partial \dot{y}}\ .$$

Let $f : E \to F$ and $\bar{\bar{\varphi}}_f$ be the second jet extension of the graph mapping φ_f of f. One says that f satisfies the Euler-Lagrange equation, if

$$[L] \circ \bar{\bar{\varphi}}_f = 0\ .$$

We now wish to consider the following problem: what are the maps that carry solutions of the Euler-Lagrange equation into solutions of the same equation? Lemma 1 suggests that we should look for these maps among automorphisms of M. Accordingly, we define a *symmetry transformation* as an automorphism ω of the bundle (M, E, π) such that

$$\text{if}\quad f : E \to F \quad\text{and}\quad [L] \circ \bar{\bar{\varphi}}_f = 0, \quad\text{then}\quad [L] \circ \bar{\bar{\omega}} \circ \bar{\bar{\varphi}}_f = 0\ . \tag{21}$$

An important example of symmetry transformations is provided by transformations leaving invariant the action corresponding to L,

$$\int_\Omega L \circ \bar{\varphi}_f$$

where $\Omega \subset E$ is a compact domain and the integral is taken with respect to the Lebesgue measure. An automorphism ω of M is called an *invariant transformation* of L if, for any Ω and f,

$$\int_{\Omega} L \circ \bar{\varphi}_f = \int_{\xi(\Omega)} L \circ \overline{\omega \circ \varphi_f \circ \xi^{-1}}$$

where, as before, $\pi \circ \omega = \xi \circ \pi$. The domain Ω and the mapping f being arbitrary, the last equation is equivalent to

$$L = J\, L \circ \bar{\omega} \tag{22}$$

where J is the jacobian of ξ,

$$J = \det \left(\frac{\partial \xi^i}{\partial x^j} \right)$$

To show that an invariant transformation is indeed a symmetry transformation, one can use the following

Lemma 2. *If* $L : \overline{M} \to \mathbf{R}$ *is a Lagrange function and* $\omega : M \to M$ *is a bundle automorphism,* $\omega(x, y) = (\xi(x), \eta(x, y))$, *then*

$$[J\, L \circ \bar{\omega}]^A = J \frac{\partial \eta_B}{\partial y_A} [L]^B \circ \bar{\bar{\omega}} . \tag{23}$$

This can be proved by a straightforward computation, using formula (17) and the obvious identities

$$\frac{\partial}{\partial y_{A\,i}} \operatorname{Grad}_j \eta_B = \frac{\partial \eta_B}{\partial y_A} \delta^i_j, \qquad \frac{\partial}{\partial y_A} \operatorname{Grad}_i \eta_B = \operatorname{Grad}_i \frac{\partial \eta_B}{\partial y_A},$$

$$\frac{\partial}{\partial x^i} \left(J \frac{\partial \xi^{-1i}}{\partial x^j} \circ \xi \right) = 0 \,.$$

Now, if ω is an invariant transformation, then Eq. (22) holds and Eq. (23) becomes

$$[L]^A = J \frac{\partial \eta_B}{\partial y_A} [L]^B \circ \bar{\bar{\omega}} . \tag{24}$$

Since ω is a transformation, the matrix $(J\, \partial \eta_B / \partial y_A)$ is non-singular and Eq. (24) implies condition (21).

An important class of symmetry transformations wider than the class of invariant transformations, consists of all those automorphisms ω of M for which Eq. (24) holds; we shall call them *generalized invariant transformations*. Their basic property is given by

Lemma 3. *A bundle automorphism* $\omega : M \to M$ *is a generalized invariant transformation if and only if there exists a mapping* $\chi : M \to E$ *such that*

$$J\, L \circ \omega + \operatorname{Div} \chi = L \,.$$

This is a direct consequence of Eqs. (23) and (24) and of a classical theorem [6]: if K is a Lagrange function, then a necessary and sufficient condition for $[K] = 0$ is that there be a mapping $\chi : M \to E$ such that $K = \operatorname{Div} \chi$.

6. Conservation Laws

Lemma 4. *If* (ω_t) *is a one-parameter group of automorphisms of the bundle* (M, E, π), $\pi \circ \omega_t = \xi_t \circ \pi$, *and* L *is a Lagrange function, then*

$$\frac{d}{dt}(J_t L \circ \overline{\omega}_t) = J_t(\overline{Z}(L) + L \operatorname{div} X) \circ \overline{\omega}_t , \tag{25}$$

where $J_t = \det(\partial \xi_t^i / \partial x^j)$ *and* $\overline{Z}$ *is the vector field induced on* $\overline{M}$ *by the group* $(\overline{\omega}_t)$ *of automorphisms, obtained by extension of* (ω_t).

This is a simple consequence of Eq. (6) and

$$\frac{dJ_t}{dt} = J_t(\operatorname{div} X) \circ \xi_t$$

where X is given by (19) and $\operatorname{div} X = \partial X^i / \partial x^i$.

Theorem 1. *A necessary and sufficient condition for the one-parameter group* (ω_t) *of automorphisms of* M *to consist of generalized invariant transformations of* L *is that there exist a mapping* $V: M \to E$ *such that*

$$\overline{Z}(L) + L \operatorname{div} X + \operatorname{Div} V = 0 . \tag{26}$$

The notation here is that of the previous lemma.

Proof. Let (ω_t) be a one-parameter group of automorphisms of M. By Lemma 3, the group consists of generalized invariant transformations of L if, for any t, there is a $\chi_t: M \to E$ such that

$$K_t = \operatorname{Div} \chi_t \tag{27}$$

where

$$K_t = L - J_t L \circ \overline{\omega}_t . \tag{28}$$

If this is so, then, putting

$$V = \left.\frac{d\chi_t}{dt}\right|_{t=0},$$

differentiating both sides of Eq. (27) with respect to t and using Lemma 4, one arrives at Eq. (26). Conversely, if we assume Eq. (26) to hold and construct the group (ω_t) generated by the vector field Z, then, by (25), (26), and (28):

$$\frac{dK_t}{dt} = J_t (\operatorname{Div} V) \circ \overline{\omega}_t .$$

Using Lemma 2 and the identity $[\operatorname{Div} V] = 0$, one obtains from the last equation

$$\frac{d}{dt}[K_t] = 0 .$$

Together with $K_0 = 0$, this implies $[K_t] = 0$ and ensures the existence of χ_t such that Eq. (27) holds.

The Noether-Bessel-Hagen [1], [2] equation (26) is, in fact, equivalent to a system of partial differential equations of the first order, linear and homogeneous in the unknowns (Z, V). If (Z_1, V_1) and (Z_2, V_2) are solutions of Eq. (26), then so is (Z_3, V_3), where

$$Z_3 = [Z_1, Z_2] \quad \text{and} \quad V_3 = Z_1(V_2) - Z_2(V_1) + \\ + V_2 \operatorname{div} X_1 - V_1 \operatorname{div} X_2 - X_1' \cdot V_2 + X_2' \cdot V_1 .$$

In particular, if Z is induced by an invariant transformation, then it satisfies the *Noether equation*

$$\overline{Z}(L) + L \operatorname{div} X = 0 . \tag{29}$$

The set of all solutions of Eq. (29) forms a subalgebra of the Lie algebra of all vector fields on M.

Let $\langle \alpha, \beta \rangle = \operatorname{Tr}(\alpha \otimes \beta)$ for $\alpha \in F$ and $\beta \in F^* \otimes B$, where B is a vector space. The following identity can be derived by a direct computation:

$$\overline{Z}(L) + L \operatorname{div} X = \langle Y - \dot{\varrho} \cdot X, [L] \rangle + \operatorname{Div} \left(XL + \left\langle Y - \dot{\varrho} \cdot X, \frac{\partial L}{\partial \dot{y}} \right\rangle \right) .$$

This, together with Eq. (18), leads to

Theorem 2. *If f is a solution of the Euler-Lagrange equation and Z generates a one-parameter group of generalized invariant transformations, then*

$$\operatorname{div}(T \circ \overline{\varphi}_f) = 0 , \tag{30}$$

where

$$T = XL + \left\langle Y - \dot{\varrho} \cdot X, \frac{\partial L}{\partial \dot{y}} \right\rangle + V .$$

Formula (30) may be interpreted, in a well-known manner [7, 8], as expressing a *conservation law* in a differential form. For example, if the base space is one-dimensional, then Eq. (30) reduces to $d(T \circ \overline{\varphi}_f)/dx = 0$, i.e., to the assertion that T is a constant of the motion.

7. Examples

1. Invariant Field Theories

A field theory whose equations follow from a Lagrange function L is said to be generally invariant (or covariant), if to any 'coordinate' transformation $\xi : E \to E$ there corresponds a transformation $\omega : M \to M$ such that the pair (ω, ξ) is a generalized invariant transformation of L. More precisely let σ denote a Lie algebra homomorphism of $\mathfrak{gl}(n) \approx E \otimes E^*$ into $\mathfrak{gl}(N) \approx F \otimes F^*$ and put

$$\sigma(e_i \otimes e^j) = \sigma_{A\,i}{}^{Bj} e^A \otimes e_B ,$$

where (e_i), (e^i), (e^A) and (e_A) are the canonical bases of E, E^*, F and F^*, respectively. One says that L describes a *generally invariant field theory of quantities of type* σ if, for any vector field X on E, with Y given by

$$Y_A(x, y) = \sigma_{A\,i}{}^{Bj} \frac{\partial X^i}{\partial x^j}(x) y_B$$

there exists a $V : M \to E$ such that Eq. (26) is satisfied. In this case, the Lie algebra of solutions of Eq. (26) is infinite-dimensional and, in addition to the 'weak' conservation law (30), one can derive a generalized Bianchi identity and a 'strong' conservation law [9].

2. *Canonical Transformations*

Let $E = \mathbf{R}$ be the base space, $F = \mathbf{R}^m$, and $M = E \times F \times F^*$ be the bundle space. Let $z = (x, q, p)$ denote a typical element of M, $q = (q^\alpha) \in F$, $p = (p_\alpha) \in F^*$, $\alpha = 1, \ldots, m$. The set $\mathscr{F}$ of all functions on M is a Lie algebra with respect to the *Poisson bracket*

$$\{A, B\} = \frac{\partial A}{\partial q^\alpha}\frac{\partial B}{\partial p_\alpha} - \frac{\partial A}{\partial p_\alpha}\frac{\partial B}{\partial q^\alpha}, \quad A, B \in \mathscr{F}.$$

An automorphism ω of the bundle (M, E, π) is called a *canonical transformation* if, for any $A, B \in \mathscr{F}$

$$\{A, B\} \circ \omega = \{A \circ \omega, B \circ \omega\}.$$

Let (ω_t) be a one-parameter group of automorphisms of M and $Z = (X, Q, P)$ the vector field induced on M by (ω_t). A necessary and sufficient condition for (ω_t) to be a one-parameter group of canonical transformations is that Z be a *derivation* of the Lie algebra $\mathscr{F}$,

$$Z(\{A, B\}) = \{Z(A), B\} + \{A, Z(B)\}.$$

From this one obtains that there is a $G \in \mathscr{F}$ such that $Q^\alpha = -\partial G/\partial p_\alpha$, $P_\alpha = \partial G/\partial q^\alpha$. Therefore, for any $A \in \mathscr{F}$,

$$Z(A) = X\frac{\partial A}{\partial x} + \{G, A\}.$$

In particular, if (ω_t) is a group of time-independent canonical transformations, then $X = 0$ and Z is an inner derivation of $\mathscr{F}$. Let the Lagrange function be of the form

$$L(\bar{z}) = \frac{1}{2}(\dot{q}^\alpha p_\alpha - q^\alpha \dot{p}_\alpha) - H(q, p), \quad H \in \mathscr{F}.$$

Assuming, for simplicity, that Z does not depend on the time x, one gets

$$\bar{Z}(L) = \{H, G\} + \operatorname{Div}\left(G - \frac{1}{2}p_\alpha\frac{\partial G}{\partial q_\alpha} - \frac{1}{2}q^\alpha\frac{\partial G}{\partial q^\alpha}\right).$$

This implies the well-known result that Z generates a (generalized) invariant, time-independent canonical transformation if and only if $\{H, G\} = 0$ and that G is the corresponding conserved quantity.

3. *A Simple Example*

The following elementary example shows how the Noether equation can be solved explicitly in specific cases. Let $E = \mathbf{R}$, $F = \mathbf{R}^m$, $\dot{F} \approx F$ and $M = E \times F$. Let the Lagrange function be $L = Q^p$, $p > 0$, where Q is the positive definite, quadratic form on $\dot{F}$,

$$Q(\dot{y}) = a_{\alpha\beta}\dot{y}^\alpha\dot{y}^\beta, \quad \alpha, \beta = 1, \ldots, m.$$

If one puts $Z = (X, Y)$, $Y = (Y^\alpha)$ and remembers Eq. (20), then one can write the Noether equation as

$$(1 - 2p)\frac{dX}{dx}Q(\dot{y}) + 2pa_{\alpha\beta}\dot{y}^\alpha\left(\frac{\partial Y^\beta}{\partial x} + \frac{\partial Y^\beta}{\partial y^\gamma}\dot{y}^\gamma\right) = 0.$$

For $p \neq \frac{1}{2}$, its general solution is

$$X = \mu x + \nu, \quad Y^\alpha = \omega^\alpha{}_\beta y^\beta + \delta^\alpha + \frac{2p-1}{2p} \mu y^\alpha$$

where μ, ν, δ^α and $\omega^\alpha{}_\beta$ are real numbers subject to

$$a_{\alpha\gamma} \omega^\gamma{}_\beta + a_{\beta\gamma} \omega^\gamma{}_\alpha = 0 .$$

For $p = \frac{1}{2}$, in addition to the solutions described above, there exists an *infinite* dimensional space of solutions, obtained by taking for X an arbitrary function on $\mathbf{R}$ and putting $Y = 0$. This is a particularly simple case of a generally invariant theory, corresponding to $n = 1$ and $\sigma = 0$.

Acknowledgments. The author wishes to thank K. Maurin for the initial encouragement, A. Lichnerowicz for having drawn his attention to Ehresmann's work on jets, P. G. Bergmann and J. N. Goldberg for their hospitality at Syracuse University and many stimulating and critical remarks.

References

1. Noether, E.: Invariante Variationsprobleme. Göttinger Nachr. **1918**, 235.
2. Bessel-Hagen, E.: Über die Erhaltungssätze der Elektrodynamik. Math. Ann. **84**, 258 (1921).
3. Nomizu, K.: Lie groups and differential geometry. Publications of the Mathematical Society of Japan, No. 2 (1956).
4. Husemoller, D.: Fibre bundles. New York: McGraw-Hill 1966.
5. Ehresmann, C.: Les prolongements d'une variété différentiable. Atti d. IV Congresso dell'Unione Mat. Italiana, Taormina 1951, Ed. Cremonese, Rome 1953.
6. Courant, R., and D. Hilbert: Methods of mathematical physics. Vol. I. New York: Interscience 1953.
7. Hill, E. L.: Hamilton's principle and conservation theorems of mathematical physics. Revs. Mod. Phys. **23**, 253 (1951).
8. Trautman, A.: Conservation laws in general relativity. In: Gravitation, ed. by L. Witten. New York: J. Wiley 1962.
9. Bergmann, P. G.: Non-linear field theories. Phys. Rev. **75**, 680 (1949).