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Types of von Neumann Algebras Associated with Extremal Invariant States

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Abstract. A globalized version of the following is proved. Let $\mathscr{R}$ be a factor acting on a Hilbert space $\mathscr{H}$, G a group of unitary operators on $\mathscr{H}$ inducing automorphisms of $\mathscr{R}$, x a vector separating and cyclic for $\mathscr{R}$ which is up to a scalar multiple the unique vector invariant under the unitaries in G. Then either $\mathscr{R}$ is of type III or ω_x is a trace of $\mathscr{R}$. The theorem is then applied to study the representations due to invariant factor states of asymptotically abelian C^*-algebras, and to show that in quantum field theory certain regions in the Minkowski space give type III factors.

1. Introduction

In the operator algebra setting for quantum field theory and quantum statistical mechanics there have been given several examples of von Neumann algebras of types III and II_1, see e.g. [1, 3, 8]. Then one has a von Neumann algebra, a group of unitary operators inducing automorphisms of it, and a unique invariant vector, and one shows the von Neumann algebra is of type II_1 if the invariant vector is a trace vector and type III otherwise. In the present paper we shall prove general theorems roughly to the same effect, and apply them to obtain generalizations of results in the quoted papers and also to describe the representations due to extremal invariant states of asymptotically abelian C^*-algebras.

2. Automorphisms of von Neumann Algebras

Our main results are proved in this section. The proof will be based on the ideas of HUGENHOLTZ [8] together with those of KOVÁCS and SZÜCS [10]. We first recall terminology and results from [10]. Let $\mathscr{R}$ be a von Neumann algebra and G a group of $*$-automorphisms of $\mathscr{R}$. A state ϱ of $\mathscr{R}$ (or more generally, a positive linear map of $\mathscr{R}$ into another von Neumann algebra) is *G-invariant* if $\varrho \circ g = \varrho$ for all $g \in G$. $\mathscr{R}$ is *G-finite* if for each non zero positive operator A in $\mathscr{R}$ there exists a normal G-invariant state ϱ of $\mathscr{R}$ such that $\varrho(A) \neq 0$. Denote by $\operatorname{conv}(g(A) : g \in G)^-$ the weakly closed convex hull of the orbit of A under G. Let $\mathscr{B}$ denote the fixed point algebra in $\mathscr{R}$, so $\mathscr{B} = \{A \in \mathscr{R} : g(A) = A$ for all $g \in G\}$. If $\mathscr{R}$ is G-finite then $\mathscr{B} \cap \operatorname{conv}(g(A) : g \in G)^-$ consists of exactly one operator $\phi(A)$ [10, Theorem 1]. The map $\phi : \mathscr{R} \to \mathscr{B}$ is the unique faithful normal G-invariant positive linear projection map (expectation) of $\mathscr{R}$ onto $\mathscr{B}$, and $\mathscr{R}$ is G-finite if and only if such a map exists [10, Theorem 2]. KOVÁCS and SZŰCS have pointed out to the author that their results also yield a shorter proof of [13, Theorem 3.1]. We first modify a well known result about strong convergence of operators.

Lemma 2.1. *Let $\mathscr{R}$ be a von Neumann algebra and ω a faithful normal semifinite trace of $\mathscr{R}$. Let $\{E_n\}_{n=1,2,\ldots}$ be a sequence of projections in $\mathscr{R}$ such that $\omega(E_n) \to 0$. Then $E_n \to 0$ strongly.*

Proof. Let F be a finite projection. Then $0 \leqq \omega(FE_nF) = \omega(E_nF) \leqq \leqq \omega(E_n) \to 0$, and $\omega(FE_nF) \to 0$. Since the functional $\omega(\cdot\, F) = \omega(F \cdot F)$ is normal and finite, $E_nF \to 0$ strongly by [6, p. 62]. Let $\varepsilon > 0$ be given. Let $\mathscr{H}$ denote the underlying Hilbert space, and let $x_1, \ldots, x_k$ be a finite set of vectors in $\mathscr{H}$. Choose a finite projection F in $\mathscr{R}$ such that $\|Fx_j - x_j\| < \varepsilon/2, j = 1, \ldots, k$, which is possible since the ideal generated by positive operators finite under ω, is strongly dense in $\mathscr{R}$. Thus

$$\|E_nx_j\| \leqq \|E_n(Fx_j - x_j)\| + \|E_nFx_j\| \leqq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for n sufficiently large, and $E_n \to 0$ strongly.

Our key result is

Theorem 2.2. *Let $\mathscr{R}$ be a von Neumann algebra with no type III portion (so $\mathscr{R}$ is semifinite) acting on a Hilbert space $\mathscr{H}$. Let $\mathscr{C}$ denote the center of $\mathscr{R}$, and let G be a group of $*$-automorphisms of $\mathscr{R}$. Let $\mathscr{B}$ denote the fixed point algebra in $\mathscr{R}$ under G. Assume $\mathscr{B} = \mathscr{C}$ and that $\mathscr{R}$ is G-finite. Let ϕ denote the unique faithful normal G-invariant positive linear projection of $\mathscr{R}$ onto $\mathscr{B}$ .Then $\mathscr{R}$ is finite, and if ψ denotes the canonical center trace of $\mathscr{R}$* [6, p. 267] *then $\phi = \psi$.*

Proof. Let ψ be a faithful normal center trace of $\mathscr{R}$ [6, p. 266]. A straightforward computation shows $\psi \circ g = g^{-1} \circ \psi \circ g$ is a faithful normal center trace of $\mathscr{R}$ for all g in G. Let ϱ be a normal G-invariant state of $\mathscr{R}$. Then $R = \operatorname{support} \varrho$ is invariant under G, so lies in $\mathscr{B}$, hence in $\mathscr{C}$. Restricting attention to $R\mathscr{R}$ we may assume $R = I$ and ϱ faithful. Identify $\mathscr{C}$ with $L^\infty_{\mathbb{C}}(Z, \nu)$ for some locally compact Hausdorff space Z, and denote by $\widetilde{\mathscr{C}}^+$ the positive ν-measurable functions on Z, cf. [6, p. 260]. Since ϱ is faithful on $\mathscr{C}$ in particular, ϱ extends uniquely to a faithful normal trace of $\widetilde{\mathscr{C}}^+$, cf. [6, p. 262]. By [6, p. 266] there exists a unique Q_g in $\widetilde{\mathscr{C}}^+$ with $0 < Q_g(\zeta) < +\infty$ locally almost everywhere on Z such that for all A in $\mathscr{R}^+$, $\psi \circ g(A) = Q_g\psi(A)$. If $g, h \in G$ then

$$\begin{aligned} Q_{gh}\psi(A) &= \psi(gh(A)) = Q_g\psi(h(A)) \\ &= Q_gQ_h\psi(A)\,, \end{aligned}$$

so by uniqueness of $Q_g, Q_{gh} = Q_g Q_h$ locally almost everywhere. Now $Q_g = I$ for all g. For if not then there exists g in G and a measurable non zero set Y in Z such that $Q_g(\zeta) < 1$ on Y. Let P be the projection in $\mathscr{C}$ corresponding to the characteristic function of Y. Choose a finite projection E in $\mathscr{R}$ such that $0 < \psi(E)(\zeta) < \infty$ on Y (cut down Y if necessary). Let $\varepsilon > 0$ be given, and choose a positive integer n so large that

$$0 < Q_g^n(\zeta)\,\psi(E)(\zeta) < \varepsilon$$

on Y, hence $0 < Q_{g^n}\psi(E)\,P < \varepsilon P$. Thus

$$0 < \varrho(Q_{g^n}\psi(EP)) = \varrho\big(\psi(g^n(EP))\big) < \varepsilon\,,$$

and $\varrho\psi(g^n(EP)) \to 0$. Since $\varrho \circ \psi$ is a normal and faithful trace of $\mathscr{R}$ and $g^n(EP)$ is a projection for each n, $g^n(EP) \to 0$ strongly by Lemma 2.1. Thus $\varrho(EP) = \varrho(g^n(EP)) \to 0$, and $EP = 0$ contrary to assumption. Thus $Q_g = I$ for all g in G, hence $\psi = \psi \circ g$ for all g in G.

Let now $A \neq 0$ be a positive operator in $\mathscr{R}$ for which $\psi(A)$ is finite. Choose by [10], a net $\{\sum_K \lambda_K^\alpha g_K^\alpha(A)\}_{\alpha \in J}$ in $\operatorname{conv}(g(A) : g \in G)$ which converges strongly to $\phi(A)$. Let E be any finite projection in $\mathscr{R}$. Then $\varrho \circ \psi(E\,\cdot)$ is ultra-weakly continuous [6, p. 80], hence weakly continuous on bounded sets. Since $\sum_K \lambda_K^\alpha g_K^\alpha(A) \to \phi(A)$ weakly,

$$\begin{aligned}
\varrho(\phi(A)\,\psi(E)) &= \varrho\big(\psi(\phi(A)E)\big)\\
&= \lim_\alpha \varrho \circ \psi\,\big(\sum_K \lambda_K^\alpha g_K^\alpha(A)E\big)\\
&= \lim_\alpha \sum_K \lambda_K^\alpha \varrho \circ \psi\big(g_K^\alpha(A\,(g_K^\alpha)^{-1}(E))\big)\\
&= \lim_\alpha \sum_K \lambda_K^\alpha \varrho \circ \psi(A\,(g_K^\alpha)^{-1}(E))\\
&\leqq \lim_\alpha \sum_K \lambda_K^\alpha \varrho \circ \psi(A)\\
&= \varrho \circ \psi(A)\,,
\end{aligned}$$

using that $\varrho \circ \psi$ is a trace, hence $\varrho \circ \psi(AF) \leqq \varrho \circ \psi(A)$ for all projections F. Let Q be a central projection in $\mathscr{R}$ for which $\psi(I)Q = +\infty\, Q$. If $Q \neq 0$ we can choose a non zero positive operator A in $\mathscr{R}$ such that $\phi(A)\,Q \neq 0$ and $0 \leqq \psi(A) \leqq Q$. In fact, if this cannot be done, then for every positive operator B in $\mathscr{R}Q$ for which $\psi(B)$ is bounded, $\phi(B)Q = 0$. Hence $\phi(Q)Q = 0$. Therefore, if $g \in G$ then $0 = g(\phi(Q)Q) = \phi(Q)\,g(Q)$, so that $\phi(Q)\,(\sum_K \lambda_K g_K(Q)) = 0$ for every element $\sum_K \lambda_K g_K(Q)$ in $\operatorname{conv}(g(Q) : g \in G)$. By [10] $\phi(Q)^2 = 0$, and $Q = 0$ since ϕ is faithful, contradicting the assumption that $Q \neq 0$. Choose A as above. Let $n > \varrho(\phi(A)Q)^{-1}$. Choose a finite projection E in $\mathscr{R}$ for which $Q\psi(E) \geqq nQ$.

Then by the inequality obtained above,

$$1 \geqq \varrho(Q) \geqq \varrho(\psi(A)) \geqq \varrho(\phi(A)\,\psi(E))$$
$$\geqq \varrho(\phi(A)\,\psi(E)Q) \geqq n\varrho(\phi(A)Q) > 1,$$

a contradiction. Thus $\psi(I)$ is finite, and $\mathscr{R}$ is finite. If ψ is the normalized center trace then since $\psi \circ g = \psi$, and since $\mathscr{B} = \mathscr{C}$, $\psi = \phi$ by uniqueness of ϕ. The proof is complete.

Corollary 2.3. *Let $\mathscr{R}$ be a von Neumann algebra with no type III portion. Let G be a group of *-automorphisms of $\mathscr{R}$ such that the fixed point algebra of $\mathscr{R}$ equals the center of $\mathscr{R}$. Then $\mathscr{R}$ is finite if and only if $\mathscr{R}$ is G-finite. Moreover, a normal state is a trace if and only if it is G-invariant.*

Proof. If $\mathscr{R}$ is G-finite then $\mathscr{R}$ is finite by Theorem 2.2. Conversely, if $\mathscr{R}$ is finite let ϕ denote the canonical center trace of $\mathscr{R}$. Then $\phi \circ g$ is a center trace of $\mathscr{R}$ for each g in G, so by uniqueness of ϕ, ϕ is G-invariant. By [10, Theorem 2] $\mathscr{R}$ is G-finite. The last statement follows from [10, Corollary 1].

In our applications the automorphisms will be implemented by a group of unitary operators, and there will be a unique cyclic vector invariant under all the unitaries. The key situation occurs when the invariant vector is also separating for the von Neumann algebra. The following result describes this situation and is a direct generalization of Hugenholtz's theorem [8].

Theorem 2.4. *Let $\mathscr{R}$ be a factor acting on a Hilbert space $\mathscr{H}$. Let G be a group and $g \to U(g)$ a unitary representation of G on $\mathscr{H}$ such that $U(g)\,\mathscr{R}\,U(g)^{-1} = \mathscr{R}$ for all g in G. Assume there exists a unit vector x which is cyclic and separating for $\mathscr{R}$ such that $U(g)x = x$ for all g in G, and that up to a scalar multiple x is the unique vector invariant under all $U(g)$. Then either $\mathscr{R}$ is of type III or ω_x is a trace of $\mathscr{R}$, in which case $\mathscr{R}$ is either of type II_1 or of type I_n, $n < \infty$.*

Furthermore, if $\mathscr{R}$ is not assumed to be a factor, and if G is a connected topological group and the representation $g \to U(g)$ is strongly continuous, then either $\mathscr{R}$ is a factor or the center of $\mathscr{R}$ has no minimal projections.

Proof. ω_x is a faithful normal G'-invariant state of $\mathscr{R}$, where G' denotes the group of $*$-automorphisms $U(g) \cdot U(g)^{-1}$ of $\mathscr{R}$. Thus $\mathscr{R}$ is G'-finite. Let $\mathscr{B}$ denote the fixed point algebra in $\mathscr{R}$. Then $\mathscr{B} = \mathbb{C}I$. In fact, if $[x]$ denotes the projection on the subspace generated by x then by the Ergodic Theorem [12, § 146], $[x] \in \mathrm{conv}(U(g) : g \in G)^-$. Thus $[x] \in \mathscr{B}'$, and the state ω_x is a homomorphism of $\mathscr{B}$. Since x is separating for $\mathscr{B}$, $\mathscr{B} \cong \mathbb{C}$ as asserted. Assume $\mathscr{R}$ is not of type III. By Theorem 2.2 $\mathscr{R}$ is finite so either of type II_1 or I_n, $n < \infty$. Let ϕ denote the unique faithful normal G-invariant positive map of $\mathscr{R}$ onto $\mathscr{B}$. Since $\mathscr{B} = \mathbb{C}I$, $\phi(A) = \omega_x(A)I$ for all A in $\mathscr{R}$. Let ψ denote the normalized trace of $\mathscr{R}$. By Theorem 2.2 $\omega_x(A) = \omega_x(\omega_x(A)I) = \omega_x(\phi(A)) = \omega_x(\psi(A))$, hence ω_x is the trace of $\mathscr{R}$, and the first part of the theorem is proved.

Assume G is a connected topological group and that the representation $g \to U(g)$ is strongly continuous. Suppose $\mathscr{R}$ is not necessarily a factor. Let $\mathscr{C}$ denote the center of $\mathscr{R}$. Consider the projection $[\mathscr{C}x]$ in $\mathscr{C}'$. If $A \in \mathscr{C}$ then $U(g)Ax = U(g)AU(g)^{-1}x \in [\mathscr{C}x]$, since $U(g)\mathscr{C}U(g)^{-1} = \mathscr{C}$ for all g in G. Thus $U(g)[\mathscr{C}x]U(g)^{-1} = [\mathscr{C}x]$, and $U(g) \cdot U(g)^{-1}$ restricts to a $*$-automorphism of the maximal abelian von Neumann algebra $\mathscr{C}[\mathscr{C}x]$, which is isomorphic to $\mathscr{C}$. Assume there is a minimal non zero projection E in $\mathscr{C}$. Then $E' = E[\mathscr{C}x]$ is one dimensional in $\mathscr{C}[\mathscr{C}x]$. Say z is a unit vector in E'. If $g \in G$ then $F' = U(g)E'U(g)^{-1}$ is a non zero projection in $\mathscr{C}[\mathscr{C}x]$, so $F'E' = 0$ or or $F'E' = E'$. Thus either $(U(g)z, z) = (F'U(g)z, z) = 0$, or $U(g)z = e^{i\theta}z$, in which case $|(U(g)z, z)| = 1$. Since the map $g \to |(U(g)z, z)|$ is continuous on G, and G is connected, its image is connected, and $|(U(g)z, z)| = 1$ for all g in G. Thus $U(g)E'U(g)^{-1} = E'$ for all g in G, hence $U(g)EU(g)^{-1} = E$ for all g, and $E \in \mathscr{B}$, $E = I$, since in this case $\mathscr{B} = \mathbb{C}I$ too. The proof is complete.

We have been unable to conclude whether there exist a factor $\mathscr{M}$ and an abelian von Neumann algebra $\mathscr{Z} \cong \mathscr{C}$ such that $\mathscr{R} \cong \mathscr{Z} \otimes \mathscr{M}$.

In order to study the group of automorphisms information may be obtained from the study of its action on the center $\mathscr{C}$ of $\mathscr{R}$. Such a result will be obtained later (Theorem 3.3). For the present we draw some immediate conclusions from the proof of the above theorem.

Corollary 2.5. *Let the assumptions and notation be as in Theorem 2.4. If $\mathscr{R}$ is not a factor and $\mathscr{C}$ denotes the center of $\mathscr{R}$ then $\mathscr{C}'$ is a homogeneous von Neumann algebra of type I. The projection $[\mathscr{C}x]$ belongs to $\{U(g) : g \in G\}'$ and is an abelian projection with central carrier I in $\mathscr{C}'$. In particular, $[\mathscr{C}x]\mathscr{R}[\mathscr{C}x] = \mathscr{C}[\mathscr{C}x] \cong \mathscr{C}$.*

In some applications the invariant vector x will not be separating for $\mathscr{R}$. As in Corollary 2.5 it is immediate that the support $[\mathscr{R}'x]$ of ω_x is invariant under the unitaries, hence Theorem 2.4 can be applied to the von Neumann algebra $\mathscr{R}_x = [\mathscr{R}'x]\mathscr{R}[\mathscr{R}'x]$.

Corollary 2.6. *Let $\mathscr{R}$ be a factor acting on a Hilbert space $\mathscr{H}$. Let G be a group and $g \to U(g)$ a unitary representation of G on $\mathscr{H}$ such that $U(g)\mathscr{R}U(g)^{-1} = \mathscr{R}$ for all g in G. Suppose there exists a unit vector x in $\mathscr{H}$ cyclic under $\mathscr{R}$ such that up to a scalar multiple x is the unique vector in $\mathscr{H}$ invariant under all $U(g)$, $g \in G$. Then $\mathscr{R}$ is of type III if and only if $\mathscr{R}'$ is not finite.*

Proof. Let $\mathscr{R}_x$ be as above and apply Theorem 2.4 to it. If $\mathscr{R}_x$ is of type III then so is $\mathscr{R}'_x = [\mathscr{R}'x]\mathscr{R}'$, which is isomorphic to $\mathscr{R}'$ since x is separating for $\mathscr{R}'$, hence $\mathscr{R}'$ is of type III and therefore $\mathscr{R}$ by [6, p. 102]. Otherwise ω_x is a faithful trace of $\mathscr{R}_x$, so by [6, p. 235] $\mathscr{R}_x$ is standard and finite. Thus $\mathscr{R}' \cong \mathscr{R}'_x$ is finite, contrary to assumption. The converse is immediate [6, p. 102].

3. Asymptotically Abelian C^*-Algebras

In the different generalizations of asymptotically abelian C^*-algebras [7, 11, 13] the extremal invariant states give rise to von Neumann algebras satisfying the assumptions of Corollary 2.6, and by Theorem 2.4 the center will with the added assumptions in the theorem either have no minimal projections or the von Neumann algebra is a factor. At least in the situations described in [11] and [13] the invariant factor states will always be extreme. Since we shall need this result and later the more restricted results in [13] we shall study the situation in that paper. If $\mathscr{A}$ is a C^*-algebra and G a group for which there is a representation $g \to \tau_g$ as $*$-automorphisms of $\mathscr{A}$, we say G is represented as a *large group of automorphisms* if for all G-invariant states ϱ of $\mathscr{A}$ and A in $\mathscr{A}$,

$$\operatorname{conv}(\pi_\varrho(\tau_g A) : g \in G)^- \cap \pi_\varrho(\mathscr{A})' \neq \emptyset ,$$

where $\varrho = \omega_{x_\varrho} \circ \pi_\varrho$ is the canonical decomposition of ϱ as a composition of a vector state ω_{x_ϱ} due to a cyclic vector, and π_ϱ is a $*$-representation of $\mathscr{A}$ on a Hilbert space $\mathscr{H}_\varrho$. For such an invariant state there is a unitary representation $g \to U_\varrho(g)$ of G on $\mathscr{H}_\varrho$ such that $U_\varrho(g)\, \pi_\varrho(A)\, U_\varrho(g)^{-1} = \pi_\varrho(\tau_g A)$ and $U_\varrho(g) x_\varrho = x_\varrho$ for all g in G. Then ϱ is an extremal G-invariant state if and only if x_ϱ is up to a scalar multiple the unique vector invariant under the $U_\varrho(g)$ [13, Theorem 3.7]. It is immediate from Theorem 2.4 that if ϱ is extremal and x_ϱ is separating for $\pi_\varrho(\mathscr{A})^-$ (cf. [7, Theorem 3] for equivalent conditions) then if it is a factor, $\pi_\varrho(\mathscr{A})^-$ is finite if and only if ϱ is a trace of $\mathscr{A}$, and $\pi_\varrho(\mathscr{A})^-$ is of type III otherwise. More generally we have

Theorem 3.1. *Let $\mathscr{A}$ be a C^*-algebra and G a group represented as a large group of $*$-automorphisms of $\mathscr{A}$. Let ϱ be a G-invariant state of $\mathscr{A}$ for which $\mathscr{R} = \pi_\varrho(\mathscr{A})^-$ is a factor. Then the following conditions may occur.*

1) *$\mathscr{R}$ is of type III if and only if $\mathscr{R}'$ is not finite.*

2) *$\mathscr{R}$ is finite if and only if ϱ is a trace, in which case $\mathscr{R}$ has coupling* 1.

3) *$\mathscr{R}$ is of type I_∞ or II_∞ if and only if $\mathscr{R}'$ is finite and ϱ is not a trace.*

Proof. From the preceding remarks $\mathscr{R}$ satisfies the conditions of Corollary 2.6, hence 1) is immediate. Let $E = [\mathscr{R}' x_\varrho]$. Then $E \in \mathscr{R}$, $E \mathscr{R} E$ is a factor, and x_ϱ is separating and cyclic for $E\mathscr{R}E$ as acting on E. As pointed out in the proof of Corollary 2.6 $E\, U_\varrho(g) = U_\varrho(g) E$ for all g in G, and Theorem 2.4 is applicable to $E\mathscr{R}E$. We may assume $\mathscr{R}$ is not of type III (so $\mathscr{R}'$ is finite by 1)). By Theorem 2.4 $E\mathscr{R}E$ is either of type II_1 or I_n with n finite and of coupling 1, and ω_{x_ϱ} is a trace of $E\mathscr{R}E$.

Suppose $\mathscr{R}$ is finite. Let tr denote the normalized trace. Then since each $U_\varrho(g) \cdot U_\varrho(g)^{-1}$ is an automorphism of $\mathscr{R}$, tr composed with it is a trace, hence by the uniqueness of the trace [6, p. 90],

$$\operatorname{tr}(A) = \operatorname{tr}(U_\varrho(g)\, A\, U_\varrho(g)^{-1})$$

for all A in $\mathscr{R}$, g in G. Thus $\mathrm{tr} \circ \pi_\varrho$ is a G-invariant state of $\mathscr{A}$. Moreover, tr is weakly continuous on bounded sets and if $\mathrm{tr} = \omega_y \circ \pi$, π is normal. Denote by $\mathscr{A}_{SA}$ and $\mathscr{R}_{SA}$ the set of self-adjoint operators in $\mathscr{A}$ and $\mathscr{R}$ respectively. Since $\pi_\varrho(\mathscr{A}_{SA})_1$ — the unit ball in $\pi_\varrho(\mathscr{A}_{SA})$ — is weakly dense in $(\mathscr{R}_{SA})_1$ by the Kaplansky Density Theorem [6, p. 46], so is $\pi(\pi_\varrho(\mathscr{A}_{SA})_1) = \pi(\pi_\varrho(\mathscr{A}_{SA}))_1$ in $\pi(\mathscr{R}_{SA})_1$. Thus $\pi(\pi_\varrho(\mathscr{A}))^- = \pi(\mathscr{R})$ is a factor, and $\mathrm{tr} \circ \pi_\varrho = \omega_y \circ \pi \circ \pi_\varrho$ is a G-invariant factor trace, hence is extreme [13, Theorem 3.7]. Now ω_{x_ϱ} restricted to $E\mathscr{R}E$ is a trace, so by the uniqueness of the trace, $\omega_{x_\varrho} \mid E\mathscr{R}E = \lambda\, \mathrm{tr} \mid E\mathscr{R}E$. Thus, for $A \geqq 0$ in $\mathscr{R}$,

$$\omega_{x_\varrho}(A) = \omega_{x_\varrho}(EAE) = \lambda\, \mathrm{tr}(EAE) = \lambda\, \mathrm{tr}(EA) \leqq \lambda\, \mathrm{tr}(A)\,,$$

and $\varrho \leqq \lambda\, \mathrm{tr} \circ \pi_\varrho$. Since $\mathrm{tr} \circ \pi_\varrho$ is extremal, $\varrho = \mathrm{tr} \circ \pi_\varrho$, and $\omega_{x_\varrho} = \mathrm{tr}$. Thus $E = I$, $\mathscr{R}$ has coupling 1, and ϱ is a trace. Conversely, if ϱ is a trace then ω_{x_ϱ} is the unique trace of $\mathscr{R}$, hence $\mathscr{R}$ is finite. This completes the proof of 2), and hence of 3).

We shall see below that under some stricter conditions if $\mathscr{R}$ is a factor then $[x_\varrho]$ — the projection on x_ϱ — is the only finite dimensional projection in $\mathscr{B}(\mathscr{H}_\varrho)$ — the bounded operators on $\mathscr{H}_\varrho$ — commuting with all the $U_\varrho(g)$, viz. ϱ is an *E_I-state*. Our next result states a similar property for $[\mathscr{R}' x_\varrho]$ as a projection in $\mathscr{R}$.

Corollary 3.2. *Let the assumptions and notation be as in Theorem* 3.1. *Suppose $\mathscr{R}$ is either of type I_∞ or II_∞. Then $E = [\mathscr{R}' x_\varrho]$ is the unique non zero finite projection in $\mathscr{R}$ invariant under the $U_\varrho(g)$, g in G.*

Proof. From the proof of the theorem $E\mathscr{R}E$ is finite of coupling 1 and with ω_{x_ϱ} as the trace. Suppose F is a finite projection in $\mathscr{R}$ invariant under the $U_\varrho(g)$. If $F \geqq E$ then the argument in the proof of the theorem applied to $F\mathscr{R}F$ yields $F = E$. In the general case let $G = E \vee F$. Then G is finite [6, p. 243]. If $y \in E$, $z \in F$ then $U_\varrho(g)\,(y + z) = U_\varrho(g)\, y + U_\varrho(g)\, z \in E \vee F = G$, so G is invariant under the $U_\varrho(g)$. From the first part of the proof $G = E$, and $F \leqq E$. Now x_ϱ is separating and cyclic for $E\mathscr{R}E$. From the proof of Theorem 2.4 the fixed point algebra for the automorphisms in $E\mathscr{R}E$ is $\mathbb{C}E$. Thus $F = 0$ or E. The proof is complete.

The remaining part of this section consists of results which are more or less known [4, 9], but which show useful characterizations of strongly clustering states in our setting, a situation which occurs for invariant factor states. If $\mathscr{A}$ is a C^*-algebra, G a group, and $g \to \tau_g$ a representation of G as $*$-automorphisms of $\mathscr{A}$ we say $\mathscr{A}$ is *asymptotically abelian* with respect to G if for all self-adjoint operators A in $\mathscr{A}$ there exists a sequence $\{g_n(A)\}_{n=1,2,\ldots}$ in G such that

$$\lim_{n\to\infty} \|[\tau_{g_n(A)}(A), B]\| = 0$$

for all B in $\mathscr{A}$. A G-invariant state ϱ is *strongly clustering* if

$$\lim_{n\to\infty} \varrho(\tau_{g_n(A)}(A)B) = \varrho(A)\,\varrho(B)$$

whenever A and B are self-adjoint in $\mathscr{A}$. ϱ is an *E_I-state* if $[x_\varrho]$ is the unique finite dimensional projection in $\mathscr{B}(\mathscr{H}_\varrho)$ commuting with all the $U_\varrho(g)$. Our first result points out the relevance of the center of $\pi_\varrho(\mathscr{A})^-$ in order to study the group of the unitaries $U_\varrho(g)$.

Theorem 3.3. *Let $\mathscr{A}$ be a C^*-algebra asymptotically abelian with respect to the abelian group G. Then every strongly clustering state of $\mathscr{A}$ is an E_{I}-state. In particular, every invariant factor state is an E_{I}-state.*

For the proof of this we shall need the argument given in

Lemma 3.4. *Let $\mathscr{A}$ be a C^*-algebra asymptotically abelian with respect to the group G. If ϱ is a strongly clustering state of $\mathscr{A}$ then $[x_\varrho]$ is the only one dimensional projection in $\mathscr{B}(\mathscr{H}_\varrho)$ commuting with all the $U_\varrho(g)$.*

Proof. Let E be a one dimensional projection different from $[x_\varrho]$, and suppose E commutes with all the $U_\varrho(g)$. Then $a = \|Ex_\varrho\| < 1$. Let y be a unit vector in E. Then $U_\varrho(g)y = \chi(g)y$ with χ a character of G, hence $U_\varrho(g)^{-1}E = \overline{\chi(g)}E$. By [13, Theorem 5.4] if $A \in \mathscr{A}$ then, with $g_n = g_n(A)$,

$$\omega_{x_\varrho}(\pi_\varrho(A))I = \text{weak}\lim_n U_\varrho(g_n)\,\pi_\varrho(A)\,U_\varrho(g_n)^{-1}\,.$$

Thus, as $n \to \infty$,

$$\begin{aligned} |\omega_{x_\varrho}(\pi_\varrho(A)\,E\,\pi_\varrho(B))| &= |\chi(g)\,\omega_{x_\varrho}(U_\varrho(g_n)\,\pi_\varrho(A)\,U_\varrho(g_n)^{-1}\,E\,\pi_\varrho(B))| \to \\ &\to |\omega_{x_\varrho}(\omega_{x_\varrho}(\pi_\varrho(A))\,E\,\pi_\varrho(B))| \\ &= |\omega_{x_\varrho}(\pi_\varrho(A))|\;|\omega_{x_\varrho}(E\,\pi_\varrho(B))| \leqq \\ &\leqq a\|\pi_\varrho(A)^* x_\varrho\|\;\|\pi_\varrho(B)x_\varrho\|\,. \end{aligned}$$

Since x_ϱ is cyclic, if z, w are vectors in $\mathscr{H}_\varrho$ then $|(Ez, w)| \leqq a\|z\|\,\|w\|$. Applying this to $z = w = y$ we obtain a contradiction. The proof is complete.

Proof of Theorem 3.3. If the theorem is false we can find a minimal finite dimensional projection E in $\{U_\varrho(g) : g \in G\}'$ orthogonal to $[x_\varrho]$. Since G is abelian, the minimality of E implies $U_\varrho(g)\,E = \chi(g)E$ with χ a character of G. A contradiction is now obtained in exactly the same way as in the proof of Lemma 3.4 (with $a = 0$). If $\pi_\varrho(\mathscr{A})^-$ is a factor then ϱ is strongly clustering by [13, Corollary 5.5], hence ϱ is an *E_I-state.*

Kastler and Robinson [9] have shown that if $\mathscr{A}$ is asymptotically abelian (in the stricter sense that $G = \mathbb{R}^n$, and $\lim\|[\tau_g A, B]\| = 0$ whenever $g \to \infty$ in $\mathbb{R}^n$) then ϱ is strongly clustering if and only if $U_\varrho(g) \to [x_\varrho]$ weakly whenever $g \to \infty$, a result shown by Borchers in quantum field theory [4]. As we shall need this result in our applications to quantum field theory for translations of space-like vectors, we include a more general proof.

Theorem 3.5. *Let $\mathscr{A}$ be a C^*-algebra asymptotically abelian with respect to the group G. Suppose the sequence $\{g_n(A)\} = \{g_n\}$ is independent of A*

for each self-adjoint A in $\mathscr{A}$. Then a G-invariant state ϱ of $\mathscr{A}$ is strongly clustering if and only if $U_\varrho(g_n) \to [x_\varrho]$ weakly.

Proof. Suppose ϱ is strongly clustering. Since the map $U \to U^*$ is weakly continuous and $[x_\varrho]$ is self-adjoint we may as well show $U_\varrho(g_n)^{-1} \to [x_\varrho]$ weakly. Let $x_1, \ldots, x_k, y_1, \ldots, y_k$ be vectors in $\mathscr{H}_\varrho$. We may assume that there exist A_j, B_j in $\pi_\varrho(\mathscr{A})$ such that $x_j = B_j x_\varrho$, $y_j = A_j x_\varrho$. Let $\varepsilon > 0$ be given. Choose n so large that

$$|(U_\varrho(g_n) A_j^* U_\varrho(g_n)^{-1} B_j x_\varrho, x_\varrho) - \omega_{x_\varrho}(A_j^*)\, \omega_{x_\varrho}(B_j)| < \varepsilon$$

for $j = 1, \ldots, k$. Then

$$\begin{aligned}
&|((U_\varrho(g_n)^{-1} - [x_\varrho])\, B_j x_\varrho, A_j x_\varrho)| \\
&\qquad = |(U_\varrho(g_n) A_j^* U_\varrho(g_n)^{-1} B_j x_\varrho, x_\varrho) - (A_j^* \omega_{x_\varrho}(B_j) x_\varrho, x_\varrho)| \\
&\qquad = |(U_\varrho(g_n) A_j^* U_\varrho(g_n)^{-1} B_j x_\varrho, x_\varrho) - \omega_{x_\varrho}(A_j^*)\, \omega_{x_\varrho}(B_j)| \\
&\qquad < \varepsilon\,,
\end{aligned}$$

and $U_\varrho(g_n)^{-1} \to [x_\varrho]$ weakly. Conversely, if $U_\varrho(g_n) \to [x_\varrho]$ weakly, let $A, B \in \pi_\varrho(\mathscr{A})$. Then, as $n \to \infty$,

$$\begin{aligned}
(U_\varrho(g_n) A\, U_\varrho(g_n)^{-1} B x_\varrho, x_\varrho) &= (B x_\varrho, U_\varrho(g_n) A^* x_\varrho) \\
&\to (B x_\varrho, [x_\varrho] A^* x) = \omega_{x_\varrho}(B)\, \omega_{x_\varrho}(A)\,,
\end{aligned}$$

and ϱ is strongly clustering.

4. Quantum Field Theory

We assume we have assigned to every bounded region $\mathcal{O}$ in the four dimensional Minkowski space $\mathscr{M}$ a C^*-algebra $\mathscr{A}(\mathcal{O})$ of operators on an infinite dimensional Hilbert space $\mathscr{H}$. We denote by $\mathscr{R}(\mathcal{O})$ its weak closure and assume $\mathscr{R}(\mathcal{O})$ contains the identity operator I on $\mathscr{H}$. Moreover, we assume there is a strongly continuous unitary representation $a \to U(a)$ of the four dimensional translation group, which we shall identify with $\mathscr{M}$, such that the following properties are satisfied.

1) The spectrum of $U(a)$ is contained in the closed forward light-cone $\overline{\mathscr{V}}^+$.

2) $\mathscr{A}(\mathcal{O})$ and $\mathscr{A}(\mathcal{O} + a)$ are related by the equation

$$\mathscr{A}(\mathcal{O} + a) = U(a)\, \mathscr{A}(\mathcal{O})\, U(a)^{-1}\,.$$

3) If two regions $\mathcal{O}$ and $\mathcal{O}'$ are space-like to one another then $\mathscr{A}(\mathcal{O}) \subset \mathscr{A}(\mathcal{O}')'$.

4) If $\mathcal{O} \subset \mathcal{O}'$ then $\mathscr{A}(\mathcal{O}) \subset \mathscr{A}(\mathcal{O}')$.

5) If $\{\mathcal{O}_n\}$ is any covering of the unbounded region $\mathcal{O} \subset \mathscr{M}$ of bounded regions $\mathcal{O}_n \subset \mathcal{O}$, then the von Neumann algebra generated by the family $\{\mathscr{A}(\mathcal{O}_n)\}$ is independent of the covering, and is denoted by $\mathscr{R}(\mathcal{O})$.

6) There exists up to a scalar multiple a unique vacuum vector x cyclic under $\mathscr{R}(\mathscr{M})$, i.e. $U(a)x = x$ for all a in $\mathscr{M}$.

It is clear that the relations 2), 3) and 4) hold for the $\mathscr{R}(\mathcal{O})$ as well. Under the above assumptions ARAKI [2, Corollary 1] has shown $\mathscr{R}(\mathscr{M}) = \mathscr{B}(\mathscr{H})$. Furthermore, he has shown [1] that if $\mathcal{O}$ is the region of points (p_0, p_1, p_2, p_3) in $\mathscr{M}$ such that $p_1 > 0$, $|p_0| < |p_1|$, then $\mathscr{R}(\mathcal{O})$ is a factor not of type I. We shall generalize this and prove a general result which implies $\mathscr{R}(\mathcal{O})$ is of type III.

Theorem 4.1. *Suppose we have a local field theory satisfying axioms* 1) – 6). *Let $\mathcal{O}$ be an open unbounded region in the Minkowski space $\mathscr{M}$ such that there exists a non zero space-like vector a in $\mathscr{M}$ for which $\mathcal{O} = \mathcal{O} + \mathbb{R}a$, and such that there exists an open non void region space-like to $\mathcal{O}$. Then $\mathscr{R}(\mathcal{O})$ is a factor of type III.*

Proof. By the Reeh-Schlieder Theorem [5, Lemma 5], since $\mathscr{R}(\mathscr{M})$ is all bounded operators, x is separating and cyclic for $\mathscr{R}(\mathcal{O})$. By hypothesis $\mathrm{U}(\lambda a)\,\mathscr{R}(\mathcal{O})\,\mathrm{U}(\lambda a)^{-1} = \mathscr{R}(\mathcal{O})$ for all real λ. Let $\mathscr{B}$ denote the fixed point algebra in $\mathscr{R}(\mathcal{O})$ of this one-parameter group of automorphisms. Since x is the unique vacuum vector $[x] = E(0)$, where by Stone's Theorem

$$U(b) = \int e^{ip\cdot b}\, dE(p)$$

for $b \in \mathscr{M}$. Let $\mathscr{A}$ denote the C^*-algebra generated by all $\mathscr{R}(\mathcal{O}')$ with $\mathcal{O}'$ bounded regions in $\mathscr{M}$. Then $\mathscr{A}$ is asymptotically abelian with respect to the translation group $\mathscr{M}$, where the sequences $\{g_n(A)\} = \{g_n\}$ are chosen to be translates of space-like vectors. Since $\mathscr{A}^- = \mathscr{R}(\mathscr{M}) = \mathscr{B}(\mathscr{H})$, ω_x is strongly clustering on $\mathscr{A}$ [13, Corollary 5.5], see also [2, Proposition 4]. By Theorem 3.5, or by [4, Lemma 4], $U(na) \to E(0)$ weakly as $n \to \infty$, hence $E(0) \in \{U(\lambda a) : \lambda \in \mathbb{R}\}''$. In particular, if E is a non zero projection in $\mathscr{B}$ then $EE(0) = E(0)E \neq 0$ as $Ex \neq 0$. Since $E(0)$ is one dimensional $E \geqq E(0)$. Hence $(I - E)x = 0$, and $E = I$ since x is separating. Thus $\mathscr{B} = \mathbb{C}I$.

In order to show $\mathscr{R}(\mathcal{O})$ is a factor we modify the argument in the proof of [2, Proposition 2]. Let $\mathscr{N}$ be a neighborhood of the origin in $\mathscr{M}$ and $\mathcal{O}'$ a bounded subregion of $\mathcal{O}$ such that $\mathcal{O}' + \mathscr{N} \subset \mathcal{O}$. Let A be a self-adjoint operator in the center of $\mathscr{R}(\mathcal{O})$. Let $B \in \mathscr{R}(\mathcal{O}')$, and let

$$F(u) = (U(u)Ax, Bx)$$
$$G(u) = (U(-u)B^*x, Ax)\,.$$

From the spectrum condition F and G are boundary values of analytic functions holomorphic in the forward and backward tubes respectively. If $u \in \mathscr{N}$ then, since $U(-u)\,B\,U(u) \in \mathscr{R}(\mathcal{O})$, $F(u) = G(u)$. From the edge-of-the-wedge theorem, see [14] the functions are analytic continuations of one another. In particular, $F(u) = G(u)$ for all u in $\mathscr{M}$. Thus, using the spectrum condition once more, $F(u)$ is the Fourier transform of a (complex) measure, whose support lies in the intersection of the forward cone with the backward cone, hence is 0. Thus $F(u)$ is a constant. In particular,

$$(U(u)Ax, Bx) = (U(0)Ax, Bx) = (Ax, Bx) .$$

Since by the Reeh-Schlieder Theorem x is cyclic under $\mathscr{R}(\mathscr{O}')$, $U(u)\,A\,U(u)^{-1}x = Ax$ for all $u \in \mathscr{M}$. In particular this holds for $u = \lambda a$, $\lambda \in \mathbb{R}$. Since x is separating for $\mathscr{R}(\mathscr{O})$, $U(\lambda a)\,A\,U(\lambda a)^{-1} = A$ for all $\lambda \in \mathbb{R}$, and $A \in \mathscr{B}$, which is the scalars by the preceding paragraph. Thus $\mathscr{R}(\mathscr{O})$ is a factor.

If $\mathscr{R}(\mathscr{O})$ is not of type III an application of Theorem 2.2 shows $\mathscr{R}(\mathscr{O})$ is finite and with ω_x as the trace. Let $\mathscr{O}'$ be a bounded non void subregion of $\mathscr{O}$. Then ω_x is a trace of $\mathscr{R}(\mathscr{O}')$, and by the Reeh-Schlieder Theorem x is separating and cyclic for $\mathscr{R}(\mathscr{O}')$. In particular ω_x is a faithful trace of $\mathscr{R}(\mathscr{O}')'$ [6, p. 89]. Let $\mathscr{O}''$ be a bounded non void region in $\mathscr{M}$. Then there is a vector b in $\mathscr{M}$ such that $\mathscr{O}'' + b$ is space-like to $\mathscr{O}'$, hence $U(b)\,\mathscr{R}(\mathscr{O}'')\,U(b)^{-1} \subset \mathscr{R}(\mathscr{O}')'$. Since $U(b)x = U(b)^{-1}x = x$, ω_x is a trace of $\mathscr{R}(\mathscr{O}'')$. A straightforward argument now shows ω_x is a trace of $\mathscr{R}(\mathscr{M})$, contradicting the fact that $\mathscr{R}(\mathscr{M}) = \mathscr{B}(\mathscr{H})$. The proof is complete.

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