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Local Field Equation for A^4-Coupling in Renormalized Perturbation Theory*

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Abstract. For the model of A^4-coupling a finite form of the local field equation is proposed and checked in renormalized perturbation theory.

1. Introduction

As is well known the canonical quantization of the Lagrange density

$$\mathscr{L} = \frac{1}{2} \partial_\mu A_0 \partial^\mu A_0 - \frac{m_0^2}{2} A_0{}^2 - \frac{\lambda_0}{4} A_0{}^4 \tag{1.1}$$

leads to a field equation and commutation relations which are meaningless, at least in perturbation theory. This difficulty is avoided in the abstract formulation of quantum field theory which is based on general principles such as Lorentz invariance, microcausality and spectrum conditions [1]. Indeed, well defined power series can be constructed which solve the basic equations of the theory to all orders [2]. Due to the general nature of the principles the abstract formulation provides a frame for all local and invariant interactions specifying only the number and types of the fields involved and the masses and spins of the stable particles. The question arises how in this framework a specific model can be characterized by imposing a simple and meaningful condition on the field operator.

By analogy to Valatin's formulation of quantum electrodynamics [3] we propose the field equation

$$\begin{aligned} -(\Box + m^2)\, A(x) &= \lambda \lim_{\xi \to 0} j(x, \xi) \\ j(x, \xi) &= \frac{:A(x+\xi)\, A(x)\, A(x-\xi): \, - \alpha(\xi)\, A(x)}{g(\xi)} \qquad \xi^2 < 0 \end{aligned} \tag{1.2}$$

as such a condition for the model (1.1) of A^4-coupling. The parameters m and λ denote the physical mass and (suitably defined) coupling constant

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of the model. The : :-product is obtained from the ordinary operator product by trivial vacuum subtractions

$$
\begin{aligned}
:A(x_1)\,A(x_2)\,A(x_3): = A(x_1)\,A(x_2)\,A(x_3) - \\
- \langle A(x_1)\,A(x_2)\rangle_0\,A(x_3) - \text{cycl. perm. terms}
\end{aligned}
\tag{1.3}
$$

for spacelike distances $(x_i - x_j)^2 < 0$, $(i \neq j)$. *Combined with the general* principles the field equation (1.2—3) provides a complete description of the model in terms of renormalized quantities, but without reference to a power series expansion[1].

The limit $\xi \to 0$ should be understood in the weak sense that (1.2) holds for each matrix element between suitable state vectors. It is essential, however, that the renormalization functions α and g are independent of the state vectors. α and g are to a large extent arbitrary, only the leading singularities at $\xi = 0$ are relevant in (1.2). The right hand side of (1.1) may be considered as an appropriate definition of the local operator product $A(x)^3$ which otherwise is not defined.

The purpose of this paper is to study the field equation (1.2) in renormalized perturbation theory. A check in perturbation theory may be regarded as a preliminary test for the consistency of the scheme. The given formulation should also be suitable for investigations independent of perturbation theory. Recently SYMANZIK [6] derived an expansion of GREEN functions in Euclidean quantum field theory which does not have the shortcomings of perturbation theory. It should be interesting to see whether the Euclidean analogue of (1.2) is satisfied by such an expansion.

Field equations involving a limiting procedure of the form (1.2) were first proposed by VALATIN for quantum electrodynamics and verified in the lowest order approximations [3]. The validity of similar field equations has been checked in a number of non-relativistic models [7]. A few years ago the discussion was taken up again for local relativistic theories [8, 9, 10]. In LEHMANN's approach [8] the renormalization functions are determined by simple assumptions for finite $\xi^2 < 0$ which reduce to the usual renormalization conditions in the limit $\xi \to 0$. WILSON [9] checked field equations of the form (1.2) for meson-nucleon interaction in some low orders of perturbation theory. He further discussed the definition of arbitrary local operator products on the basis of dimensional arguments. In an earlier version of the present work [10] the field equation (1.2) and similar equations for meson-nucleon inter-

[1] Other finite formulations of models have been discussed in the literature. We mention two examples. For the model of A^3-coupling in six dimensions SYMANZIK [4] derived finite equations for the field operator which involve non-local integral transformations. T. T. WU [5] gave a finite formulation of Dyson's integral equations for the model of A^4-coupling.

action were checked in all orders of perturbation theory. However, the calculations were based on a rather intricate method of renormalization.

For a check of the proposal in perturbation theory the power series

$$A(x, \lambda) = \sum_{n=0}^{\infty} \lambda^n A_n(x) \tag{1.4}$$

$$\alpha(\xi, \lambda) = \sum_{n=0}^{\infty} \lambda^n \alpha_n(\xi), \quad g(\xi, \lambda) = \sum_{n=0}^{\infty} \lambda^n g_n(\xi) \tag{1.5}$$

must be constructed such that to all orders of λ

(i) $A(x)$ is a local relativistic field in the sense of the general postulates

(ii) $A(x)$ satisfies the field equation (1.2).

The expansions (1.5) are determined by (1.4) if — as in this paper — explicit representations of α and g in terms of Green's functions of $A(x)$ are used.

For the construction of the power series (1.4) the following three methods are available.

(1) In perturbation theory STEINMANN [2] gave a systematic treatment of certain integral equations which form necessary and sufficient conditions for (i). By appropriate choice of the parameters one obtains an expansion (1.4) which should correspond to the model (1.1). In this approach (i) is obviously satisfied. However, the structure of the expressions obtained is complicated and it should be difficult to check (ii).

(2) The straightforward way of constructing (1.4) is to iterate the Yang-Feldman equation[2]

$$A(x) = A_{in}(x) - \lambda \int \Delta_{\mathrm{Ret}}(x - x') \lim_{\xi \to 0} j(x', \xi)\, dx' . \tag{1.6}$$

If the iterated integrals exist, (ii) is automatically satisfied. Unfortunately, the iterated integrals are so complicated that it would be tedious to prove the finiteness of each iteration step[3/4].

(3) The most convenient method of constructing (1.4) is based on DYSON's renormalization theory [13]. Starting from the Lagrangian (1.1) the Gell-Mann-Low expansion of the time-ordered functions

$$\tau(x_1, \ldots x_r) = \langle T A(x_1) \ldots A(x_r) \rangle_0 \tag{1.7}$$

is formally derived. In its final form the renormalized Gell-Mann-Low expansion is a power series

$$\tau(x_1 \ldots x_r \lambda) = \sum_{n=0}^{\infty} \lambda^n \tau_n(x_1 \ldots x_r) \tag{1.8}$$

[2] In ref. [11] HEPP proved the Yang-Feldman equation in Wightman's framework for a dense set of collision states.

[3] For a discussion of the formal iteration solution of the conventional field equations see [12].

[4] Slightly more convenient is the analogous integral equation with Δ_F as Green's function.

with finite coefficients τ_n. Putting all momenta but one of the Fourier transform of (1.7) on the mass shell one obtains expansion (1.4) for the matrix elements of the field operator. It is generally believed that the renormalized Gell-Mann-Low expansion corresponds to a local relativistic theory in the sense that (i) and the relation (1.7) between field operator and τ-functions are satisfied for (1.4)[5].

In this paper method (3) is used for the construction of $A(x)$. Assuming that (i) and (1.7) are satisfied it is shown that the field equation holds for every order in λ.

The problem of overlapping divergencies presents a serious difficulty in defining (1.8) for the model of A^4-coupling. In conventional renormalization theory this problem was solved by T. T. WU [5] using differentiation with respect to the momentum variables. A most elegant method of renormalizing arbitrary interactions was developed by BOGOLIUBOV [14] working with a regularized Gell-Mann-Low expansion. As was proved recently by HEPP [15] the coefficients of (1.8) constructed according BOGOLIUBOV's rules approach well defined limits when the regularization is removed.

Since the presence of a regularization parameter is inconvenient for our purpose BOGOLIUBOV's method is reformulated without regularization. The τ-functions are defined by the expansion (1.8) with the coefficients related to renormalized Feynman integrals. The integrand of each Feynman integral is constructed algebraically by rules which resemble those of BOGOLIUBOV[6]. With this definition a system of equations for the τ-functions is derived (see equ. (3.24)) which implies (1.2) for the field operator.

The method can be extended to other renormalizable theories, except for models where infrared divergencies cause additional complications. The pseudoscalar meson-nucleon interaction has been worked out and will be treated in a forthcoming paper.

In section 2 some notations are collected which will be used throughout the paper. Section 3 contains a heuristic motivation of the field equation and the formulae for α and g. The definition of renormalized Feynman integrals is given in section 4 and applied in section 5 to the derivation of (1.9).

Closely related to the present approach is the work by BRANDT of which only the first of a series of papers is available how [16]. In this first paper BRANDT derives a set of integral equations from WILSON's

[5] A proof of this statement will be given in a separate paper.

[6] Orginally I defined the integrand of a renormalized integral recursively by algebraic relation (4.33) of section 4. In this approach it is quite tedious to show that the result is independent of the way the recursion is carried out. Using BOGOLIUBOV's definition the relations (4.33) become algebraic identities.

form [9] of the nucleon field equation which are found to be equivalent to corresponding integral equations of renormalized perturbation theory.

2. Notation of Green's Functions

In this section we collect some definitions of Green's functions and their Fourier transforms which will be used in the work that follows

$$\begin{gathered} T(x_1 \ldots x_r) = T A(x_1) \ldots A(x_r) \\ \tau(x_1 \ldots x_r) = \langle T A(x_1) \ldots A(x_r) \rangle_0 \\ \Delta'_F(x-y) = \tau(x, y), \quad \hat{\Delta}'_F(p) = \int e^{ipx} \Delta'_F(x)\, dx \end{gathered} \tag{2.1}$$

$\tilde{f}(p_1 \ldots p_r)$ denotes the Fourier transform of a function $f(x_1 \ldots x_r)$ of r four vectors

$$\tilde{f}(p_1 \ldots p_r) = \frac{1}{(2\pi)^{2r}} \int e^{i \Sigma p_j x_j} f(x_1 \ldots x_r)\, dx_1 \ldots dx_r . \tag{2.2}$$

The truncated part of $\tau(x_1 \ldots x_r)$ will be denoted by

$$\begin{gathered} \eta(x_1 \ldots x_r) = \tau(x_1 \ldots x_r)^T \\ \tilde{\eta}(p_1 \ldots p_r) = \hat{\eta}(p_1 \ldots p_r)\, \delta(\textstyle\sum P_j) . \end{gathered} \tag{2.3}$$

We further introduce

$$\zeta(p_1 \ldots p_r) = (-i)^r (p_1^2 - m^2) \ldots (p_r^2 - m^2)\, \hat{\eta}(p_1 \ldots p_r) \tag{2.4}$$

$$\Lambda(p_1 \ldots p_r) = \frac{\hat{\eta}(p_1 \ldots p_r)}{\hat{\Delta}'_F(p_1) \ldots \hat{\Delta}'_F(p_r)} \tag{2.5}$$

$\Lambda(p_1 \ldots p_4)$ is the conventional vertex function of A^4-coupling. We will also need partially truncated Green's functions such as

$$\tau([x_1 \ldots x_n]\, y_1 \ldots y_m) = \tau(x_1 \ldots y_m) - \textstyle\sum_x \eta(Z_1) \ldots \eta(Z_N) . \tag{2.6}$$

The sum $\sum_x$ extends over all partitions of $(x_1 \ldots y_m)$ into classes $Z_1, \ldots, Z_N$ for which at least one class Z_α contains x_j only, i.e.

$$Z_\alpha \subseteqq (x_1, \ldots, x_n)$$

for at least one α. Further we define

$$\tau([x_1 \ldots x_n]\, [y_1 \ldots y_m]) = \tau(x_1 \ldots y_m) - \textstyle\sum_{xy} \eta(Z_1) \ldots \eta(Z_N). \tag{2.7}$$

The sum $\sum_{xy}$ extends over all partitions of $(x_1 \ldots y_m)$ into classes $Z_1 \ldots Z_N$ for which at least one class Z_α contains x_i or y_j only, i.e.

$$Z_\alpha \subseteqq (x_1 \ldots x_n) \quad \text{or} \quad Z_\alpha \subseteqq (y_1 \ldots y_m)$$

for at least one α.

The following related functions will frequently be used in this work

$$\begin{aligned} &H(\xi p p_1 \ldots p_r) \\ &= \frac{1}{(2\pi)^4} \int dl_1 \ldots dl_3\, \delta(\textstyle\sum l_j - p)\, e^{-i(l_1 - l_2)\xi}\, \tilde{\tau}([l_1 \ldots l_3]\, p_1 \ldots p_r) \end{aligned} \tag{2.8}$$

$$\begin{aligned} &\delta(p + \textstyle\sum p_j)\, G(\xi p p_1 \ldots p_r) \\ &= \frac{1}{(2\pi)^4} \int dl_1 \ldots dl_3\, \delta(\textstyle\sum l_j - p)\, e^{-i(l_1 - l_2)\xi}\, \tilde{\tau}([l_1 \ldots l_3]\, [p_1 \ldots p_r]). \end{aligned} \tag{2.9}$$

For $n = 1$ we use the notation

$$G(\xi p p_1) = G(\xi p)\,. \tag{2.10}$$

Apparently

$$G(\xi p) = \frac{1}{(2\pi)^4}\int dl_1 \ldots dl_3\, \delta(\textstyle\sum l_j - p)\, e^{-i(l_1-l_2)\xi}\, \hat{\eta}(l_1 \ldots l_3 p). \tag{2.11}$$

The following definitions introduce functions with a particular one-particle pole separated off. For $n, m \geqq 2$ we define

$$\begin{aligned} &\hat{\eta}(p_1 \ldots p_n | q_1 \ldots q_m) \\ &\quad = \hat{\eta}(p_1 \ldots p_n q_1 \ldots q_m) - \frac{\hat{\eta}(p_1 \ldots p_n, -p)\, \hat{\eta}(p q_1 \ldots q_m)}{\hat{\Delta}'_F(p)} \\ &\qquad\qquad p = \textstyle\sum p_j = -\sum q_j\,. \end{aligned} \tag{2.12}$$

As is well known these functions do not have a one-particle pole at $p^2 = m^2$. Related functions are introduced by

$$\begin{aligned} &\tilde{\tau}([p_1 \ldots p_n]\,|[q_1 \ldots q_m]) = \tilde{\tau}([p_1 \ldots p_n]\,[q_1 \ldots q_m]) - \\ &\quad - \delta(\textstyle\sum p_j + \sum q_j)\, \dfrac{\hat{\eta}(p_1 \ldots p_n, -p)\, \hat{\eta}(p q_1 \ldots q_m)}{\hat{\Delta}'_F(p)} \qquad (n, m \geqq 2) \end{aligned} \tag{2.13}$$

$$\begin{aligned} &\delta(p + \textstyle\sum p_j)\, Q(\xi p p_1 \ldots p_r) \qquad (r \geqq 3) \\ &= \frac{1}{(2\pi)^4}\int dl_1 \ldots dl_3\, \delta(\textstyle\sum l_j - p)\, e^{-i(l_1-l_2)\xi}\, \dfrac{\tilde{\tau}([l_1 \ldots l_3]/[p_1 \ldots p_r])}{\hat{\Delta}'_F(p_1) \ldots \hat{\Delta}'_F(p_1)}\,. \end{aligned} \tag{2.14}$$

It follows

$$Q(\xi p p_1 \ldots p_r) = \frac{G(\xi p p_1 \ldots p_r)}{\hat{\Delta}'_F(p_1) \ldots \hat{\Delta}'_F(p_r)} - G(\xi p)\, A(p p_1 \ldots p_r)\,. \tag{2.15}$$

For $r = 1$ we define

$$Q(\xi p p_1) = Q(\xi p) = \frac{G(\xi p)}{\hat{\Delta}'_F(p)}\,. \tag{2.16}$$

The normal product of field operators

$$N(x_1 \ldots x_r) = {:}A(x_1) \ldots A(x_r){:}$$

is as usual defined recursively by Wick's rule

$$T(x_1 \ldots x_r) = N(x_1 \ldots x_r) + \sum \prod_{s=1}^{N} \Delta'_F(x_{i_s} - x_{j_s})\, N(X) \tag{2.17}$$

with the contraction function Δ'_F. The sum extends over all partitions of $(x_1 \ldots x_r)$ into classes

$$(x_{i_1} x_{j_1}), \ldots (x_{i_N} x_{j_N}),\, X = (x_{h_1} \ldots x_{h_l})\,, \quad N \geqq 1$$

with empty X included. The mixed products

$$\begin{aligned} T({:}x_1 \ldots x_n{:}\, y_1 \ldots y_m) &= T({:}A(x_1) \ldots A(x_n){:}\, A(y_1) \ldots A(y_m)) \\ T({:}x_1 \ldots x_n{:}\,&y_1 \ldots y_m\, {:}z_1 \ldots z_r{:}) \\ = T({:}A(x_1) \ldots A(x_n){:}\, &A(y_1) \ldots A(y_m)\, {:}A(z_1) \ldots A(z_r){:}) \end{aligned}$$

are defined by (2.17), but omitting the terms containing factors $\Delta'_F(x_i - x_j)$ or $\Delta'_F(z_i - z_j)$. The vacuum expectation values are denoted by

$$\tau(:x_1 \ldots x_n : y_1 \ldots y_m) = \langle T(:x_1 \ldots x_n : y_1 \ldots y_m)\rangle_0$$
$$\tau(:x_1 \ldots x_n : y_1 \ldots y_m : z_1 \ldots z_r :) = \langle T(:x_1 \ldots x_n : y_1 \ldots y_m : z_1 \ldots z_r :)\rangle_0 .$$

3. Heuristic Motivation

The model of a scalar field with A^4-coupling is formally described by the field equations

$$-(\Box + m^2)\, A(x) = \lambda \frac{:A(x)^3: - \alpha A(x)}{\beta + \gamma} \tag{3.1}$$

and the commutation relations

$$\begin{aligned} [A(x)\, \dot{A}(y)] &= \frac{\gamma}{\beta + \gamma}\, i\, \delta_3(x - y) \qquad x^0 = y^0 \\ [A(x)\, A(y)] &= [A(x)\, \dot{A}(y)] = 0 . \end{aligned} \tag{3.2}$$

The operator product $:A(x_1)\, A(x_2)\, A(x_3):$ is defined by (see (2.17))

$$\begin{aligned} &:A(x_1)\, A(x_2)\, A(x_3): \\ &= T A(x_1)\, A(x_2)\, A(x_3) - \Delta'_F(x_1 - x_2)\, A(x_3) - \text{cycl. perm. terms.} \end{aligned} \tag{3.3}$$

For spacelike coordinate differences this reduces to (1.3). α, β, and γ are related to the conventional renormalization constants by

$$Z_1 = \gamma^{-1}, \quad Z_2 = 1 + \frac{\beta}{\gamma}, \quad \delta m^2 = \lambda \frac{\alpha}{\beta + \gamma} . \tag{3.4}$$

As is well known the definition of the renormalized coupling constant is to some extent arbitrary. For the purpose of this work it is convenient to define λ by the value of the vertex function at zero momenta

$$\lambda = \frac{(2\pi)^4 i}{6} \Lambda(0, 0, 0, 0) . \tag{3.5}$$

In addition to (3.1), (3.2) it is assumed in this section that the field operator $A(x)$ satisfies the usual postulates of quantum field theory such as Lorentz invariance, spectrum condition and asymptotic condition.

By formal manipulations of field equations and commutation relations we will obtain expression for α, β and γ in terms of Green's functions. The formulae obtained will suggest how to modify the field equation in case of divergent renormalization constants.

In order to determine α we take the matrix element of (3.1) between the vacuum state Ω and the one particle state Φ_p of momentum p. Källen's renormalization condition

$$(\Omega,\, A(x)\, \Phi_p) = \frac{1}{(2\pi)^{3/2}}\, e^{ipx}$$

yields

$$\alpha = (2\pi)^{3/2} (\Omega, :A(0)^3:\Phi_p) \,. \tag{3.6}$$

Using the reduction technique we can express α in terms of Green's functions

$$\begin{aligned} \alpha &= Q(m^2), \quad Q(p^2) = \frac{G(p)}{\hat{\Delta}'_F(p)} \\ G(p) &= \int e^{-ipy} \langle T:A(0)^3: A(y)\rangle_0 dy \\ &= \int e^{-ipy} \eta(0\,0\,0\,y)\, dy \,. \end{aligned} \tag{3.7}$$

The constants β and γ will be determined from the field equations of the time ordered function of two or four arguments resp. As is well known equ. (3.1—2) imply formally[7]

$$\begin{aligned} &-(\Box_x + m^2)\, T(x x_1 \ldots x_r) \\ &= \frac{\lambda}{\beta+\gamma} \{T(xxx:x_1 \ldots x_r) - \alpha T(x x_1 \ldots x_r)\} + \\ &+ i \sum_{i=1}^{r} \frac{\gamma}{\beta+\gamma}\, \delta(x - x_j)\, T(x_1 \ldots x_{j-1} x_{j+1} \ldots x_r) \end{aligned} \tag{3.8}$$

for the time-ordered products. The vacuum expectation value of (3.8) gives the field equations of the τ-functions.

The constant β is easily determined from the field equation of the propagator

$$\begin{aligned} (p^2 - m^2)\, \hat{\Delta}'_F &= \frac{\lambda}{\beta+\gamma} (G - \alpha \hat{\Delta}'_F) + i \frac{\gamma}{\beta+\gamma} \\ &= \frac{\lambda}{\beta+\gamma} \frac{Q(p^2) - Q(m^2)}{p^2 - m^2} (p^2 - m^2)\, \hat{\Delta}'_F + i \frac{\gamma}{\beta+\gamma} \end{aligned} \tag{3.9}$$

Setting $p^2 = m^2$ we get

$$\beta = \lambda \frac{\partial Q}{\partial p^2} \text{ at } p^2 = m^2 \tag{3.10}$$

since

$$(p^2 - m^2)\, \hat{\Delta}'_F = i \quad \text{at} \quad p^2 = m^2 \,.$$

In order to determine γ we use the field equation of $\tau(x_1 \ldots x_4)$ which for the truncated part implies

$$\begin{aligned} &-(\beta+\gamma)(\Box_{x1} + m^2)\, \eta(x_1 \ldots x_4) \\ &\qquad = \lambda(\tau(:x_1 x_1 x_1: :x_2 x_3 x_4:) - \alpha \eta(x_1 x_2 x_3 x_4)). \end{aligned} \tag{3.11}$$

With the notation

$$\delta(\textstyle\sum p_j)\, G(p_1 \ldots p_4) = \frac{1}{(2\pi)^8} \displaystyle\int e^{i\Sigma p_j x_j} \tau(:x_1 x_1 x_1: :x_2 x_3 x_4:)\, dx_1 \ldots dx_4 \tag{3.12}$$

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[7] For $r = 1$ the last term of the right hand side is $i \frac{\gamma}{\gamma+\beta} \delta(x - x_1)$.

the Fourier transform of equ. (3.11) becomes

$$(\beta+\gamma)(p_1^2-m^2)\,\eta(p_1\ldots p_4)=\lambda\{G(p_1\ldots p_4)-\alpha\eta(p_1\ldots p_4)\}. \tag{3.13}$$

Defining

$$Q(p_1\ldots p_4)=\frac{G(p_1\ldots p_4)}{\hat{\Delta}_F'(p_2)\ldots\hat{\Delta}_F'(p_4)}-G(p_1)\,\Lambda(p_1\ldots p_4) \tag{3.14}$$

and using (3.9) we obtain the relation

$$i\gamma\Lambda(p_1\ldots p_4)=\lambda Q(p_1\ldots p_4)\,.$$

With (3.5) the constant γ becomes

$$\gamma=\frac{(2\pi)^4}{6}\,Q(0,0,0,0)\,. \tag{3.15}$$

Hence we obtain for γ the expression

$$\begin{aligned}\gamma=\frac{1}{6\,\hat{\Delta}_F'(0)^3}\int dy_1\,dy_2\,dy_3\langle T:A(0)^3::A(y_1)\,A(y_2)\,A(y_3):\rangle_0+\\ +i\lambda\int dy\langle T:A(0)^3:A(y)\rangle_0\,.\end{aligned} \tag{3.16}$$

By (3.7), (3.10) and (3.16) the constants α, β and γ are all expressed in terms of Green's functions involving the formal interaction term $:A(x)^3:$ of the field equation (3.1). This result suggests to modify the field equation according to

$$-(\Box+m^2)\,A(x)=\lambda\lim_{\xi\to 0}\frac{:A(x+\xi)\,A(x)\,A(x-\xi):-\alpha(\xi)\,A(x)}{\beta(\xi)+\gamma(\xi)},\quad \xi=(0,\boldsymbol{\xi}) \tag{3.17}$$

where the functions $\alpha(\xi)$, $\beta(\xi)$ and $\gamma(\xi)$ are obtained by the substitutions

$$:A(0)^3:\;\to\;:A(\xi)\,A(0)\,A(-\xi):\,.$$

We thus get from (3.7), (3.10) and (3.16) the following representations of $\alpha(\xi)$, $\beta(\xi)$ and $\gamma(\xi)$ in terms of Green's functions

$$\alpha(\xi)=\left\{\frac{1}{\hat{\Delta}_F'(p)}\int dy\,e^{-ipy}\langle T:A(\xi)\,A(0)\,A(-\xi):A(y)\rangle_0\right\}_{p^2=m^2} \tag{3.18}$$

$$\beta(\xi)=\lambda\frac{\partial}{\partial p^2}\left\{\frac{1}{\hat{\Delta}_F'(p)}\int dy\,e^{-ipy}\langle T:A(\xi)\,A(0)\,A(-\xi):A(y)\rangle_0\right\}_{p^2=m^2} \tag{3.19}$$

$$\begin{aligned}\gamma(\xi)=\frac{1}{6\hat{\Delta}_F'(0)^3}\int dy_1\,dy_2\,dy_3\langle T:A(\xi)\,A(0)\,A(-\xi)::A(y_1)\,A(y_2)\,A(y_3):\rangle_0+\\ +i\lambda\int dy\langle T:A(\xi)\,A(0)\,A(-\xi):A(y)\rangle_0\,.\end{aligned} \tag{3.20}$$

Comparing with the definition (2.15) and (2.16) we have

$$\alpha(\xi)=Q(\xi,p)\quad p=(m,0,0,0,0) \tag{3.21}$$

$$\beta(\xi)=\lambda\left.\frac{\partial Q(\xi,p)}{\partial p^2}\right|_{p^2=m^2}\quad\text{for}\quad p=(\sqrt{p^2},0,0,0) \tag{3.22}$$

$$\gamma(\xi)=\frac{(2\pi)^4}{6}\,Q(\xi\,0\,0\,0\,0)\,. \tag{3.23}$$

We thus arrive at the proposal that the field operator satisfy the field equation (1.2) with spacelike limit $\xi^2 < 0$ and the functions $\alpha(\xi)$ and

$$g(\xi) = \beta(\xi) + \gamma(\xi)$$

expressed in terms of $A(x)$ by (3.21—23).

In a similar manner one writes the field equations of the time ordered products in the modified form

$$-(\Box_x + m^2)\, T(x x_1 \ldots x_r) = \lambda \lim_{\xi \to 0} \frac{N(\xi x x_1 \ldots x_r)}{\beta(\xi) + \gamma(\xi)}, \quad \xi^2 \neq 0$$

$$N(\xi x x_1 \ldots x_r) = T(:x+\xi, x, x-\xi: x_1 \ldots x_r) - \alpha(\xi)\, T(x x_1 \ldots x_r) + \tag{3.24}$$

$$+\, i\,\gamma(\xi) \sum_{j=1}^{r} \delta(x - x_j)\, T(x_1 \ldots x_{j-1} x_{j+1} \ldots x_r)\,.$$

We finally give the Fourier transform of the vacuum expectation value of (3.24)

$$(p^2 - m^2)\, \tilde{\tau}(p p_1 \ldots p_r) = \lambda \lim_{\xi \to 0} \frac{\tilde{n}(\xi p p_1 \ldots p_r)}{\beta(\xi) + \gamma(\xi)} \tag{3.25}$$

with

$$\tilde{n}(\xi p p_1 \ldots p_r) = H(\xi p p_1 \ldots p_r) - \alpha(\xi)\, \tilde{\tau}(p p_1 \ldots p_r) +$$

$$+\, i\,\gamma(\xi) \sum_{j=1}^{r} \delta(p + p_j)\, \tilde{\tau}(p_1 \ldots p_{j-1} p_{j+1} \ldots p_r)\,.$$

In the following sections it will be shown that (3.25) is indeed satisfied by the power series of the τ-functions. The equation (3.1) of the field operator and (3.24) then follow for matrix elements between suitable state vectors.

4. Renormalized Gell-Mann-Low Expansion

In this section the τ-functions will be defined by a power series with respect to the renormalized coupling constant λ. Setting

$$\tilde{\tau}(p_1 \ldots p_r) = \lim_{\varepsilon \to +0} \tilde{\tau}(p_1 \ldots p_r \varepsilon) \tag{4.1}$$

we express $\tau(p_1 \ldots p_r \varepsilon)$ in terms of the truncated part

$$\tilde{\tau}(p_1 \ldots p_r \varepsilon)^T = \hat{\eta}(p_1 \ldots p_r \varepsilon)\, \delta(\textstyle\sum p_j) \tag{4.2}$$

with

$$\hat{\eta}(p_1 \ldots p_r \varepsilon) = \prod_{j=1}^{r} \frac{i}{p_j^2 - m^2 + i\varepsilon(\mathbf{p}_j^2 + m^2)}\, \zeta(p_1 \ldots p_r \varepsilon)\,. \tag{4.3}$$

For ζ we give a defining power series which we write in the form

$$\zeta(p_1 \ldots p_r \varepsilon) = \sum_{\Gamma \in \mathscr{C}_r} \frac{1}{\mathscr{S}(\Gamma)} J_\Gamma(p_1 \ldots p_r \varepsilon)\,. \tag{4.4}$$

The sum extends over the class $\mathscr{C}_r$ of Feynman diagrams Γ with the following properties. ($\mathscr{S}(\Gamma)$ and J_Γ will be defined below.)

The vertices of Γ are denoted by $V_1 \ldots, V_N$ ($N \geqq 1$). We distinguish external and internal lines of Γ. Each external line is attached to a vertex of Γ, each internal line connects two vertices of Γ. Γ is assumed to be connected and to have r external lines denoted by $E_1, \ldots, E_r$. To each external line E_j a momentum four-vector p_j is assigned. Each vertex of Γ is either a 4-vertex or a 2-vertex. At a 4-vertex join four (internal or external) lines, at a 2-vertex join two lines[8].

The intrinsic symmetry number $\mathscr{S}(\Gamma)$ is defined as follows [5]. A permutation Π of the vertices of Γ is called an internal automorphism if

(i) ΠV_a and ΠV_b are connected by the same number of internal lines as V_a and V_b.

(ii) $\Pi V_a = V_a$ if an external line is attached to V_a. Then $\mathscr{S}(\Gamma)$ is defined by

$$\mathscr{S}(\Gamma) = g\, 2^{\alpha} (3\,!)^{\beta}\, 2^{\gamma}$$

where g is the number of internal automorphisms of Γ, α is the number of vertex pairs V_a, V_b which are connected by two lines, β is the number of vertex pairs which are connected by three lines and γ is the number of lines which connect a vertex with itself.

The terms $J_\Gamma(p_1 \ldots p_r)$ of the expansion (4.4) will be defined as the finite part of the Feynman integral J^0_Γ belonging to the diagram Γ. We first give the rules of constructing the unsubtracted Feynman integral J_Γ. Γ need not be connected for the following definitions.

The internal lines connecting V_a and V_b, with direction[9] from V_b to V_a, are denoted by $L_{ab\sigma}$ ($\sigma = 1, \ldots, \nu(ab)$). To each vertex V_a corresponds the external momentum

$$q(V_a) = q_a = \sum_{\Gamma_\nu \in \mathscr{E}_a} p_\nu \tag{4.5}$$

where the sum extends over the set $\mathscr{E}_a$ of the external lines L_ν which are attached to the vertex V_a. Particularly we have $q_a = 0$ for a vertex to which no external line is attached. To each internal line $L_{ab\sigma}$ we assign the momentum $l_{ab\sigma}$ and require

$$l(L_{ab\sigma}) = l_{ab\sigma} = -l_{ab\sigma} \quad \text{for} \quad a \neq b\,.$$

The Feynman integral J^0_Γ of the diagram Γ is constructed according to the following rules. To each line $L_{ab\sigma}$ corresponds the factor

$$\hat{\Delta}_F(l_{ab\sigma}) = \frac{i}{l^2_{ab\sigma} - m^2 + i\varepsilon(\mathbf{l}^2_{abc} + m^2)}\,. \tag{4.6}$$

To each vertex V_a corresponds the factor

$$P_a \delta\Big(\sum_{b\sigma}^{a} l_{ab\sigma} - q_a\Big)\,. \tag{4.7}$$

[8] Internal lines connecting a vertex with itself are counted twice in this context.

[9] For the assignment of momentum variables it is convenient to introduce directed internal lines.

Here $\sum\limits_{b\sigma}^{a}$ denotes the sum over all internal lines $L_{ab\sigma}$ ($b \neq a$) having V_a as one of its endpoints. The factor P_a is

$$P_a = -\frac{6i\lambda}{(2\pi)^4} \quad \text{for a 4-vertex} \tag{4.8}$$

$$P_a = A(\lambda) + B(\lambda)\,(l^2 - m^2) \quad \text{for a 2-vertex (Fig. 1)}$$

$A(\lambda) + B(\lambda)\,(l^2 - m^2)$

Fig. 1. Trivial self-energy diagram

$\pm l$ denotes the internal or external momentum corresponding to the lines joining at the 2-vertex. $A(\lambda)$ and $B(\lambda)$ are power series with respect to λ with (finite) coefficients to be defined later.

With these insertion rules the Feynman integral becomes

$$J_\Gamma^0(p_1 \ldots p_r) = \lim_{\varepsilon \to +0} J_\Gamma^0(p_1 \ldots p_r\,\varepsilon)$$

$$\delta\left(\sum_{a=1}^{r} p_a\right) J_\Gamma^0(p_1 \ldots p_r\,\varepsilon) = \tag{4.9}$$

$$= \int \prod_{ab\sigma} dl_{ab\sigma} \prod_{ab\sigma} \hat{\Delta}_F(l_{ab\sigma}) \prod_{a=1}^{r} P_a\,\delta\left(\sum_{b\sigma}^{a} l_{ab\sigma} - q_a\right)$$

or

$$J_\Gamma^0(p_1 \ldots p_r\,\varepsilon) = \int dk_1 \ldots dk_m I_\Gamma(k_1 \ldots k_m p_1 \ldots p_r) \tag{4.10}$$

$$I_\Gamma = \prod_{ab\sigma} \hat{\Delta}_F(l_{ab\sigma}) \prod_{a=1}^{r} P_a\,. \tag{4.11}$$

$\prod\limits_{ab\sigma}$ denotes the product over all internal lines $L_{ab\sigma}$ of Γ. In (4.10) the trivial integrations over internal momenta have been carried out in the following manner. The arguments $l_{ab\sigma}$ are written as

$$l_{ab\sigma} = k_{ab\sigma} + r_{ab\sigma}(q_1 \ldots q_N) \tag{4.12}$$

$$k_{ab\sigma} = k_{ab\sigma}(k_1 \ldots k_m)\,, \tag{4.13}$$

The $r_{ab\sigma}$ are linear combinations of $q_1, \ldots, q_N$ and form a particular solution of[10]

$$\sum_{b,\sigma}^{a} r_{ab\sigma} = q_a\,, \quad r_{ab\sigma} + r_{ba\sigma} = 0\,. \tag{4.14}$$

The $r_{ab\sigma}$ are called basic internal momenta or r-momenta. The $k_{ab\sigma}$ are linear combinations of $k_1, \ldots, k_m$ and form the general solution of

$$\sum_{b\sigma}^{a} k_{ab\sigma} = 0, \quad k_{ab\sigma} + k_{ba\sigma} = 0 \tag{4.15}$$

with m of the forms $k_{ab\sigma}$ chosen to be identical with $k_1, \ldots, k_m$. The $k_{ab\sigma}$ are called homogenous internal momenta or k-momenta, the set $(k_1, \ldots, k_m)$ is called a basis of independent k-momenta. Occasionally we

[10] The momenta q_a must satisfy momentum conservation for each connected component of Γ.

use the notation

$$r_{ab\sigma} = r(L_{ab\sigma}), \quad k_{ab\sigma} = k(L_{ab\sigma}) .$$

We introduce the following abbreviations

$$k = (k_1 \ldots k_m), \quad p = (p_1 \ldots p_r) \tag{4.16}$$

$$K = \{k_{ij\sigma}\}_{L_{ij\sigma} \in \mathscr{L}(\Gamma)}, \quad q = (q_1 \ldots q_N) . \tag{4.17}$$

Equ. (4.5), (4.13) are written as

$$q = q(p), \quad K = K(k) . \tag{4.18}$$

We introduce the I-function $I_\Gamma(K, q)$ of the diagram Γ as the integrand of the corresponding Feynman integral in terms of the variables (4.17). Hence

$$I_\Gamma = I_\Gamma(K, q) \tag{4.19}$$

is defined by (4.11) with (4.6), (4.8) and (4.12). In this notation the Feynman integral becomes

$$J^0_\Gamma(p, \varepsilon) = \int dk_1 \ldots dk_m \, I_\Gamma(K(k), q(p)) . \tag{4.20}$$

The finite part of a Feynman integral will be defined in the form

$$J_\Gamma(p) = \lim_{\varepsilon \to +0} J_\Gamma(p, \varepsilon) \tag{4.21}$$

$$J_\Gamma(p\varepsilon) = \int dk_1 \ldots dk_m \, R_\Gamma(K(k), q(p)) . \tag{4.22}$$

where the R-function $R_\Gamma(K, q)$ is obtained from $I_\Gamma(K, q)$ by a suitable number of subtractions. For the precise formulation of the rules we will need some definitions concerning the I-functions of reduced diagrams and subdiagrams of Γ.

Let $\mathscr{L}_{\text{int}}(\gamma)$ denote the set of internal lines and $\mathscr{V}(\gamma)$ denote the set of vertices of a diagram γ. To any set $\mathscr{L} \subseteqq \mathscr{L}_{\text{int}}(\Gamma)$ we define a subdiagram γ of Γ by the internal lines $L \in \mathscr{L}$ and the vertices which are endpoints of a line L in $\mathscr{L}$. The external lines of γ at the vertex $V \in \mathscr{V}(\gamma)$ are the external lines attached to V in Γ and in addition as many external lines as there are internal lines[8] L of Γ with endpoint V and $L \notin \mathscr{L}$.

A diagram is called a renormalization part if it is a selfenergy part (proper diagram with two external lines) or a vertex part (proper diagram with four external lines). The dimension $d(\Gamma)$ of a proper diagram Γ is defined as the dimension of the multiple integral (4.20). The renormalization parts Γ form all proper diagrams with dimension $d(\Gamma) \geqq 0$.

Let $\gamma_1, \ldots, \gamma_c$ be mutually disjoint renormalization parts of Γ. $\Gamma/\gamma_1 \ldots \gamma_c$ denotes the reduced diagram which is obtained from Γ by reducing each γ_j to a point. The I-function $I_{\Gamma/\gamma_1 \ldots \gamma_c}(K, q)$ is defined again by (4.6), (4.8), (4.11) and (4.12), but with the products in (4.11) restricted to the lines and vertices of $\Gamma/\gamma_1 \ldots \gamma_c$.

Let γ be a renormalization part of Γ. We define the I-function of γ by

$$I_\gamma(K^\gamma, q^\gamma) = \prod_{\mathscr{L}_{\mathrm{int}}(\gamma)} \hat{\Delta}_F(l^\gamma_{ab\sigma}) \prod_{\mathscr{V}(\gamma)} P_a \tag{4.23}$$

with (4.6), (4.8) and

$$l^\gamma_{ab\sigma} = k^\gamma_{ab\sigma} + r^\gamma_{ab\sigma}(q^\gamma) \quad q^\gamma(V_a) = q^\gamma_a \tag{4.24}$$

$$K^\gamma = \{k^\gamma_{ab\sigma}\}_{L_{ab\sigma}\in\mathscr{L}_{\mathrm{int}}(\gamma)}, \quad q^\gamma = \{q^\gamma_a\}_{V_a\in\mathscr{V}(\gamma)}. \tag{4.25}$$

The products in (4.23) extend over all vertices and internal lines of γ.

We next express $k^\gamma_{ab\sigma}$, q^γ_a as linear combinations of $k_{ab\sigma}$, q_a by requiring

$$l^\gamma_{ab\sigma}(K^\gamma, q^\gamma) \equiv l_{ab\sigma}(K, q) \tag{4.26}$$

for all $L_{ab\sigma} \in \mathscr{L}(\gamma)$. According to

$$q^\gamma_a = \sum^a_{L_{ab\sigma}\in\mathscr{L}_{\mathrm{int}}(\gamma)} l_{ab\sigma}$$

$$k^\gamma_{ab\sigma} = l_{ab\sigma} - r^\gamma_{ab\sigma}$$

the $k^\gamma_{ab\sigma}$, q^γ_a are uniquely determined by this requirement. The choice of basic internal momenta $q_{ab\sigma}$, $q^\gamma_{ab\sigma}$ selected for every connected subdiagram γ of Γ is called admissable if the $k^\gamma_{ab\sigma}$ depend on the $k_{ab\sigma}$ only:

$$k^\gamma_{ab\sigma} = k^\gamma_{ab\sigma}(K), \quad q^\gamma_a = q^\gamma_a(K, q). \tag{4.27}$$

With the substitution (4.27) I_γ becomes a function of K and q.

After these preparations we define now — following BOGOLIUBOV — the R-function of Γ by

$$R_\Gamma(K, q) = \overline{R}_\Gamma(K, q) \tag{4.28}$$

if Γ is no renormalization part and

$$R_\Gamma(K, q) = (1 - t^{d(\Gamma)}_q)\, \overline{R}_\Gamma(K, q) \tag{4.29}$$

if Γ is a renormalization part of dimension $d(\Gamma) \geqq 0$. The function $\overline{R}_\Gamma$ is defined recursively by

$$\overline{R}_\Gamma(K, q) = I_\Gamma(K, q) +$$

$$+ \sum_{\gamma_1\cdots\gamma_c} I_{\Gamma/\gamma_1\ldots\gamma_c}(K, q) \prod_{\tau=1}^{c} P_\tau^{-1} O_{\gamma_\tau}(K^{\gamma_\tau}, q^{\gamma_\tau}) \tag{4.30}$$

$$K^{\gamma_\tau} = K^{\gamma_\tau}(K), \quad q^{\gamma_\tau} = q^{\gamma_\tau}(K, q)$$

$$O_\gamma(K^\gamma, q^\gamma) = -t^{d(\gamma)}_{q^\gamma} \overline{R}_\gamma(K^\gamma, q^\gamma). \tag{4.31}$$

$t^{d(\Gamma)}_q$, if applied to a function of q, denotes the Taylor series with respect to $q_1 \ldots q_r$ up to and including terms of order $d(\Gamma)$. The sum in (4.30) extends over all sets $s = (\gamma_1 \ldots \gamma_c)$ of mutually disjoint renormalization parts of Γ with $\gamma_j \neq \Gamma$. Let W_τ be the vertex of $\Gamma/\gamma_1 \ldots \gamma_c$ which is obtained by reducing γ_τ. Then P_τ in equ. (4.30) denotes the factor (4.8) which is assigned to W_τ in $\Gamma/\gamma_1 \dots \gamma_c$[11].

As will be shown in a forthcoming paper [17] the finite part (4.22) of a Feynman integral thus defined is an absolutely convergent integral for $\varepsilon > 0$. The integral is independent of the choice of the basic internal momenta $q_{ab\sigma}, q^{\gamma}_{ab\sigma}$. In the limit $\varepsilon \to +0$ $J(p\varepsilon)$ converges strongly towards a covariant distribution (4.21) in $p_1, \dots, p_r$.

Finally we define the power series $A(\lambda)$, $B(\lambda)$ recursively by

$$A(\lambda) = -\sum_{S \in \mathscr{K}} \frac{1}{\mathscr{S}(S)} J_s(m^2), \quad J_s(p^2) = J_s(p, -p) \tag{4.32}$$

$$B(\lambda) = -\sum_{S \in \mathscr{K}} \frac{1}{\mathscr{S}(S)} J_s'(m^2), \quad J_s'(p^2) = \frac{\partial J_s(p^2)}{\partial p^2}. \tag{4.32}$$

The sum extends over the class $\mathscr{K}$ of all proper self-energy parts S excluding the trivial diagram (Fig. 1) with one vertex only. This prescription guarantees that the propagator satisfies the renormalization condition

$$-i(p^2 - m^2)\,\Delta_F'(p) = 1 \quad \text{at} \quad p^2 = m^2 .$$

Concluding this section we state an algebraic identity for R-functions which will be crucial for the derivation of local field equations.

Theorem. *Let Γ be a diagram, W be a 4-vertex of Γ with the property that W does not belong to any renormalization part $\gamma \neq \Gamma$ of dimension $d(\gamma) > 0$. Under this hypothesis the identity*

$$\begin{aligned}\overline{R}_\Gamma(K, q) = &-\frac{6i\lambda}{(2\pi)^4} F_\Gamma(K, q)\, R_{\hat{\Gamma}}(K^{\hat{\Gamma}}, q^{\hat{\Gamma}}) - \\ &- \sum_{\gamma \in T(\Gamma, W)} \overline{R}_{\Gamma/\gamma}(K, q)\, F_\gamma(K^\gamma, 0)\, R_{\hat{\gamma}}(K^{\hat{\gamma}}, 0) \end{aligned}\tag{4.33}$$

$$K^\delta = K^\delta(K) \quad q^\delta = q^\delta(K, q) \ \ \textit{for} \quad \delta = \hat{\Gamma}, \gamma, \hat{\gamma}$$

holds.

$\hat{\gamma}$ is defined as subdiagram of Γ by

$$\mathscr{L}(\hat{\gamma}) = \mathscr{L}(\gamma) - \mathscr{L}(\gamma, W)$$

where $\mathscr{L}(\gamma, W)$ is the set of all internal lines of γ with endpoint W. F_γ is given by

$$\begin{aligned} F_\Gamma(K, q) &= \Pi_{\Gamma, W} \hat{\Delta}_F(l_{ab\sigma}) \\ F_\gamma(K^\gamma, q^\gamma) &= \Pi_{\gamma, W} \hat{\Delta}_F(l^\gamma_{ab\sigma}) \end{aligned}\tag{4.34}$$

with the products extending over all lines of $\mathscr{L}(\gamma, W)$. $T(\Gamma, W)$ denotes the set of all renormalization parts γ of Γ with $W \in \mathscr{V}(\gamma)$ and $\gamma \neq \Gamma$.

---

[11] It should be noted that in general these rules give more subtractions than Bogoliubov's original prescription. This is for instance the case for the second order selfenergy diagram. Apparently less subtractions are required if the subtractions are made before the regularization limit is taken. I am indebted to Dr. Hepp for many interesting discussions concerning this point and the equivalence of both methods.

We also write (4.33) in the abbreviated form

$$\overline{R}_\Gamma = -\frac{6i\lambda}{(2\pi)^4} F_\Gamma R_{\hat{\Gamma}} - \sum_{\gamma\in T} \overline{R}_{\Gamma/\gamma} F_\gamma^0 R_{\hat{\gamma}}^0 \tag{4.35}$$

where the superscript ^{0} denotes the value at zero external momenta.

The proof of this identity is elementary, and will be given in a separate paper [17] for the general case of arbitrary coupling.

The theorem is related to HEPP's Lemma 2.3 in ref. [15] .One difference is that in (4.33) the degrees of the subtractions in $\overline{R}_{\Gamma/\gamma}$, $R_{\hat{\gamma}}$ refer to Γ/γ and $\hat{\gamma}$ rather than to the original diagram Γ. This is desirable for the derivation of field equations, but requires a restriction on Γ and W as stated in the hypothesis. In the general case one has more complicated identities.

5. Check of Field Equations in Perturbation Theory

a) Relation for Λ-Functions

In this section we will verify that the equation (3.25) for the τ-functions are satisfied by the renormalized Gell-Mann-Low expansion (4.1—4) to every order of λ.

We begin checking the relation ($r \geqq 3$)

$$\gamma(\xi)\,\Lambda(pp_1\ldots p_r) = -i\lambda Q(\xi pp_1\ldots p_r) + \omega(\xi pp_1\ldots p_r) \tag{5.1}$$

where Λ and Q are defined by (2.5), (2.14) and the function ω satisfies

$$\lim_{\xi\to 0} \omega(\xi pp_1\ldots p_r) = 0 \tag{5.2}$$

$\Lambda(pp_1\ldots p_r)$ has the expansion

$$\Lambda(pp_1\ldots p_r) = \lim_{\varepsilon\to +0} \Lambda(pp_1\ldots p_r\varepsilon) \tag{5.3}$$

$$\Lambda(pp_1\ldots p_r\varepsilon) = \sum_{\Gamma\in\mathscr{K}_r} \frac{1}{\mathscr{S}(\Gamma)} J_\Gamma(pp_1\ldots p_r\varepsilon) \tag{5.4}$$

where $\mathscr{K}_r$ is the class of all connected diagrams with $r+1$ external lines, but no external self-energy insertions. The momenta $p, p_1, \ldots, p_r$ are assigned to the external lines denoted by $E, E_1, \ldots, E_r$. The corresponding external vertices will be denoted by $W, W_1, \ldots, W_r$.

The hypothesis of the theorem (page 175) applies to W for any $\Gamma \in \mathscr{K}_r$ since by definition W does not belong to any self-energy part of Γ. Identity (4.33) then implies

$$R_\Gamma + \sum_{\gamma\in C(\Gamma,W)} R_{\Gamma/\gamma} F_\gamma^0 R_{\hat{\gamma}}^0 = -\frac{6i\lambda}{(2\pi)^4} F_\Gamma R_{\hat{\Gamma}}\,. \tag{5.5}$$

Here $C(\Gamma, W)$ is the set of all vertex parts of Γ with $W \in \mathscr{V}(\gamma)$. The case $\gamma = \Gamma$ is included provided $N = 3$. F_γ^0, $R_{\hat{\gamma}}^0$ denote F_γ, $R_{\hat{\gamma}}$ at zero external momenta.

The elements of C are totally ordered by inclusion, i.e.

$$\gamma \subseteqq \gamma' \quad \text{or} \quad \gamma' \subseteqq \gamma$$

for any two elements γ, γ' of C. Let γ_0 denote the smallest element of C

$$\gamma_0 \subseteqq \gamma \quad \text{for all} \quad \gamma \in C\,.$$

We distinguish the following two types of diagrams in $\mathscr{K}_r$. Γ is called degenerate if two or more external lines are attached to the vertex W

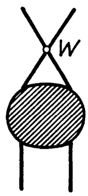

Fig. 2. Exemple of degenerate diagram

(Fig. 2). Γ is called non-degenerate if only one external line is attached to W. Γ is called trivial if it has one vertex only. All non-trivial degenerate diagrams have two external lines attached to W.

We further introduce a numbering ν of the elements of $\mathscr{L}(\Gamma, W)$, i.e. the internal lines with endpoint W. Let Γ be non-degenerate. In this case we consider all numberings ν which assign the numbers 1, 2 and 3 to the lines of $\mathscr{L}(\Gamma, W)$ which are accordingly denoted L_1, L_2 and L_3 with direction towards W. If Γ is degenerate and non-trivial we consider all numberings ν which assign the numbers (1, 2), (2, 3) or (3, 1) to the two lines of $\mathscr{L}(\Gamma, W)$.

A relation for J_Γ will be derived for every $\Gamma \in \mathscr{K}_r$ and any numbering ν. Finally summing over all $\Gamma \in \mathscr{K}_r$ and ν we will obtain the desired relation (5.1).

We begin discussing the simple example of a non-degenerate diagram which is further restricted by the condition that L_1, L_2 and L_3 all be internal lines of γ_0. In this case

$$p^\gamma(W) = p \quad \text{for any} \quad \gamma \in C\,.$$

It is convenient to choose the r-momenta of L_1, L_2, L_3 as

$$r(L_i) = r^\gamma(L_i) = \frac{p}{3}\,, \quad \gamma \in C\,. \tag{5.6}$$

We further select a basis $k_1, \ldots, k_m$ of the momenta $k_{ab\sigma}$ $(L_{ab\sigma} \in \mathscr{L}_{\text{int}}(\Gamma))$ such that

(i) k_1, k_2 correspond to L_1, L_2 resp.

(ii) $k_1^\gamma, \ldots, k_{m(\gamma)}^\gamma$ form a basis of the momenta $k_{ab\sigma}^\gamma$

$$(L_{ab\sigma} \in \mathscr{L}_{\text{int}}(\gamma)) \quad \text{for any} \quad \gamma \in C\,.$$

With this notation we have

$$k_1^\gamma = k_1\,, \ k_2^\gamma = k_2 \quad \text{for} \quad \gamma \in C\,.$$

Multiplying (5.5) by $e^{-i\xi(k_1-k_2)}$ and integrating over $k_1, \ldots, k_m$ we obtain

$$J_\Gamma + \sum_{\gamma \in D} J_{\Gamma/\gamma} X_\gamma = -\frac{6i\lambda}{(2\pi)^4} W_\Gamma + Y_\Gamma \tag{5.7}$$

with

$$X_\gamma = \int dk_1^\gamma \ldots dk_{m(\gamma)}^\gamma \, e^{-i\xi(k_1^\gamma - k_2^\gamma)} F_\gamma^0 R_{\hat\gamma}^0 \tag{5.8}$$

$$W_\gamma = \int dk_1 \ldots dk_m \, e^{-i\xi(k_1-k_2)} F_\Gamma \, R_{\hat\Gamma} \tag{5.9}$$

$$Y_\Gamma = \int dk_1 \ldots dk_m (e^{-i\xi(k_1-k_2)} - 1) \, R_\Gamma \tag{5.10}$$

In deriving (5.7) it was used that $R_{\Gamma/\gamma}$ does not depend on the variables k_1, k_2.

Next we extend equ. (5.7) to the general case of a non-degenerate diagram $\Gamma \in \mathscr{K}_r$. Let $C_0(\Gamma)$ be the set of all $\gamma \in C(\Gamma)$ with

$$L_1, L_2, L_3 \in \mathscr{L}_{\text{int}}(\gamma)\,.$$

Let further $C_i(\Gamma, \nu)$ be the set of all $\gamma \in C(\Gamma)$ with

$$L_i \notin \mathscr{L}_{\text{int}}(\gamma)\,.$$

Note that for given Γ and ν at most one of the sets $C_i(\Gamma, \nu)$ is not empty. We choose the r-momenta of L_1, L_2, L_3 as

$$r(L_i) = r^\gamma(L_i) = \frac{p}{3}\,, \quad \gamma \in C_0 \tag{5.11}$$

$$r^\gamma(L_i) = \frac{p^\gamma(W)}{2} \quad \text{for} \quad \gamma \in C_j\,, \quad j = 1, 2, 3\,. \tag{5.12}$$

Further we select a basis $k_1, \ldots, k_m$ of the momenta $k_{ab\sigma}$ ($L_{ab\sigma} \in \mathscr{L}_{\text{int}}(\Gamma)$) such that

(i) k_1, k_2 correspond to L_1, L_2 resp.

(ii) $k_1^\gamma, \ldots, k_{(m)\gamma}^\gamma$ form a basis of the momenta $k^\gamma{}_{ab\sigma}$

$$(L_{ab\sigma} \in \mathscr{L}_{\text{int}}(\gamma)) \quad \text{for any} \quad \gamma \in C_0)\,.$$

(iii) $k_1^\gamma, k_3^\gamma, \ldots k_{m(\gamma)+1}^\gamma$ form a basis of the momenta $k_{ab\sigma}^\gamma$

$$(L_{ab\sigma} \in \mathscr{L}_{\text{int}}(\gamma)) \quad \text{for any} \quad \gamma \in C_2 \quad \text{or} \quad \gamma \in C_3\,.$$

(iv) $k_2^\gamma, k_3^\gamma, \ldots, k_{m(\gamma)+1}^\gamma$ form a basis of the momenta $k_{ab\sigma}^\gamma$ for $\gamma \in C_1$.

With this notation we have

$$p^\gamma(W) = p\,, \quad k_1^\gamma = k_1\,, \quad k_2^\gamma = k_2 \quad \text{for} \quad \gamma \in C_0 \tag{5.13.0}$$

$$p^\gamma(W) = \frac{2p}{3} k_1\,, \quad k_2^\gamma = k_2 + \frac{k_1}{2} \quad \text{for} \quad \gamma \in C_1 \tag{5.13.1}$$

$$p^\gamma(W) = \frac{2p}{3} - k_2\,, \quad k_1^\gamma = k_1 + \frac{k_2}{2} \quad \text{for} \quad \gamma \in C_2 \tag{5.13.2}$$

$$p^\gamma(W) = \frac{2p}{3} + k_1 + k_2\,, \quad k_1^\gamma = \frac{k_1 - k_2}{2} \quad \text{for} \quad \gamma \in C_3\,. \tag{5.13.3}$$

Multiplying both sides of equ. (5.5) by $e^{-i\xi(k_1 - k_2)}$ and integrating over $k_1, \ldots, k_m$ we obtain

$$J_S + \sum_{i=0}^{3} \sum_{\gamma \in C_i} J_{\Gamma/\gamma} X_{i\gamma\nu} = -\frac{6i\lambda}{(2\pi)^4} W_{\Gamma\nu} + Y_{\Gamma\nu} + \sum_{i=1}^{2} \sum_{\gamma \in C_i} Y_{i\gamma\nu}\,. \tag{5.14}$$

Here

$$\begin{aligned}
X_{0\gamma\nu} &= \int dk_1^\gamma \ldots dk_{m(\gamma)}^\gamma\, e^{-i\xi(k_1^\gamma - k_2^\gamma)}\, F_\gamma^0 R_{\hat\gamma}^0 \\
X_{1\gamma\nu} &= \int dk_2^\gamma \ldots dk_{m(\gamma)+1}^\gamma\, e^{i\xi k_2^\gamma}\, F_\gamma^0 R_{\hat\gamma}^0 \\
X_{2\gamma\nu} &= \int dk_1^\gamma\, dk_3^\gamma \ldots dk_{m(\gamma)+1}^\gamma\, e^{-i\xi k_2^\gamma} F_\gamma^0 R_{\hat\gamma}^0 \\
X_{3\gamma\nu} &= \int dk_1^\gamma\, dk_3^\gamma \ldots dk_{m(\gamma)+1}^\gamma\, e^{-2i\xi k_1^\gamma} F_\gamma^0 R_{\hat\gamma}^0 \\
W_{\Gamma\nu} &= \int dk_1 \ldots dk_m\, e^{-i\xi(k_1 - k_2)}\, F_\Gamma R_{\hat\Gamma} \\
Y_{\Gamma\nu} &= \int dk_1 \ldots dk_m (1 - e^{-i\xi(k_1 - k_2)})\, R_\Gamma \\
Y_{1\gamma\nu} &= \int dk_1\, dk_{m(\gamma)+2} \ldots dk_m (1 - e^{-\frac{3i}{2}\xi k_1})\, R_{\Gamma/\gamma} \times \\
&\quad \times \int dk_2^\gamma \ldots dk_{m(\gamma)+1}^\gamma\, e^{i\xi k_2^\gamma}\, F_\gamma^0 R_{\hat\gamma}^0 \\
Y_{2\gamma\nu} &= \int dk_2\, dk_{m(\gamma)+2} \ldots dk_m (1 - e^{\frac{3i}{2}\xi k_1})\, R_{\Gamma/\gamma} \times \\
&\quad \times \int dk_1^\gamma\, dk_3^\gamma \ldots dk_{m(\gamma)+1}^\gamma\, e^{-i\xi k_1^\gamma} F_\gamma^0 R_{\hat\gamma}^0\,.
\end{aligned} \tag{5.15}$$

Next we derive similar relations for degenerate diagrams $\Gamma \in \mathscr{K}_r$ with $W = W_a$. Starting point is again the algebraic identity (5.5). If 2 and 3 are assigned to the lines of $\mathscr{L}(\Gamma, W)$ we choose the momenta corresponding to L_2, L_3 as

$$r(L_i) = r^\gamma(L_i) = \frac{p + p_a}{2} \qquad i = 2, 3,\ \gamma \in C\,. \tag{5.16}$$

We select a basis $k_2, \ldots, k_{m+1}$ of the momenta $k_{ab\sigma}(L_{ab\sigma} \in \mathscr{L}_{\text{int}}(\Gamma))$ such that

(i) k_2 corresponds to L_2

(ii) $k_2^\gamma, \ldots, k_{m(\gamma)+1}^\gamma$ form a basis of the momenta $k_{ab\sigma}^\gamma$

$$(L_{ab\sigma} \in \mathscr{L}_{\text{int}}(\gamma)) \quad \text{for any} \quad \gamma \in C(\Gamma)\,.$$

Equ. (5.5) is then multiplied by

$$e^{+i\xi\left(\frac{3p_a}{2} + \frac{p}{2} + k_2\right)}$$

and integrated over k_j with the result

$$J_\Gamma + \sum_{\gamma \in C} J_{\Gamma/\gamma} X_{/\gamma\nu} = -\frac{6i\lambda}{(2\pi)^4} W_\Gamma + Y_{0\Gamma\nu} + \sum_{\gamma \in C} Y_{1\gamma\nu}. \tag{5.17}$$

Here

$$\begin{aligned}
X_{1\gamma\nu} &= \int dk_2^\gamma \ldots dk_{m(\gamma)+1}^\gamma \, e^{i\xi k_2^\gamma} F_\gamma^0 R_{\hat\gamma}^0 \\
W_{\Gamma\nu} &= \int dk_2 \ldots dk_{m+1} \, e^{i\xi\left(\frac{3p_a}{2} + \frac{p}{2} + k_2\right)} F_\Gamma R_{\hat\Gamma} \\
Y_{0\Gamma\nu} &= \int dk_2 \ldots dk_{m+1} \left(1 - e^{i\xi\left(\frac{3p_a}{2} + \frac{p}{2} + k_2\right)}\right) R_\Gamma \\
Y_{1\gamma\nu} &= \left(1 - e^{i\xi\left(\frac{3p_a}{2} + \frac{p}{2}\right)}\right) J_{\Gamma/\gamma} \times \\
&\quad \times \int dk_2^\gamma \ldots dk_{m(\gamma)+1}^\gamma \, e^{i\xi k_2^\gamma} F_\gamma^0 R_{\hat\gamma}^0 .
\end{aligned} \tag{5.18}$$

If ν assigns the numbers 1, 3 to $\mathscr{L}(\Gamma, W)$ the corresponding relation is obtained from (5.17) by interchanging the indices 1 and 2 and the substitution $\xi \to -\xi$.

If ν assigns the numbers 1, 2 to $\mathscr{L}(\Gamma, W)$ we set

$$r(L_i) = r^\gamma(L_i) = \frac{p + p_a}{2} \qquad i = 1, 2 . \tag{5.19}$$

We select a basis $k_1, k_3, k_4, \ldots, k_{m+1}$ of the momenta $k_{ab\sigma}$ ($L_{ab\sigma} \in \mathscr{L}_{\text{int}}(\Gamma)$) such that

(i) k_1 corresponds to L_1

(ii) $k_1^\gamma, k_3^\gamma, \ldots, k_{m(\gamma)+1}^\gamma$ form a basis of the momenta $k_{ab\sigma}^\gamma$

$$(L_{ab\sigma} \in \mathscr{L}_{\text{int}}(\gamma)) \quad \text{for any} \quad \gamma \in C(\Gamma) .$$

Multiplying (5.5) by $e^{-2ik_1\xi}$ and integrating over k_j we get

$$\begin{aligned}
J_\Gamma + \sum_{\gamma \in C} J_{\Gamma/\gamma} X_{3\gamma\nu} &= -\frac{6i\lambda}{(2\pi)^4} W_{\Gamma\nu} + Y_{0\Gamma\nu} \\
X_{3\gamma\nu} &= \int dk_1^\gamma \ldots dk_\gamma^m(\gamma) \, e^{-2ik_1^\gamma \xi} F_\gamma^0 R_{\hat\gamma}^0
\end{aligned} \tag{5.20}$$

$$\begin{aligned}
W_{\Gamma\nu} &= \int dk_1 \, dk_3 \ldots dk_{m+1} \, e^{-2ik_1\xi} F_\Gamma R_{\hat\Gamma} \\
Y_{0\Gamma\nu} &= \int dk_1 \, dk_3 \ldots dk_{m+1} (1 - e^{-2ik_1\xi}) R_\Gamma .
\end{aligned} \tag{5.21}$$

Summing now both sides of the relations (5.7), (5.14), (5.17) and (5.20) over all possible $\Gamma \in \mathscr{K}_r$ and assignments ν we obtain

$$\begin{aligned}
&\gamma(\xi\varepsilon) A(pp_1 \ldots p_\nu \varepsilon) \\
&\qquad = -i\lambda Q(\xi pp_1 \ldots p_\nu \varepsilon) + \omega(\xi p_1 \ldots p_\nu \varepsilon) \\
&\qquad \varepsilon > 0
\end{aligned} \tag{5.22}$$

where γ and Q are given by the following power series in λ

$$\gamma(\xi\varepsilon) = \frac{(2\pi)^4}{\lambda} Q(\xi, 0, 0, 0, 0, \varepsilon) \tag{5.23}$$

$$Q(\xi pp_1 \ldots p_\nu \varepsilon) = \frac{6}{(2\pi)^4} + \sum_{a=1}^{r} \sum_{i=1}^{3} Q_{ai} + Q_0 \tag{5.24}$$

$$Q_0 = \frac{1}{(2\pi)^4} \sum_{\Delta \in \mathscr{A}_r} \frac{1}{\mathscr{S}(\Delta)} \int dl_1 \ldots dl_3\, dk\, \delta\left(\sum_{j=1}^{3} l_j - p\right) e^{-i\xi(l_1 - l_2)} \prod_{j=1}^{3} \hat{\Delta}_F(l_j \varepsilon)\, R_\Delta(l_1 \ldots l_3\, p_1 \ldots p_r\, k\, \varepsilon)$$

$$Q_{a1} = \frac{3}{(2\pi)^4} \sum_{\Delta \in \mathscr{B}_r} \frac{1}{\mathscr{S}(\Delta)} \int dl_2\, dl_3\, dk\, \delta(l_2 + l_3 - p_a - p)\, e^{i\xi(p_a + l_2)} \prod_{j=1}^{3} \hat{\Delta}_F(l_j \varepsilon)\, R_\Delta(l_2 l_3 p_1 \ldots p_{a-1}\, p_{a+1} \ldots p_r k \varepsilon) \tag{5.25}$$

$$Q_{a2} = \frac{3}{(2\pi)^4} \sum_{\Delta \in \mathscr{B}_r} \frac{1}{\mathscr{S}(\Delta)} \int dl_1\, dl_3\, dk\, \delta(l_1 + l_3 - p_a - p)\, e^{i\xi(p_a + l_1)} \hat{\Delta}_F(l_1 \varepsilon)\, \hat{\Delta}_F(l_3 \varepsilon)\, R_\Delta(l_1 l_3 p_1 \ldots p_{a-1}\, p_{a+1} \ldots p_r k \varepsilon)$$

$$Q_{a3} = \frac{3}{(2\pi)^4} \sum_{\Delta \in \mathscr{B}_r} \frac{1}{\mathscr{S}(\Delta)} \int dl_1\, dl_2\, dk\, \delta(l_1 + l_2 - p_a - p)\, e^{-i\xi(l_1 - l_2)} \prod_{j=1}^{2} \hat{\Delta}_F(l_j \varepsilon)\, R_\Delta(l_1 l_2 p_1 \ldots p_{a-1} p_{a+1} \ldots p_r k \varepsilon)$$

$$dk = dk_1 \ldots dk_{m(\Delta)}, \quad k = (k_1, \ldots, k_{m(\Delta)}) .$$

$\mathscr{A}_r$ is the class of all diagrams Δ with the following properties

(i) Δ has $N+3$ external lines. $E_1, \ldots, E_3$ are called upper lines (with momenta $l_1, \ldots, l_3$), $E_4, \ldots, E_{N+3}$ are called lower lines (with momenta $p_1, \ldots, p_N$).

(ii) Δ contains no self-energy insertions into lower lines.

(iii) To each connected component of Δ at least one upper and at least one lower line are attached.

(iv) There is no one-particle cut of Δ which separates the upper lines from the lower ones.

$\mathscr{B}_r$ is the class of all diagrams Δ with the properties

(i) Δ has $r+1$ external lines. E_1, E_2 are called upper lines, $E_3, \ldots, E_{N+3}$ are called lower lines.

(ii) and (iii) as above.

In the limit $\varepsilon \to +0$ (5.24) becomes the expansion of $Q(\xi p p_1 \ldots p_r)$ as defined by (2.15). Hence taking the limit $\varepsilon \to +0$ we obtain the relation (5.1) which was stated in the beginning of this section.

ω is given by the expansion

$$\omega(\xi p p_1 \ldots p_r \varepsilon) = \sum_{\Gamma \in \mathscr{K}_r} \frac{1}{\mathscr{S}(\Gamma)} Y_\Gamma + \frac{1}{2} \sum_{\Gamma \in \mathscr{K}_r} \frac{1}{\mathscr{S}(\Gamma)} \sum_{\nu} \sum_{\gamma \in C_1(\Gamma)} Y_{1\gamma\nu} + \frac{1}{2} \sum_{\Gamma \in \mathscr{K}_\nu} \frac{1}{\mathscr{S}(\Gamma)} \sum_{\nu} \sum_{\gamma \in C_2(\Gamma)} Y_{2\gamma\nu} \tag{5.26}$$

with Y_Γ, $Y_{1\gamma\nu}$, $Y_{2\gamma\nu}$ defined by equ. (5.15).

Each term in the expansion of $\alpha(\xi)$ is a continuous function of ξ for $\xi^2 \neq 0$. The expansion terms of $Q(\xi p p_1 \ldots p_r)$ and $\omega(\xi p p_1 \ldots p_r)$ are distributions of $\mathscr{S}'_{4r+4}$ in the variables $p, p_1, \ldots, p_r$ and depend continuously on ξ for $\xi^2 \neq 0$. Moreover, each term of the expansion of ω

vanishes in the limit $\xi \to 0$

$$\lim_{\xi \to 0} \lim_{\varepsilon \to +0} \omega(\xi p p_1 \ldots p_r \varepsilon) = 0 . \tag{5.27}$$

b) Relation for Proper Self-Energy Parts

In this section we will check the relation

$$\begin{aligned}\gamma(\xi)\, \Pi^*(p^2) = -i\lambda Q(\xi, p^2) + i\lambda(\alpha(\xi) + o(\xi, p)) + \\ + i(p^2 - m^2)\, \beta(\xi)\end{aligned} \tag{5.28}$$

where

$$\lim_{\varepsilon \to +0} o(\xi, p) = 0 . \tag{5.29}$$

The function Π^* is as usual defined by

$$\hat{\Delta}'_F = \hat{\Delta}_F + \hat{\Delta}_F \Pi^* \hat{\Delta}'_F \tag{5.30}$$

and has the expansion

$$\Pi^*(p^2) = \sum_{S \in \mathscr{K}} \frac{1}{\mathscr{S}(S)} J_S(p^2) - A(\lambda) - B(\lambda)\,(p^2 - m^2) . \tag{5.31}$$

The sum extends over the class $\mathscr{K}$ of all proper self-energy diagrams excluding the trivial diagram Fig. 1 with one vertex only. We have

$$\Pi^*(m^2) = \Pi^{*\prime}(m^2) = 0 \tag{5.32}$$

which implies the renormalization condition

$$(p^2 - m^2)\hat{\Delta}'_F = i \quad \text{at} \quad p^2 = m^2 .$$

$J_S(p^2)$ is given by

$$J_S(p^2) = \lim_{\varepsilon \to +0} J_S(p, -p, \varepsilon) \tag{5.33}$$

$$J_S(p, -p, \varepsilon) = \int dk\, R_S(p, -p, k, \varepsilon)$$

with R_S as defined in section 4. We rewrite (5.33) in terms of the even part of R_S with respect to p. Let $f(p)$ be a function of the four vector p, then we define $f^+(p^2)$ by

$$f^+(p^2) = \frac{1}{2}(f(p) + f(-p))$$

with

$$\begin{aligned} p &= (\sqrt{p^2}, 0, 0, 0) \quad &\text{for} \quad p^2 \geqq 0 \\ p &= (0, \sqrt{-p^2}, 0, 0) \quad &\text{for} \quad p^2 \leqq 0 . \end{aligned}$$

Since $J_S(p^2)$ depends on p^2 only we obtain

$$J_S(p^2) = \lim_{\varepsilon \to +0} J_S^+(p^2, \varepsilon) \tag{5.34}$$

where

$$J_S^+(p^2, \varepsilon) = \int dk\, R_S^+(p^2 k \varepsilon) \tag{5.35}$$

$$R_S^+(p^2, k, \varepsilon) = (1 - t^1_{p^2})\, \overline{R}_S^+(p^2 k \varepsilon) .$$

Using (4.32) the expansion of $\Pi^*(p^2)$ may be written as

$$\Pi^*(p^2) = \lim_{\varepsilon \to +0} \Pi^*(p^2 \varepsilon) \tag{5.36}$$

where

$$\Pi^*(p^2 \varepsilon) = \sum_{S \in \mathscr{K}} \frac{1}{\mathscr{S}(S)} G_S(p^2 \varepsilon) \tag{5.37}$$

$$G_S(p^2 \varepsilon) = (1 - t^1_{p^2 - m^2})\, J_S^+(p^2 \varepsilon) .$$

A proper self-energy part S is called degenerate if both external lines are attached to the same vertex (Fig. 3). Otherwise S is called non-degenerate. S is called trivial if it has one vertex only (Fig. 1). If S is

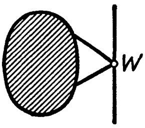

Fig. 3. Degenerate self-energy diagram

degenerate and non-trivial the corresponding integral J_S vanishes. Therefore, we may restrict ourselves to non-degenerate proper self-energy parts. Let W be one of the vertices of S. The hypothesis of the theorem (section 4) applies to W for any $S \in \mathscr{K}$ since W does not belong to any self-energy part other than S itself. Identity (4.33—35) states in this case

$$\overline{R}_S = -\frac{6i\lambda}{(2\pi)^4} F_S R_{\hat{S}} - \sum_{\gamma \in T(S, W)} R_{S/\gamma} F_\gamma^0 R_{\hat{\gamma}}^0 \tag{5.38}$$

which implies for R_S^+ the relation

$$R_S^+ + \sum_{\gamma \in T(S, W)} R_{S/\gamma}^+ F_\gamma^0 R_{\hat{\gamma}}^0 = -\frac{6i\lambda}{(2\pi)^4} (1 - t_{p^2}^1)\, E_S^+ \tag{5.39}$$

$$E_S(pk\varepsilon) = F_S(pk\varepsilon)\, R_{\hat{S}}(pk\varepsilon)\,.$$

We further introduce a numbering ν of the elements of $\mathscr{L}(S, W)$, the internal lines with endpoint W. All numberings ν are considered which assign the numbers 1, 2 and 3 to the lines of $\mathscr{L}(S, W)$. A relation for G_S will be derived for every $S \in \mathscr{K}$ and any ν. Finally summing over S and ν we will obtain the desired relation (5.28).

We define the following subsets of $T(S, W)$. $C'(S, W)$ denotes the set of all $\gamma \in C(S, W)$ for which S/γ is non-degenerate. The elements of $C'(S, W)$ are totally ordered by inclusion. $C''(S, W)$ is the set of all $\gamma \in C(S, W)$ for which S/γ is degenerate. Further we introduce $C_0'(S, W)$ as the set of all $\gamma \in C'(S, W$ with

$$L_1, L_2, L_3 \in \mathscr{L}_{\text{int}}(\gamma)\,.$$

Let $C_i'(S, W, \nu)$ be the set of all $\gamma \in C(S)$ with

$$L_i \notin \mathscr{L}_{\text{int}}(\gamma)\,.$$

Similarly let $C_0''(S)$ denote the set of all $\gamma \in C''(S)$ with

$$L_1, L_2, L_3 \in \mathscr{L}_{\text{int}}(\gamma)$$

and $C_i''(S, W, \nu)$ be the set of all $\gamma \in C''(S)$ with

$$L_i \notin \mathscr{L}_{\text{int}}(\gamma)\,.$$

It should be noted that for a given diagran S at most one of the sets $C_i'(S)$ $(i = 1, 2, 3)$ is not empty.

Apparently, the total set T is the union of the subsets C_i', C_i''. Accordingly identity (5.39) takes the form

$$R_S^+ + \sum_{i=0}^{3} \sum_{\gamma \in C_i'} R_{S/\gamma}^+ F_\gamma^0 R_\gamma^0 + \sum_{i=0}^{3} \sum_{\gamma \in C_i''} R_{S/\gamma}^+ F_\gamma^0 R_\gamma^0 = -\frac{6 i \lambda}{(2\pi)^4} (1 - t_{p^2}^1) E_S^+ . \quad (5.40)$$

We choose the q-momenta of L_1, L_2, L_3 such that relation (5.11) holds for $\gamma \in C_0'$, C_0'' and (5.12) holds for all other $\gamma \in C$. Further we select a basis $B = (k_1, \ldots, k_m)$ of the momenta $k_{ab\sigma}$ $(L_{ab\sigma} \in \mathscr{L}_{\text{int}}(S))$ such that

(i) k_1, k_2 correspond to the lines L_1, L_2 resp.

(ii) $k_1^\gamma, \ldots, k_{m(\gamma)}^\gamma$ form a basis of the momenta $k_{ab\sigma}^\gamma$

$$(L_{ab\sigma} \in \mathscr{L}_{\text{int}}(\gamma)) \quad \text{for any } \gamma \in C_0')$$

(iii) $k_1^\gamma, k_3^\gamma, \ldots, k_{m(\gamma)+1}^\gamma$ form a basis of the momenta $k_{ab\sigma}^\gamma$

$$(L_{ab\sigma} \in \mathscr{L}_{\text{int}}(\gamma)) \quad \text{for any} \quad \gamma \in C_2' \quad \text{or} \quad \gamma \in C_3'$$

(iv) $k_2^\gamma, k_3^\gamma, \ldots, k_{m(\gamma)+1}^\gamma$ form a basis of the momenta $k_{ab\sigma}^\gamma$

$$(L_{ab\sigma} \in \mathscr{L}_{\text{int}}(\gamma)) \quad \text{for any} \quad \gamma \in C_1'(S) .$$

(v) Let $\gamma \in C_i''(S)$ with $i = 0, 1, 2$. B_γ denotes the set of all $k_{ab\sigma}^\gamma \in B$ with $L_{ab\sigma} \in \mathscr{L}_{\text{int}}(\gamma)$. Then B_γ form a basis of the momenta

$$k_{ab\sigma}^\gamma (L_{ab\sigma} \in \mathscr{L}_{\text{int}}(\gamma)) .$$

(vi) Let $\gamma \in C_3''(S)$. B_γ denotes the set of all $k_{ab\sigma}^\gamma \in B$ with $L_{ab\sigma} \in \mathscr{L}_{\text{int}}(\gamma)$. but excluding k_2^γ. Then B_γ form a basis of the momenta

$$k_{ab\sigma}^\gamma (L_{ab\sigma} \in \mathscr{L}_{\text{int}}(\gamma)) .$$

With this notation we have the relation (5.13.i) for $\gamma \in C_i'(S)$ or $C_i''(S)$.

Multiplying the identity (5.40) by $\exp(-i\xi(k_1 - k_2))$, integrating over the internal momenta and applying the operator $(1 - t_{p^2-m^2}^1)$ we obtain

$$\begin{aligned} G_S(p^2 \varepsilon) &+ \sum_{i=0}^{3} \sum_{\gamma \in C_i'} G_{S/\gamma}(p^2 \varepsilon) X_{i\gamma}'(\xi \varepsilon) \\ &= -\frac{6 i \lambda}{(2\pi)^4} (1 - t_{p^2-m^2}^1) \mathscr{E}_S^+ (p^2 \xi \varepsilon) + U_S(p^2 \xi \varepsilon) + \qquad (5.41) \\ &\quad + \sum_{i=1}^{2} \sum_{\gamma \in C_i'} U_{i\gamma}'(p^2 \xi \varepsilon) + \sum_{i=1}^{2} \sum_{\gamma \in C_i''} U_{i\gamma}''(p^2 \xi \varepsilon) . \end{aligned}$$

Here

$$
\begin{aligned}
X'_{0\gamma} &= \int dk_1^\gamma \dots dk_{m(\gamma)}^\gamma \, e^{-i\xi(k_1^\gamma - k_2^\gamma)} \, F_\gamma^0 R_{\hat{\gamma}}^0 \\
X'_{1\gamma} &= \int dk_2^\gamma \dots dk_{m(\gamma)+1}^\gamma \, e^{i\xi k_2^\gamma} F_\gamma^0 R_{\hat{\gamma}}^0 \\
X'_{2\gamma} &= \int dk_1^\gamma \, dk_3^\gamma \dots dk_{m(\gamma)+1}^\gamma \, e^{-i\xi k_1^\gamma} F_\gamma^0 \, R_{\hat{\gamma}}^0 \\
X'_{3\gamma} &= \int dk_1^\gamma \, dk_3^\gamma \dots dk_{m(\gamma)+1}^\gamma \, e^{-2i\xi k_1^\gamma} F_\gamma^0 \, R_{\hat{\gamma}}^0 \\
\mathscr{E}_S^+ &= \int dk_1 \dots dk_m \, e^{-i\xi(k_1-k_2)} \, E_S^+ \\
Y_S &= \int dk_1 \dots dk_m (1 - e^{-i\xi(k_1-k_2)}) \, R_S^+ \\
Y'_{1\gamma} &= \int dk_1 \, dk_{m(\gamma)+2} \dots dk_m \big(1 - e^{-\frac{3i}{2}\xi k_1}\big) \, R_{S/\gamma}^+ \times \\
&\quad \times \int dk_2^\gamma \dots dk_{m(\gamma)+1}^\gamma \, e^{i\xi k_2^\gamma} F_\gamma^0 \, R_{\hat{\gamma}}^0 \\
Y'_{2\gamma} &= \int dk_2 \, dk_{m(\gamma)+2} \dots dk_m \big(1 - e^{\frac{3i}{2}\xi k_2}\big) \, R_{S/\gamma}^+ \times \\
&\quad \times \int dk_1^\gamma \, dk_3^\gamma \dots dk_{m(\gamma)+1}^\gamma \, e^{-i\xi k_1^\gamma} F_\gamma^0 R_{\hat{\gamma}}^0 \\
Y''_{1\gamma} &= \int \prod_{B-B_\gamma} dk_{ab\sigma} \big(1 - e^{-\frac{3i}{2}\xi k_1}\big) \, R_{S/\gamma}^+ \times \\
&\quad \times \int \prod_{B_\gamma} dk_{ab\sigma}^\gamma \, e^{i\xi k_2^\gamma} F_\gamma^0 \, R_{\hat{\gamma}}^0 \\
Y''_{2\gamma} &= \int \prod_{B-B_\gamma} dk_{ab\sigma} \big(1 - e^{\frac{3i}{2}\xi k_2}\big) \, R_{S/\gamma}^+ \times \\
&\quad \times \int \prod_{B_\gamma} dk_{ab\sigma}^\gamma \, e^{-i\xi k_1^\gamma} F_\gamma^0 R_{\hat{\gamma}}^0 \, .
\end{aligned}
\tag{5.42}
$$

The quantities U are obtained from the corresponding Y by

$$U = (1 - t^1_{p^2-m^2}) \, Y \, .$$

In deriving (5.43) it was used that $J_{S/\gamma} = 0$ for $\gamma \in C_i''(S)$. Finally we sum over all proper self-energy parts and all possible ν with the result

$$
\begin{aligned}
\gamma(\xi, \varepsilon) \, \Pi^*(p^2, \varepsilon) &= -i\lambda Q(\xi, p, \varepsilon) + i\lambda(\alpha(\xi, \varepsilon) + o(\xi, p, \varepsilon)) + \\
&\quad + i(p^2 - m^2) \, \beta(\xi, \varepsilon) \, .
\end{aligned}
\tag{5.43}
$$

α, β and Q are given by the power series (5.23—25) with

$$
\begin{aligned}
Q(\xi p^2 \varepsilon) &= Q(\xi, p, -p, \varepsilon) \\
& \qquad p = (\sqrt{p^2}, 0, 0, 0) \quad \text{for} \quad p^2 \geqq 0 \\
& \qquad p = (0, \sqrt{-p^2}, 0, 0) \text{ for } \quad p^2 \leqq 0
\end{aligned}
\tag{5.44}
$$

and

$$
\begin{aligned}
\alpha(\xi \varepsilon) &= Q(\xi m^2 \varepsilon) \\
\beta(\xi \varepsilon) &= \lambda \frac{\partial Q(\xi p^2 \varepsilon)}{\partial p^2} \quad \text{at} \quad p^2 = m^2 \, .
\end{aligned}
\tag{5.45}
$$

For (5.44—45) we used symmetry in k_1, k_2 and the fact that E_S is invariant under the substitution

$$p \to -p, \quad k_j \to -k_j .$$

Taking the limit $\varepsilon \to +0$ we obtain the relation (5.28). For $o(\xi, p, \varepsilon)$ we have the expansion

$$\begin{aligned} o(\xi, p, \varepsilon) = \sum_{S \in \mathscr{K}} \frac{1}{\mathscr{S}(S)} U_S + \frac{1}{2} \sum_{S \in \mathscr{K}_2} \frac{1}{\mathscr{S}(S)} \sum_{i=1}^{2} \sum_{\nu} U'_{i\gamma\nu} + \\ + \frac{1}{2} \sum_{S \in \mathscr{K}} \frac{1}{\mathscr{S}(S)} \sum_{i=1}^{2} \sum_{\gamma \in C'_i(S)} \sum_{\nu} U'_{i\gamma\nu} . \end{aligned} \tag{5.46}$$

Every term of this expansion vanishes in the limit $\xi \to 0$.

c) Field Equations for τ-Functions and Field Operators

We begin deriving the field equation for the propagator. From (5.28) we obtain

$$\begin{aligned} \gamma(\xi)\, \Pi^*(p^2)\, \hat{\Delta}'_F(p) = -i\lambda G(\xi, p) + i\lambda(\alpha(\xi) + o(\xi p))\, \hat{\Delta}'_F(p) + \\ + i(p^2 - m^2)\, \beta(\xi)\, \hat{\Delta}'_F(p) . \end{aligned} \tag{5.47}$$

This yields

$$\begin{aligned} (\beta(\xi) + \gamma(\xi))\, (p^2 - m^2)\, \hat{\Delta}'_F(p) \\ = \lambda Q(\xi, p) - \lambda(\alpha(\xi) + o(\xi, p))\, \hat{\Delta}'_F(p) + i\,\gamma(\xi) . \end{aligned} \tag{5.48}$$

Dividing by $\beta(\xi) + \gamma(\xi)$ and taking the limit $\xi \to 0$ we obtain

$$(p^2 - m^2)\, \hat{\Delta}'_F(p) = \lim_{\xi \to 0} \frac{\lambda(Q(\xi, p) - \alpha(\xi))\, \hat{\Delta}'_F(p) + i\,\gamma(\xi)}{\beta(\xi) + \gamma(\xi)} \tag{5.49}$$

This is equ. (3.25) in case of the propagator.

In order to derive the field equation of an arbitrary τ-function we multiply both sides of (5.28) by $\hat{\Delta}'_F \Lambda$. Using (5.1) and (2.15) we get

$$\begin{aligned} (\beta(\xi) + \gamma(\xi))\, (p^2 - m^2)\, \hat{\Delta}'_F(p)\, \Lambda(p p_1 \ldots p_r) \\ = \lambda \frac{G(\xi p p_1 \ldots p_r)}{\hat{\Delta}'_F(p_1) \ldots \hat{\Delta}'_F(p_r)} - \lambda(\alpha(\xi) + o(\xi p))\, \hat{\Delta}'_F(p)\, \Lambda(p p_1 \ldots p_r) \\ + i\,\omega(\xi p p_1 \ldots p_r) . \end{aligned} \tag{5.50}$$

For $\tau(p p_1 \ldots p_r)$ this implies

$$\begin{aligned} (\beta(\xi) + \gamma(\xi))\, (p^2 - m^2)\, \tau(p p_1 \ldots p_r) \\ = \lambda H(\xi p p_1 \ldots p_r) - \lambda(\alpha(\xi) + o(\xi p))\, \tilde{\tau}(p p_1 \ldots p_r) + \\ + i\,\gamma(\xi) \sum_{j=1}^{r} \delta(p + p_j)\, \tilde{\tau}(p_1 \ldots p_{j+1}\, p_{j+1} \ldots p_r) + \psi(\xi p p_1 \ldots p_r) \end{aligned} \tag{5.51}$$

where

$$\lim_{\xi \to 0} \psi(\xi p p_1 \ldots p_r) = 0 .$$

Dividing by $\beta(\xi) + \gamma(\xi)$ and taking the limit $\xi \to 0$ we obtain (3.25). Hence we have proved that the system (3.25) is satisfied by the renormalized expansions of time ordered functions.

From (5.51) it is not difficult to derive the field equation (3.24) for the matrix elements

$$(\Phi^G_{\mathrm{out}\, b}, T(x x_1 \ldots x_r) \Phi^F_{\mathrm{in}\, a}) \tag{5.52}$$

between state vectors $\Phi^F_{\mathrm{in}\, a} \in D_{\mathrm{in}}$, $\Phi^G_{\mathrm{out}\, b} \in D_{\mathrm{out}}$ of the form[12]

$$\begin{aligned} \Phi^F_{\substack{\mathrm{in}\\ \mathrm{out}}\, a} &= \int \frac{d_3 k_1}{2\omega_1} \cdots \frac{d_3 k_a}{2\omega_a} F(\mathbf{k}_1 \ldots \mathbf{k}_a)\, \Phi^{k_1 \ldots k_a}_{\substack{\mathrm{in}\\ \mathrm{out}}} \\ \Phi^{k_1 \ldots k_a}_{\substack{\mathrm{in}\\ \mathrm{out}}} &= \tilde{A}^*_{\substack{\mathrm{in}\\ \mathrm{out}}}(k_1) \ldots \tilde{A}^*_{\substack{\mathrm{in}\\ \mathrm{out}}}(k_a)\, \Omega \\ k_i^2 &= m^2, \quad k_i^0 = \omega_i = \sqrt{\mathbf{k}_i^2 + m^2}\,. \end{aligned} \tag{5.53}$$

The matrix elements (5.52) can be expressed in terms of τ-functions by using the reduction formula

$$\begin{aligned} &\langle [\ldots [S\, T(x x_1 \ldots x_r), A^*_{\mathrm{in}}(k_1)] \ldots A^*_{\mathrm{in}}(k_n)] \rangle_0 \\ &= i^n \varepsilon(k_1) \ldots \varepsilon(k_n) (2\pi)^{n/2} \int e^{-i(px + \Sigma p_j x_j)} \tilde{\tau}(p p_1 \ldots p_r : k_1 \ldots k_n :)\,. \end{aligned} \tag{5.54}$$

On the other hand (5.51) yields[13]

$$\begin{aligned} &(\beta(\xi) + \gamma(\xi))\, (p^2 - m^2)\, \tilde{\tau}(p p_1 \ldots p_r : k_1 \ldots k_n :) \\ &\quad = \lambda H(\xi p p_1 \ldots p_r : k_1 \ldots k_n :) - \\ &\quad - \lambda(\alpha(\xi) + o(\xi p))\, \tilde{\tau}(p p_1 \ldots p_r : k_1 \ldots k_n :) + \\ &\quad + i\gamma(\xi) \sum_{j=1}^{r} \delta(p + p_j)\, \tilde{\tau}(p_1 \ldots p_{j-1}\, p_{j+1} \ldots p_r : k_1 \ldots k_n :) + \\ &\quad + \psi(\xi p p_1 \ldots p_r : k_1 \ldots k_n :) \end{aligned} \tag{5.55}$$

where

$$\begin{aligned} &H(\xi p p_1 \ldots p_r : k_1 \ldots k_n :) \\ &\quad = \frac{1}{(2\pi)^4} \int d l_1 \ldots d l_3\, \delta(\textstyle\sum l_j - p)\, e^{-i(l_1 - l_2)\xi} \times \\ &\qquad \times \tilde{\tau}(i l_1 \ldots l_3 : p p_1 \ldots p_r\, k_1 \ldots k_n :)\,. \end{aligned} \tag{5.56}$$

Putting the momenta k_j on the mass shell (with appropriate signs of k_j^0) and taking the Fourier transform with respect to x_j we obtain

$$-(\Box_x + m^2)\, v(x x_1 \ldots x_r) = \lambda \lim_{\xi \to 0} \frac{w(\xi x x_1 \ldots x_r)}{\beta(\xi) + \gamma(\xi)} \tag{5.57}$$

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[12] D_{in} and D_{out} denote the free field domains for the incoming and outgoing fields as defined in ref. [18]. It is understood that the tested field operators $A(f)$ are defined on a domain D with the properties $D \subseteqq D_{\mathrm{in}}$, $D \subseteqq D_{\mathrm{out}}$ and $A(f)\, D \subseteqq D$.

[13] For $r = 0$ the third term on the rigth hand side is missing, for $r = 1$ it is $i\gamma(\xi)\, \delta(p + p_1)\, \tilde{\tau}(: k_1 \ldots k_n :)$.

with

$$\begin{aligned} v(x x_1 \dots x_r) &= \langle [\dots [S\,T(x x_1 \dots x_r), A^*_{\mathrm{in}}(k_1)] \dots A^*_{\mathrm{in}}(k_n)] \rangle_0 \\ w(\xi x x_1 \dots x_r) &= \langle [\dots [S\,N(\xi x x_1 \dots x_r), A^*_{\mathrm{in}}(k_1)] \dots A^*_{\mathrm{in}}(k_n)] \rangle_0 \end{aligned} \tag{5.58}$$

(for the definition of N see equ. (3.24)). Working out the multiple commutators and folding by test functions F and G one finally obtains (3.24) for matrix elements (5.52). The field equation (1.2) of the field operator $A(x)$ is the special case of (2.24) with $r = 0$,

$$g(\xi) = \beta(\xi) + \gamma(\xi)$$

and spacelike limit $\xi \to 0$.

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