

Asymptotically Abelian Systems

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Abstract. We study pairs $\{\mathfrak{A}, \alpha\}$ for which \mathfrak{A} is a C^* -algebra and α is a homomorphism of a locally compact, non-compact group G into the group of $*$ -automorphisms of \mathfrak{A} . We examine, especially, those systems $\{\mathfrak{A}, \alpha\}$ which are (weakly) asymptotically abelian with respect to their invariant states (i.e. $\langle \Phi | A \alpha_g(B) - \alpha_g(B) A \rangle \rightarrow 0$ as $g \rightarrow \infty$ for those states Φ such that $\Phi(\alpha_g(A)) = \Phi(A)$ for all g in G and A in \mathfrak{A}). For concrete systems (those with \mathfrak{A} acting on a Hilbert space and $g \rightarrow \alpha_g$ implemented by a unitary representation $g \rightarrow U_g$ on this space) we prove, among other results, that the operators commuting with \mathfrak{A} and $\{U_g\}$ form a commuting family when there is a vector cyclic under \mathfrak{A} and invariant under $\{U_g\}$. We characterize the extremal invariant states, in this case, in terms of “weak clustering” properties and also in terms of “factor” and “irreducibility” properties of $\{\mathfrak{A}, U_g\}$. Specializing to amenable groups, we describe “operator means” arising from invariant group means; and we study systems which are “asymptotically abelian in mean”. Our interest in these structures resides in their appearance in the “infinite system” approach to quantum statistical mechanics.

Introduction

In the general frame of quantum mechanics the physical observables are described as self-adjoint operators on a Hilbert space \mathcal{H} and the bounded observables (corresponding to bounded operators) therefore generate a C^* -algebra acting on \mathcal{H} . The algebraic approach to field theory [1, 2] proposes to consider as physical only the *local* observables i.e. those corresponding to measurements performed within finite regions of space during a finite time. These observables are described mathematically as the self-adjoint elements of an incomplete C^* -algebra whose completion \mathfrak{A} , called *the quasi-local algebra*, is considered as the main

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mathematical entity whose abstract algebraic structure contains the whole physical information (the choice of a particular faithful representation being a matter of arbitrariness without physical implications). Field theory is thus viewed mathematically as the investigation of the quasi-local algebra \mathfrak{A} which is supposed to satisfy appropriate axioms of physical origin (locality, Lorentz invariance etc.). Whilst the physical (local) observables are described as elements of \mathfrak{A} , the physical states are represented as the normalized positive linear forms over \mathfrak{A} (*states* over \mathfrak{A}). In this framework the invariance groups of the physical theory (group of space and time translations, Lorentz transformations etc.) are homomorphically mapped into the automorphism group of the C^* -algebra \mathfrak{A} . The automorphism α_g corresponding to the group element g describes physically the "shifting by g " of the local observables.

One of the first problems arising in this approach is the investigation of the translationally invariant states over the algebra \mathfrak{A} , those being natural candidates for the description of the equilibrium states of statistical mechanics (more generally the states invariant under any group of physical transformations are of interest) [3, 4, 4a].

In this connection recent studies have revealed that a number of general results depend upon only a very weak locality assumption about the quasi-local algebra (entailing only a small part of the structure required by physics) namely a property of asymptotic abelianness (cf. Definitions 1 and 2 below) [5, 6, 7, 8, 9, 10]. It seems therefore interesting to investigate mathematically the systems $\{\mathfrak{A}, \alpha\}$ of a C^* -algebra \mathfrak{A} and a homomorphic mapping α of a group G into the automorphism group of \mathfrak{A} which possess such an asymptotic abelian property. This is the purpose of the present paper.

Section I is concerned with general groups G and an abstract notion of mean of elements of \mathfrak{A} . Section II describes the case of an amenable group possessing invariant means and discusses the connection between the above notion and the invariant means over G .

Owing to the delay between the conclusion of this research and the writing of this paper (due to geographical dispersion of the authors) some results overlapping with ours meanwhile appeared in the literature [9, 9a].

I. General Case

Theorem 1. *Let $g \rightarrow U_g$ be a unitary representation of the group G on the Hilbert space \mathcal{H} , \mathfrak{A} a C^* -algebra acting on \mathcal{H} such that $\alpha_g(A) = U_g A U_g^{-1}$ is in \mathfrak{A} for each g in G . We denote furthermore by E_0 the orthogonal projection in \mathcal{H} on the subspace $\{\psi \in \mathcal{H} \mid U_g \psi = \psi \text{ for all } g \in G\}$; by \mathfrak{B} the von Neumann algebra generated by \mathfrak{A} and $\{U_g \mid g \in G\}$; by P the central carrier of*

E_0 in \mathcal{R} ; and we assume that, for all $\varphi \in E_0 \mathcal{H}$ and $A, B \in \mathfrak{Q}$

$$\text{Inf}_{g \in G} |(\varphi| A \cdot \alpha_g(B) - \alpha_g(B) \cdot A |\varphi)| = 0. \quad (1)$$

Then

(i) $E_0 \mathcal{R} E_0$ is abelian¹ so that E_0 is an abelian projection in \mathcal{R} and $\mathcal{R} P$ is of type I with its center isomorphic to $E_0 \mathcal{R} E_0$.

(ii) If $\mathcal{R} E_0 \mathcal{H}$ is dense in \mathcal{H} (or equivalently, $\mathfrak{Q} E_0 \mathcal{H}$ is dense in \mathcal{H} ; or if $P = I$; or if $T \in \mathcal{R}' \rightarrow T E_0 \in \mathcal{R}' E_0$ is an isomorphism between von Neumann algebras) then \mathcal{R} is of type I with its center isomorphic to $E_0 \mathcal{R} E_0$; and there exists a normal positive mapping $A \rightarrow M(A)$ from \mathcal{R} onto its center determined uniquely by

$$M(A) E_0 = E_0 A E_0, \quad A \in \mathcal{R}.$$

(iii) If E_1 is an abelian projection in \mathcal{R} with central carrier I , then E_1 contains a vector cyclic under \mathcal{R} (or \mathfrak{Q} , if $E_0 = E_1$) if and only if \mathcal{R}' is abelian and \mathcal{R}' (or $E_1 \mathcal{R} E_1$) is countably decomposable.

Proof. Since $U_g A = U_g A U_g^{-1} U_g = B U_g$ with $B (= U_g A U_g^{-1})$ in \mathfrak{Q} , the elements of the form $A_1 U_{g_1} + \cdots + A_n U_{g_n}$ comprise a strong-operator dense subalgebra of \mathcal{R} , where A_j is in \mathfrak{Q} and g_j in G .

As $U_g E_0 = E_0 = E_0 U_g$ for g in G , $E_0 (A_1 U_{g_1} + \cdots + A_n U_{g_n}) E_0 = E_0 (A_1 + \cdots + A_n) E_0$ so that it will suffice to prove that $E_0 A E_0 B E_0 = E_0 B E_0 A E_0$ for each A and B in \mathfrak{Q} to establish that E_0 is an abelian projection in \mathcal{R} .

With $\varphi = E_0 \psi$, $\psi \in \mathcal{H}$, we have

$$(\psi| E_0 A E_0 B E_0 - E_0 B E_0 A E_0 |\psi) = (\varphi| A E_0 B - B E_0 A |\varphi).$$

Now E_0 lies in the von Neumann algebra generated by the U_g , g in G . Suppose that it lies in the strong-operator closure of the set of real linear combinations of U_g 's, $g \in G$. For each $\varepsilon > 0$, we can choose elements $g_j \in G$ and real constants λ_j , $j = 1, 2, \dots, n$, such that

$$\|(E_0 - \sum_{j=1}^n \lambda_j U_{g_j}) B \varphi\| = \|(E_0 - \sum_{j=1}^n \lambda_j U_{g_j}) B \varphi\| \leq \varepsilon/2 \|A^* \varphi\|$$

and

$$\|(E_0 - \sum_{j=1}^n \lambda_j U_{g_j}) B^* \varphi\| = \|(E_0 - \sum_{j=1}^n \lambda_j U_{g_j}) B^* \varphi\| \leq \varepsilon/2 \|A \varphi\|$$

where g can still be arbitrarily chosen in G . We then have

$$\begin{aligned} |(\varphi| A E_0 B - B E_0 A |\varphi)| &\leq \\ &\leq \varepsilon + \text{Inf}_{g \in G} \left| \left(\varphi \left| A \left(\sum_{j=1}^n \lambda_j U_{g_j} \right) B - B \left(\sum_{j=1}^n \lambda_j U_{g_j} \right)^* A \right| \varphi \right) \right| \end{aligned}$$

¹ This fact and its connection with reduction theory was first noted in [26] in the framework of Wightman field theory.

where the second term on the right hand side vanishes by assumption being equal to

$$\inf_{g \in G} |(\varphi | (A \cdot \alpha_g(B') - \alpha_g(B')A) | \varphi)| \quad \text{with} \quad B' = \sum_{j=1}^n \lambda_j \alpha_{g_j}(B).$$

Thus E_0 is abelian, $\mathcal{R}P$ is a von Neumann algebra containing an abelian projection with central carrier the unit and is therefore of type I [12, Théorème 1 p. 123]. As $E_0\mathcal{R}E_0$ is abelian it is the center of its commutant $\mathcal{R}'E_0$. Since the latter is isomorphic to $\mathcal{R}'P$ [12, Proposition 2 p. 19], $E_0\mathcal{R}E_0$ is isomorphic with the center of $\mathcal{R}'P$ which is obviously also the center of $\mathcal{R}P$.

It remains to prove that E_0 is approximable in the strong operator topology by sums of the form $\sum_{j=1}^n \lambda_j U_{g_j}$ with the λ_j real. More specifically let \mathcal{T} be the strong operator closure of convex linear combinations of the U_{g_j} , $g_j \in G$; and let E_g , $g \in G$, be the projector with range those vectors ψ such that $U_g\psi (= U_{g^{-1}}\psi) = \psi$. The mean ergodic theorem [11; Cor. 2 and 4, p. 662] yields the fact that E_g is the strong operator limit of $\frac{1}{n} \sum_{j=1}^n U_g^j$ in \mathcal{T} as $n \rightarrow \infty$. If E and F are two projections which are the strong operator limits of operators of \mathcal{T} , then their intersection $E \wedge F$ is such a limit, since $E \wedge F$ is the strong operator limit of $EEEE \dots EF$ (Method of Images) and multiplication of operators is strong operator continuous on bounded sets. Since $E_0 = \bigwedge_{g \in G} E_g$ and $\bigwedge_{g \in G} E_g$ is strong operator limit of finite intersections of the projections E_g , E_0 lies in \mathcal{T} .

Thus (i) is proven and the specialization $P = I$ yields the first assertion of (ii). We then obtain the mapping $A \rightarrow M(A)$ by composition of the mapping $A \in \mathcal{R} \rightarrow E_0 A E_0 \in \mathcal{R}'E_0$ with the isomorphism $T E_0 \in \mathcal{R}'E_0 \leftrightarrow \leftrightarrow T \in \mathcal{R}'$, both of which are positive and normal. As the range of the first mapping is the center of $\mathcal{R}'E_0$ the range of M is the center of \mathcal{R}' (and \mathcal{R}). If M_0 is another mapping from \mathcal{R} to \mathcal{R}' such that $M_0(A) = E_0 A E_0$, $[M_0(A) - M(A)]E_0 = 0$; so that $0 = [M_0(A) - M(A)]P = M_0(A) - M(A)$ since $P = I$ and $M_0(A) - M(A) \in \mathcal{R} \wedge \mathcal{R}'$.

Assuming now that ψ_0 is a unit vector in the range of the abelian projection E_1 such that $\mathcal{R}\psi_0$ is dense in \mathcal{H} , E_1 has central carrier I as does the projection E on the closure of $\mathcal{R}\psi_0$. Now $E \leq E_1$, and abelian projections can be characterized as being minimal in the set of projections with the same central carrier [25; § 31 p. 332]. Thus $E = E_1$ and ψ_0 is cyclic under \mathcal{R}' in $E_1\mathcal{H}$. Hence I , the projection on the closure of $\mathcal{R}\psi_0$ is an abelian projection in \mathcal{R}' [12; Prop. 3 p. 242], so that \mathcal{R}' is abelian and is the center of $\mathcal{R}(= \mathcal{R}'')$. Thus $E_1\mathcal{R}'E_1$ is the center of the abelian algebra $E_1\mathcal{R}E_1$ and $\mathcal{R}'E_1 = E_1\mathcal{R}E_1$. Further $\mathcal{R}'(= \mathcal{R}'P)$ is isomorphic with $E_0\mathcal{R}E_0$ and is the range of the mapping M .

With \mathcal{R}' abelian, we have just noted that $\mathcal{R}' E_1 (= E_1 \mathcal{R}' E_1)$ is maximal abelian on $E_1 \mathcal{H}$. If in addition \mathcal{R}' (or $E_1 \mathcal{R}' E_1$) is countably decomposable, $\mathcal{R}' E_1$ has a vector ψ_1 cyclic for $E_1 \mathcal{H}$ [12; Corollaire p. 20]. Since $\mathcal{R}' \psi_1$ is dense in $E_1 \mathcal{H}$, $\mathcal{R}' \psi_1$ is dense in the range of an abelian projection in \mathcal{R}' with central carrier I . This projection must be I , since \mathcal{R}' is abelian.

Remark. In the foregoing proof, if (1) is valid for a single vector $\varphi \in E_0 \mathcal{H}$ we have established that φ is a trace vector for $E_0 \mathcal{R}' E_0$. If in addition φ is cyclic under \mathfrak{A} , then $E_0 \mathcal{R}' E_0$, $\mathcal{R}' E_0$ and \mathcal{R}' are finite von Neumann algebras.

In place of (1), the assumption that

$$\inf_{g \in G} |(\varphi | A \alpha_g(B) - \alpha_g(B) A | \varphi)| = 0$$

for a single φ in $E_0 \mathcal{H}$ such that the closure of $\mathfrak{A} \varphi$ contains $E_0 \mathcal{H}$, and all $\varphi \in E_0 \mathcal{H}$ would suffice to establish (i), (ii) and (iii).

Theorem 1 is especially interesting in connection with ‘‘asymptotically abelian systems’’ defined as follows

Definition 1. *If $\{\mathfrak{A}, \alpha\}$ is a pair consisting of a C^* -algebra \mathfrak{A} and a homomorphism $g \rightarrow \alpha_g$ of the locally compact non compact group G into the automorphism group of \mathfrak{A} we call $\{\mathfrak{A}, \alpha\}$ an asymptotically abelian system (a weakly asymptotically abelian system) whenever, to each $\varepsilon > 0$ (to each $\varepsilon > 0$ and state Φ over \mathfrak{A}) with $A, B \in \mathfrak{A}$ there is a compact $K \subset G$ such that $g \notin K$ implies $\|A \alpha_g(B) - \alpha_g(B) A\| < \varepsilon$ ($|\langle \Phi, A \alpha_g(B) - \alpha_g(B) A \rangle| < \varepsilon$).*

Corollary 1. *Let $g \rightarrow U_g$ be a unitary representation of the locally compact non compact group G on a Hilbert space \mathcal{H} , \mathfrak{A} a C^* -algebra acting on \mathcal{H} such that $\alpha_g(A) = U_g A U_g^{-1}$ is in \mathfrak{A} for each g in G and let E_0 and \mathcal{R} be as in Theorem 1. The conclusions (i) (ii) and (iii) of Theorem 1 are valid whenever $\{\mathfrak{A}, \alpha\}$ is a weakly asymptotically abelian system or whenever this is the case for $\{\mathfrak{A}, \alpha | G_0\}$, where G_0 is a non compact subgroup of G .*

Theorem 1 is mainly useful for the study of invariant states over C^* -algebras, in the following manner:

Corollary 2. *Let \mathfrak{A} be a C^* -algebra, $g \rightarrow \alpha_g$ a homomorphism of the group G into the automorphism group of \mathfrak{A} , Φ a state over \mathfrak{A} invariant under G i.e. such that $\Phi(\alpha_g(A)) = \Phi(A)$ for all $A \in \mathfrak{A}$ and $g \in G$, π_Φ the $*$ -representation of \mathfrak{A} on a Hilbert space \mathcal{H} with cyclic vector Ω , U_Φ the unitary representation of G on \mathcal{H} determined by*

$$\begin{cases} (\Omega | \pi_\Phi(A) | \Omega) = \langle \Phi, A \rangle & A \in \mathfrak{A} \\ \pi_\Phi(\alpha_g(A)) = U_\Phi(g) \pi_\Phi(A) U_\Phi(g)^{-1}, & \\ U_\Phi(g) \Omega = \Omega & g \in G \end{cases} \quad (2)$$

\mathcal{R} the von Neumann algebra generated by the set $\pi_\Phi(\mathfrak{A}) \cup U_\Phi(G)$ and E_0 the orthogonal projection in \mathcal{H} on the set $\{\psi \in \mathcal{H} | U_\Phi(g) \psi = \psi \text{ for all } g \in G\}$. Suppose that, for all $A, B \in \mathfrak{A}$ and all vector states Φ over \mathfrak{A} from \mathcal{H}

invariant under G

$$\inf_{g \in G} |\langle \Phi, A \cdot \alpha_g(B) - \alpha_g(B) \cdot A \rangle| = 0. \tag{3}$$

Then \mathcal{R}' and $E_0 \mathcal{R} E_0$ are abelian von Neumann algebras and the mapping $T \in \mathcal{R}' \rightarrow T E_0$ is a *-isomorphism from \mathcal{R}' onto $E_0 \mathcal{R} E_0$. Thus, for each $A \in \mathfrak{A}$ there is a unique $M_\Phi(A) \in \mathcal{R}'$ such that

$$M_\Phi(A) E_0 = E_0 \pi_\Phi(A) E_0, \tag{4}$$

the mapping $\pi_\Phi(A) \rightarrow M_\Phi(A) \in \mathcal{R}'$ being linear, positive and weak-operator continuous from $\pi_\Phi(\mathfrak{A})$ into \mathcal{R}' and the set $\{M_\Phi(A) \mid A \in \mathfrak{A}\}$ being weak-operator dense in \mathcal{R}' .

It is interesting that under quite general conditions, in addition to the above structure, one has the feature that $\mathcal{R}' \subseteq \mathfrak{A}''$. This is the case whenever G is amenable and $g \rightarrow \alpha_g$ is strongly continuous as will be shown in Section II below but also in other contexts as shown by

Theorem 2. $\{\mathfrak{A}, \alpha\}$ be as in Corollary 1 above with $\{\mathfrak{A}, \alpha\}$ a weakly asymptotically abelian system and G locally compact, non compact and connected. Then \mathcal{R}' is contained in the weak-operator closure \mathfrak{A}'' of \mathfrak{A} (or equivalently $\mathfrak{A}' \subseteq \mathcal{R}$).

Proof. From [13] some cyclic subgroup G_0 is not relatively compact; and from the definition $\{\mathfrak{A}, U|G_0\}$ is a concrete weakly asymptotically abelian system with Ω invariant under $U|G_0$ and cyclic for \mathfrak{A} . Now G_0 has an invariant mean (see section II below) and \mathcal{R}_0 , the von Neumann algebra generated by \mathfrak{A} and $U|G_0$ is contained in \mathcal{R} . Thus $\mathcal{R}' \subseteq \mathcal{R}_0' \subseteq \mathfrak{A}''$.

We note that if $\mathcal{R}' \subset \mathfrak{A}''$ one then has $\mathcal{R}' = \mathfrak{A}'' \wedge \mathfrak{A}' \wedge U(G)$. \mathcal{R}' then consists of those elements in the center of \mathfrak{A}'' which commute with all $U_g, g \in G$. In particular the central reduction of \mathcal{R} (see below) then yields mutually disjoint representations of \mathfrak{A} .

A further specialization of particular interest in the study of ground states at finite temperature [14] has equivalent formulations stated in the

Theorem 3. Let again $\{\mathfrak{A}, \alpha\}$ be as in Corollary 1 above with $\{\mathfrak{A}, \alpha\}$ a weakly asymptotically abelian system, Ω a vector cyclic for \mathcal{R} invariant under U_g , and \mathcal{R}' contained in \mathfrak{A}'' .

The following are equivalent

- (i) Ω is cyclic for \mathfrak{A}' (or separating for \mathfrak{A}'').
- (ii) $\{\mathfrak{A}' \cup U(G)\}' = \mathcal{R}$.
- (iii) $\mathfrak{A}'' \cap U(G)' \subseteq \mathfrak{A}'' \wedge \mathfrak{A}'$.

Proof. Writing $\mathcal{R}_1 = \{\mathfrak{A}' \cup U(G)\}'$, (ii) reads $\mathcal{R}_1 = \mathcal{R}$ and (iii) $\mathcal{R}_1 \subseteq \mathfrak{A}'' \cap \mathfrak{A}'$. Since $\mathcal{R}' \subset \mathfrak{A}''$ or $\mathfrak{A}' \subset \mathcal{R}$ one has $\mathcal{R}_1 \subset \mathcal{R}$. Therefore (cf. proof of Theorem 1) $E_0 \mathcal{R}_1 E_0$ is abelian and since $E_0 \in U(G)'$, $E_0 \mathcal{R}_1 E_0$ is a von Neumann algebra. From (i) $E_0 \mathcal{R}_1 E_0$ is maximal abelian on $E_0 \mathcal{H}$ so that $E_0 \mathcal{R}_1 E_0 = E_0 \mathcal{R} E_0$. Thus their commutants $\mathcal{R}'_1 E_0$ and $\mathcal{R}' E_0$ coincide and $\mathcal{R}'_1 = \mathcal{R}'$ or $\mathcal{R}_1 = \mathcal{R}$ (since E_0 has central carrier I). On the other hand (iii) trivially results from (ii) since $\mathcal{R}' = \mathfrak{A}'' \cap \mathfrak{A}' \cap U(G)'$ as

we noticed above. In turn (iii) implies $\mathcal{R}_1 \supseteq \mathfrak{Q}$ and hence that Ω is cyclic under \mathcal{R}_1 . Since Ω is invariant under $U(G)$ and each U_g induces an automorphism of \mathfrak{Q}' , Ω is then cyclic for \mathfrak{Q}' i.e. one has (i).

We note that the cyclicity of Ω for \mathfrak{Q}' is a property which is conserved under the central reduction of \mathcal{R} discussed below. This follows from the fact that the reduction of \mathcal{R} is a reduction of \mathcal{R}_1 because of equality (ii) of the Theorem.

The extremal invariant states (E-states) over \mathfrak{Q} . The states over \mathfrak{Q} invariant under G (the G -invariant states) form a convex w^* -relatively compact subset of the topological dual \mathfrak{Q}^* of \mathfrak{Q} . Special interest is attached to the extreme points of this convex set to which we give the name of *E-states*. *E-states* can be characterized in a variety of ways expressed in

Theorem 4. *With the assumptions and notation of Corollary 2 the following are equivalent for the G -invariant state Φ over \mathfrak{Q}*

(i) *for all $A, B \in \mathfrak{Q}$ $(\Omega | \pi_\Phi(A) M_\Phi(B) | \Omega) = \langle \Phi, A \rangle \langle \Phi, B \rangle$.*

(ii) *Ω is the only G -invariant vector of \mathcal{H} i.e. $E_0 = E_\Omega$ where E_Ω denotes the orthogonal projection in \mathcal{H} on the vector Ω .*

(iii) *For all $A_1, A_2, B \in \mathfrak{Q}$*

$$(\Omega | \pi_\Phi(A_1) M_\Phi(B) \pi_\Phi(A_2) | \Omega) = \langle \Phi, A_1 A_2 \rangle \langle \Phi, B \rangle.$$

(iv) *$M_\Phi(A)$ is a multiple of the identity for all $A \in \mathfrak{Q}$.*

(v) *The set of operators $\pi_\Phi(\mathfrak{Q}) \cup U_\Phi(G)$ is irreducible (i.e. \mathcal{R} consists of all bounded operators on \mathcal{H}).*

(vi) *\mathcal{R} is a factor.*

(vii) *Φ is an extremal element of the convex set of G -invariant states over \mathfrak{Q} .*

Proof. We have, by the definition of M_Φ

$$(\Omega | \pi_\Phi(A) M_\Phi(B) | \Omega) = (\Omega | \pi_\Phi(A) E_0 \pi_\Phi(B) | \Omega), \quad (5)$$

thus since Ω is cyclic for $\pi_\Phi(\mathfrak{Q})$ and

$$\langle \Phi, A \rangle \langle \Phi, B \rangle = (\Omega | \pi_\Phi(A) | \Omega) (\Omega | \pi_\Phi(B) | \Omega) \quad (6)$$

(i) is equivalent to $E_0 = |\Omega\rangle\langle\Omega|$ ($|\Omega\rangle\langle\Omega| = E_\Omega$). On the other hand (iii) is obtained from (i) by setting $A = A_1 A_2$ and permuting A_2 with $M_\Phi(B) \in \mathcal{R}'$. Conversely we obtain (i) from (iii) by setting $A_1 = A$ and taking for A_2 an approximate identity in \mathfrak{Q} . Now (iv) is evidently equivalent to (iii) due to the cyclicity of Ω for $\pi_\Phi(\mathfrak{Q})$. (iv) and (vi) are equivalent because the $M_\Phi(A)$, $A \in \mathfrak{Q}$, are dense in \mathcal{R}' . (vi) is equivalent to (v) because \mathcal{R}' is abelian. Finally (vii) is equivalent to (v), for, with E' a projection in \mathcal{R}' , $\omega_{E'\Omega} \pi_\Phi \leq \Phi = \omega_\Omega \pi_\Phi$ and $\omega_{E'\Omega} \pi_\Phi$ is invariant under α_g . Thus $\omega_{E'\Omega} \pi_\Phi = a\Phi$ and

$$(\pi(A)\Omega | \alpha\Omega) = (\pi(A) E' \Omega | E' \Omega) = (\pi(A)\Omega | E' \Omega) \quad \text{for all } A \text{ in } \mathfrak{Q}. \quad (7)$$

Since Ω is cyclic under \mathfrak{Q} , $(aI - E')\Omega = 0$ and, since Ω is separating

for \mathcal{R}' , $E' = aI$. Hence $E' = 0$ or I and $\mathcal{R}' = \{bI\}$. Conversely, if \mathcal{R} is all bounded operators and ω is an invariant linear functional such that $0 < \omega < \Phi$, from [16; Lemma 2.2] $\omega = \omega_{T'\Omega}\pi_\Phi$ with T' a positive operator in \mathcal{R}' . Since \mathcal{R}' is the scalars $\omega = a\Phi$ and Φ is extremal among the invariant states of \mathfrak{A} .

Decomposition of an invariant state into E-states. With the assumptions and notation of Corollary 2, suppose, in addition, that \mathcal{R}' , which we know to be abelian, is generated by its minimal projections $\{F_\lambda\}$ (i.e. those non-zero projections in \mathcal{R}' which dominate no other non-zero projection in \mathcal{R}'). In this case, $\mathcal{R}F_\lambda$ has commutant $F_\lambda\mathcal{R}'F_\lambda = \{aF_\lambda\}$ on $F_\lambda(\mathcal{H})$; so that $\mathcal{R}F_\lambda$ is the algebra of all bounded operators on $F_\lambda(\mathcal{H})$. The U_gF_λ induce automorphisms of $\mathcal{R}F_\lambda$ and $\Omega_\lambda (= F_\lambda\Omega)$ is invariant under all U_gF_λ and cyclic for $\pi_\Phi(\mathfrak{A})F_\lambda$. From Theorem 4, (v) and (vii), Ω_λ gives rise to an E -state Φ_λ of \mathfrak{A} , where $\Phi_\lambda(A) = (\pi_\Phi(A)\Omega'_\lambda | \Omega'_\lambda)$, with $\Omega'_\lambda = \Omega_\lambda/\|\Omega_\lambda\|$ — noting that $\Omega_\lambda \neq 0$ since Ω is separating for \mathcal{R}' being cyclic for \mathcal{R} .

We have

$$\begin{aligned} \Phi(A) &= (\pi_\Phi(A)\Omega | \Omega) = (\pi_\Phi(A) (\sum_\lambda F_\lambda)\Omega | (\sum_\lambda F_\lambda)\Omega) = \sum_\lambda (\pi_\Phi(A)\Omega_\lambda | \Omega_\lambda) \\ &= \sum_\lambda \|\Omega_\lambda\|^2 (\pi_\Phi(A)\Omega'_\lambda | \Omega'_\lambda) = \sum_\lambda \|\Omega_\lambda\|^2 \Phi_\lambda(A); \end{aligned} \tag{8}$$

and $\sum_\lambda \|\Omega_\lambda\|^2 = \|\Omega\|^2 = 1$. In this way we have expressed Φ as a convex sum of E -states Φ_λ . If $\Phi = \sum_\gamma a_\gamma \Phi'_\gamma$ with Φ'_γ an E -state for each γ and $a_\gamma > 0$, then $\Phi'_\gamma(A) = (\pi_\Phi(A)T'\Omega | T'\Omega)$ for some T' in \mathcal{R}' by [16, Lemma 2.2] and all A in \mathfrak{A} . Thus $F_\lambda T'\Omega = b_\lambda \Omega'_\lambda$ and $\Phi'_\gamma = \sum_\lambda |b_\lambda|^2 \Phi_\lambda$. By extremality $\Phi'_\gamma = \Phi_{\lambda_0}$ for some λ_0 , $|b_{\lambda_0}| = 1$, and $b_\lambda = 0$ for all $\lambda \neq \lambda_0$. It remains to note that if $\sum_\lambda c_\lambda \Phi_\lambda = 0$, where $\sum |c_\lambda| < \infty$, then $c_\lambda = 0$ for all λ , in order to establish the uniqueness of our decomposition of Φ into E -states.

For this note that $((\pi_\Phi(A_1)U_{g_1} + \dots + \pi_\Phi(A_n)U_{g_n})\Omega'_\lambda | \Omega'_\lambda) = \Phi_\lambda(A_1 + \dots + A_n)$ so that $\sum c_\lambda \omega_{\Omega'_\lambda}$ vanishes on the $*$ -algebra generated by $\pi_\Phi(\mathfrak{A})$ and $\{U_{g_j}\}$. Since $\sum |c_\lambda| < \infty$, $\sum c_\lambda \omega_{\Omega'_\lambda}$ is normal and vanishes on the weak-operator closure, \mathcal{R} , generated by this algebra. Thus $0 = \sum c_\lambda \omega_{\Omega'}(F_\lambda) = c_\lambda$ for all λ ; and the uniqueness of our decomposition follows.

With suitable separability assumptions, the analogous argument using direct integral decompositions of \mathcal{R} and \mathcal{H} relative to \mathcal{R}' , yields a (unique) direct integral decomposition of an arbitrary G -invariant state of \mathfrak{A} as a “convex” integral of E -states.

II. The Case of an Amenable Group

We first recall a few definitions and results concerning amenable groups. For a locally compact group G let us denote by $\mathcal{C}(G)$, and $\mathcal{C}_0(G)$, the C^* -algebras of continuous complex valued functions on G respectively

bounded, and vanishing at infinity, with the sup norm $\|\cdot\|_\infty$. We will sometimes denote a function $f \in \mathcal{C}(G)$ by the symbol $f(\hat{g})$ where the roofed letter indicates a dummy variable. A *mean over G* is a positive linear form η over the C^* -algebra $\mathcal{C}(G)$ of unit norm (i.e. $\eta(1) = 1$ where 1 denotes the constant unit function). The mean η is respectively *left- or right-invariant* if $\eta\{f(h\hat{g})\} = \eta\{f(\hat{g})\}$ or $\eta\{f(\hat{g}h)\} = \eta\{f(\hat{g})\}$ for all $f \in \mathcal{C}(g)$ and $h \in G$. It is (*two sided*) *invariant* if it is both left and right-invariant. The existence of a left-invariant, right-invariant or invariant mean over G are equivalent requirements which characterize a subclass of locally compact groups which we will call *amenable* (for the properties of amenable groups see [17]). Amenable groups can be characterized by the existence of *M-filters* as defined in [7; Definition 1]. Every abelian or solvable locally compact group is amenable. Every compact group is trivially amenable. However, no non-compact semi-simple Lie group is amenable [18] (for instance $SL(2, R)$ is not amenable). The Lorentz group is not amenable. The group of euclidean motions in 3-space is amenable since each extension of an amenable group by an amenable group is amenable [17]. If G is amenable and non compact the restriction of every left- or right-invariant mean to $\mathcal{C}_0(G)$ vanishes. In what follows we denote by $\mathfrak{A}_G^{\mathcal{C}}$ the linear space of all weakly-continuous, weakly bounded functions from G to \mathfrak{A} (i.e. for each Φ in the topological dual \mathfrak{A}^* of \mathfrak{A} and $X \in \mathfrak{A}_G^{\mathcal{C}}$, $\langle \Phi, X(\hat{g}) \rangle \in \mathcal{C}(G)$).

We note that each $X \in \mathfrak{A}_G^{\mathcal{C}}$ is norm-bounded (i.e. $\text{Sup}_{g \in G} \|X(g)\| = \|X\|_\infty$ is finite) from the uniform boundedness principle [11; Chapt. II § 1 p. 49]. We denote by \mathfrak{A}^{**} the von Neumann enveloping algebra of \mathfrak{A} (the topological dual of \mathfrak{A}^* with its strong topology, see [15, Chapter 12]).

In the three next paragraphs we describe means of vector or operator valued functions. Then we introduce the notion of *η -asymptotic abelianness* relevant to C^* -algebras acted upon by amenable groups which allows us to derive properties analogous to those of the previous section.

*The mapping M_η from \mathfrak{A} to \mathfrak{A}^{**} .* This paragraph deals with averages of the functions of the type $g \rightarrow \alpha_g(A)$, $A \in \mathfrak{A}$, from G to \mathfrak{A} with invariant means over G . The following lemma first gathers elementary facts on “vectorial means”.

Lemma 1. *Let \mathfrak{A} be a C^* -algebra and G a locally compact group. To each mean η over G there exists a unique mapping $\tilde{\eta}$ from $\mathfrak{A}_G^{\mathcal{C}}$ to \mathfrak{A}^{**} such that*

$$\langle \Phi, \tilde{\eta}(X) \rangle = \eta\{\langle \Phi, X(\hat{g}) \rangle\} \quad \text{for all } \Phi \in \mathfrak{A}^* . \quad (11)$$

This mapping $\tilde{\eta}$ is linear, bounded of norm not exceeding 1 (i.e. $\|\tilde{\eta}(X)\| \leq \|X\|_\infty$ for all $X \in \mathfrak{A}_G^{\mathcal{C}}$), positive (i.e. $X(g) \geq 0$ for all $g \in G$

implies $\tilde{\eta}(X) \geq 0$) and such that, for all $A \in \mathfrak{A}$

$$\tilde{\eta}(A) = A \tag{12}$$

$$\left. \begin{aligned} \tilde{\eta}(A \cdot X) &= A \cdot \tilde{\eta}(X) \\ \tilde{\eta}(X \cdot A) &= \tilde{\eta}(X) \cdot A \end{aligned} \right\} \tag{13}$$

(here A , $A \cdot X$ and $X \cdot A$ are the elements of \mathfrak{A}_0^G respectively defined by $A(g) = A$, $\{A \cdot X\}(g) = A \cdot X(g)$, $\{X \cdot A\}(g) = X(g) \cdot A$).

If G is amenable and non compact and η is right-invariant, $\tilde{\eta}(X)$ vanishes whenever X vanishes weakly at infinity (i.e. whenever, for each $\Phi \in \mathfrak{A}^*$, $\langle \Phi, X(\hat{g}) \rangle \in \mathcal{C}_0(G)$). If G is amenable and η is right-invariant then $\tilde{\eta}$ is right-invariant (i.e. $\tilde{\eta}(X(\hat{g}h)) = \tilde{\eta}(X(\hat{g}))$ for all $g \in G$).

Proof. One has

$$|\eta\{\langle \Phi, X(\hat{g}) \rangle\}| \leq \text{Sup}_{g \in G} |\langle \Phi, X(g) \rangle| \leq \|\Phi\| \|X\|_\infty$$

therefore $\Phi \in \mathfrak{A}^* \rightarrow \eta\{\langle \Phi, X(\hat{g}) \rangle\}$ is a linear form on \mathfrak{A}^* of norm $\leq \|X\|_\infty$. For positivity, note that $\eta\{\langle \Phi, X(\hat{g}) \rangle\} \geq 0$ if Φ is a state of \mathfrak{A} and $X \geq 0$; while (12) stems from the fact that $\eta(1) = 1$. For $X \in \mathfrak{A}_0^G$ and $A \in \mathfrak{A}$ one has

$$\eta\{\langle \Phi, A \cdot X(\hat{g}) \rangle\} = \eta\{\langle A^t \Phi, X(\hat{g}) \rangle\} = \langle A^t \Phi, \tilde{\eta}(X) \rangle = \langle \Phi, A \tilde{\eta}(X) \rangle$$

and analogously for right-multiplication by A (we denote by A^t the transpose of left-multiplication by A and note that multiplication by $A \in \mathfrak{A}$ in \mathfrak{A}^{**} can be defined by double transposition). The last assertion of the Lemma follows immediately from the definition (11) and the fact that $\eta | \mathcal{C}_0(G) = 0$. As an immediate consequence we now have

Lemma 2. *Let \mathfrak{A} be a C^* -algebra, $g \in G \rightarrow \alpha_g$ a strongly continuous homomorphism of the locally compact group G into the automorphism group of \mathfrak{A} (i.e. $g \in G \rightarrow \alpha_g(A)$ is a norm-continuous function of g for all $A \in \mathfrak{A}$). Let η be a mean over G ; and let*

$$M_\eta(A) = \tilde{\eta}(\alpha_{\hat{g}}(A)), \quad A \in \mathfrak{A}. \tag{14}$$

*Then M_η is a linear positive bounded mapping from \mathfrak{A} to \mathfrak{A}^{**} of norm not exceeding 1. If G is amenable and η is right invariant M_η is such that*

$$M_\eta(\alpha_g(A)) = M_\eta(A), \quad A \in \mathfrak{A}, g \in G. \tag{15}$$

*Moreover the means $M_\eta(A)$, $A \in \mathfrak{A}$ are G -invariant elements of \mathfrak{A}^{**} in the sense that, for all $g \in G$, and each left invariant mean η over G ,*

$$\alpha_g(M_\eta(A)) = M_\eta(A) \tag{15a}$$

*where α_g denotes the action of G on \mathfrak{A}^{**} obtained by double transposition.*

Proof. M is evidently linear, positive since $A \geq 0$ implies $\alpha_g(A) \geq 0$ for all $g \in G$, and bounded by 1 because $\|\alpha_g(A)\| = \|A\|$, $g \in G$. The invariance of M_η for a right-invariant η follows from the definition (14)

and the last assertion of Lemma 1. For (15a) note that

$$\begin{aligned} \langle \Phi, \alpha_g(M_\eta(A)) \rangle &= \eta \{ \langle \alpha_g^t(\Phi), \alpha_{\hat{g}}(A) \rangle \} = \eta \{ \langle \Phi, \alpha_{g\hat{g}}(A) \rangle \} \\ &= \eta \{ \langle \Phi, \alpha_{\hat{g}}(A) \rangle \} = \langle \Phi, M_\eta(A) \rangle \end{aligned}$$

for each Φ in \mathfrak{Q}^* .

The mapping M_η on the dual space \mathfrak{Q}^* . In the same setting as for the last proposition we now describe the transpose of the operator M_η , which we again denote by M_η .

Lemma 3. *Let \mathfrak{A} be a C^* -algebra and $g \rightarrow \alpha_g$ a strongly continuous homomorphism of the locally compact group G into the automorphism group of \mathfrak{A} . To each mean η over G there exists a unique mapping M_η of \mathfrak{Q}^* into \mathfrak{Q}^* such that*

$$\langle M_\eta(\Phi), A \rangle = \langle \Phi, M_\eta(A) \rangle = \eta \{ \langle \Phi, \alpha_{\hat{g}}(A) \rangle \}, \quad A \in \mathfrak{A}, \Phi \in \mathfrak{Q}^*. \quad (16)$$

This mapping M_η is linear, positive and bounded of norm not exceeding 1. If G is amenable and η is right-invariant M_η is idempotent and projects onto the set of G -invariant elements of \mathfrak{Q}^* (i.e. those $\Phi \in \mathfrak{Q}^*$ such that $\langle \Phi, \alpha_g(A) \rangle = \langle \Phi, A \rangle$ for all $A \in \mathfrak{A}$ and $g \in G$). If η is an arbitrary mean and Φ is G -invariant $M_\eta(\Phi) = \Phi$.

Proof. One has, for all $\Phi \in \mathfrak{Q}^*$ and $A \in \mathfrak{A}$

$$|\langle \Phi, M_\eta(A) \rangle| \leq \|\Phi\| \|M_\eta(A)\| \leq \|\Phi\| \cdot \|A\|$$

therefore $\Phi \rightarrow M_\eta(\Phi)$ is a linear mapping in \mathfrak{Q}^* for which $\|M_\eta(\Phi)\| \leq \|\Phi\|$, $\Phi \in \mathfrak{Q}^*$. Further, for $\Phi \geq 0$ and $A \geq 0$ one has $M_\eta(A) \geq 0$ and $\langle \Phi, M_\eta(A) \rangle \geq 0$ whence $M_\eta(\Phi) \geq 0$ and M_η is positive. If η is right-invariant $M_\eta(\Phi)$ is G -invariant for all $\Phi \in \mathfrak{Q}^*$ as a consequence of (16) and (15). On the other hand G -invariant elements of \mathfrak{Q}^* are invariant under M_η for an arbitrary mean η due to (16) and the definition of a mean. Thus for arbitrary Φ one has $M_\eta(M_\eta(\Phi)) = M_\eta(\Phi)$.

We note that, since M_η projects onto the G -invariant elements of \mathfrak{Q}^* , the fact that $M_\eta = 0$ for some right-invariant mean η implies that there does not exist any G -invariant state (= positive linear form of unit norm) in \mathfrak{Q}^* . Conversely the absence of G -invariant states entails the vanishing of all operators M_η (in \mathfrak{Q}^* or from \mathfrak{A} to \mathfrak{Q}^{**}) for all right-invariant means η over G . A last remark is that, if \mathfrak{A} has a unit, the mapping M_η is of unit norm (obvious by setting $A = I$ in (16)). But if $\mathfrak{A} = \mathcal{I} \oplus \{\lambda I\}$, with \mathcal{I} a two-sided ideal, the M_η might be trivial in the sense that $M_\eta(\Phi) \{\lambda I \oplus \lambda I\} = \lambda$ for all states Φ over \mathfrak{A} and all η .

Means in covariant representations. We consider now as in Section I covariant representations (π, U) of the pair (\mathfrak{A}, G) and describe means of such operator valued functions as $g \rightarrow U_g$ or $g \rightarrow U_g \pi(A) U_g^{-1}$. We begin with a lemma analogous to Lemma 1 describing means of operator-valued functions on G .

Lemma 4. *Let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on a Hilbert space \mathcal{H} , G a locally compact group and \mathcal{X} the set of functions $g \rightarrow X(g)$ from G to $\mathcal{L}(\mathcal{H})$ such that $g \rightarrow (\varphi | X(g) | \psi)$ is a continuous function on G for all $\varphi, \psi \in \mathcal{H}$ and $\text{Sup}_{g \in G} \|X(g)\| = \|X\|_\infty$ is finite. To each mean η over G there exists a unique operator $\tilde{\eta}$ from \mathcal{X} to $\mathcal{L}(\mathcal{H})$ such that*

$$(\varphi | \tilde{\eta}(X) | \psi) = \eta\{(\varphi | X(\hat{g}) | \psi)\}, \quad \varphi, \psi \in \mathcal{H}. \tag{17}$$

The operator $\tilde{\eta}$ is linear, bounded of norm not exceeding 1 (i.e. $\|\tilde{\eta}(X)\| \leq \|X\|_\infty$ for all $X \in \mathcal{X}$), positive (i.e. $X(g) \geq 0$ for all $g \in G$ implies $\tilde{\eta}(X) \geq 0$) and such that, for all $A \in \mathcal{L}(\mathcal{H})$, with notations analogous to those of Lemma 1

$$\tilde{\eta}(A) = A \tag{18}$$

$$\tilde{\eta}(A \cdot X) = A \cdot \tilde{\eta}(X) \tag{19}$$

$$\tilde{\eta}(X \cdot A) = \tilde{\eta}(X) \cdot A.$$

Furthermore $\tilde{\eta}(X)$, $X \in \mathcal{X}$, is contained in the bicommutant of the range of X (i.e. $\tilde{\eta}(X) \in \{X(g) | g \in G\}''$).²

Proof. Analogous to the proof of Lemma 1. We have

$$|\eta\{(\varphi | X(\hat{g}) | \psi)\}| \leq \text{Sup}_{g \in G} |(\varphi | X(g) | \psi)| \leq \|\varphi\| \|\psi\| \|X\|_\infty$$

whence, by Riesz's Theorem, the existence of $\tilde{\eta}(X)$ of norm $\leq \|X\|_\infty$ follows. Positivity and property (18) are proved as in Lemma 1 and properties (19) in an analogous manner: one has, for left multiplications

$$\begin{aligned} (\varphi | \tilde{\eta}(A \cdot X) | \psi) &= \eta\{(\varphi | A X(\hat{g}) | \psi)\} = \eta\{(A^* \varphi | X(\hat{g}) | \psi)\} \\ &= (A^* \varphi | \tilde{\eta}(X) | \psi) = (\varphi | A \tilde{\eta}(X) | \psi). \end{aligned}$$

The fact that $\tilde{\eta}(X)$ is contained in the bicommutant of the range of X is then an immediate consequence of (19).

Proposition 1. *Let G be an amenable locally compact group, $g \rightarrow U_g$ a strongly continuous unitary representation of G on a Hilbert space, E_0 the orthogonal projector onto the G -invariant vectors of \mathcal{H} (i.e. those $\psi \in \mathcal{H}$ such that $U_g \psi = \psi$ for all $\psi \in \mathcal{H}$). Using the notation of the preceding Lemma we then have that $\tilde{\eta}(U_{\hat{g}}) = E_0$ for all right or left-invariant means η over G .*

Proof. We first observe that, due to (19), for each $h \in G$ and a right-invariant η

$$\tilde{\eta}(U) U_h = \tilde{\eta}(U_{\hat{g}} U_h) = \tilde{\eta}(U_{\hat{g}h}) = \tilde{\eta}(U). \tag{20}$$

By taking means, using (19) and (18) again, it follows that

$$\tilde{\eta}\{\tilde{\eta}(U) U_h\} = \tilde{\eta}(\tilde{\eta}(U)) = \tilde{\eta}(U)^2 = \tilde{\eta}(U),$$

² The above Lemma 1 is essentially identical with Lemma 2 from which it can be deduced by application to the universal representation of \mathfrak{A} . However we prefer to present straightforward independent proofs.

thus $\tilde{\eta}(U)$ is idempotent. To prove that it is self adjoint we note that, taking adjoints in (20), $U_h \tilde{\eta}(U)^* = \tilde{\eta}(U)^*$, whence we derive as above

$$\tilde{\eta}(U_h \tilde{\eta}(U)^*) = \tilde{\eta}(\tilde{\eta}(U)^*) = \tilde{\eta}(U) \tilde{\eta}(U)^* = \tilde{\eta}(U)^* .$$

Hence $\tilde{\eta}(U)^*$ is self adjoint and $\tilde{\eta}(U)$ is a self adjoint projection such that $U_h \tilde{\eta}(U) = \tilde{\eta}(U) U_h = \tilde{\eta}(U)$, $h \in G$, i.e. such that its range is contained in that of E_0 . Thus $\tilde{\eta}(U) E_0 = \tilde{\eta}(U)$ and, using (19),

$$\tilde{\eta}(U) E_0 = \tilde{\eta}\{U_{\hat{g}} E_0\} = \tilde{\eta}(E_0) = E_0 .$$

Therefore $\tilde{\eta}(U) = E_0$. Finally, for a left invariant η , η_1 given by $\eta_1(\hat{f}(\hat{g})) = \eta(\hat{f}(\hat{g}^{-1}))$ is right invariant and $\tilde{\eta}(U) = \tilde{\eta}_1(U^*) = \tilde{\eta}_1(U)^* = E_0^* = E_0$.

Corollary. *With G, U, \mathcal{H} and η as in the preceding Proposition and χ a continuous character of G , $\tilde{\eta}(\tilde{\chi}U) = E_{\chi}$, the orthogonal projection on the space of vectors $\psi \in \mathcal{H}$ such that $U_g \psi = \chi(g) \psi$ for all $g \in G$.*

Proof. Apply the Proposition to the representation $g \rightarrow \tilde{\chi}(g) U_g$.

Lemma 5. *With \mathfrak{A} a C^* -algebra and $g \rightarrow \alpha_g$ a strongly continuous homomorphism of the locally compact amenable group G into the automorphism group of \mathfrak{A} , we recall that a covariant representation (π, U) of the system $\{\mathfrak{A}, \alpha\}$ is a pair consisting of a $*$ -representation π of \mathfrak{A} and a strongly continuous unitary representation U of G such that $\pi(\alpha_g(A)) = U_g \pi(A) U_g^{-1}$ for all $A \in \mathfrak{A}$ and $g \in G$. Using the preceding notation we then have that, for a left invariant mean η over G ,*

$$\tilde{\eta}\{U_{\hat{g}} \pi(A) U_{\hat{g}}^{-1}\} = \pi\{M_{\eta}(A)\} \quad (= m_{\eta}^{\pi}(A)) \quad (21)$$

where the ultraweakly continuous extension of π to \mathfrak{A}^{**} is again denoted by π . The $m_{\eta}^{\pi}(A)$, $A \in \mathfrak{A}$, are elements of $\pi(\mathfrak{A})'$ invariant under G in the sense that

$$U_g m_{\eta}^{\pi}(A) U_g^{-1} = m_{\eta}^{\pi}(A) .$$

Proof. Equation (21) results immediately from the comparison of definitions (16) and (17). The rest of the lemma follows from

$$\begin{aligned} (\varphi | U_g \pi(M_{\eta}(A)) U_g^{-1} | \psi) &= \eta\{\langle \omega_{U_g^{-1}\psi, U_g^{-1}\psi} \pi | \alpha_{\hat{g}}(A) \rangle\} \\ &= \eta\{\langle \varphi | U_g \pi(\alpha_{\hat{g}}(A)) U_g^{-1} | \psi \rangle\} \\ &= \eta\{\langle \varphi | \pi(\alpha_{g\hat{g}}(A)) | \psi \rangle\} = \eta\{\langle \varphi | \pi(\alpha_{\hat{g}}(A)) | \psi \rangle\} \\ &= \eta\{\langle \omega_{\psi, \varphi} \pi | \alpha_{\hat{g}}(A) \rangle\} = (\varphi | m_{\eta}^{\pi}(A) | \psi) . \end{aligned}$$

Asymptotic abelianness in mean. We begin with the

Definition 2. *Let $\{\mathfrak{A}, \alpha\}$ be a pair consisting of a C^* -algebra \mathfrak{A} and a strongly continuous homomorphism $g \rightarrow \alpha_g$ of the locally compact amenable group G into the automorphism group of \mathfrak{A} . Let η be a right or left invariant mean over G . The system $\{\mathfrak{A}, \alpha\}$ is called an (abstract) η -asymptotically abelian system if, for all $A, B \in \mathfrak{A}$ and each state Φ over \mathfrak{A}*

$$\eta\{\langle \Phi, A \cdot \alpha_{\hat{g}}(B) - \alpha_{\hat{g}}(B) \cdot A \rangle\} = 0 .$$

If \mathfrak{A} is a concrete C^ -algebra acting on a Hilbert space \mathcal{H} and $g \rightarrow U_g$ is a*

strongly continuous unitary representation of G on \mathcal{H} such that $\alpha_g(A) = U_g A U_g^{-1}$ for all $A \in \mathfrak{A}, g \in G, \{\mathfrak{A}, \alpha\}$ is called a concrete η -asymptotically abelian system whenever the condition above holds for all vector states Φ of \mathfrak{A} .

Remark 1. Each covariant representation of an abstract η -asymptotically abelian system $\{\mathfrak{A}, \alpha\}$ yields a concrete η -asymptotically abelian system.

Remark 2. For a non-compact amenable group G weak asymptotic abelianness of the abstract system $\{\mathfrak{A}, \alpha\}$ obviously implies η -asymptotic abelianness for all right or left invariant means η over G .

Remark 3. The set of right (left) invariant means η for which the abstract or concrete system $\{\mathfrak{A}, \alpha\}$ is η -asymptotically abelian is a w^* -closed (and thus w^* -compact) convex subset of the topological dual space of $\mathcal{C}(G)$.

Lemma 6. *Let η be a right or left invariant mean over the amenable group G . The concrete system $\{\mathfrak{A}, \alpha\}$ is η -asymptotically abelian if and only if $\eta\{U_{\hat{g}} A U_{\hat{g}}^{-1}\}$ (which we denote by $m_\eta(A)$) is contained in the center of the weak closure \mathfrak{A}'' of \mathfrak{A} for all $A \in \mathfrak{A}$. Analogously the abstract system $\{\mathfrak{A}, \alpha\}$ is η -asymptotically abelian if and only if $M_\eta(A)$ is in the center of \mathfrak{A}^{**} for all $A \in \mathfrak{A}$.*

Proof. From (19), $\tilde{\eta}\{\langle \Phi, B\alpha_{\hat{g}}(A) - \alpha_{\hat{g}}(A)B \rangle\} = \langle \Phi, BM_\eta(A) - M_\eta(A)B \rangle$; so that the abstract system $\{\mathfrak{A}, \alpha\}$ is η -asymptotically abelian if and only if $M_\eta(A)$ lies in the center of \mathfrak{A}^{**} for each A in \mathfrak{A} . Replacing M_η above by m_η , we conclude that the concrete system $\{\mathfrak{A}, \alpha\}$ is η -asymptotically abelian if and only if $m_\eta(A) \in \mathfrak{A}' \cap \mathfrak{A}''$ for each $A \in \mathfrak{A}$.

Theorem 5. *Let $\{\mathfrak{A}, \alpha\}$ be a concrete η -asymptotically abelian system acting on the Hilbert space \mathcal{H} with η a left invariant mean over the group G . We denote by E_0 the projector onto the subspace of vectors $\psi \in \mathcal{H}$ such that $U_g \psi = \psi$ for all $g \in G$ and by \mathfrak{R} the von Neumann algebra generated by \mathfrak{A} and $U(G)$. Then*

(i) $E_0 \mathfrak{R} E_0$ is abelian with the consequences stated in (i), (ii), (iii) of Theorem 1.

(ii) If E_0 has central carrier I in \mathfrak{R} the mapping m_η of Lemma 6 coincides on \mathfrak{A} with the mapping M of Theorem 1 (thus the restriction of m_η to \mathfrak{A} does not depend upon the choice of η within the set considered in Remark 3 above)³.

(iii) M and m_η coincide also on \mathfrak{A}'' if and only if the conditions of Theorem 3 are realized or equivalently if the system $\{\mathfrak{A}'', \alpha\}$ is η -asymptotically abelian.

³ This explains why the M could be obtained in [7] by use of an arbitrary M -filter (most statements and proofs in [7] are — except for the last two sections — valid for amenable groups rather than only abelian groups).

Proof. By Lemma 6 we have that $m_\eta(A) \cdot B - B \cdot m_\eta(A) = 0$ for all $A, B \in \mathfrak{A}$. Now $E_0 m_\eta(A) B E_0 = E_0 \tilde{\eta}(U_{\hat{g}} A U_{\hat{g}}^{-1}) B E_0 = \tilde{\eta}(E_0 A U_{\hat{g}}^{-1}) B E_0 = E_0 A E_0 B E_0$ from (19) and Proposition 1. We conclude that $E_0 A E_0 B E_0 - E_0 B E_0 A E_0 = 0$ whence (i). In order to prove (ii) we observe that, for $A \in \mathfrak{A}$, using (19) again,

$$m_\eta(A) E_0 = E_0 A E_0 = M(A) E_0 .$$

Thus $m_\eta(A) = M(A)$, since $m_\eta(A)$ and $M(A)$ are in \mathcal{H} . To prove (iii) we first observe that Lemma 6 entails the equivalence of condition (iii) of Theorem 3, namely $\mathfrak{A}'' \cap U(G)' \subseteq \mathfrak{A}'$, and the η -asymptotic abelianness of the system $\{\mathfrak{A}'', \alpha\}$. However the latter implies that $m_\eta = M$ over \mathfrak{A}'' using the equation above. Now this last property implies in turn property (iii) of Theorem 3 since $M(\mathfrak{A}'') \subseteq \mathcal{H}$ and $m_\eta(\mathfrak{A}'') = \mathfrak{A}'' \cap U(G)'$.

Definition 3. Let \mathfrak{A} be a C^* -algebra, G a locally compact amenable group, $g \rightarrow \alpha_g a$ a strongly continuous homomorphism of G into the automorphism group of \mathfrak{A} . A state Φ over \mathfrak{A} is called η -weakly clustering whenever $\eta\{\langle \Phi, A \alpha_{\hat{g}}(B) \rangle\} = \Phi(A) \cdot \eta\{\langle \Phi, \alpha_{\hat{g}}(B) \rangle\}$ for all $A, B \in \mathfrak{A}$ with η a right or left invariant mean over G . A G -invariant state Φ is called weakly clustering whenever there exists a right or left invariant mean η over G for which it is η -weakly clustering i.e. such that

$$\langle \Phi, A \cdot M_\eta(B) \rangle = \langle \Phi, A \rangle \langle \Phi, B \rangle . \quad (22)$$

Remark. For a general state Φ and denoting by the same symbol the ultraweakly continuous extension of Φ to \mathfrak{A}^{**} , η -weakly clustering can be formulated as the requirement that $\langle \Phi, M_\eta(B)^2 \rangle = \langle \Phi, M_\eta(B) \rangle^2$ for all self adjoint $B \in \mathfrak{A}$ (the last condition is known to be equivalent to $\langle \Phi, A \cdot M_\eta(B) \rangle = \langle \Phi, A \rangle \langle \Phi, M_\eta(B) \rangle$ for all $A \in \mathfrak{A}^{**}$ [see, for example, 27, Lemma]).

We can now rephrase Theorem 4 in the following way:

Theorem 4a. With $\{\mathfrak{A}, \alpha\}$ an η -asymptotically abelian abstract system and Φ an invariant state over \mathfrak{A} the following are all equivalent to (v), (vi), (vii) of Theorem 4;

- (i) Φ is weakly clustering.
- (ii) Ω is the only G -invariant vector in \mathcal{H} .
- (iii) Condition (22) holds for all right or left invariant means over G .
- (iv) $M_\eta^{\pi\Phi}(A) = \langle \Phi, A \rangle \cdot I$ for all $A \in \mathfrak{A}$ and all η such that $\{\mathfrak{A}, \alpha\}$ is η -asymptotically abelian.

III. Examples

A. A System where the Group is compact

Let \mathfrak{A} be the full algebra of $n \times n$ complex matrices, G be the compact group of unitary $n \times n$ matrices with Haar measure μ and define

$\alpha_U(A) = UAU^{-1}$, $A \in \mathfrak{A}$, $U \in G$. We then have

$$\int \alpha_U(A) d\mu(U) = \frac{1}{n} \text{Tr}(A) \cdot I$$

so that (cf. Lemma 6) the system $\{\mathfrak{A}, \alpha\}$ is μ -asymptotically abelian.

Remark. If we let H be the direct sum of G and any locally compact, non compact group, then composing α with the quotient mapping modulo this group yields a system which is η -asymptotically abelian for each invariant mean η over H but not weakly asymptotically abelian.

*B. The Algebra of Canonical Anticommutation Relations:
a System which is weakly but not Norm asymptotically Abelian*

With \mathfrak{A} the C^* -algebra of c.a.r. on $L_2(R^n)$ (generated by the smeared out bounded creation field operators $\psi(f)$, f in L_2), let α_x be the automorphism of \mathfrak{A} determined by $\alpha_x(\psi(f)) = \psi(f_x)$, f in L_2 where $f_x(y) = f(y - x)$. Then α is a strongly continuous representation of the translation group of R^n by $*$ -automorphisms of \mathfrak{A} . We show first that the system $\{\mathfrak{A}, \alpha\}$ is weakly asymptotically abelian.

We recall that \mathfrak{A} is the linear space sum of \mathfrak{A}_e and \mathfrak{A}_0 where \mathfrak{A}_e is the norm closure of the linear span of monomials in $\psi(f)$ and $\psi^*(f)$ having an even number of terms with \mathfrak{A}_0 the same for monomials having an odd number of terms. From the commutation relations it follows that $\lim_{x \rightarrow \infty} \|[\alpha_x(A), B]_-\| = 0$ for each A in \mathfrak{A}_e and all B in \mathfrak{A} , while $\lim_{x \rightarrow \infty} \|[\alpha_x(A), B]_+\| = 0$ for all A and B in \mathfrak{A}_0 .

If we show that $\varphi(\alpha_x(A)) \rightarrow 0$ as $x \rightarrow \infty$ ⁴ for each A in \mathfrak{A}_0 and each state φ of \mathfrak{A} , then $\varphi([\alpha_x(A), B]_-) = \varphi_B(\alpha_x(A)) - \varphi^B(\alpha_x(A))$, where $\varphi_B(C) = \varphi(CB)$ and $\varphi^B(C) = \varphi(BC)$, tends to 0 as $x \rightarrow \infty$ for A in \mathfrak{A}_0 , while $[\alpha_x(A), B]_- = [\alpha_x(A_0), B]_- + [\alpha_x(A_e)B]_-$ with A_0 in \mathfrak{A}_0 and A_e in \mathfrak{A}_e . Of course it will suffice to deal with self-adjoint A .

Passing to the representation associated with φ , we may assume that \mathfrak{A} acts on \mathcal{H} with cyclic vector Ω and that $\varphi(A) = (A\Omega | \Omega)$. Suppose that $\alpha_{x_n}(A)\Omega$ tends weakly to ψ in \mathcal{H} where $x_n \rightarrow \infty$ with n . Then

$$\begin{aligned} \|\psi\|^2 &= \lim_{n = \infty} (\alpha_{x_n}(A)\Omega | \lim_{m = \infty} \alpha_{x_m}(A)\Omega) = \lim_{n = \infty} \left(\lim_{m = \infty} (\alpha_{x_m}(A) \alpha_{x_n}(A)\Omega | \Omega) \right) \\ &= \lim_{n = \infty} \left(\lim_{m = \infty} \varphi(\alpha_{x_m}(A) \alpha_{x_n}(A)) \right) \\ &= - \lim_{n = \infty} \left(\lim_{m = \infty} \varphi(\alpha_{x_n}(A) \alpha_{x_m}(A)) \right) = -\|\psi\|^2. \end{aligned}$$

It follows that $\psi = 0$ and that $\lim_{x \rightarrow \infty} (\alpha_x(A)\Omega | \Omega) = \lim_{x \rightarrow \infty} \varphi(\alpha_x(A)) = 0$.

⁴ This fact follows from an argument of R. POWERS (Princeton thesis to appear). We wish to thank R. POWERS for helpful discussion concerning this example.

If $\{\mathfrak{A}, \alpha\}$ was norm asymptotically abelian, $\|\psi(f_x)\psi(g) - \psi(g)\psi(f_x)\| \rightarrow 0$ as $x \rightarrow \infty$ while $\|\psi(f_x)\psi(g) + \psi(g)\psi(f_x)\| = 0$ for all x ; so that $\|\psi(f_x)\psi(g)\| \rightarrow 0$ as $x \rightarrow \infty$. If f_x and g have compact disjoint supports, however, $\|\psi(f_x)\psi(g)\| = \|f\| \|g\| \neq 0$ if both f and g are non zero. Thus $\{\mathfrak{A}, \alpha\}$ is not norm asymptotically abelian.

In the last examples all M_η are zero, so that there are no G -invariant states (cf. the remark following Lemma 3).

C. The C^ -Algebra of a locally compact Abelian Group
Acted upon by the dual Group*

With G a locally compact non discrete abelian group and \hat{G} its non compact dual group topologized as usual, the C^* -algebra of G is the algebra $\mathcal{C}_0(\hat{G})$ of continuous functions on \hat{G} vanishing at infinity with pointwise multiplication, complex conjugation and the Sup norm $\|\cdot\|_\infty$. For each $k \in \hat{G}$ we define $\{\alpha_k(f)\}(p) = f(p - k)$, $f \in \mathcal{C}_0(\hat{G})$, $p \in \hat{G}$. Since $\mathcal{C}_0(\hat{G}) = \mathfrak{A}$ is abelian we have a trivially asymptotically abelian system $\{\mathfrak{A}, \alpha\}$. Now the set $A(\hat{G})$ of Fourier transforms of L_1 -functions on G is a dense sub- $*$ -algebra of \mathfrak{A} and therefore, to prove that all means on \mathfrak{A} and on \mathfrak{A}^* vanish, it suffices to verify that they vanish on $A(\hat{G})$. Now for $f \in A(\hat{G})$ and μ a bounded measure over \hat{G} one has

$$\langle \mu, M_\eta(f) \rangle (= \langle M_\eta(\mu), f \rangle) = \eta \{ \langle \mu \alpha_k(f) \rangle \} = 0;$$

since $\langle \mu, \alpha_k(f) \rangle (= f * \mu)$ is an element of $A(\hat{G}) \subset \mathcal{C}_0(\hat{G})$.

D. The Twisted Convolution Algebra $\overline{L_1(E, \sigma)}$

Acted upon by the Translations of the underlying Symplectic Space E

Let E be a finite-dimensional real vector space equipped with a non-degenerate bilinear form σ . The twisted convolution algebra $L_1(E, \sigma)$ over E (see [19]) is obtained by taking the set of Lebesgue-integrable functions over E with its L_1 -norm $\|\cdot\|_1$ and the following $*$ -operation and product \times :

$$\begin{cases} f^*(\psi) = \overline{f(-\psi)}, & f, g \in L_1(E, \sigma) \\ (f \times g)(\psi) = \int e^{-i\sigma(\xi, \psi)} f(\xi) g(\psi - \xi) d\xi, & \psi \in E \end{cases} \quad (23)$$

$L_1(E, \sigma)$ is shown to be a Banach $*$ -algebra possessing a unique irreducible representation up to unitary equivalence. Since this representation is faithful, the completion $\overline{L_1(E, \sigma)} = \mathfrak{A}$ of $L_1(E, \sigma)$ in the operator norm is a C^* -algebra which is shown to be isomorphic to the C^* -algebra of compact operators on the irreducible representation space. Furthermore the $*$ -operation and product (23) can be extended to the set $M_1(E, \sigma)$ of bounded complex measures over E by defining, for each continuous function f on E with compact support

$$\begin{cases} \langle \mu^*, f \rangle = \overline{\langle \mu, f^* \rangle}, & \mu, \nu \in M_1(E, \sigma) \\ \langle \mu \times \nu, f \rangle = \int d\mu(\eta) \int d\nu(\xi) e^{-i\sigma(\eta, \xi)} f(\xi + \eta). \end{cases} \quad (24)$$

$M_1(E, \sigma)$ with the $\|\cdot\|_1$ -norm of measures thus becomes a *-Banach algebra in which $L_1(E, \sigma)$, identified with the set of measures absolutely continuous with respect to Lebesgue-measure, is a (two-sided) *-ideal. Furthermore the definitions (24) are compatible with the extension of the operation (23) on $L_1(E, \sigma)$ to its C^* -completion \mathfrak{A} .

We now assign to each $u \in E$ the *-automorphism α_u of \mathfrak{A} defined by

$$\alpha_u(a) = \delta_u \times a \times \delta_{-u} \quad a \in \mathfrak{A} \tag{25}$$

where δ_u is the Dirac measure on E at u . Since, according to (24), one has

$$\begin{cases} \delta_u^* = \delta_{-u} = \delta_u^{-1} \\ \delta_u \times \delta_v = e^{i\sigma(u, v)} \delta_{u+v} \end{cases} \tag{26}$$

$u \rightarrow \alpha_u$ is a homomorphic mapping of the additive group of E into the group of *-automorphism of \mathfrak{A} . Further, one easily calculates from (24) that, for $f \in L_1(E, \sigma)$,

$$\{\alpha_u(f)\}(\psi) = e^{-2i\sigma(u, \psi)} f(\psi) \quad \text{a.e. in } \psi \in E. \tag{27}$$

We now show that for each $a \in \mathfrak{A}$ and each continuous linear form Φ over \mathfrak{A} one has

$$\langle \Phi, \alpha_u(a) \rangle \rightarrow \langle \Phi, a \rangle \quad \text{as } u \rightarrow 0 \tag{28}$$

and

$$\langle \Phi, \alpha_u(a) \rangle \rightarrow 0 \quad \text{as } u \rightarrow \infty, \tag{29}$$

that is,

$$\langle \Phi, \alpha_{\hat{u}}(a) \rangle \in \mathcal{C}_0(E).$$

Since each continuous linear form on \mathfrak{A} is the difference of two positive forms and since $L_1(E, \sigma)$ is dense in \mathfrak{A} it suffices to prove (28) and (29) for a positive form Φ and for $a = f, f \in L_1(E, \sigma)$. Now each positive form Φ on $L_1(E, \sigma)$ corresponds to a function $\varphi \in L_\infty(E)$; so that, by (27),

$$\langle \Phi, \alpha_u(f) \rangle = \int e^{-2i\sigma(u, \psi)} f(\psi) \varphi(\psi) d\psi = \widehat{f\varphi}(\sigma u) \tag{30}$$

where $\widehat{f\varphi}$ is the Fourier-transform of the L_1 -function $f\varphi$.

As a consequence of the weak continuity of the mapping $u \in E \rightarrow \alpha_u(a) \in \mathfrak{A}$ expressed by (28) it is known that this mapping is also continuous in the norm topology of \mathfrak{A} [28; 10.2. Corollary]. On the other hand it follows immediately from (28) that one has the following asymptotic abelian property:

$$\langle \Phi, \alpha_u(a) \cdot b - b \cdot \alpha_u(a) \rangle \rightarrow 0, \quad \text{as } u \rightarrow \infty, \quad \begin{matrix} a, b \in \mathfrak{A} \\ \Phi \in \mathfrak{A}^* \end{matrix} \tag{31}$$

and that, for every mean η on E and all $a \in \mathfrak{A}$ and $\Phi \in \mathfrak{A}^*$

$$\eta(\langle \Phi, \alpha_{\hat{u}}(a) \rangle) = \langle \Phi, M_\eta(a) \rangle = \langle M_\eta(\Phi), a \rangle = 0. \tag{32}$$

Therefore all means on \mathfrak{A} (or on \mathfrak{A}^*) vanish.

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