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# The Yukawa Coupling of Quantum Fields in Two Dimensions. II*

JAMES GLIMM
Massachusetts Institute of Technology
Cambridge, Massachusetts

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**Abstract.** The Yukawa coupling in two dimensional space time is considered. A space cutoff is introduced in the interaction term $V$, so that the renormalized Hamiltonian $H_{\text{ren}}$ is a rigorously defined bilinear form in the Fock Hilbert space. The main result is that $H_{\text{ren}}$ is positive provided the finite part of the renormalization terms are suitably chosen. As a consequence, the Schrödinger equation $(i\partial/\partial t - H_{\text{ren}})\Phi = 0$ can be solved.

## Contents



## § 1. Discussion of the Results

We study the energy operator in a model of Quantum Field Theory. The model consists of bosons and fermions interacting with a Yukawa coupling. We consider this model in two dimensional space time and we introduce a space cut-off in the interaction energy $V$. Thus we write

$$V = \int_{t=0} : \Psi^\dagger(x)\, \Psi(x) : \Phi(x)\, h(x)\, dx\,, \tag{1.1}$$

for $h$ a function which is zero when $|x|$ is large. (The limit, $h \to 1$, would remove the space cutoff.) We studied the same problem in [2]; hereafter we refer to this paper as I. We showed that the renormalized total energy operator

$$H_{\text{ren}} = H_0 + V + (\text{infinite counter terms}) \tag{1.2}$$

was rigorously defined as a bilinear form on a domain $\mathscr{D} \times \mathscr{D}$, with $\mathscr{D}$ dense in Fock space. The counter terms depend on two parameters, and these parameters take on infinite values. The parameters can be changed

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by any finite amount and the resulting new $H_{\text{ren}}$ will still be a rigorously defined bilinear form. Such a change will be called a finite renormalization.

We wish to show that the bilinear form $H_{\text{ren}}$ is positive. We can show this only after a finite renormalization has been performed.

**Theorem A.** *There are constants $\delta_0$ and (for $\delta \geqq \delta_0$) $c_0(\delta)$ such that the bilinear form*

$$H' = H_{\text{ren}} - \delta \int : \Phi^2(x) : h^2(x)\, dx + c\, I \tag{1.3}$$

*is positive if*

$$\delta \geqq \delta_0\,, \quad c \geqq c_0(\delta)\,. \tag{1.4}$$

In this theorem $\delta_0 = \delta_0(h)$ and $c_0(\delta) = c_0(\delta, h)$ depend on $h$. As $h \to 1$, $\delta_0(h) \to \infty$ and $c_0(\delta_0(h), h) \to \infty$. For $h$ fixed, the theorem gives us a great deal of information. With the help of Friedrich's extension theorem we obtain

**Theorem B.** *Suppose $\delta \geqq \delta_0$ and $c \geqq c_0(\delta)$. Then there is a positive self adjoint operator whose bilinear form is an extension of $H'$.*

We call this positive operator $H'$ also. The spectral theorem gives us

**Theorem C.** *Suppose $\delta \geqq \delta_0$ and $c \geqq c_0(\delta)$. The Schrödinger equation*

$$i \frac{\partial}{\partial t} \varphi(t) = H' \varphi(t)$$

*has*

$$\varphi(t) = e^{-itH'} \varphi(0)$$

*as its solution.*

E. NELSON suggested using this approach to Theorem C.

We remark that if the coupling constant is small (depending on $h$) then we can choose $\delta_0 = 0 = \delta$. In the relativistic Heisenberg picture ($h = 1$), influence propagates at a finite speed. Thus the existence of solutions to the Heisenberg equations of motion is a local question, and one can use Theorem C to construct possible candidates for solutions to the relativistic Heisenberg equations. See also remarks in the introduction to I.

Theorem A does not by itself imply Theorem B because there are positive bilinear forms which do not come from positive selfadjoint operators. For example the bilinear form

$$\bar{f}, g \to f(0)\, g(0)$$

defined for $f$ and $g$ continuous on $[0, 1]$ does not come from a self adjoint operator on $L_2([0, 1])$. Thus we need to improve on Theorem A. Let $F_\tau$ be the operator

$$F_\tau = \int \mu(k)^\tau a^*(k)\, a(k)\, dk + \\ + \int \omega(p)^\tau [b^*(p)\, b(p) + b'^*(p)\, b'(p)]\, dp$$

introduced in I. In I we split the interaction energy $V$ into two parts,

$$V = V_1 + V_2$$

and we proved that $H' - V_2$ is a symmetric densely defined operator and that $V_2$ is infinitely small with respect to $F_\tau$, $2^{-1} < \tau < 1$. Our improvement on Theorem A follows.

**Theorem A$'$.** *Assume the hypothesis* (1.4) *of Theorem A. Then*

$$H' - 2^{-1} F_\tau \tag{1.5}$$

*is positive if $\tau$ is near enough to one,* $\tau < 1$.

*Proof of Theorem B.* We assume Theorem A$'$. The bilinear form $H' - V_2$ is closable. In fact $H' - V_2$ is a symmetric operator bounded from below (by I and Theorem A$'$) so the statement follows from [3, p. 318]. Also $V_2$ is bounded with respect to $F_\tau$ and $F_\tau$ in turn is bounded with respect to $H' - V_2$ by I and Theorem A$'$. Thus $H' = H' - V_2 + V_2$ is closable ([3, p. 320]) and Theorem B follows ([3, p. 322–323]).

## § 2. The Method of Proof

The remainder of this paper will be devoted to the proof of Theorem A$'$. In formal perturbation theory or in the rigorous treatment of very simple models one constructs a unitary equivalence of the total energy $H'$ with the free energy operator $H_0$,

$$H' U = U H_0 . \tag{2.1}$$

Since $H_0$ is positive (in momentum space it is multiplication by a positive function), $H'$ is seen to be positive also. The unitary transformation tells us much more than this. If we set

$$\hat{a}(k) = U a(k) U^{-1}$$
$$\hat{a}^*(k) = U a^*(k) U^{-1} , \quad \text{etc.}$$

then the operators $\hat{a}(k)$, $\hat{a}^*(k')$, etc. satisfy the same commutation and anticommutation relations as the annihilation and creation operators $a$, $a^*$, etc. We regard $a^*$, $a$ as creating or annihilating free particles and $\hat{a}^*$, $\hat{a}$, etc. as creating or annihilating physical particles. Furthermore $H'$ can be obtained by substituting $\hat{a}$ for $a$ etc. in the definition of $H_0$. In fact

$$H' = \int \hat{a}^*(k)\, \mu(k)\, \hat{a}(k)\, dk + \int \omega(p)\, [\hat{b}^*(p)\, \hat{b}(p) + \hat{b}'^*(p)\, \hat{b}'(p)]\, dp . \tag{2.2}$$

We used an approximate version of (2.1) to define $H'$, but this equation does not seem to be helpful in showing that $H'$ is positive. Instead we use an approximate version of (2.2). There is a formal expression for $\hat{a}$, etc. in terms of powers of $V$. The zero and first order terms are given by

$$\hat{a}(k) = a(k) + \frac{\delta \Gamma V}{\delta a^*(k)} + \cdots \tag{2.3}$$

$$\hat{a}^*(k) = \hat{a}(k) - \frac{\delta \Gamma V}{\delta a(k)} + \cdots, \tag{2.4}$$

and there are similar expressions for $\hat{b}$, $\hat{b}^*$, $\hat{b}'$ and $\hat{b}'^*$. In essence our method is the following. We substitute part of the first two terms from (2.3), etc. in the right member of (2.2). The resulting bilinear form $H''$ is obviously positive. Also $H''$ is approximately equal to $H'$, so the proof is completed by obtaining a bound for $H' - H''$.

Before giving the approximate expressions for the operators which create and annihilate physical particles, we introduce some definitions. Let

$$\bar{\mu}(k) = \mu(k) - (3/4)\,\mu^\tau(k) \tag{2.5}$$

$$\bar{\omega}(p) = \omega(p) - (3/4)\,\omega^\tau(p) \tag{2.6}$$

$$\overline{H}_0 = H_0 - (3/4)\,F_\tau\,. \tag{2.7}$$

In this definition we suppose that $2^{-1} < \tau < 1$ and that $\tau$ is close enough to one so that $\bar{\mu}$ and $\bar{\omega}$ are always positive. We recall from I, Sec. 4 the definition of $\Xi$ as the region in the $p_1$, $p_2$ plane for which

$$|\eta| \leqq |\xi|^{2(1-\tau)}$$

$$\eta = p_1 + p_2\,, \quad \xi = p_1 - p_2\,.$$

We introduce an operation $\Gamma$ which is an approximate inverse to $\mathrm{ad}\,\overline{H}_0$. We do not make the simplest possible choice of $\Gamma$. Instead we choose a $\Gamma$ which will simplify some later computations. If

$$Q_i = \int q_i\, dk\, dp_1\, dp_2$$

($i = 1, 2$) is defined as in I, Sec. 1, then we set

$$Q_i(\Xi) = \int_{\Xi} q_i\, dk\, dp_1\, dp_2$$

$$Q_i(\sim\Xi) = \int_{\sim\Xi} q_i\, dk\, dp_1\, dp_2\,.$$

The infinite mass renormalization is associated with $Q_i(\Xi)$. Let

$$\Gamma Q_i(\Xi) = \int_{\Xi} (\bar{\omega}_1 + \bar{\omega}_2)^{-1}\, q_i\, dk\, dp_1\, dp_2$$

$$\Gamma Q_i(\sim\Xi) = \int_{\sim\Xi} (\bar{\omega}_1 + \bar{\omega}_2 + \bar{\mu})^{-1}\, q_i\, dk\, dp_1\, dp_2$$

$$\Gamma(Q_i(\Xi)^*) = -(\Gamma Q_i(\Xi))^*$$

$$\Gamma(Q_i(\sim\Xi)^*) = -(\Gamma Q_i(\sim\Xi))^*\,.$$

The operator $\bar{\Gamma} V_1$ is a polynomial in the operators $a$, $a^*$, etc. Thus the functional derivatives

$$\frac{\delta \bar{\Gamma} V_1}{\delta a^*(k)}\,, \text{ etc.}$$

have an obvious meaning. For example

$$\frac{\delta \bar{\Gamma} Q_1(\Xi)}{\delta a^*(k)} = \int\limits_{\Xi} (\bar{\omega}_1 + \bar{\omega}_2)^{-1} \tilde{q}_1(p_1, p_2, k)\, b^*(p_1)\, b'^*(p_2)\, dp_1\, dp_2$$

if $q_1(p_1, p_2, k) = \tilde{q}_1(p_1, p_2, k)\, a^*(k)\, b^*(p_1)\, b'^*(p_2)$ as in I. Actually $\bar{\Gamma} V_1$ is linear in the boson operators, in the nucleon operators and in the anti-nucleon operators. Thus

$$\bar{\Gamma} V_1 = \int \left[a^*(k) \frac{\delta \bar{\Gamma} V_1}{\delta a^*(k)} + a(k) \frac{\delta \bar{\Gamma} V_1}{\delta a(k)}\right] dk$$

and there are similar formulas for the $b(p)$'s and $b'(p)$'s. Of more importance to us is a related formula. We set

$$W_1 = V_1(\Xi) = Q_1(\Xi) + Q_2(\Xi) + Q_1(\Xi)^* + Q_2(\Xi)^*$$

$$W_2 = Q_1(\sim \Xi) + Q_1(\sim \Xi)^* \,.$$

$$W = V_1(\Xi) + W_2 \,.$$

Then

$$\begin{aligned} W = &\int \bar{\mu}(k) \left[a^*(k) \frac{\delta \bar{\Gamma} W_2}{\delta a^*(k)} - \frac{\delta \bar{\Gamma} W_2}{\delta a(k)} a(k)\right] dk + \\ &+ \int \bar{\omega}(p) \left[b^*(p) \frac{\delta \bar{\Gamma} W}{\delta b^*(p)} - \frac{\delta \bar{\Gamma} W}{\delta b(p)} b(p)\right] dp + \qquad (2.8) \\ &+ \int \bar{\omega}(p) \left[-b'^*(p) \frac{\delta \bar{\Gamma} W}{\delta b'^*(p)} + \frac{\delta \bar{\Gamma} W}{\delta b'(p)} b'(p)\right] dp \,. \end{aligned}$$

This formula has only formal significance, since $V_1(\Xi)$ is not an operator. To obtain formulas with a rigorous meaning we use the cutoff operators $W_{\varrho\sigma}$, etc. introduced as in I, Sec. 1. The parameter $\varrho$ is a lower cutoff on the fermion momenta. Because of this cutoff, fermions do not appear in $W_{\varrho\sigma}$ with momenta of magnitude smaller than $\varrho$. The parameter $\sigma$ is a pair, $\sigma = \sigma_b, \sigma_f$, and this cutoff is an upper cutoff. Thus the boson in $W_{\varrho\sigma}$ has a momentum at most $\sigma_b$ in magnitude and the fermion momenta are at most $\sigma_f$ in magnitude. We choose later a large fixed value for $\varrho$; we will let $\sigma \to \infty$. This will introduce an error $W - W_\varrho$ which can be estimated easily.

We define

$$\hat{a}^*_\sigma(k) = a^*(k) - \frac{\delta \bar{\Gamma} W_{2\varrho\sigma}}{\delta a(k)}$$

$$\hat{a}_\sigma(k) = a(k) + \frac{\delta \bar{\Gamma} W_{2\varrho\sigma}}{\delta a^*(k)}$$

$$\hat{b}^*_\sigma(p) = b^*(p) - \frac{\delta \bar{\Gamma} W_{\varrho\sigma}}{\delta b(p)}$$

$$\hat{b}_\sigma(p) = b(p) + \frac{\delta \bar{\Gamma} W_{\varrho\sigma}}{\delta b^*(p)}$$

$$\hat{b}'^*_\sigma(p) = b'^*(p) + \frac{\delta \bar{\Gamma} W_{\varrho\sigma}}{\delta b'(p)}$$

$$\hat{b}'_\sigma(p) = b'(p) - \frac{\delta \bar{\Gamma} W_{\varrho\sigma}}{\delta b'^*(p)}$$

As an approximate expression for $H' - 2^{-1}F_\tau$, let $H''(\sigma)$ be the result of substituting $\hat{a}^*_\sigma$, etc. for $a^*$ in the definition of $\overline{H}_0$. Thus

$$H''(\sigma) = \int \hat{a}^*_\sigma(k)\, \bar{\mu}(k)\, \hat{a}_\sigma(k)\, dk + \tag{2.9}$$
$$+ \int \bar{\omega}(p)\, [\hat{b}^*_\sigma(p)\, \hat{b}_\sigma(p) + \hat{b}'^*_\sigma(p)\, \hat{b}'_\sigma(p)]\, dp$$
$$= \overline{H}_0 + W_{\varrho\sigma} + 2^{nd} \text{ order terms}\,. \tag{2.10}$$

To obtain the second equality, we used (2.8). The terms of second order in (2.10) are second order in the coupling constant, or in other words in $V$. They will be dominated either by $\varepsilon F_\tau + KI$ or by the (finite) cutoff renormalization counterterms. This is a little surprising since these terms in (2.10) are fourth order in $a^*$, $a$, etc. while $F_\tau$ and the counterterms are of second order in $a^*$, $a$, etc.

## § 3. Estimate for the Error Terms

If we substitute

$$a^*(k) - \frac{\delta \bar{\Gamma} W_{2\varrho\sigma}}{\delta a_\sigma(k)}$$

etc. for $\hat{a}^*_\sigma(k)$, etc. in (2.9) and then expand we get $H''_1(\sigma)$ represented as a sum of terms of order zero, one and two in the coupling constant. The sum of the zero order terms is exactly $\overline{H}_0$ and the sum of the first order terms is exactly $W_{\sigma\varrho}$. This is precisely the content of formula (2.10). The error $V_\sigma - W_{\varrho\sigma}$ can be estimated easily by use of I, Sec. 2.4 and I, Sec. 4.

**Lemma 3.1.** *The operator $R(\sigma) = V_\sigma - W_{\varrho\sigma}$ is infinitely small with respect to $F_\tau$, uniformly in $\sigma$. Thus if $\varepsilon > 0$ there is a $K = K(\varepsilon)$ which does not depend on $\sigma$ such that $R(\sigma)$ is dominated by $\varepsilon F_\tau + K(\varepsilon) I$.*

*Proof*. By the definition of $W$,

$$V_\sigma - W_{\varrho\sigma} = V_{2\sigma} + (W_\sigma - W_{\varrho\sigma}) + Q_{2\sigma}(\sim \Xi) + Q_{2\sigma}(\sim \Xi)^*\,.$$

The terms $V_{2\sigma}$ and $Q_{2\sigma}(\sim \Xi)$ were treated in I, Sec. 4. It follows that the adjoint $Q_{2\sigma}(\sim \Xi)^*$ is also infinitely small with respect to $F_\tau$. The term $W_\sigma - W_{\varrho\sigma}$ has an $L_2$ kernel which depends continuously in $L_2$ on $\sigma$ and which converges in $L_2$ to a limit as $\sigma \to \infty$. By the corollary to Theorem 2.4.3 of I, $W_\sigma - W_{\varrho\sigma}$ is dominated by $\varepsilon F_\tau + K(\varepsilon, \sigma) I$. Because of the continuity properties of the kernel, $K(\varepsilon, \sigma)$ can be chosen to be continuous in $\sigma$ and to approach a limit as $\sigma \to \infty$. It is thus bounded uniformly in $\sigma$.

Next we consider the second order term in (2.9) which comes from the meson integration. There is only one such term and it is

$$- \int \bar{\mu}(k) \frac{\delta \bar{\Gamma} W_{2\varrho\sigma}}{\delta a(k)} \frac{\delta \bar{\Gamma} W_{2\varrho\sigma}}{\delta a^*(k)} dk . \tag{3.1}$$

This expression is not Wick ordered, but it can be written as a sum of four Wick ordered terms. The totally contracted term, one of these four, contributes to the constant renormalization counter term. This term has the form $c_{1\varrho\sigma}(\sim \Xi) I$, where

$$c_{1\varrho\sigma}(\sim \Xi) = \int_{\sim \Xi} \chi_{\varrho\sigma} \bar{\mu} (\bar{\omega}_1 + \bar{\omega}_2 + \bar{\mu})^{-2} |\tilde{q}_1|^2 \, dp_1 \, dp_2 \, dk$$

and where $\chi_{\varrho\sigma}$ is the appropriate cutoff function.

**Lemma 3.2.** *The operator* (3.1) *has the form*

$$c_{1\varrho\sigma}(\sim \Xi) \, I + R(\sigma)$$

*where $R(\sigma)$ is an operator which is infinitely small with respect to $F_\tau$, uniformly in $\sigma$.*

*Proof*. Let $R_1(\sigma)$ be the term in (3.1) which contains no contractions (beyond the given meson integration), and let $R_2(\sigma) = R(\sigma) - R_1(\sigma)$. The two terms in $R_2(\sigma)$ each contain a fermion contraction. Let

$$S_1(\sigma, k) = \bar{\mu}(k)^{1/2} \frac{\delta \bar{\Gamma} Q_{1\varrho\sigma}(\sim \Xi)}{\delta a^*(k)}$$

and let

$$R_1(\sigma, k) = S_1(\sigma, k) \, S_1(\sigma, k)^* .$$

Then

$$R_1(\sigma) = \int R_1(\sigma, k) \, dk .$$

In the same fashion we can write

$$R_2(\sigma) = \int R_2(\sigma, k, p) \, dk \, dp$$

where

$$R_2(\sigma, k, p) = S_2(\sigma, k, p) \, S_2(\sigma, k, p)^*$$

$$S_2(\sigma, k, p) = \int_{\xi \notin \Xi} \chi_{\varrho\sigma} (\bar{\omega}_1 + \bar{\omega}_2 + \bar{\mu}(k))^{-1} \bar{\mu}(k)^{1/2} [\tilde{q}_1(k, q, p) \, b^*(q) + \\ + \tilde{q}_1(k, p, q) \, b'^*(q)] \, dq .$$

The kernel of $S_1(\sigma, k)$ is $\bar{\mu}^{1/2}(\bar{\omega}_1 + \bar{\omega}_2 + \bar{\mu})^{-1} \tilde{q}_1 \chi_{\varrho\sigma} \chi(\sim \Xi)$, where $\chi(\sim \Xi)$ is the function which is equal to one off $\Xi$ and is equal to zero on $\Xi$. The limiting function (as $\sigma \to \infty$) is not in $L_2$, but its product with $\omega^{-\tau/2}$ is in $L_2$,

$$\omega_1^{-\tau/2} \bar{\mu}^{1/2} (\bar{\omega}_1 + \bar{\omega}_2 + \mu)^{-1} \tilde{q}_1 \chi(\sim \Xi) \in L_2 .$$

Thus we can find a smooth function $s_0 = s_0(k, p_1, p_2)$ not depending on $\sigma$ such that

$$\|\omega_1^{-\tau/2}(s_0 - \bar{\mu}^{1/2}(\bar{\omega}_1 + \bar{\omega}_2 + \bar{\mu})^{-1} \tilde{q}_1 \chi_{\varrho\sigma}) \chi(\sim \Xi)\|_2 < \varepsilon$$

for large $\sigma$, and we can also require that $s_0$ have compact support. Let $S_0(k)$ be the operator

$$S_0(k) = \int s_0(k, p_1, p_2)\, b^*(p_1)\, b'^*(p_2)\, dp_1\, dp_2 .$$

Then

$$S_0 = \int S_0(k)\, S_0(k)^*\, dk$$

is infinitely small with respect to $F_\tau$, by $I$, Theorem 2.4.2. Also

$$R_1(\sigma) - S_0 = \int [S_1(\sigma, k) - S_0(k)]\, S_1(\sigma, k)^*\, dk + \int S_0(k)\, [S_1(\sigma, k)^* - S_0(k)^*]\, dk .$$

Let $s_{1\sigma}(k, p_1, p_2)$ be the kernel of $S_1(\sigma, k)$. We estimate the integrand in the first term as follows:

$$\|(F_\tau + I)^{-1/2}\, [S_1(\sigma, k) - S_0(k)]\, S_1(\sigma, k)^*\, (F_\tau + I)^{-1/2}\| \leqq \leqq \text{const.}\, \|\omega_1^{-\tau/2}(s_{1\sigma}(k, \,.\,, .) - s_0(k, \,.\,, .))\|_2\, \|\omega_1^{-\tau/2} s_{1\sigma}(k, \,.\,, .)\|_2 ,$$

because of $I$, Theorem 2.4.3 and the remarks following it. There is a corresponding estimate for the second term. We integrate over $k$ in these estimates and use the Schwartz inequality together with identities such as

$$(\int \|\bar{\omega}_1^{-\tau/2} s_{1\sigma}(k, \,.\,, .)\|_2^2\, dk)^{1/2} = \|\omega_1^{-\tau/2} s_{1\sigma}\|_2 .$$

This gives us the bound

$$\|(F_\tau + I)^{-1/2}\, (R_1(\sigma) - S_0)\, (F_\tau + I)^{-1/2}\| \leqq \leqq \text{const.} \|\omega_1^{-\tau/2}(s_{1\sigma} - s_0)\|_2\, \|\omega_1^{-\tau/2} s_{1\sigma}\|_2 + + \text{const.} \|\omega_1^{-\tau/2} s_0\|_2\, \|\omega_1^{-\tau/2}(s_{1\sigma} - s_0)\|_2 ,$$

and so $R_1(\sigma) - S_0$ is bounded by

$$\text{const.}\, \varepsilon (\|\omega_1^{-\tau/2} s_{1\sigma}\|_2 + 1)\, (F_\tau + I) .$$

The estimates on $R_2(\sigma)$ are similar since the kernel of $S_2(\sigma, k, p)$ is essentially the same as the kernel of $S_1(\sigma, k)$ if each is considered as a function of three variables. This completes the proof.

Next we consider second order contributions to (2.9) from the fermion (for example, nucleon) integration. First we consider terms of this form which are also linear or quadratic in $W_2$. These terms are not Wick ordered, but they can be written as a sum of Wick ordered terms. One of these terms is totally contracted and so it has the form of a constant multiple of the identity. The constant is

$$c_{2\varrho\sigma}(\sim\Xi) = \int_{\sim\Xi} \chi_{\varrho\sigma}\bar{\omega}_1(\bar{\omega}_1 + \bar{\omega}_2 + \bar{\mu})^{-2}\,|\tilde{q}_1|^2\,dp_1\,dp_2\,dk\,.$$

This term contributes to the constant renormalization counter term. The remaining terms are not all infinitely small with respect to $F_\tau$ (because some terms are quadratic in the boson operators $a$ and $a^*$), but they can be dominated by a multiple of $F_\tau + I$. This multiple is small if $\varrho$ is large.

**Lemma 3.3.** *The second order terms in* (2.9) *which come from the nucleon integration and which are linear or quadratic in $W_2$ have the form*

$$c_{2\varrho\sigma}(\sim\Xi)\,I + R_2(\sigma, \varrho)\,.$$

*Here $R_2(\sigma, \varrho)$ is an operator and $R_2(\sigma, \varrho)$ is dominated by $o(1)\,(F_\tau + I)$ where $o(1)$ is a constant depending on $\varrho$ but not on $\sigma$ and*

$$o(1) \to 0 \quad as \quad \varrho \to \infty\,.$$

We remark that there is an analogous lemma concerning the antinucleon integration in (2.9). The constant term becomes

$$c_{3\varrho\sigma}(\sim\Xi)I = \int_{\sim\Xi} \chi_{\varrho\sigma}\bar{\omega}_2(\bar{\omega}_1 + \bar{\omega}_2 + \bar{\mu})^{-2}\,|\tilde{q}_1|^2\,dk\,dp\,I\,.$$

Let $R_3(\sigma, \varrho)$ be the sum of the remaining terms and set $R(\sigma, \varrho) = R_2(\sigma, \varrho) + R_3(\sigma, \varrho)$. Evidently

$$\sum_{i=1}^{3} c_{i\varrho\sigma}(\sim\Xi) = \int_{\sim\Xi} \chi_{\varrho\sigma}(\bar{\omega}_1 + \bar{\omega}_2 + \bar{\mu})^{-1}\,|\tilde{q}_1|^2\,dk\,dp$$

and this is a good approximation to

$$c_\sigma(\sim\Xi) = \int_{\sim\Xi} \chi_\sigma(\omega_1 + \omega_2 + \mu)^{-1}\,|\tilde{q}_1|^2\,dk\,dp\,.$$

We also define

$$c_\sigma(\Xi) = \int_{\Xi} \chi_\sigma(\omega_1 + \omega_2 + \mu)^{-1}\,|\tilde{q}_1|^2\,dk\,dp$$

and recall that

$$c_\sigma I = (c_\sigma(\Xi) + c_\sigma(\sim\Xi))\,I$$

is the constant renormalization counter term for our cutoff Hamiltonian $H_{\mathrm{ren}}(\sigma)$.

**Lemma 3.4.** *There is a constant $K$ depending on $\varrho$ but not on $\sigma$ such that*

$$d_\sigma = \Big|\sum_{i=1}^{3} c_{i\varrho\sigma}(\sim\Xi) - c_\sigma(\sim\Xi)\Big| < K\,.$$

We summarize our progress to this point by rewriting (2.9)

$$\begin{aligned} H''(\sigma) &= \overline{H}_0 + V_\sigma + c_\sigma(\sim \Xi) I - \\ &- \int \overline{\omega}(p) \left[ \frac{\delta \overline{\Gamma} W_{1\varrho\sigma}}{\delta b(p)} \frac{\delta \overline{\Gamma} W_{1\varrho\sigma}}{\delta b^*(p)} + \frac{\delta \overline{\Gamma} W_{1\varrho\sigma}}{\delta b'(p)} \frac{\delta \overline{\Gamma} W_{1\varrho\sigma}}{\delta b'^*(p)} \right] dp + \qquad (3.2) \\ &+ R(\sigma) + R(\sigma, \varrho) + d_\sigma I . \end{aligned}$$

*Proof of Lemma* 3.3. Let $\chi(\Xi)$ be the function which is identically one on $\Xi$ and identically zero off $\Xi$. First we observe that the functions

$$\begin{aligned} s_1 &= \bar{\mu}^{-\tau/2} (\overline{\omega}_1 + \overline{\omega}_2 + \bar{\mu})^{-1} \, \overline{\omega}_1^{(1/2)+\gamma} \tilde{q}_1 \chi(\sim \Xi) \chi_{\varrho\sigma} \\ s_2 &= \overline{\omega}_2^{-\tau/2} (\overline{\omega}_1 + \overline{\omega}_2 + \bar{\mu})^{-1} \, \overline{\omega}_1^{(1/2)+\gamma} \tilde{q}_1 \chi(\sim \Xi) \chi_{\varrho\sigma} \\ s_3 &= (\overline{\omega}_1 + \overline{\omega}_2 + \bar{\mu})^{-1} \, \overline{\omega}_1^{(1/2)-\gamma} \tilde{q}_1 \chi(\sim \Xi) \chi_{\varrho\sigma} \\ s_4 &= (\overline{\omega}_1 + \overline{\omega}_2)^{-1} \, \overline{\omega}_1^{(1/2)-\gamma} \tilde{q}_1 \chi(\Xi) \chi_{\varrho\sigma} \\ s_5 &= (\overline{\omega}_1 + \overline{\omega}_2)^{-1} \, \overline{\omega}_1^{(1/2)-\gamma} \tilde{q}_2 \chi(\Xi) \chi_{\varrho\sigma} \end{aligned}$$

are in $L_2$ for some $\gamma > 0$ and their $L_2$ norms are bounded uniformly in $\sigma$. Also their $L_2$ norms tend to zero as $\varrho \to \infty$, uniformly in $\sigma$. Consider a term in $R_2$ which is quadratic in the $a^{\#}$ ($= a$ or $a^*$) and quadratic in the $b'^{\#}$ also. This term or its adjoint has the form

$$\begin{aligned} \int dp_1 \big(\int s_1(k, p_1, p_2)\, \bar{\mu}^{\tau/2} a^*(k)\, b'^*(p_2)\, dk\, dp_2\big) \times \\ \times \big(\int \bar{s}_j(l, p_1, p')\, a^{\#}(l)\, b'(p')\, dl\, dp'\big) , \end{aligned}$$

and $j \geqq 3$. We apply *I*, Theorem 2.4.3 to the integrand for each value of $p_1$ and conclude that this operator is bounded by

$$\begin{aligned} \text{const.} (F_\tau + I) \int dp_1 \big(\int |s_1|^2 dk\, dp_2\big)^{1/2} \big(\int |s_j|^2 dl\, dp'\big)^{1/2} &\leqq \\ \leqq \text{const.} (F_\tau + I) \big(\int |s_1|^2 dk\, dp_1\, dp_2\big)^{1/2} \big(\int |s_j|^2 dl\, dp_1\, dp'\big)^{1/2} \\ = \text{const.} \|s_1\| \, \|s_j\| \, (F_\tau + I) . \end{aligned}$$

This proves the lemma as far as the fourth degree contributions to $R_2$ are concerned.

Next consider a term in $R_2$ which is quadratic in the $a^{\#}$ and contains no fermion operators. This term has the form

$$\int dp_1\, dp_2 \big(\int s_1(k, p_1, p_2)\, \bar{\mu}^{\tau/2} a^*(k)\, dk\big) \big(\int s_3(l, p_1, p_2)\, a(l)\, dl\big)$$

and as before this term is bounded by

$$\text{const.} \|s_1\| \, \|s_3\| \, (F_\tau + I) .$$

Finally consider a term which is quadratic in the $b'^{\#}$ and which contains no meson operators. This term or its adjoint has the form

$$\int dk\, dp_1 \big(\int s_2(k, p_1, p_2)\, \overline{\omega}_2^{\tau/2} b'^*(p_2)\, dp_2\big) \big(\int s_j(k, p_1, p')\, b'(p')\, dp'\big)$$

and as before this term can be bounded by

$$\text{const.}\,\|s_2\|\,\|s_j\|\,(F_\tau + I)\,,$$

for $j \geqq 3$.

*Proof of Lemma* 3.4. The difference between $c_\sigma(\sim \Xi)$ and

$$c_{\varrho\sigma}(\sim \Xi) = \int_{\sim \Xi} \chi_{\varrho\sigma}(\omega_1 + \omega_2 + \mu)^{-1}\,|\tilde{q}_1|^2\,dk\,dp_1\,dp_2$$

is bounded uniformly in $\sigma$ because of the rapid decrease in the factor $\hat{h}(\eta + k)$ which appears in $\tilde{q}_1$. We compute

$$\sum_{i=1}^{3} c_{i\varrho\sigma}(\sim \Xi) - c_{\varrho\sigma}(\sim \Xi)$$
$$= (3/4) \int_{\sim \Xi} (\omega_1^\tau + \omega_2^\tau + \mu^\tau)\,(\bar{\omega}_1 + \bar{\omega}_2 + \bar{\mu})^{-1}\,(\omega_1 + \omega_2 + \mu)^{-1}\,\chi_{\varrho\sigma}|\tilde{q}_1|^2\,.$$

This integral is bounded uniformly in $\sigma$.

## § 4. Estimates for the Error Terms, Cont'd

It is clear that the operator $H''(\sigma)$ is a positive bilinear form. The bilinear form

$$H' - 2^{-1}F_\tau$$

is the form which we are trying to show to be positive. We have made some progress in showing the relation between $H''(\sigma)$ and $H' - 2^{-1}F_\tau$, see (3.2). It remains for us to examine the integral in (3.2) and to show that it can be dominated by

$$(\delta m_0 - \delta) \int :\Phi_\sigma^2(x):\,h^2(x)\,dx + c_\sigma(\Xi)\,I + \varepsilon(F_\tau + I)\,. \tag{4.1}$$

This will be done in § 5. In order to facilitate this we modify and simplify (4.1) and the integral in (3.2). These changes introduce new error terms and in this section we show that the new terms can be dominated by a small multiple of $F_\tau + I$. It is precisely in order to keep this multiple small that we choose $\delta$ and $\varrho$ large.

We let $\Xi_{\varrho\sigma}$ be the set

$$\{p_1, p_2 \in \Xi : 2\varrho \leqq |\xi| \leqq 2\sigma_f\}\,.$$

Let

$$\begin{aligned} :\Delta_{\varrho\sigma}(\Xi): = 2 \int_{\Xi_{\varrho\sigma}} &(1 + |\xi|)^{-1} :\Phi_\sigma(k)\,\Phi_\sigma(l):\,\hat{h}(\eta - k)\,\hat{h}(-\eta - l)\,\times \\ &\times\,\mu(k)^{-1/2}\,\mu(l)^{-1/2}\,d\xi\,d\eta\,dk\,dl \end{aligned} \tag{4.2}$$

where $\Phi(k) = a(k) + a^*(-k)$. Also we let $\Delta_{\varrho\sigma}(\Xi)$ be the corresponding integral in which the Wick product $:\Phi_\sigma(k)\,\Phi_\sigma(l):$ has been replaced by an ordinary product.

**Lemma 4.1.** *Let*

$$\delta = 4\ln(2\varrho + 1)\,. \tag{4.3}$$

*Then the first two terms in* (4.1) *can be written in the form*

$$\Delta_{\varrho\sigma}(\Xi) + P(\sigma) + e_\sigma I \tag{4.4}$$

*where $e_\sigma$ is a number bounded uniformly as $\sigma \to \infty$, $P(\sigma)$ is an operator bounded by $o(1)\,(F_\tau + I)$,*

$$o(1) \to 0 \quad as \quad \varrho \to \infty$$

*and $o(1)$ does not depend on $\sigma$.*

*Proof.* Let

$$s = (1 + |\xi|)^{-1/2}\,\mu^{-1/2}\hat{h}(\eta - k)\,.$$

Then $\mu^{-\gamma}s \in L_2$ on the set $\sim\Xi$, for any $\gamma > 0$. Let $(\sim\Xi)_{\varrho\sigma}$ be the set

$$\{p_1, p_2 \in \sim\Xi, 2\varrho \leqq |\xi| \leqq 2\sigma_f\}$$

and let

$$:\Delta_{\varrho\sigma}(\sim\Xi):\, = 2 \int_{(\sim\Xi)_{\varrho\sigma}} :\left(\int \Phi_\sigma(k)\, s(p_1, p_2, k)\, dk\right) \times \tag{4.5}$$
$$\times \left(\int \Phi_\sigma(l)\, s(p_1, p_2, l)^*\, dl\right): d\xi\, d\eta\,.$$

Now the sum $:\Delta_{\varrho\sigma}:$ of (4.2) and (4.5) is just the first term in (4.1). In fact the integrands in (4.2) and (4.5) are identical because of the identities $\Phi_\sigma(-l) = \Phi_\sigma(l)^*$ and $\hat{h}(-a) = \hat{h}(a)^-$. Furthermore the first term in (4.1) is an integral with the same integrand. This follows from the definitions of $\xi$, $\eta$ and $\delta m_\sigma$; see I, Sec. 3.5. $P(\sigma)$ is defined to be (4.5). We use formula (2.4.7) from Theorem 2.4.3 of I to bound each factor in the integrand of (4.5). We conclude that (4.5) is bounded by

$$\text{const.} \int_{(\sim\Xi)_{\varrho\sigma}} \|\mu^{-\tau/4}s(p_1, p_2, \cdot)\|^2\, dp_1\, dp_2 (F_\tau + I)\,.$$

The coefficient of $(F_\tau + I)$ is small uniformly in $\sigma$, as $\varrho \to \infty$.

We have shown that the sum of the first two terms in (4.1) is equal to

$$:\Delta_{\varrho\sigma}(\Xi): + P(\sigma) + c_\sigma(\Xi) I\,.$$

Let

$$e_\sigma = c_\sigma(\Xi) - 2 \int_{|k| \leqq \sigma_b} \int_{\Xi_{\varrho\sigma}} (1 + |\xi|)^{-1}\,\mu^{-1}|\hat{h}(\eta + k)|^2\, d\xi\, d\eta\, dk\,. \tag{4.6}$$

The second term is the fully contracted part of $\Delta_{\varrho\sigma}(\Xi)$ and is thus exactly the difference

$$:\Delta_{\varrho\sigma}(\Xi): - \Delta_{\varrho\sigma}(\Xi)\,.$$

This proves (4.4).

Let $S$ be defined by formula (1.1) of I and recall that $S^2 = 4 + O(|\xi|^{-2})$ in $\Xi$. Then $e_\sigma$ is bounded by two integrals. The first integral contains the effect of the differing regions of integration in the two terms of (4.6). The other integral is

$$\int_{|k| \leqq \sigma_b} \int_{\Xi_{\varrho\sigma}} |4(1 + |\xi|)^{-1} - (\omega_1 + \omega_2 + \mu)^{-1}\, S^2|\,\mu^{-1}|\hat{h}|^2\, dp_1\, dp_2\, dk\,.$$

This integral is bounded uniformly in $\sigma$ because

$$|4(1+|\xi|)^{-1}-(\omega_1+\omega_2+\mu)^{-1}S^2| \leqq \text{const.}\,[|\xi|^{-3}+4\mu(1+|\xi|)^{-2}]$$

in $\Xi$ and because

$$(1+|\xi|)^{-2}\,|\hat{h}|^2$$

is in $L_1$ on $\Xi$. This completes the proof.

Again we let

$$s=(1+|\xi|)^{-1/2}\mu^{-1/2}\hat{h}(\eta-k)$$

and we let

$$Z_{\varrho\sigma}(p)=\int\limits_{\Xi_{\varrho\sigma}}-2\,\operatorname{sgn}\xi\, s(p_1,p,k)\, b^*(p_1)\,\Phi_\sigma(k)\,dk\,dp_1\,.$$

The next lemma simplifies one term in the integral in (3.2). There is an analogous lemma dealing with the other term in the integral.

**Lemma 4.2.**

$$-\int\bar{\omega}(p)\frac{\delta\bar{\Gamma}W_{1\varrho\sigma}}{\delta b'(p)}\frac{\delta\bar{\Gamma}W_{1\varrho\sigma}}{\delta b'^*(p)}\,dp=2^{-1}Z_{\varrho\sigma}(p)^*Z_{\varrho\sigma}(p)\,dp+P_1(\sigma)\,.$$

*The operator* $P_1(\sigma)$ *is bounded by* $o(1)\,(F_\tau+I)$,

$$o(1)\to 0 \quad \textit{as} \quad \varrho\to\infty$$

*and* $o(1)$ *does not depend on* $\sigma$.

*Proof.* Let

$$Y_{\varrho\sigma}(p)=(2\bar{\omega}(p))^{1/2}\frac{\delta\bar{\Gamma}W_{1\varrho\sigma}}{\delta b'^*(p)}\,.$$

Then $Y_{\varrho\sigma}(p)$ and $Z_{\varrho\sigma}(p)$ each have kernels dominated by a multiple of $|s(\cdot,p,\cdot)|$ on the set $\Xi$. The kernels of $Y_{\varrho\sigma}(p)-Z_{\varrho\sigma}(p)$ are much smaller and are dominated by multiples of

$$t(p_1,p,k)=|[2(1+|\xi|)^{-1/2}-|S|\,(2\bar{\omega}_2)^{1/2}\,(\bar{\omega}_1+\bar{\omega}_2)^{-1}]|\,\mu^{-1/2}|\hat{h}(\eta+k)|\,.$$

For some $\gamma>0$ the functions $\omega_2^\gamma t$ and $\omega_2^{-\gamma}s$ are in $L_2$ on $\Xi$. Moreover

$$2P_1(\sigma)=\int[Y_{\varrho\sigma}(p)-Z_{\varrho\sigma}(p)]^*\,Y_{\varrho\sigma}(p)\,dp+ \\ +\int Z_{\varrho\sigma}(p)^*\,[Y_{\varrho\sigma}(p)-Z_{\varrho\sigma}(p)]\,dp\,.$$

Thus $P_1(\sigma)$ is dominated by

$$\text{const.}\,(N+I)\int\|\omega_2^\gamma\chi(\Xi_{\varrho\sigma})\,t(\cdot,p,\cdot)\|_2\,\|\omega_2^{-\gamma}\chi(\Xi_{\varrho\sigma})\,s(\cdot,p,\cdot)\|_2\,dp$$

by I, Theorem 2.4.1. However this operator is bounded by

$$\text{const.}\,\|\omega_2^\gamma\chi(\Xi)_{\varrho\sigma}t\|_2\,\|\omega_2^{-\gamma}\chi(\Xi_{\varrho\sigma})s\|_2\,(F_\tau+I)\,.$$

The coefficient of $F_\tau+I$ tends to zero as $\varrho\to\infty$, uniformly in $\sigma$. This completes the proof.

Let

$$Z'_{\varrho\sigma}(p)=\int\limits_{\Xi_{\varrho\sigma}}-2\,\operatorname{sgn}\xi\, s(p,p_2,k)\, b'^*(p_2)\,\Phi_\sigma(k)\,dk\,dp_2$$

and let $P_2(\sigma)$ be the error term arising from the simplification of the other term in the integral in (3.2).

## § 5. Completion of the Proof

We have shown that the positive bilinear form $H''(\sigma)$ is given by the formula

$$\begin{aligned} H''(\sigma) = H_0 - (3/4)\, F_\tau + V_\sigma + c_\sigma(\sim \Xi) I + \\ + 2^{-1} \int Z_{\varrho\sigma}(p)^* Z_{\varrho\sigma}(p)\, dp + \\ + 2^{-1} \int Z'_{\varrho\sigma}(p)^* Z'_{\varrho\sigma}(p)\, dp + \\ + R(\sigma) + R(\sigma, \varrho) + P_1(\sigma) + P_2(\sigma) + d_\sigma I\,. \end{aligned} \tag{5.1}$$

The bilinear form we wish to show to be positive is $H' - 2^{-1}F_\tau$, where $H'$ is defined by (1.3). We let

$$H'(\sigma) = H_{\mathrm{ren}}(\sigma) - \delta \int :\Phi_\sigma(x)^2: h^2(x)\, dx + cI\,.$$

The bilinear form of $H'(\sigma)$ converges to the bilinear form of $H'$, so it is sufficient to show that $H'(\sigma) - 2^{-1}F_\tau$ is positive. According to Lemma 4.1 and the definition of $H_{\mathrm{ren}}(\sigma)$,

$$\begin{aligned} H'(\sigma) - 2^{-1}F_\tau = H_0 - 2^{-1}F_\tau + V_\sigma + c_\sigma(\sim \Xi)\, I + \\ + \Delta_{\varrho\sigma}(\Xi) + P(\sigma) + (c + e_\sigma) I \end{aligned} \tag{5.2}$$

if $\varrho$ and $\delta$ are related by (4.3). We assert that

$$\begin{aligned} \Delta_{\varrho\sigma}(\Xi) \geqq 2^{-1} \int Z_{\varrho\sigma}(p)^* Z_{\varrho\sigma}(p)\, dp + \\ + 2^{-1} \int Z'_{\varrho\sigma}(p)^* Z'_{\varrho\sigma}(p)\, dp\,. \end{aligned} \tag{5.3}$$

If $\varrho$ and $\delta$ are sufficiently large it will follow from our estimates of the error terms $R(\sigma)$, $R(\sigma, \varrho)$, etc. that

$$H''(\sigma) \leqq H'(\sigma) - 2^{-1}F_\tau + KI$$

for all $\sigma$. Here $\varrho$, $\delta$ and $K$ do not depend on $\sigma$. If $c$ is sufficiently large, $c \geqq c_0(\delta)$, then

$$0 \leqq H''(\sigma) \leqq H'(\sigma) - 2^{-1}F_\tau\,.$$

Thus the theorems of this paper will be completely proved once we prove (5.3).

We will prove a proposition more general than (5.3). Let $s = s(p_1, p_2, k)$ be in $L_2$ and let

$$S(p_1, p_2) = \int s(p_1, p_2, k)\, \Phi(k)\, dk \tag{5.4}$$

$$S(p_2) = \int s(p_1, p_2, k)\, \Phi(k)\, b^*(p_1)\, dk\, dp_1\,. \tag{5.5}$$

**Proposition 5.1.**

$$\int S(p_1, p_2)^* S(p_1, p_2)\, dp_1\, dp_2 \geqq \int S(p_2)^* S(p_2)\, dp_2\,. \tag{5.6}$$

*Proof*. First we observe that both sides of (5.6) are operators defined on the domain $\mathscr{D}_N$ of $N$, since

$$\begin{aligned}\|\textstyle\int S(p_1, p_2)^* S(p_1, p_2)\, dp_1\, dp_2 (N + I)^{-1}\| &\leqq \\ &\leqq \text{const.} \textstyle\int \|s(p_1, p_2, \cdot)\|_2^2\, dp_1\, dp_2 \leqq \\ &\leqq \text{const.} \|s\|_2^2\end{aligned}$$

and

$$\|\textstyle\int S(p_2)^* S(p_2)\, dp_2 (N + I)^{-1}\| \leqq \text{const.} \|s\|_2^2$$

by I, Theorem 2.4.1.

In order to prove (5.6) it will be sufficient to prove that

$$\textstyle\int S(p_1, p_2)^* \int S(p_1, p_2)\, dp_1 \geqq S(p_2)^* S(p_2) \tag{5.7}$$

for almost every $p_2$. Since $p_2$ is held fixed we may write $s$ as a function of two variables, $s(p, k) = s(p, p_2, k)$, and we omit writing the $p_2$ dependence of operators (5.4) and (5.5). With this change, (5.7) becomes

$$\textstyle\int S(p)^* S(p)\, dp \geqq S^* S\,. \tag{5.8}$$

Each side of (5.8) depends continuously on the function $s$. Here the topology on $s$ is the $L_2$ topology and the topology on the operators in (5.8) is the weak topology of bilinear forms defined on $\mathscr{D}_N \times \mathscr{D}_N$. Thus it is sufficient to consider the case in which $s$ is a finite sum,

$$s(p, k) = \sum_{j=1}^{J} f_j(p)\, g_j(k)\,,$$

and each $f_i$ and each $g_j$ is in $L_2$ (and is piecewise constant, for example).

Let

$$\Phi_j = \left(\textstyle\int \Phi(k)\, g_j(k)\, dk\right)^-\,.$$

Then $\{\Phi_1, \ldots, \Phi_J\}$ is a commuting family of unbounded normal operators. The $\Phi_j$'s have a simultaneous spectral resolution

$$dE_\lambda,\ \lambda = \lambda_1, \ldots, \lambda_J, \quad \text{and} \quad \Sigma \alpha_j \Phi_j = \textstyle\int \Sigma \alpha_j \lambda_j\, dE_\lambda\,.$$

Also the fermion operators commute with the spectral projections $E_\lambda$. We can compute

$$\begin{aligned} S &= \textstyle\int \Sigma_j f_j(p)\, b^*(p)\, \Phi_j\, dp \\ &= \textstyle\int \left(\int \Sigma_j f_j(p)\, \lambda_j b^*(p)\, dp\right) dE_\lambda \end{aligned}$$

and

$$S^* S = \textstyle\int \left(\int \Sigma_i \lambda_i f_i(q)\, b^*(q)\, dq\right)^* \left(\int \Sigma_j \lambda_j f_j(p)\, b^*(p)\, dp\right) dE_\lambda\,.$$

The equalities above are asserted to hold only on a suitable dense domain. It is a general property of fermion operators that

$$\left(\textstyle\int f(q)\, b^*(q)\, dq\right)^* \left(\textstyle\int f(p)\, b^*(p)\, dp\right) \leqq \|f\|_2^2 I\,.$$

In fact this follows from the anticommutation relations

$$b(q)\, b^*(p) + b^*(p)\, b(q) = \delta(p-q)\,.$$

Thus

$$S^* S \leqq \int \left(\int |\Sigma_j \lambda_j f_j(p)|^2\, dp\right) dE_\lambda\,.$$

We interchange order of the $p$ and $\lambda$ integrations. We then use the formula

$$\int \varphi_1(\lambda)\, \varphi_2(\lambda)\, dE_\lambda = \int \varphi_1(\lambda)\, dE_\lambda \int \varphi_2(\lambda)\, dE_\lambda$$

to place a spectral integral in each factor. This gives us the inequality

$$\begin{aligned} S^* S &\leqq \int \left(\int \Sigma_i \lambda_i f_i(p)\, dE_\lambda\right)^* \left(\int \Sigma_j \lambda_j f_j(p)\, dE_\lambda\right) dp \\ &= \int S(p)^*\, S(p)\, dp \end{aligned}$$

and this proves the proposition.

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