Commun. math. Phys. 4, 331—348 (1967)

Attempt of an Axiomatic Foundation of Quantum Mechanics and More General Theories, II

GÜNTHER LUDWIG
Institut für theoretische Physik, Universität Marburg

Received December 15, 1966

Abstract. The consequences of an axiomatic formulation of physical probability fields established in a first paper [1] are investigated in case of a finite dimensional ensemble-space.

It will be shown that the stated number of axioms can be diminuished essentially. Further the structure of an ortho-complemented orthomodular lattice for the decision effects (also often called "properties" or still more misunderstandingly "propositions") and the orthoadditivity of the probability measures upon this lattice, both, can be essentially inferred from the axioms 3 and 4, *only.* This seems to give a better comprehension of the lattice structure defined by the decision effects.

Particularly, it is pointed out that no assumption (axiom) concerning the commensurability of two decision effects E_1 E_2 with $E_1 \leqq E_2$ must be made but that this commensurability is a theorem of the theory.

I. Fundamental axioms

Since in a preceding paper [1] we briefly discussed the heuristic aspects having led to the statement of the axioms, these axioms shall be quoted very briefly in this paper and, from the first, will be restricted on the case of a finite dimensional ensemble-space.

We will start from two sets:

Let $\underline{K}$ be the set of all ensembles V,

let $\underline{L}$ be the set of all effects F.

Axiom 1. *Over* $\underline{K} \times \underline{L}$ *(cartesian product) a real-valued function* μ *is defined, satisfying:*

α) $o \leqq \mu(V, F) \leqq 1$,

β) $\mu(V_1, F) = \mu(V_2, F)$ *for all* $F \in \underline{L}$ *implies* $V_1 = V_2$,

γ) $\mu(V, F_1) = \mu(V, F_2)$ *for all* $V \in \underline{K}$ *implies* $F_1 = F_2$,

δ) *for each* V *there exists a* F *with* $\mu(V, F) = 1$,

ε) *there exists a* F *(denoted by* 0*) with* $\mu(V, 0) = o$ *for all* $V \in \underline{K}$.

Definition 1. Let B be the set of all functions $X(F)$ on $\underline{L}$ with

$$X(F) = \sum_{i=1}^{n} a_i \mu(V_i, F), \quad V_i \in \underline{K} \tag{1}$$

a_i real numbers and n any finite integer.

Obviously B is a vector space over the field of real numbers.

The following axiom is additional, *only in order to simplify the investigations of this part*. Nevertheless, it may be expected to get all the typical structures. The more general case might be a limit of that which will be investigated in this paper, the correlation resembling to representing integrals as limits of sums.

Axiom 1 ad.. *B is finite dimensional; we introduce the abbreviation* $\dim B \underset{\mathrm{df}}{=} N$.

By the correspondence $X \leftrightarrow V$, if

$$X(F) = \mu(V, F) \quad \text{for all} \quad F \in \underline{L},$$

$\underline{K}$ becomes a subset of B. When we define

$$\mu(X, F) \underset{\mathrm{df}}{=} X(F)$$

μ can be uniquely extended to the entire $B \times \underline{L}$ and, for fixed F, μ is a linear functional on B.

Definition 2. $\|X\| \underset{\mathrm{df}}{=} \sup\{|X(F)| \, |F \in \underline{L}\}$ is a norm of B.

An immediate consequence of axiom 1β and definition 1 is that the normtopology in B is identical with the Euclidean topology of the N-dimensional space B and that $\underline{K}$ has exactly N linearly independent V's. Because of axiom 1α and δ a ray $\{\lambda V \,|\, V \in \underline{K} \text{ fixed and } \lambda \in \mathbf{R}\}$ has only one element of $\underline{K}$, namely V.

Definition 3. K denotes the closed convex closure of $\underline{K}$ in B. The elements of K are called mixtures of the elements of $\underline{K}$.

As an immediate inference, the conditions $\alpha, \beta, \gamma, \varepsilon$ of axiom 1 are satisfied on $K \times \underline{L}$, too.

Definition 4. By B' we denote the vector space consisting of all linear functionals $Y(X)$, i.e. the conjugate space of B.

As is well-known, B' is also a N-dimensional vector space over the field of real numbers. $\mu(X, F)$ being a linear functional of B for fixed F, the elements F of $\underline{L}$ can be canonically identified with certain elements of B'. Then it is possible to extend the definition of μ to all elements $Y \in B'$ by $\mu(X, Y) \underset{\mathrm{df}}{=} Y(X)$.

Definition 5. L is the closure of $\underline{L}$ in B'.

II. Axioms of sensitivity-increase of effects

Definition 6. For any $F \in L$ we define $K_-(F) = \{V \,|\, V \in K$ and $\mu(V, F) = o$ for fixed $F \in L\}$.

For any subset $l \subseteqq L$ we define $K_-(l) = \{V \,|\, V \in K$ and $\mu(V, F) = o$ for all $F \in l\}$.

Definition 7. A closed subset $K' \subseteqq K$ is said to be a *completely convex subset (c.c.s.)* (also called: extremal subset), if for $o \leqq \lambda \leqq 1$

α) $V = \lambda V_1 + (1-\lambda) V_2$ with $V_1, V_2 \in K'$ implies $V \in K'$

β) $V = \lambda V_1 + (1-\lambda) V_2$ with $V \in K'$ implies $V_1, V_2 \in K'$.

We repeat the following theorems deduced in [1]:

Theorem 1. *$K_-(F)$ and $K_-(l)$ are c.c.s.*

Theorem 2. $K_-(l) = \bigcap_{F \in l} K_-(F)$.

Theorem 3. $K_-\left(\bigcup_{\lambda \in \Lambda} l_\lambda\right) = \bigcap_{\lambda \in \Lambda} K_-(l_\lambda)$, *$\Lambda$ any indexing set.*

Definition 8. For $Y_1, Y_2 \in B'$ we define $Y_1 \leqq Y_2$ if and only if $\mu(V, Y_1) \leqq \mu(V, Y_2)$ for all $V \in \underline{K}$.

An immediate consequence is $\mu(V, Y_1) \leqq \mu(V, Y_2)$ for all $V \in K$. Thereby B' is a partially ordered vector space.

From $F_1 \leqq F_2$ follows directly $K_-(F_1) \supseteqq K_-(F_2)$. Hence, in the subsequent axiom 3, $K_-(F_3) \subseteqq K_-(F_1) \cap K_-(F_2)$ is valid because of $F_1 \leqq F_3$ and $F_2 \leqq F_3$. Consequently, $K_-(F_3) \supseteqq K_-(F_1) \cap K_-(F_2)$ may be replaced by $K_-(F_3) = K_-(F_1) \cap K_-(F_2)$ in this axiom.

Definition 9. $L_-(V) \underset{\text{df}}{=} \{F \mid F \in L \text{ and } \mu(V, F) = o \text{ for fixed } V \in K\}$ and for any subset $k \subseteqq K$

$$L_-(k) \underset{\text{df}}{=} \{F \mid F \in L \text{ and } \mu(V, F) = o \text{ for all } V \in k\}.$$

Definition 10. $W = \{K_-(l) \mid l \subseteqq L\}$.

Definition 11. $U = \{L_-(k) \mid k \subseteqq K\}$.

Similar to the theorems 2 and 3 the following theorems can be easily proved:

Theorem 4. $L_-(k) = \bigcap_{V \in k} L_-(V)$.

Theorem 5. $L_-\left(\bigcup_{\lambda \in \Lambda} k_\lambda\right) = \bigcap_{\lambda \in \Lambda} L_-(k_\lambda)$, *$\Lambda$ any indexing set.*

The subsets $l \subseteqq L$ leading to the same $K_-(l)$ form a directed set with a maximal element $l_m = \cup\, l$ (join of all l with the same $K_-(l)$); this results from theorem 3. Because of $l_m = L_-K_-(l)$ these maximal elements l_m are elements of U. Conversely, if we consider all the sets k leading to the same $L_-(k)$, then, because of theorem 5, k_m can be defined in an analogous manner and $k_m = K_-L_-(k)$ holds; hence:

Theorem 6. *The mappings $K_-: W \to U$ and $L_-: U \to W$ are bijective and invert the order relation; i.e. they are dual-isomorphisms from W and U, respectively, satisfying $K_-L_- = 1_W$ and $L_-K_- = 1_U$.*

Directly from theorem 3 and 5 results the

Theorem 7. *W and U are complete lattices.*

So, again by theorem 6,

Theorem 8. *W and U are dual-isomorphic. K_-, L_- are such dual-isomorphisms from the lattice U on W and from W on U, respectively, K_- being the inverse mapping of L_-.*

Axiom 2. *For any neighbourhood of each $F \in L$ there exists a $F' \in \underline{L}$ with $F' \leqq F$.*

This axiom expresses the "idealized" effects $F \in L$ with $F \notin \underline{L}$ to be idealized just because their sensitivity is reached by less sensitive F's only approximately.

The most basic aciom of the whole theory is

Axiom 3. *For each $F_1, F_2 \in \underline{L}$ there exists a $F_3 \in \underline{L}$ such that $F_1 \leqq F_3$, $F_2 \leqq F_3$ and $K_-(F_3) \supseteqq K_-(F_1) \cap K_-(F_2)$.*

The axioms 2 and 3 only contain the sets $\underline{K}$ and $\underline{L}$, although, for a moment, K and L seem to be important for these axioms. First of all, the sets $K_-(F)$ are used in axiom 3. This is only a simplifying writing because by definition 3 each $V \in K_-(F)$ is only a mixture of elements of $\underline{K}$. Likewise in axiom 2 a Cauchy sequence of $\underline{L}$ may be substituted for an element of L. Then the axiom would run: every Cauchy sequence of $\underline{L}$ is equivalent to an isotonely increasing sequence (F_ν), $(F_\nu \leqq F_{\nu+1})$.

Theorem 9. *From axiom 2 and 3 there may be inferred that axiom 3 is also valid for L (instead of $\underline{L}$).*

Proof. Let $\overline{F}_1$ and $\overline{F}_2$ be two elements of L. According to axiom 2 there exist two $F_1, F_2 \in \underline{L}$ with $F_1 \leqq \overline{F}_1$, $F_2 \leqq \overline{F}_2$ which, however, approximate $\overline{F}_1$ and $\overline{F}_2$ at any accuracy. By axiom 3 there exists a F_3 with $F_3 \geqq F_1$, $F_3 \geqq F_2$ and $K_-(F_3) \supseteqq K_-(F_1) \cap K_-(F_2)$. Since $F_1 \leqq \overline{F}_1$ and $F_2 \leqq \overline{F}_2$ also imply $K_-(F_1) \supseteqq K_-(\overline{F}_1)$ and $K_-(F_2) \supseteqq K_-(\overline{F}_2)$, so $K_-(F_3) \supseteqq K_-(\overline{F}_1) \cap K_-(\overline{F}_2)$. Instead of F_1, F_2 select two sequences $(F_1^\nu), (F_2^\nu)$ converging to $\overline{F}_1, \overline{F}_2$, respectively. (F_3^ν) selected for this purpose being a bounded point set ($\|F_3^\nu\| \leqq 1$) it has at least one accumulation point $\overline{F}_3 \in L$ for which $\overline{F}_3 \geqq \overline{F}_1$, $\overline{F}_3 \geqq \overline{F}_2$ and $K_-(\overline{F}_3) \supseteqq \bigcap_\nu K_-(F_3^\nu) \supseteqq$ $\supseteqq K_-(\overline{F}_1) \cap K_-(\overline{F}_2)$ have to be demonstrated: from $F_3^\nu \geqq F_1^\nu$ there results

$$\mu(V, F_3^\nu) \geqq \mu(V, F_1^\nu) \quad \text{for all} \quad V \in K\,.$$

For any $\varepsilon > o$ a ν can be selected such that for all $V \in K$

$$|\mu(V, F_3^\nu) - \mu(V, \overline{F}_3)| < \varepsilon \quad \text{and} \quad |\mu(V, F_1^\nu) - \mu(V, \overline{F}_1)| < \varepsilon\,.$$

Hence $\mu(V, \overline{F}_3) \geqq \mu(V, \overline{F}_1) - 2\varepsilon$ for any $\varepsilon > o$ and so $\mu(V, \overline{F}_3) \geqq$ $\geqq \mu(V, \overline{F}_1)$. Likewise, $\mu(V, F_3^\nu) = o$ for all ν implies $\mu(V, \overline{F}_3) < \varepsilon$ for any $\varepsilon > o$, i.e., $\mu(V, \overline{F}_3) = o$, so $K_-(\overline{F}_3) \supseteqq \bigcap_\nu K_-(F_3^\nu)$. This completes the proof.

In what will follow, theorem 9 will be used instead of the axioms 2 and 3. So, from the first, it would have been possible to postulate axiom 3 (instead of the axioms 2 and 3) for L instead of $\underline{L}$.

Theorem 10. *The following two conditions are equivalent:*

1) *axiom 3 is valid for L*

2) *the sets $l \in U$ are ascending directed sets, i.e. for any $F_1, F_2 \in l$ there exists a $F_3 \in l$ with $F_1 \leqq F_3$, $F_2 \leqq F_3$.*

Proof. 1) $\Rightarrow$ 2). According to 1) there exists a F_3 for F_1 and F_2 such that $F_1 \leqq F_3$, $F_2 \leqq F_3$ and $K_-(F_3) \supseteq K_-(F_1) \cap K_-(F_2) = K_-(\{F_1, F_2\})$. Because of $\{F_1, F_2\} \subseteq l$ and hence $K_-(l) \subseteq K_-(\{F_1, F_2\})$ the condition 2) results.

2) $\Rightarrow$ 1). 2) and $l = L_-K_-(\{F_1, F_2\})$ imply the existence of $F_3 \in l$ such that $F_3 \geqq F_1$ and $F_3 \geqq F_2$. From $F_3 \in l$ results

$$K_-(F_3) \supseteq K_-(l) = K_-(\{F_1, F_2\}) = K_-(F_1) \cap K_-(F_2) .$$

The latter formulation of 2) was postulated in [1] as an axiom. The elements $l \in U$ being equal to $L_-(k)$ for any set $k \subseteq K$, 2) has the intuitive meaning:

For any two effects F_1, F_2 satisfying $\mu(V, F_1) = \mu(V, F_2) = o$ for all $V \in k$ there is a $F_3 \geqq F_1$, $F_3 \geqq F_2$ with $\mu(V, F_3) = o$ for all $V \in k$.

Thus, with the secondary condition "$\mu(V, F) = o$ for all $V \in k$" the experimental problem would be to construct more and more sensitive effect-apparatuses.

Theorem 11. *For each $l \in U$ an element $E_l \in B'$ is uniquely defined by* $E_l(V) \underset{\text{df}}{=} \sup_F \{\mu(V, F) | F \in l$ *and* $V \in K$ *(or* $\underline{K}$*)*$\}$.

If $l_1 \neq l_2$, then $E_{l_1} \neq E_{l_2}$. $l_1 \supseteq l_2$ implies $E_{l_1} \geqq E_{l_2}$. E_l belongs to l and is the maximal element of the directed set l, i.e. $F \leqq E_l$ holds for all $F \in l$.

This was proved in [1].

Definition 12. The elements E_l introduced by theorem 11 are called *decision effects.* Let G denote the set of all decision effects.

Theorem *12. According to definition* 8 *G is a complete lattice with respect to the order and G is isomorphic with the set-lattice U but dual-isomorphic with the set lattice W. The isomorphism of G on W is defined by $E \leftrightarrow K_-(E)$.*

The proof of this theorem, too, was given in [1].

From $K_-(E_1) \cap K_-(E_2) = K_-(E_1 \cup E_2)$ may be inferred at once: $\mu(V, E_1) = o$ and $\mu(V, E_2) = o$ imply $\mu(V, E_1 \cup E_2) = o$.

Theorem 13. *There exists a $F \in L$ with $\mu(V, F) = 1$ for all $V \in K$.*

This element F may be denoted by **1**. **1** is the "unit element" of the lattice G.

Proof. $L_-K_-(l) \supseteq l$ and $L_-K_-(l) \subseteq L$ imply $L_-K_-(L) = L$, i.e. L is the unit element of the lattice U. Consequently, the unit element **1** of G is

$$\mathbf{1} = E_L$$

and therefore determined by

$$\mu(V, \mathbf{1}) = \sup_F \{\mu(V, F) | F \in \underline{L} \text{ and } V \in K\} .$$

For $V \in \underline{K}$ $\mu(V, \mathbf{1}) = 1$ is immediately deduced from axioms $1\alpha, \delta$). K being the closed convex hull of $\underline{K}$, $\mu(V, \mathbf{1}) = 1$ is valid for all $V \in K$, too.

α) to ε) of axiom 1 hold consequently on $K \times L$, too. The null effect according to ε) is the null element of G. Because of the validity of $\mu(V, \mathbf{1}) = 1$ and α) a ray $\{\lambda V \mid V \in K$ fixed and $\lambda \in \mathbf{R}\}$ contains one and only one point, namely the fixed $V \in K$. As a consequence K is a $(N-1)$ dimensional convex set in the plane $\mu(X, \mathbf{1}) = 1$ contained in B.

Theorem 14. *B' is the span of $\underline{L}$, i.e. $\underline{L}$ contains N linearly independent elements.*

Proof. It is sufficient to prove the theorem for L instead of $\underline{L}$. Assume L spans a proper subspace $T \subset B'$ only, then there is $X \neq 0$ of B with $\mu(X, Y) = o$ for all $Y \in T$. K being the span of B, there holds $X = \sum_{i=1}^{N} \alpha_i V_i$, $V_i \in K$. K being convex even admits the selection of $V_1, V_2 \in K$ (by adding the positive and negative elements of $\sum_{i=1} \alpha_i V_i$, respectively!) such that $X = \alpha_1 V_1 - \alpha_2 V_2$ with $\alpha_1 \geqq o$, $\alpha_2 \geqq o$ is valid. Then $\mu(X, Y) = o$ implies $\alpha_1 \mu(V_1, F) = \alpha_2 \mu(V_2, F)$ for all $F \in L$. From $F = \mathbf{1}$ results $\alpha_1 = \alpha_2$ and hence, because of $X \neq 0$, $\mu(V_1, F) = \mu(V_2, F)$, so, by axiom 1 β),

$$V_1 = V_2$$

but this contradicts $X \neq 0$.

Definition 13. By $\hat{L}$ we denote the closed convex set generated by L in B'.

The following theorem expresses well-known properties of a convex set generated by a topologically closed set L:

Theorem 15. *The extreme points of $\hat{L}$ belong to L. Each $Y \in \hat{L}$ may be written as*

$$Y = \sum_{\nu=1}^{n} \lambda_\nu F_\nu, \quad \lambda_\nu > o, \quad \sum_{\nu=1}^{n} \lambda_\nu = 1, \quad F_\nu \in L\,.$$

Using this we can prove the

Theorem 16. *Axiom* 3 *and theorem* 10 *still hold if $\hat{L}$ is substituted for L.*

Proof. The proof of theorem 10 can be immediately transferred to $\hat{L}$ instead of L. The only item consists in proving b) of theorem 10 for $\hat{L}$. We consider the sets $l = L_{-}(k)$ and $\hat{l} = \hat{L}_{-}(k) = \{Y \mid Y \in L$ and $\mu(V, Y) = o$ for all $V \in k\}$; therefore $\hat{l} \supseteqq l$. Since any element of $\hat{l}$ has a representation by some $F_\nu \in L$ according to theorem 15 the F_ν's must be elements of l. Thus, $\hat{l}$ is the convex set generated by l. l being ascending directed, so is $\hat{l}$ (by theorem 15) and l and $\hat{l}$ have the same maximal element $E_l = E_{\hat{l}}$. E_l is an extreme point of $\hat{l}$.

By theorem 16 we are henceforth permitted always to use $\hat{L}$ instead of L and so to substitute the set $\hat{U}$ of all $\hat{L}_{-}(k)$ for the set U of all $L_{-}(k)$. U and $\hat{U}$ are order-isomorphic.

Axiom 4. *For every V of a c.c.s. $k \subset K$ there exists at least one $F \neq 0$ of L with $\mu(V, F) = o$.*

While axiom 3 expresses the existence of "sufficiently sensitive" effects, the axiom 4 expresses the existence of "sufficiently insensitive" effects. Every $V \in K$ being not a boundary point of K in the plane $\mu(X, \mathbf{1}) = 1$, i.e. being not an element of a c.c.s. $\neq K$, can be written with any other $V_1 \in K$ as a mixture

$$V = \lambda V_1 + (1 - \lambda) V_2, \quad o \leqq \lambda \leqq 1, \quad V_2 \in K .$$

The extreme points of K (being elements of the closure of $\underline{K}$ and hence being arbitrarily precisely approximable by ensembles of $\underline{K}$!) allow a representation as a mixture only by the triviality $V_1 = V_2 = V$. The axiom 4 consequently signifies that

1) such ensembles cannot produce *every* effect, and that

2) for each set of extreme points generating a c.c.s. $k \neq K$ only, there is a F being not produced by these ensembles.

Thus nothing but $\underline{K}$ and $\underline{L}$ (for experimental verification) enter axiom 4, too.

Theorem 17. *a) any* $X \in B$ satisfying $\mu(X, F) \geqq o$ *for all* $F \in \underline{L}$ *and* $\mu(X, \mathbf{1}) = 1$ *is an element of* K.

b) any $Y \in B'$ *satisfying* $o \leqq \mu(V, Y) \leqq 1$ *for all* $V \in \underline{K}$ *is an element of* $\hat{L}$.

Proof. a) The intersection $K \cap S$ of K with a supporting hyperplane S of the cone generated by K is, as easily to be seen, a c.c.s. unequal to K. Through every boundary point of K there exist such supporting hyperplanes. According to axiom 4, for every boundary point V of K a $F \neq 0$ of L is such a supporting hyperplane. These F's, however, form a sufficient system of supporting hyperplanes of the cone generated by K, i.e. the intersection of the positive half-space determined by these supporting hyperplanes is the cone generated by K. But the intersection of this cone with the plane $\mu(X, \mathbf{1}) = 1$ is K. Any X, however, is in the positive half-space of each of these supporting hyperplanes hence in the cone generated by K and, because of $\mu(X, \mathbf{1}) = 1$, X is in K; this proving a).

b) By the definition

$$K' \underset{\text{df}}{=} \{X \mid X \in B \text{ and } o \leqq \mu(X, F) \leqq 1 \text{ for all } F \in L\} ,$$

K' is the positive cone generated by K and cut off by the plane $\mu(X, \mathbf{1}) = 1$, i.e. $K' = \{\lambda V \mid V \in K \text{ and } o \leqq \lambda \leqq 1\}$. So, K' being the intersection of all the half-spaces $\mu(X, F) \geqq o$, $\mu(X, F) \leqq 1$ for all $F \in L$, conversely, $\hat{L}$ is the set $\{Y \mid \mu(X, Y) \geqq o \text{ and } \mu(X, Y) \leqq 1 \text{ for all } X \in K'\}$.

With $X = \lambda V$ and $o \leqq \lambda \leqq 1$, $V \in K$ there follows:

$$\mu(X, Y) \geqq o \quad \text{equivalent with} \quad \mu(V, Y) \geqq o ,$$
$$\mu(X, Y) \leqq 1 \quad \text{equivalent with} \quad \mu(V, Y) \leqq 1 ,$$

this proving b), too.

Theorem 18. *Every exposed point of K is as singleton an element of W.*

Proof. If V is an exposed point then $K_-\hat{L}_-(V) = \{V\}$ unless $L_-(V) = \Theta$. Since V is a boundary point there consequently exists at least one supporting hyperplane $F \in \hat{L}$ containing V such that $\hat{L}_-(V) \neq \Theta$.

Theorem 19. *The set of all extreme points of $\hat{L}$ is equal to G.*

To prove this we verify the following lemma at first:

Theorem 20. *Any $F \in \hat{L}$ has the representation*

$$F = \sum_{i=1}^{n} \lambda_i (E_i - E_{i+1}) \tag{2}$$

with $o < \lambda_i \leqq 1$, $\lambda_i \neq \lambda_j$ if $i \neq j$, $E_i \in G$, $E_{n+1} = o$ and $E_i > E_{i+1}$.

Proof. Let α_1 be the maximum of $\mu(V, F)$ on K and let E_1 be determined by $K_-(F) = K_-(E_1)$. Then $\alpha_1^{-1} F \leqq E_1$. Putting

$$E_1 = \alpha_1^{-1} F + F_2 \tag{3}$$

we infer $F_2 \in \hat{L}$ from theorem 17b. (3) directly implies $K_-(F_2) \supseteqq K_-(E_1)$. Since there is $V \in K$ with $\mu(V, \alpha_1^{-1}F) = 1$ and therefore (because of $\mu(V, E_1) \leqq 1$) with $\mu(V, F_2) = o$ so $K_-(F_2) \neq K_-(E_1)$, hence $K_-(F_2) \supset K_-(E_1)$.

Defining E_2 by $K_-(F_2) = K_-(E_2)$ we obtain $E_2 < E_1$.

To F_2 the same procedure as before to F is applied till an element $F_{n+1} = 0$ is obtained. In this way a set of equations of the form (3) is got from which by elimination of the F_i's the statement (2) results together with the λ_i's satisfying the conditions stated in theorem 20.

Proof of theorem 19. Every $E \in G$ is an extreme point, for $E = \lambda F_1 + (1 - \lambda) F_2$ with $F_1, F_2 \in \hat{L}$ and $o < \lambda < 1$ implies $K_-(F_1) \supseteqq K_-(E)$ and $K_-(F_2) \supseteqq K_-(E)$ and hence $F_1 \leqq E$ and $F_2 \leqq E$. Suppose $F_1 < E$ then with a suitable $V \in K$

$$\mu(V, E) = \lambda \mu(V, F_1) + (1 - \lambda) \mu(V, F_2) < \\ < \lambda \mu(V, E) + (1 - \lambda) \mu(V, E) = \mu(V, E) .$$

Consequently, $F_1 = F_2 = E$ and hence E is an extreme point.

Conversely, suppose F to be an extreme point and $F \notin G$, then there is a $\lambda_i \neq 1$ in (2), say $\lambda_K \neq 1$. Then a $\varepsilon > o$ is possible to be selected such that $\lambda_K + \varepsilon \leqq 1$ and $\lambda_K - \varepsilon \geqq o$. Then

$$o \leqq \sum_{i \neq k} \lambda_i (E_i - E_{i+1}) + (\lambda_k \pm \varepsilon)(E_k - E_{k+1}) \leqq \sum_i (E_i - E_{i+1}) = E_1 .$$

With $F_1 = F + \varepsilon(E_k - E_{k+1})$ and $F_2 = F - \varepsilon(E_k - E_{k+1})$ we then obtain

$$F_1, F_2 \in \hat{L} \quad \text{and} \quad F = \frac{1}{2} F_1 + \frac{1}{2} F_2 ,$$

this contradicts to be an extreme point.

From theorems 19 and 20 there may be easily inferred:

Theorem 21. *For every* $F \in \hat{L}$ *there exists a* λ *with* $o < \lambda \leqq 1$ *such that* $F \geqq \lambda E$ *with* $K_{-}(E) = K_{-}(F)$.

Proof. Substitute all λ_i in equation (2) by the minimum of the λ_i.

Theorem 22. *Every* $Y \in B'$ *can be written as*

$$Y = \sum_{i=1}^{n} \alpha_i (E_i - E_{i+1})$$

with $\alpha_i \neq \alpha_j$ *if* $i \neq j$, $E_i \in G$, $E_{n+1} = 0$ *and* $E_i > E_{i+1}$.

Proof. With $\beta < \min\{\mu(V, Y) \mid V \in K\}$ theorem 20 can be applied to $F = Y + \beta 1$.

Theorem 23. *Every* $F \in \hat{L}$ *has the representation as*

$$F = \sum_{i=1}^{m} \lambda_i E_i \quad \text{with} \quad \lambda_i > o, \sum_{i=1}^{m} \lambda_i = 1 \quad \text{and} \quad E_i \in G .$$

Proof. An immediate consequence of theorem 19.

III. Commensurability

In [1] the concepts "coexistence" and "commensurability" having been discussed in detail, only their definitions shall be stated here. As is well-known, a Boolean ring (without 1-element) is defined to be a ring with the additional property $a \cdot a = a$ for all elements of the ring. Because of

$$a \dotplus a = (a \dotplus a) \cdot (a \dotplus a) = a \dotplus a \dotplus a \dotplus a$$

$$a \dotplus a = 0 \quad \text{holds then} .$$

The sign $\dotplus$ is selected to distinguish addition in Boolean ring from that in B or B'.

Definition 14. $A \subseteqq \hat{L}$ is said to be *coexistent* if there is al Boolean ring R with $A \subseteqq R \subseteqq \hat{L}$ and

$$\mu(V, F_1) + \mu(V, F_2) = 2\mu(V, F_1 \cdot F_2) + \mu(V, F_1 \dotplus F_2)$$

for all $F_1, F_2 \in R$ and all $V \in \underline{K}$.

Definition 15. A set A of decision effects E is called to be *commensurable* if there is a Boolean ring R with $A \subseteqq R \subseteqq G$ (G the set of all decision effects) and

$$\mu(V, E_1) + \mu(V, E_2) = 2\mu(V, E_1 \cdot E_2) + \mu(V, E_1 \dotplus E_2)$$

for all $E_1, E_2 \in R$ and all $V \in \underline{K}$ (and hence $\in K$).

If A consists of exactly two decision effects then these two decision effects are called to be commensurable.

Next we will show the important

Theorem 24. *If there holds $E_1 \leqq E_2$ for two decision effects E_1, E_2 then they are commensurable.*

This theorem was postulated in [1] to be an axiom. This is unnecessary because the assertion of theorem 24 is inferable from the preceding axioms 1—4. As lemmata for the proof of theorem 24, we show

Theorem 25. *With $E_1, E_2 \in G$ such that $E_1 \leqq E_2$ and with E_{12} defined by $K_-(E_{12}) = K_-(F)$ there follows from $E_2 = E_1 + F$ that $E_2 = E_1 \cup E_{12}$ and $E_1 \cap E_{12} = 0$, i.e. E_{12} is a relative complement of E_1 for E_2.*

Proof. Let be $\overline{F} \in L$ with $\overline{F} \leqq E_1$ and $\overline{F} \leqq F$. $\alpha \underset{\text{df}}{=} \max\{\mu(V, E_1 + \overline{F}) \mid V \in K\}$ yields $1 \leqq \alpha \leqq 2$ and $F' = \frac{1}{\alpha}(E_1 + \overline{F}) \in \hat{L}$ and, because of $K_-(F') = K_-(E_1)$, $F' \leqq E_1$, too. With V such that $\mu(V, E_1 + \overline{F}) = \alpha$ there holds

$$1 = \mu(V, F') = \frac{1}{\alpha}\mu(V, E_1) + \frac{1}{\alpha}\mu(V, \overline{F}) \leqq \frac{1}{\alpha} + \frac{1}{\alpha}\mu(V, F)\,.$$

Because of $F' \leqq E_1$ there is $\mu(V, E_1) = 1$ and hence $\mu(V, E_2) = 1$ and $\mu(V, F) = o$, therefore $1 \leqq \frac{1}{\alpha}$, this implying $\alpha = 1$ because of $1 \leqq \alpha \leqq 2$. Thus $E_1 + \overline{F} \leqq E_1$ and hence $\overline{F} = 0$. Select a λ according to theorem 21 such that $\lambda E_{12} \leqq F$ and hence a fortiori $\lambda(E_1 \cap E_{12}) \leqq F$. $\overline{F} = \lambda(E_1 \cap E_{12})$ implies $\overline{F} \leqq E_1$ and $\overline{F} \leqq F$, thus $\overline{F} = 0$ and hence $E_1 \cap E_{12} = 0$.

From $E_2 = E_1 + F$ there results

$$K_-(E_1) \cap K_-(E_{12}) = K_-(E_1) \cap K_-(F) = K_-(E_2)$$

and hence, because of

$$K_-(E_1 \cup E_{12}) = K_-(E_1) \cap K_-(E_{12})\,, \quad E_2 = E_1 \cup E_{12}$$

finally.

Theorem 26. *F from theorem 25 is a decision effect, i.e. $E_2 - E_1 \in G$ if $E_1, E_2 \in G$ and $E_1 \leqq E_2$.*

Proof. We verify $F = E_{12}$ with E_{12} defined by $K_-(E_{12}) = K_-(F)$.

$$K_-(E_{12}) = K_-(F) \supseteqq K_-(E_2) \quad \text{imply} \quad E_{12} \leqq E_2\,.$$

Therefore there also holds $E_2 = E_{12} + \overline{F}$, theorem 25 being valid for E_2 and E_{12}, too. $E_{12} + \overline{F} = E_1 + F$ implies $E_{12} - F = E_1 - \overline{F}$. Because of $F \leqq E_{12}$ there hold $F_1 \underset{\text{df}}{=} E_{12} - F \in \hat{L}$ and $F_1 \leqq E_{12}$. But $F_1 = E_1 - \overline{F}$ has as an implication $F_1 \leqq E_1$, too, thus $K_-(F_1) \supseteqq K_-(E_{12})$ and $K_-(F_1) \supseteqq K_-(E_1)$. Therefore a consequence of $K_-(F_1) \in W$ is also

$$K_-(F_1) \supseteqq K_-(E_1) \cup K_-(E_{12}) = K_-(E_1 \cap E_{12})\,, \quad \text{i.e.} \quad F_1 \leqq E_1 \cap E_{12}\,.$$

Since according to theorem 25 $E_1 \cap E_{12} = 0$ holds, $F_1 = 0$ is also valid, i.e. $F = E_{12}$.

Proof of theorem 24. $E_2 = E_1 + E_{12}$ according to theorem 26. By definition 15 select a Boolean ring R:

$$0, E_1, E_2, E_1 \dotplus E_2 = E_{12}, \quad E_1 \cdot E_2 = E_1 .$$

Because of $E_2 = E_1 + E_{12}$ all the necessary conditions can be easily verified.

The following theorems were then proved in [1]:

Theorem 27. *The lattice G is orthocomplemented, i.e. there exists a dual-automorphism $E \to E^*$ on G such that*

$$E^{**} = E \quad \textit{and } E \cap E^* = 0 .$$

E^* being defined by $\mathbf{1} = E + E^*$.

Two elements E_1, E_2 are called *orthogonal*, $E_1 \perp E_2$, if and only if $E_2 \leqq E_1^*$.

Definition 16. An orthocomplemented lattice is said to be *orthomodular*, if and only if

$$E_1 \cup E_2 = E_1 \cup E_2' \quad \text{with} \quad E_2 \perp E_1,\ E_2' \perp E_1$$

implies $E_2 = E_2'$.

Theorem 28. *The lattice G is orthomodular.*

Theorem 29. *The measures $\mu(V, \cdot\,)$ on G, $V \in K$, are orthoadditive, i.e. for mutually orthogonal E_i's*

$$\mu\Big(V, \bigcup_{i=1}^{n} E_i\Big) = \sum_{i=1}^{n} \mu(V, E_i) .$$

Theorem 30. *If there is a set of orthoadditive measures $\{m \mid m(E) \geqq o$ for all $E \in G$, $m(E_1) = m(E_2)$ for all m implies $E_1 = E_2\}$ on an orthocomplemented lattice G then G is orthomodular.*

Theorem 31. *The following two statements are equivalent*

1) *G is orthomodular and $\mu(V, \cdot\,)$ are orthoadditive measures on G for all $V \in K$.*

2) *Two elements $E_1, E_2 \in G$ such that $E_1 \leqq E_2$ are commensurable.*

Theorem 32. *Two decision effects E_1, E_2 are commensurable if and only if*

$$E_1 = (E_1 \cap E_2^*) \cup (E_1 \cap E_2) .$$

So far, the theorems proved in [1].

As is simply to be verified the set

$$K_+(l) \underset{\text{df}}{=} \{V \mid V \in K \quad \text{and} \quad \mu(V, F) = 1 \quad \text{for all} \quad F \in l \subseteqq \hat{L}\}$$

is completely convex. Likewise we define

$$\hat{L}_+(k) \underset{\text{df}}{=} \{F \mid F \in \hat{L} \quad \text{and} \quad \mu(V, F) = 1 \quad \text{for all} \quad V \in k \subseteqq K\} .$$

There exists a maximal decision effect E belonging uniquely to $\hat{L}_- K_+(l)$ such that $K_- \hat{L}_- K_+(l) = \{V \mid V \in K \text{ and } \mu(V, E) = o\}$.

Hence, because of $\mu(V, E) + \mu(V, E^*) = 1$

$$K_-\hat{L}_-K_+(l) = \{V \mid V \in K \quad \text{and} \quad \mu(V, E^*) = 1\},$$

i.e.

$$K_-\hat{L}_-K_+(l) = K_+(E^*) = K_-(E)$$

and thus, because of

$$K_+(l) \subseteqq K_-\hat{L}_-K_+(l),$$

$$K_+(l) \subseteqq K_+(E^*),$$

too.

From $F \in \hat{L}_+K_+(l)$, $\mathbf{1} - F \in \hat{L}_-K_+(l)$ is deducible hence $\mathbf{1} - F \leqq E = \mathbf{1} - E^*$ and so $F \geqq E^*$. Consequently, E^* is the minimal element of $\hat{L}_+K_+(l)$. So, there holds, because of $l \leqq \hat{L}_+K_+(l)$, $F \geqq E^*$ for all $F \in l$ and hence $K_+(l) \supseteqq K_+(E^*)$.

Thus the following theorems are obtained:

Theorem 33. *Any element of W can be represented as*

$$K_+(E^*) = K_-(E).$$

Theorem 34. *Every set $K_+(l)$ is an element of W.*

Theorem 35. *If $F \in \hat{L}_-(k)$ then $\mathbf{1} - F \in \hat{L}_+(k)$ and conversely. In addition, $\hat{L}_+(k)$ has a minimal element belonging to G.*

Theorem 36. *Any $F \in \hat{L}$ can be uniquely represented by*

$$F = \sum_\nu \lambda_\nu E_\nu, \; o \leqq \lambda_\nu \leqq 1, \; \lambda_\nu \neq \lambda_\mu$$

if $\nu \neq \mu$ and mutually orthogonal $E_\nu \in G$.

Proof. According to theorems 20 and 26

$$F = \sum_i \lambda_i E'_i \quad \text{with} \quad E'_i = E_i - E_{i+1}$$

is valid.

Because of $E'_i = E_i \cap E^*_{i+1}$ and $E_i > E_{i+1}$ there holds $E'_i \perp E_j$ for $j > i$ and hence a fortiori $E'_i \perp E'_j$ for $j > i$. Therefore the E'_i's are mutually orthogonal.

Given two representations

$$F = \sum_\nu \lambda_\nu E_\nu = \sum_\mu \lambda'_\mu E'_\mu$$

it can be easily deduced that one of the λ_ν's and one of the λ'_μ's are the maximum of $\mu(V, F)$ on K and that the corresponding E_ν respectively E'_μ are equal to E_1 determined from above. So, step by step, uniqueness too, results from the procedure of theorem 20.

Theorem 37. *Any $Y \in B'$ can be uniquely represented by*

$$Y = \sum_\nu \alpha_\nu E_\nu, \; \alpha_\nu \neq \alpha_\mu \quad \text{if} \quad \nu \neq \mu, \; E_\nu \in G \quad \text{mutually orthogonal.}$$

Proof. An immediate conclusion from theorems 36 and 22.

IV. Decomposition of G into irreducible components

Definition 17. The *centre* Z of the lattice G is the set of those elements of G which are commensurable with every element of G.

As a consequence $0, \mathbf{1} \in G$ belong to Z in any case.

Definition 18. G is called to be *irreducible* if Z consists of the elements 0 and $\mathbf{1}$ only.

Theorem 38. *The centre Z consists of all the elements which can be obtained as union of elements of a set z the elements of which are mutually orthogonal. z is said to be the basis of Z. The union of all elements of z is equal to* $\mathbf{1}$.

Proof. By an *atom* of Z an element ($\neq 0$) of Z is denoted for which there does not exist a smaller element $\neq 0$ of Z. Let z be the set of all atoms of Z. According to theorem 32 any two elements E_1, E_2 with $E_1 \neq E_2$ are mutually orthogonal because for atoms there holds $E_1 \cap E_2 = 0$. If E is any element of Z there must be an atom $E_1 \leqq E$ for, otherwise, there would be an infinite sequence (E_ν) with $E > E_1 > E_2 > \cdots$ By theorem 32 there would be thus an infinite sequence of mutually orthogonal elements $E_\nu \cap E^*_{\nu+1}$, this contradicting the finite dimension of B'. Applying theorem 32 and using the atom E_1 we can write $E = E_1 \cap (E \cap E_1^*)$. In the same way $E \cap E_1^*$ can be proceeded with. B' being finite dimensional, so $E = \bigcup_\nu E_\nu$ is obtained by some $E_\nu \in z$. Thus $\bigcup_{E_\alpha \in z} E_\alpha$ is the greatest element of Z which contains all the other elements, i.e. $\bigcup_{E_\alpha \in z} E_\alpha = \mathbf{1}$.

Any $E \in G$ being commensurable with all elements of Z and so a fortiori with all elements of z there holds with $E_\nu \in z$ $(\nu = 1, 2, \ldots, n)$

$$E = \bigcup_\nu (E \cap E_\nu),$$

this being easily derived by repeated application of theorem 32. For all $E \in G$ the elements $E \cap E_\nu$, ν fixed, form an orthocomplemented orthomodular sublattice G_ν of G satisfying

$$(E \cap E') \cap E_\nu = (E \cap E_\nu) \cap (E' \cap E_\nu)$$

and

$$(E \cap E') \cap E_\nu = \Big[\bigcup_{\mu,\sigma} (E \cap E_\mu) \cup (E' \cap E_\sigma)\Big] \cap E_\nu = (E \cap E_\nu) \cup (E' \cap E_\nu).$$

Theorem 39. *The space B' is the direct sum of spaces $(B')_\nu$, each $(B')_\nu$ being spanned by elements of G_ν.*

Proof. The elements of G spanning the entire B', it is sufficient to prove that the elements of G (as elements of B') can be decomposed into a sum of elements of G_ν with the additional condition

$$(B')_\nu \cap (B')_\mu = \{0\} \quad \text{if} \quad \nu \neq \mu \quad \text{(set theoretical intersection)}.$$

Since mutually orthogonal $E_\nu \in G$ generally satisfy the identity $\bigcup_\nu E_\nu = \sum_\nu E_\nu$ ($\sum$ denotes summing in B') which is clearly equivalent with

$$\mu\left(V, \bigcup_\nu E_\nu\right) = \sum_\nu \mu(V, E_\nu) \quad \text{for all} \quad V \in K,$$

so there is the validity of

$$E = \bigcup_\nu (E \cap E_\nu) = \sum_\nu (E \cap E_\nu) \quad \text{with} \quad E_\nu \in z.$$

From theorem 37 uniqueness of such a decomposition results, this proving the assertion.

A decomposition of B into $B = \sum_\nu \oplus B_\nu$ corresponds to a decomposition $B' = \sum_\nu \oplus (B')_\nu$. This can be derived most evidently if for every $X \in B$ a component X_ν is defined by $\mu(X_\nu, F) = \mu(X, F_\nu)$, F_ν being a component of F in $(B')_\nu$. Because F_ν is uniquely determined by F and the mapping $F \to F_\nu$ is linear, the definition is meaningful. $X = \sum_\nu X_\nu$ can be directly inferred from $F = \sum_\nu F_\nu$. If $\nu \neq \mu$ then $\mu(X_\nu, F_\mu) = o$. Suppose $X_\nu = X_\mu$ if $\nu \neq \mu$, then $\mu(X_\nu, F_\nu) = o$, too. This implies $\mu(X_\nu, \sum_\mu F_\mu) = o, F = \sum_\mu F_\mu$ being arbitrary. Thus the final result $X_\nu = 0$.

If the X_ν's (as linear functionals over B') are restricted to $(B')_\nu$, so each $(B_\nu, (B')_\nu)$ is a pair of dual spaces with just the properties belonging to (B, B'). Therefore, the structure of a decomposable system is clarified as soon as the structure of irreducible systems (i.e. such with $Z = \{0, 1\}$) has been clarified. A subsequent part III will treat of this.

V. Coexistent effects

By definition 14 two effects $F_1, F_2 \in L$ are coexistent if there are other effects of $\hat{L}$:

$$F_1 \cdot F_2, \; F_1 \dot{+} F_2, \; F_1 \dot{+} F_1 \cdot F_2, \; F_2 \dot{+} F_1 \cdot F_2, \; F_1 \dot{+} F_2 \dot{+} F_1 \cdot F_2$$

which together with 0 form the Boolean ring R and satisfy the condition of definition *14*. By abbreviating $F_1 \cdot F_2 \underset{\text{df}}{=} F$, $F_1 \dot{+} F_1 \cdot F_2 = F'$ and $F_2 \dot{+} F_1 \cdot F_2 = F''$ we can satisfy all the conditions of definition 14 if there hold

$$F_1 = F' + F, \quad F_2 = F'' + F \quad \text{and} \quad F' + F'' + F \in \hat{L}.$$

Thus the following theorem has been proved:

Theorem 40. *F_1 and F_2 are coexistent if and only if there exist three elements $F', F'', F \in \hat{L}$ with*

$$F_1 = F' + F, \quad F_2 = F'' + F \quad \textit{and} \quad F' + F'' + F \in \hat{L}$$

all being elements of L.

Consequently, two effects F_1, F_2 with $F_1 \leqq F_2$ are always coexistent.

Definition 19. Let be $F \in \hat{L}$, $E \in G$ and $F = F_| + F_\perp$, where $F_|$, $F_\perp \in \hat{L}$ and $F_| \leqq E$, $F_\perp \leqq E^*$.

Then E is said to *reduce* F.

Theorem 41. *An effect $F \in \hat{L}$ and a decision effect $E \in G$ are coexistent if and only if E reduces F.*

Proof. From theorem 40 there follows

$F_1 = E_1$, $E_1 = F' + F$, $F_2 = F'' + F$ and $F' + F'' + F = E_1 + F'' \in L$, i.e.

$$o \leqq \mu(V, E_1) + \mu(V, F'') \leqq 1 \quad \text{for all} \quad V \in K .$$

If $V \in K_+(E_1) = K_-(E_1^*)$, then $\mu(V, F'') = o$, i.e. $F'' \leqq E_1^*$. Because of $E_1 = F' + F$ there holds $F \leqq E_1$ and hence E_1 reduces F_2. If E reduces the effect F, so with $E - F_| = F'$: $E = F' + F_|$, $F = F_| + F_\perp$ and $F' + F_| + F_\perp = E + F_\perp \in \hat{L}$, since $E + F_\perp \leqq E + E^* = \mathbf{1}$.

Theorem 42. *Two coexistent decision effects are commensurable, too.*

Proof. According to theorem 41 E_1 and E_2 are coexistent whenever E_1 reduces E_2 and vice versa. E_1 reduces E_2 if $E_2 = F_{|1} + F_{\perp 2}$, where $F_{|1} \leqq E_1$ and $F_{\perp 1} \leqq E_1^*$. Since $F_{\perp 1} \leqq E_2$ and $F_{|1} \leqq E_2$, so $F_{|1} \leqq E_1 \cap E_2$ and $F_{\perp 1} \leqq E_2 \cap E_1^*$. We will show that $F_{|1} = E_1 \cap E_2$ and $F_{\perp 1} = E_2 \cap E_1^*$.

1) let be $E_1 \cap E_2 \neq 0$. We select a $V \in K_+(E_1 \cap E_2) \subseteq K_+(E_2)$. Then $1 = \mu(V, E_2) = \mu(V, F_{|1})$, because $\mu(V, F_{\perp 1}) \leqq \mu(V, E_1^*) = o$ on account of $K_+(E_1 \cap E_2) = K_-(E_1^* \cap E_2^*) \subseteq K_-(E_1^*)$. Since $\mu(V, F_{|1}) = 1$ for all $V \in K_+(E_1 \cap E_2)$, thus $F_{|1} = E_1 \cap E_2$; for otherwise, there holds $E_1 \cap E_2 = F_{|1} + F$ with $F \leqq E_1 \cap E_2$ and hence

$$1 = \mu(V, E_1 \cap E_2) = \mu(V, F_{|1}) + \mu(V, F) = 1 + \mu(V, F) ,$$

i.e.

$$\mu(V, F) = o \quad \text{for all} \quad V \in K_+(E_1 \cap E_2) = K_-(E_1^* \cap E_2^*) .$$

Therefore, $F \leqq E_1^* \cap E_2^* = (E_1 \cap E_2)^*$ and hence, because of $F \leqq E_1 \cap E_2$, $F \leqq (E_1 \cap E_2) \cap (E_1 \cap E_2)^* = 0$ finally.

1a) Furtheron, if $E_2 \cap E_1^* \neq 0$ then choose a $V \in K_+(E_2 \cap E_1^*)$. Likewise $F_\perp = E_2 \cap E_1^*$ is deduced and hence $E_2 = (E_2 \cap E_1) + (E_2 \cap E_1^*)$. From $E_2 \cap E_1 \perp E_2 \cap E_1^*$ we finally obtain

$$E_2 = (E_2 \cap E_1) \cup (E_2 \cap E_1^*) .$$

1b) If $E_2 \cap E_1^* = 0$ then $F_\perp = 0$ and thus $E_2 = E_2 \cap E_1$.

2) If $E_1 \cap E_2 = 0$ then $E_2 = E_2 \cap E_1^*$.

This completes the proof of theorem 41 from theorem 32.

In a work[1] by Mr. Hellwig it was shown that, concerning the measuring process with respect to quantum mechanics, effects F_1, F_2 can be produced coexistent by a measuring apparatus, if the conditions of theorem 40 are satisfied.

---

[1] Preprint Marburg.

VI. Reflections on the basic suppositions of the theory

Disregarding the formulation of the axioms 1—4 suggested by physics, we can characterize the mathematical contents of the theory by the following facts of the case:

There are given a vector space B and in B a convex cone $\mathfrak{K}$ (with vertex 0) containing no entire line through the origin but spanning B. A plane S without 0 cuts $\mathfrak{K}$ in such a way that the intersection is a bounded set K. By the convex cone $\mathfrak{K}$ a partial ordering is determined on the space B' dual with B in a well-known manner (see definition 8). All $Y \in B'$ with $Y \geqq 0$ determine a convex cone $\mathfrak{K}_0$ in B' which is the *polar* of $\mathfrak{K}$ in B. The set $\{Y \mid Y \in B'$ and $\mu(V, Y) \leqq 1, V \in K\}$ determines another convex cone $\mathfrak{K}_1$ in B' the vertex of which is fixed by the plane $S = \{X \mid X \in B$ and $\mu(X, Y_1) = 1\}$. The intersection of $\mathfrak{K}_0$ with $\mathfrak{K}_1$ is $\hat{L}$. So far, the mathematical structure is consequently defined only by an (almost) "arbitrary" convex cone $\mathfrak{K}$ and an (almost) "arbitrary" plane S. In this *still very general* mathematical state the only limiting additional supposition is the axiom 3 assumed in the form:

Axiom 3. *For any two $F_1, F_2 \in \hat{L}$ there exists a $F_3 \in \hat{L}$ such that $F_1 \leqq F_3$, $F_2 \leqq F_3$ and $\mu(V, F_1) = \mu(V, F_2) = o$ implies $\mu(V, F_3) = o$, too.*

Thus we are authorized to say that axiom 3 is the only decisively important supposition. It is the reason why the decision effects form an orthocomplemented orthomodular lattice and why the probability measures ($V \in K$) are orthoadditive over this lattice. It also comprehends the reason for certain decision effects to be commensurable.

A strengthening of axiom 3 leads easily to reparation of the "classical" case characterized by $G = Z$:

Axiom 3c. *The set $\{F \mid F \geqq F_1,\ F \geqq F_2$ and $K_-(F) = K_-(F_1) \cap \cap K_-(F_2)$ for $F_1, F_2 \in \hat{L}\}$ has a greatest lower bound.*

For the present assume $G = Z$. Let E_ν be the atoms of $G = Z$. Then G consists of all elements $\sum_i E_{\nu_i}$, $\sum_i$ being extended to a subset of the E_ν's. All the elements of $\hat{L}$ have the representation $\sum_\nu \lambda_\nu E_\nu$ where $o \leqq \lambda_\nu \leqq 1$. All $V \in K$ are uniquely given by $v_\nu \underset{\text{df}}{=} \mu(V, E_\nu)$ where $\sum_\nu v_\nu = 1$. Likewise, all $V \in K$ are obtained by arbitrary $v_\nu \geqq o$ with $\sum_\nu v_\nu = 1$. Clearly, the set $\{F \mid F \geqq F_1,\ F \geqq F_2$ and $K_-(F) = K_-(F_1) \cap \cap K_-(F_2)$ for $F_1, F_2 \in \hat{L}\}$ has a greatest lower bound obtained by $\lambda_\nu = \max\{\lambda_\nu^1, \lambda_\nu^2\}$ with λ_ν^1 by F_1 and λ_ν^2 by F_2.

Axiom 3c being now supposed, any two $E_1, E_2 \in G$ will be shown to have to be commensurable. It sufices to prove this for $E_1 \cap E_2 = 0$.

Considering the two effects $F_1 = \frac{1}{2} E_1$ and $F_2 = \frac{1}{2} E_2$ we get an effect $F = \frac{1}{2}(E_1 \cup E_2)$ satisfying

$$K_-(F) = K_-(E_1 \cup E_2) = K_-(E_1) \cap K_-(E_2) = K_-(F_1) \cap K_-(F_2) .$$

Next, the nonexistence of an effect F_3 with $F_3 \geqq \frac{1}{2} E_1$, $F_3 \geqq \frac{1}{2} E_2$ and $F_3 < F$ shall be verified. On account of $F_3 < F$, in the canonical decomposition of F according to theorem 36 there exists at least one λ_ν such that $\lambda_\nu < \frac{1}{2}$; further, for the E_ν's $E_\nu \leqq E_1 \cap E_2$. Yet $V \in K_+(E_1)$ implies $\mu(V, F_3) = \frac{1}{2}$, so does $V \in K_+(E_2)$. For such a V, however, there would be (in the canonical decomposition of F) $\mu(V, E_\nu) = o$ for all E_ν with $\lambda_\nu < \frac{1}{2}$, i.e. $E_\nu \perp E_1$ and $E_\nu \perp E_2$. This contradicts $E_\nu \leqq E_1 \cup E_2$.

If, however, E_1 and E_2 are incommensurable so there holds, indeed,

$$\frac{1}{2}(E_1 + E_2) \geqq F_1 \quad \text{and} \quad \frac{1}{2}(E_1 + E_2) \geqq F_2$$

but *not*

$$\frac{1}{2}(E_1 + E_2) \geqq \frac{1}{2}(E_1 \cup E_2) ,$$

this contradicting axiom 3c.

To prove $\frac{1}{2}(E_1 + E_2) \ngeqq \frac{1}{2}(E_1 \cup E_2)$ we consider the canonical decomposition of $\frac{1}{2}(E_1 + E_2)$ according to theorem 36. In this there must be at least one $\lambda_\nu < \frac{1}{2}$ $\left(\text{the greatest } \lambda_\nu \text{ is } > \frac{1}{2}\,!\right)$.

This results as follows:

For one $V \in K_+(E_1)$ $\mu(V, E_2) \neq o$ must be valid since, otherwise, $E_2 \perp E_1$; this contradicts E_1 and E_2 being not commensurable. Consequently,

$$\max\left\{\mu\left(V, \frac{1}{2} E_1 + \frac{1}{2} E_2\right) \mid V \in K\right\} > \frac{1}{2}$$

and hence one $\lambda_\nu > \frac{1}{2}$.

Since $E_2 \perp E_1$, $\mu(V, E_2) < 1$ is deduced from $V \in K_+(E_1^* \cap (E_1 \cup E_2))$, hence we get $\frac{1}{2} > \sum_\nu \lambda_\nu \mu(V, E_\nu)$ where $\sum_\nu \mu(V, E_\nu) = 1$. Thus $\left(\text{since one } \lambda_\nu > \frac{1}{2}\right)$ at least one λ_ν is $< \frac{1}{2}$. So we have the relation for that $V \in K_+(E_\nu)$ belonging to the ν with $\lambda_\nu < \frac{1}{2}$:

$$\mu\left(V, \frac{1}{2} E_1 + \frac{1}{2} E_2\right) = \lambda_\nu < \frac{1}{2} = \mu\left(V, \frac{1}{2}(E_1 \cup E_2)\right)$$

hence

$$\frac{1}{2} E_1 + \frac{1}{2} E_2 \ngeqq \frac{1}{2}(E_1 \cup E_2) .$$

In the general infinite dimensional case, too, the mathematical theory of partially ordered topological vector spaces and of the correlations between the cones $\mathfrak{K}$, $\mathfrak{K}_0$, $\mathfrak{K}_1$ has been investigated [2]. It will be decisive to investigate the limitation of the theory by axiom 3.

I thank Mr. G. DÄHN for critical perusal of a preprint from which this version has originated and for the English translation.

References

[1] LUDWIG, G.: Versuch einer axiomatischen Grundlegung der Quantenmechanik und allgemeinerer physikalischer Theorien. Z. Physik **181**, 233—260 (1964). (English translation as preprint).

[2] SCHAEFER, H.: Topological vector spaces. New York: The MacMillan Company 1966.