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On the Mathematical Structure of the B. C. S.-Model*

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Abstract. It is investigated in which sense the Bogoliubov-Haag treatment of the B.C.S.-model gives the correct solution in the limit of infinite volume. We find that in a certain subspace of the infinite tensor product space the field operators show the correct time behaviour in the sense of strong convergence.

§ 1. Introduction

In spite of the vast amount of papers on the many body problem there have been very few nontrivial problems where a result could be established with certainty. One notable exception is the B.C.S. theory of superconductivity where through the work of BOGOLIUBOV [1] and HAAG [2] it turned out that H_{BCS} may be replaced by a H_B which can be diagonalized by a Bogoliubov transformation. The present paper studies the question under which circumstances and in what sense this statement is correct.

We shall use the quasi-spin formulation [3] in which the BCS-Hamiltonian is

$$H_\Omega = \sum_{p=1}^{\Omega} \varepsilon_p (1 - \sigma_p^z) - \frac{2T_c}{\Omega} \sum_{p=1}^{\Omega} \sigma_p^- \sum_{p^1=1}^{\Omega} \sigma_{p'}^+ \, . \tag{1}$$

Ω is the number of pair states and T_c an interaction constant which, in suitable units, is the critical temperature. H_Ω acts in a 2^Ω dimensional space and the $\boldsymbol{\sigma}_p$ are Pauli matrices

$$\left(\sigma^\pm = \frac{1}{2} (\sigma_x \pm i \sigma_y)\right) .$$

If the kinetic energy ε_p is independent of p (strong coupling), H_Ω can be trivially diagonalized since with

$$\mathrm{S} = \frac{1}{2} \sum_p \boldsymbol{\sigma}_p \tag{2}$$

it becomes

$$H_\Omega = \varepsilon(\Omega - 2S_z) - \frac{2T_c}{\Omega} (\mathrm{S}^2 - S_z(S_z + 1)) \, . \tag{3}$$

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Since S has eigenvalues $S(S+1)$, $0 \leqq S \leqq \Omega/2$, $-S < S_z < S$, we see that the eigenvalues $E(S, S_z)$ define two characteristic frequencies which are for $S, S_z \gg 1$

$$\Delta = E(S, S_z) - E(S-1, S_z) = -4T_c \frac{S}{\Omega} \tag{4}$$

and

$$2\mu = E(S, S_z) - E(S, S_z - 1) = -2\varepsilon + 4T_c \frac{S_z}{\Omega} \tag{5}$$

S_z gives the number of pairs so that the second frequency is directly related to the chemical potential and becomes zero for $\tilde{H} = H - \mu N$. In the general case the problem is solved in the limit $\Omega \to \infty$ by observing that the commutator of

$$\mathrm{s}_\Omega = \frac{1}{\Omega} \mathrm{S} \tag{6}$$

with an operator of the algebra of the σ's goes to zero. Thus in an irreducible representation s_∞ is a c-number. Now H_Ω can be replaced by

$$\left(s_\pm = \frac{1}{2}(s_x \pm i s_y)\right)$$

$$H_B = -\sum_p \left(\varepsilon_p \sigma_p^{(z)} + 4T_c(\sigma_p^- s_\infty^+ + s_\infty^- \sigma_p^+)\right) + \text{const} . \tag{7}$$

Since H_B gives the same commutators with all σ's as H_∞, they should differ only by a c-number. H_B is linear in the σ's and can be written

$$H_B = \sum_p \sqrt{\varepsilon^2 + 4T_c^2(s_x^2 + s_y^2)}\, \boldsymbol{\sigma}_p \mathbf{n}_p + \text{const}; \quad \mathbf{n}^2 = 1 . \tag{8}$$

Thus H_B contains only one frequency which will turn out to be Δ.

To investigate in what sense the arguments for the general case are correct it is useful to consider H_Ω as operator in the infinite tensor product space (C.D.P.S.) [4] of the problem with $\Omega = \infty$. Then one can study in which topology the various operators converge for $\Omega \to \infty$. It turns out that the densities s_Ω and $\frac{1}{\Omega} H_\Omega$ converge strongly in a rather large subspace of the C.D.P.S. If E_Ω^0 is a suitable c-number $H_\Omega - E_\Omega^0$ converges only weakly and only in a small subspace towards H_B.

Thus the operator H_∞ does not seem to be a useful object. One may argue that this was to be expected and that these topological questions are of no physical interest.

What one actually wants to know is whether

$$\boldsymbol{\sigma}_p(t) = e^{iH_\Omega t} \boldsymbol{\sigma}_p e^{-iH_\Omega t}$$

converges towards $e^{iH_B t} \boldsymbol{\sigma}_p e^{-iH_B t}$ and this should exist everywhere in the C.D.P.S.

It turns out that in general they are quite different although H_B and H_Ω give the same commutators with the σ's. However the s_∞ in H_B is treated as a c-number and thus a constant in time.

But

$$\mathrm{s}_\infty(t) = \lim_{\Omega\to\infty} e^{iH_\Omega t}\mathrm{s}\,e^{-iH_\Omega t} \tag{9}$$

turns out not to be constant in time since

$$\lim_{\Omega\to\infty}[H_\Omega, s_\Omega] \neq 0$$

and it rotates with the frequency 2μ of (5) around the z-axis. Just in these representations where $\mu = 0$, H_B gives the correct time development and these are the subspaces where $H_\Omega - E_\Omega$ converges. Generally the situation can be saved if $\tilde{H} = H - \mu N$ is used for the description of the time development. The chemical potential μ has a value such that $\tilde{H}$ has no second frequency and thus

$$\mathrm{s}_\infty(t) = \lim_{\Omega\to\infty} e^{i\tilde{H}_\Omega t}\mathrm{s}\,e^{-i\tilde{H}_\Omega t} \tag{10}$$

is actually a constant. Then it follows immediately that $\tilde{H}_B$ gives the correct time behaviour in the sense that $e^{i\tilde{H}_\Omega t}\boldsymbol{\sigma} e^{-i\tilde{H}_\Omega t}$ converges towards $e^{i\tilde{H}_B t}\boldsymbol{\sigma} e^{-i\tilde{H}_B t}$ in the strong operator topology.

§ 2. The infinite tensor product

In this section we shall briefly review the theory of the infinite tensor [4] product specialized to our problem. For each p we have a twodimensional complex space, in which we characterize the unit vectors $|n)$ by a (real) unit three-vector $\mathbf{n}$ by

$$(\boldsymbol{\sigma}\cdot\mathbf{n})\,|n) = |n)\,. \tag{11}$$

This determines $|n)$ up to a phase $e^{i\varphi}$. The scalar product of two such vectors is given by

$$(n'|n) = e^{i\varphi}\sqrt{\frac{1+(\mathbf{n}\cdot\mathbf{n}')}{2}}\,. \tag{12}$$

The Hilbert space we shall consider is the product

$$H = \prod_p \otimes\, C_p^{(2)}\,.$$

The unit vectors

$$|\{n\}\rangle = \prod_p \otimes\, |n_p) \tag{13}$$

are characterized by the set $\{n\}$ of the three-vectors $\mathbf{n}_p$. The scalar product of two such vectors is

$$\langle\{n'\}|\{n\}\rangle = \prod_p (n'_p|n_p) = \prod_p e^{i\varphi_p}\sqrt{\frac{1+(\mathbf{n}_p\cdot\mathbf{n}'_p)}{2}} \tag{14}$$

Since $p = 1, 2 \ldots \Omega$ for $\Omega = \infty$ the question of the convergence of $\prod_p$ arises. We distinguish three cases:

1. $\prod_p$ converges absolutely, i.e.

$$\sum_p \left| e^{i\varphi_p} \sqrt{\frac{1+(\mathbf{n}_p \cdot \mathbf{n}'_p)}{2}} - 1 \right| < \infty\,. \tag{15}$$

(15) can be seen to define an equivalence relation which we denote by $|\{n'\}\rangle \approx |\{n\}\rangle$ and we shall say that if (15) holds $|\{n'\}\rangle$ and $|\{n\}\rangle$ are in the same equivalence class $C\{n\}$.

2. $\prod_p$ does not converge absolutely but does so without the phase factors $e^{i\varphi_p}$

$$\sum_p \left| \sqrt{\frac{1+(\mathbf{n}_p \cdot \mathbf{n}'_p)}{2}} - 1 \right| < \infty\,. \tag{16}$$

(16) defines another equivalence relation which we denote by

$$|\{n'\}\rangle \overset{W}{\approx} |\{n\}\rangle$$

and all $|\{n'\}\rangle$ of (16) form the weak equivalence class $C_w\{n\}$. One sees easily $C_w\{n\} \supset C\{n\}$. If $|\{n'\}\rangle \overset{W}{\approx} |\{n\}\rangle$ but $|\{n'\}\rangle \not\approx |\{n\}\rangle$ (14) is meaningless and then we define

$$\langle\{n'\}\,|\,\{n\}\rangle = 0\,. \tag{17}$$

3. $\prod_p$ diverges to zero, i.e.

$$\sum_p \left| \sqrt{\frac{1+(\mathbf{n}_p \cdot \mathbf{n}'_p)}{2}} - 1 \right| \to \infty \tag{18}$$

in which case (14) gives $\langle\{n'\}\,|\,\{n\}\rangle = 0$.

By linear extension one now obtains from the $|\{n\}\rangle$ vectors the (nonseparable) Hilbert space $\mathscr{H}$. Similarly the $|\{n\}\rangle$ of an equivalence class $C\{n\}$ span the (separable) product space (I.D.P.S)

$$\mathscr{H}_{\{n\}} = \prod_p^c \otimes\, C_p^{(2)}\,. \tag{19}$$

One can show that the vectors obtained from $|\{n\}\rangle$ by flipping a finite number of "spins" are dense in $\mathscr{H}_{\{n\}}$. Explicitely we can construct these vectors as follows. Denote by $2\mathbf{n}^+$ a vector orthogonal to $\mathbf{n}$ plus i times the vector orthogonal to the two others. Then $|\{n\}\rangle$ is characterized by

$$(\boldsymbol{\sigma}_p \cdot \mathbf{n}_p)\,|\{n\}\rangle = |\{n\}\rangle\ \forall\, p \quad \text{or} \quad (\boldsymbol{\sigma}_p \cdot \mathbf{n}_p^+)\,|\{n\}\rangle = 0\ \forall\, p\,. \tag{20}$$

Define $\mathbf{n}^- = (\mathbf{n}^+)^*$ and

$$|\{m\},\{n\}\rangle = \prod_p (\boldsymbol{\sigma}_p \cdot \mathbf{n}_p^-)^{m_p}\,|n\rangle \tag{21}$$

where $m_p = 0{,}1$. Those p in (21) where $m_p = 1$ have their spins flipped and the dense set is characterized by

$$\sum_p m_p < \infty\,. \tag{22}$$

Regarding weak equivalence classes it is clear that the unitary operator $U_{\{\varphi\}}$ defined by

$$U_{\{\varphi\}} \prod_p \otimes |n_p) = \prod_p \otimes e^{i\varphi_p}|n_p) \tag{23}$$

does not lead out of $C_w\{n\}$. However for $\sum_p |\varphi_p| \to \infty$ it leads out of $C(n)$. Conversely for $|'\rangle \overset{W}{\approx} |\rangle$, but $|'\rangle \not\approx |\rangle$ there is always *a* $U_{\{\varphi\}}$ such that

$$U_{\{\varphi\}}|'\rangle \approx |\rangle .$$

The operators we get from the $\boldsymbol{\sigma}_p$ by algebraic processes and weak closure form the algebra $B^{\#} = \left(\bigcup_p \boldsymbol{\sigma}_p\right)''$.

They are, however, not all the bounded operators but one has v. NEUMANNS Theorem:

$$\forall A \in B^{\#} \leftrightarrow [A, P_c] = [A, U] = 0 . \tag{I}$$

Here P_c is the projection operator onto an equivalence class and U is a unitary operator of the form (23). (I) means that $B^{\#}$ does not lead out of an equivalence class and hence the $\mathscr{H}_{\{n\}}$ reduce the representation of the σ's. One can see easily that within an $\mathscr{H}_{\{n\}}$ the representation is irreducible and an operator commuting with all σ_p's is a c-number in $\mathscr{H}_{\{n\}}$[1]. Furthermore, since the U's transform from one C to another C within one C_w, (I) shows that the representation of $B^{\#}$ in all C's of a C_w is the same. This is also the reason for our notation $\mathscr{H}_{\{n\}}$ although the n's characterize only C_w and not C. To define the C one still has to know the phase factors φ_p but (I) tells us that the representation of $B^{\#}$ does not depend on them.

§ 3. Operator convergence for $\Omega \to \infty$

As it is to be expected [5] the intensive quantities show the largest domain of convergence. This is described by[2]

Lemma 1.

$$\lim_{\Omega\to\infty} 2\mathrm{s}_\Omega = \lim_{\Omega\to\infty} \frac{1}{\Omega} \sum_{p=1}^{\Omega} \boldsymbol{\sigma}_p \to \lim_{\Omega\to\infty} \frac{1}{\Omega} \sum_{p=1}^{\Omega} \mathrm{n}_p = \eta \mathrm{n}, \qquad \mathrm{n}^2 = 1, \quad 0 \leqq \eta \leqq 1 \tag{24}$$

in these $\mathscr{H}_{\{n\}}$ *in which* $\lim_{\Omega\to\infty} \frac{1}{\Omega} \sum_{p=1}^{\Omega} \mathrm{n}_p$ *exists.*

Proof.

$$2\mathrm{s}_\Omega\,|\{n\},\{m\}\rangle = \frac{1}{\Omega}\sum_{p=1}^{\Omega}[\mathrm{n}_p(\boldsymbol{\sigma}_p\cdot\mathrm{n}_p) + 2\mathrm{n}_p^-(\boldsymbol{\sigma}_p\cdot\mathrm{n}_p^+) + + 2\mathrm{n}_p^+(\boldsymbol{\sigma}_p\cdot\mathrm{n}_p^-)]\,|\{n\},\{m\}\rangle . \tag{25}$$

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[1] This number may be different in $\mathscr{H}_{\{n\}}$ if $|\{n\}\rangle \overset{w}{\not\approx} |\{n'\}\rangle$.

[2] Our notation for convergence is $\Rightarrow$ uniform, $\to$ strong, $\rightharpoonup$ weak, the two latter in the extended sense explained below.

Now

$$\frac{1}{\Omega}\sum_{p=1}\mathfrak{n}_p(\boldsymbol{\sigma}_p\mathfrak{n}_p)\,|\{n\},\{m\}\rangle=\frac{1}{\Omega}\sum_{p=1}^{\Omega}(-)^{m_p}\,\mathfrak{n}_p|\{n\},\{m\}\rangle\to\eta\,\mathfrak{n}\,|\{n\},\{m\}\rangle\,,$$

$$\frac{1}{\Omega}\sum_{p=1}\mathfrak{n}_p^-(\boldsymbol{\sigma}_p\mathfrak{n}_p^+)\,|\{n\},\{m\}\rangle=\frac{1}{\Omega}\sum_{p=1}^{\Omega}\frac{1-(-)^{m_p}}{2}\,\mathfrak{n}_p^-(\boldsymbol{\sigma}_p\mathfrak{n}_p^+)\,|\{n\},\{m\}\rangle\to 0$$

$$\frac{1}{\Omega}\sum_{p=1}^{\Omega}\mathfrak{n}_p^+(\boldsymbol{\sigma}_p\mathfrak{n}_p^+)\,|\{n\},\{m\}\rangle=\frac{1}{\Omega}\sum_{p=1}^{\Omega}\frac{1+(-)^{m_p}}{2}\,\mathfrak{n}_p^+(\boldsymbol{\sigma}_p\mathfrak{n}_p^-)\,|\{n\},\{m\}\rangle\to 0.$$

Since $\|\mathfrak{s}_\Omega\|\leqq 1\;\forall\,\Omega$ and the $|\{n\},\{m\}\rangle$ are dense in $\mathscr{H}_{\{n\}}$, this proves lemma 1.

Remark. One might have expected since $[s_\Omega^\alpha,\sigma_p^\beta]=i\,\varepsilon^{\alpha\beta\gamma}\frac{\sigma_p^\gamma}{\Omega}\Rightarrow 0$ that $s_\Omega\Rightarrow$ towards a c-number. This is not the case because

$$\|2s_\Omega-\eta\,\mathfrak{n}\|\geqq 1\;\forall\,\Omega\,. \tag{26}$$

Since the product of strongly converging bounded operators converges strongly to the product of the limits, lemma 1 gives the limits of intensive quantities, f.i.

Corollary.

$$\frac{H_\Omega}{\Omega}\to\lim_{\Omega\to\infty}\frac{1}{\Omega}\sum_{p=1}^{\Omega}\varepsilon_p(1-n_p^{(z)})-\frac{T_c\eta^2}{2}(n_x^2+n_y^2)\,,\;e^{it\mathfrak{s}_\Omega\cdot\mathfrak{b}}\to e^{it\eta\mathfrak{n}\cdot\mathfrak{b}/2} \tag{27}$$

where $\mathfrak{b}$ is an arbitrary vector. We shall always assume that the ε_p remain bounded and vary sufficiently slowly so that the lim in (27) exists.

Whereas the limits for intensive quantities exist in a reasonably large class of $\mathscr{H}_{\{n\}}$, one can give a meaning to the limits of extensive quantities only in more restricted domains:

Lemma 2.

$$\sum_{p=1}^{\Omega}\frac{1}{2}(1-\boldsymbol{\sigma}_p\mathfrak{n}_p)\to N\quad in\quad D_N\subset\mathscr{H}_{\{n\}} \tag{28}$$

where $N|\{n\},\{m\}\rangle=\sum_{p=1}^{\infty}m_p|\{n\},\{m\}\rangle$
and D_N *is the domain of* N.

Proof.

$$\left\|\left(\sum_{p=1}^{\Omega}\frac{1}{2}(1-\boldsymbol{\sigma}_p\mathfrak{n}_p)-N\right)|\{n\},\{m\}\rangle\right\|=\sum_{p=\Omega+1}^{\infty}m_p\to 0\,.$$

Since N is self-adjoint one immediately sees the

Corollary.

$$U_\Omega=e^{it\sum_{p=1}^{\Omega}(1-(\boldsymbol{\sigma}_p\cdot\mathfrak{n}_p))}\to e^{2itN}\quad\text{in}\quad\mathscr{H}_{\{n\}}\,. \tag{29}$$

Remark. $\sum_{p=1}^{\infty}(1-\boldsymbol{\sigma}_p\cdot\mathbf{n}_p)$ has a meaning only in the equivalence class determined by the $\{n_p\}$. For non-equivalent $\{n'_p\}$ this sum has domain zero, i.e. it makes any non-zero vector infinitely long. Similarly f.i. $e^{it\sum_{p=0}^{\Omega}(1-\sigma_p^{(z)})}$ converges only in the equivalence class where all n's are in the z-direction. One might wonder why one cannot define a rotation U_∞ around the z-axis for other equivalence classes since this is certainly a unitary operation in $\mathscr{H}$. However, for other equivalence classes U_∞ leads from one $\mathscr{H}_{\{n\}}$ to another for any $t \neq 0$. Hence, by v. NEUMANNS theorem this cannot be effected by limits of operators from $B^{\#}$ like $e^{it\sum_{p=1}^{\Omega}\sigma_p^{(z)}}$. Such a rotation is discontinous in the rotation angle t and hence Stone's theorem is not applicable. Thus we cannot write U_∞ as e^{itN} in agreement with the observation that the formal expression

$$N=\sum_{p=1}^{\infty}(\sigma_p^{(z)}-1)$$

has no meaning in other equivalence classes.

Extensive quantities of the type of the interaction energy have even worse convergence properties.

One might expect that by subtracting the expectation value, now in each factor, convergence can be achieved. What happens is described by

Lemma 3.

$$\frac{1}{\Omega}\sum_{p,p'}\left(\sigma_p^{+}-\frac{1}{2}(n_p^{(x)}+i\,n_p^{(y)})\right)\left(\sigma_{p'}^{-}-\frac{1}{2}(n_{p'}^{(x)}-i\,n_{p'}^{(y)})\right)\rightarrow\frac{1}{4\Omega}\sum_{p}(1+n_p^{(z)})^2. \quad (30)$$

Proof. Again it is expedient to use the decomposition

$$\sigma_p^{+}=\frac{1}{2}(n_p^{(x)}+i\,n_p^{(y)})(\boldsymbol{\sigma}_p\cdot\mathbf{n}_p)+(n_p^{+(x)}+i\,n_p^{+(y)})(\mathbf{n}_p^{-}\cdot\boldsymbol{\sigma}_p)+$$
$$+(n_p^{-(x)}+i\,n_p^{-(y)})\cdot(\boldsymbol{\sigma}_p\cdot\mathbf{n}_p^{+}).$$

On inserting and using the commutation relations of the σ's we see that for $p=p'$ the term of the r.h.s. of (29) remains without operators as factors. The other contributions can be seen to converge to zero as in (25). However, now we have a double sum and only one factor $1/\Omega$ in front. In brief these are terms of the form

$$|\rangle=\frac{1}{\Omega}\sum_{p,p'}(\boldsymbol{\sigma}_p\mathbf{n}_p^{-})(\boldsymbol{\sigma}_{p'}\mathbf{n}_{p'}^{-})\,|\{n\},\{m\}\rangle\,. \quad (31)$$

They converge weakly to zero but their norm (for $m_p=0$, f.i.)

$$\langle|\rangle=\frac{1}{\Omega^2}\sum_{p,p'}1=1 \quad (32)$$

does not go to zero.

Remark. The right hand side R_Ω of (30) converges weakly on the dense set $|\{m\}, \{n\}\rangle$ towards a c-number c. However it cannot converge everywhere since $\|R_\Omega\| \propto \Omega$ and does not remain bounded uniformely in Ω. One finds that for those states $|f\rangle$ for which $R_\Omega|f\rangle$ does not converge weakly, $\|R_\Omega|f\rangle\| \to \infty$ and thus the limit is not defined. We shall call c the generalized weak limit in the same way as we interpreted N in (28) as a strong limit in a generalized sense. The difference is that N is already selfadjoint and thus there is no further selfadjoint extension.

A c-number defined on a dense set is not self-adjoint but c^* is c everywhere in $\mathscr{H}$. But $c^* = c^{**}$ and thus it is essentially selfadjoint. That is to say c everywhere in $\mathscr{H}$ is the unique selfadjoint extension of the limit of the R_Ω and contains all symmetric extensions.

Collecting our findings so far we are in the position to state the convergence properties of H_Ω in form of

Theorem 1.

$$\begin{aligned} H_\Omega - E_\Omega = &-\frac{2T_c}{\Omega}\sum_{p=1}^{\Omega}\left(\sigma_p^+ - \frac{1}{2}(n_p^{(x)} + i n_p^{(y)})\right) \times \\ &\times \sum_{p'=1}^{\Omega}\left(\sigma_{p'}^- - (n_{p'}^{(x)} - i n_{p'}^{(y)})\right) + \frac{T_c}{2}\sum_{p,p'}(n_p^{(x)} + i n_p^{(y)})(n_{p'}^{(x)} - i n_{p'}^{(y)}) - \\ &-\sum_p \sigma_p^{(x)}\sum_{p'}\frac{T_c n_{p'}^{(x)}}{\Omega} - \sum_p \sigma_p^{(y)}\sum_{p'}\frac{T_c n_{p'}^{(y)}}{\Omega} - \sum_p \sigma_p^{(z)}\varepsilon_p + \sum_p \varepsilon_p - \\ &- E_\Omega \to \sum_p \sqrt{\varepsilon_p^2 + T_c^2\eta^2(n^{(x)2} + n^{(y)2})}\;.\times \\ &\times \eta_p(1 - \boldsymbol{\sigma}_p \mathbf{n}_p) \equiv H_B \quad \textit{with} \quad \eta_p = \pm 1 \end{aligned} \tag{33}$$

if the "gap equation"

$$1 = \frac{T_c}{\Omega}\sum_p \frac{\eta_p}{\sqrt{\varepsilon_p^2 + T_c^2\eta^2(n^{(x)2} + n^{(y)2})}} \tag{34}$$

holds.

$$\begin{aligned} E_\Omega = &\sum_p \varepsilon_p + \frac{T_c}{2\Omega}\sum_{p,p'}(n_p^{(x)} + i n_p^{(y)})(n_{p'}^{(x)} - i n_{p'}^{(y)}) - \\ &-\sum_{p=1}^{\Omega}\sqrt{\varepsilon_p^2 + T_c^2\eta^2(n^{(x)2} + n^{(y)2})}\;\eta - \frac{T_c}{2\Omega}\sum_p (1 + n_p^{(z)})^2 \end{aligned} \tag{35}$$

Proof. Equating the coefficients of $\boldsymbol{\sigma}_p$ in (33) we see

$$\begin{aligned} \sqrt{\varepsilon_p^2 + T_c^2\eta^2(n^{(x)2} + n^{(y)2})}\; n_p^{(x,y)}\eta_p &= T_c \eta n^{(x,y)} \\ \sqrt{\varepsilon_p^2 + T_c^2\eta^2(n^{(x)2} + n^{(y)2})}\; n_p^{(z)}\eta_p &= \varepsilon_p \end{aligned} \tag{36}$$

from which we deduce (34). The rest then follows from the lemmata 2 and 3.

The condition (34) simplifies in the strong coupling limit ($\varepsilon_p = \text{const.}$ hence all $\mathbf{n}_p$ parallel) to

$$n^{(z)} = \frac{\varepsilon}{T_c \eta}, \quad \eta = \lim_{\Omega \to \infty}\frac{1}{\Omega}\sum_{p=1}^{\Omega}\eta_p, \quad \mathbf{n}_p = \eta_p \mathbf{n} \tag{37}$$

For the ground state (all $\eta_p = 1$) only in those $\mathscr{H}_{\{n\}}$ with $n^{(z)} = \varepsilon/T_c$ the Hamiltonian tends towards a limit. These are the representations for which the second frequency (5) goes to zero.

$$-\Delta = 4T_c S/\Omega = 2T_c\eta = \sqrt{\varepsilon^2 + T_c^2\eta^2(1 - n^{(z)2})}$$

Fixing $n^{(z)}$ (and hence the number of pairs) we are forced into those $\mathscr{H}_{\{n\}}$ where η and thus the percentage of p-modes which contribute negatively to the energy is given by (37). This peculiar situation disappears if one uses $\tilde{H} = H - \mu N$ where $\varepsilon \to \varepsilon - \mu$. In this case μ is to be chosen such that (37) is satisfied for the given value of $n^{(z)}$ and η.

Remark. Since $H_\Omega - E_\Omega \not\rightarrow H_B$ we have $e^{it(H_\Omega - E_\Omega)} \not\rightarrow e^{itH_B}$. We see this explicitely in the degenerate model

$$\varepsilon = 0, \quad \tilde{H}_\Omega - E_\Omega = -\frac{2T_c}{\Omega}\sum_{p'p'} \sigma_p^+\sigma_{p'}^- + \Omega T_c/2$$

where the ground state is all $\mathbf{n}_p$ in the $x-y$-plane, for instance $\mathbf{n}_p = \mathbf{i}$ = unit vector in x-direction. Then $H_B|\{0\},\{\mathbf{i}\}\rangle = 0$ and hence

$$\langle\{\mathbf{i}\},\{0\}|\, e^{itH_B}|\{0\},\{\mathbf{i}\}\rangle = 1\,.$$

However using well-known formulae for matrix elements of rotation matrices we find

$$\langle\{\mathbf{i}\},\{0\}|\, e^{it(H_\Omega - E_\Omega)}\,|\{0\},\{\mathbf{i}\}\rangle = \sum_{m=-s}^{m=s} 2^{-2s}\frac{(2s)!}{(s-m)!\,(s+m)!}\, e^{itT_c\frac{m+m^2}{s}} \to$$

$$\to \int_{-\infty}^{\infty} \frac{dm}{\sqrt{\pi s}}\, e^{-\frac{m^2}{s}(1+itT_c)} = \frac{1}{\sqrt{1+itT_c}}\,.$$

Generally the situation is that weak convergence of unitary operators towards a unitary limit implies strong convergence. However if $(it)^{-1}(e^{iH_nt} - 1) = G_n(t)$ posess uniform boundedness properties on a certain domain, $G_n(t) \to G(t)\ \forall\, t \neq 0$ implies $G_n(0) \to G(0)$ on this domain. In our case these boundedness conditions are met and thus the weak convergence of the time translation operators would imply $H_\Omega - E_\Omega \to H_B$ which we know not to be the case. Nevertheless we shall now see that H_B describes in its domain the correct time dependence.

§ 4. Time dependence of operators

The crucial question for the time-dependence of the $\boldsymbol{\sigma}_p$'s is whether the

$$\mathbf{s}_n = \frac{1}{2\Omega}\sum_{p=1}^{\Omega} \varepsilon_p^n \boldsymbol{\sigma}_p \tag{38}$$

actually tend towards constant c-numbers or show a time dependence. Defining as usual

$$A(t) = e^{itH_\Omega}A(0)e^{-itH_\Omega}$$
$$i\frac{d}{dt}A = [A, H_\Omega] \tag{39}$$

we find for fixed Ω

$$i \frac{d}{dt} s_n^+ = 2 s_{n+1}^+ - 4 T_c s^+ s_n^{(z)} \tag{40}$$

$$i \frac{d}{dt} s_n^{(z)} = 8 T_c (s_n^+ s^- - s^+ s_n^-). \tag{41}$$

In general these time derivatives are $\neq 0$. The reader who is surprised by this fact should remember that $\lim_{\Omega \to \infty} \sigma_n$ commutes with the operators from $B^{\#}$. However $\lim_{\Omega \to \infty} (H_\Omega - E_\Omega)$ in general does not exist and hence is not in $B^{\#}$. Only for those $\mathscr{H}_{\{n\}}$ which satisfy (34) and thus $H_\Omega = E_\Omega$ converges σ_n tends toward zero. With $E_p = \sqrt{\varepsilon_p^2 + T_c^2\,{}^2(n^{(x)\,2} + n^{(y)\,2})}$ we note that for $\Omega \to \infty$ the r.h.s. of (40) converges strongly towards

$$\begin{aligned} &\lim \left\{ \frac{1}{\Omega} \sum_{p=1}^{\Omega} \varepsilon_p^{n+1} n_p^+ - \frac{T_c}{\Omega^2} \sum_{p,p'=1}^{\Omega} n_p^+ \varepsilon_{p'}^n n_{p'}^{(z)} \right\} \\ &\qquad = \lim \left\{ \frac{1}{\Omega} \sum_p \frac{\varepsilon_p^{n+1} \eta_p}{E_p} \left(1 - \frac{T_c}{\Omega} \sum_{p'} \frac{\eta_{p'}}{E_{p'}} \right) \right\} = 0\,. \end{aligned} \tag{42}$$

Similarly (41) becomes

$$2 T_c \lim_{\Omega \to \infty} \left\{ \eta^2 T_c^2 n^+ n^- \sum_{p p'=1}^{\Omega} \frac{\eta_p \eta_{p'}}{E_p E_{p'}'} (\varepsilon_p^n - \varepsilon_{p'}^n) \right\} = 0\,. \tag{43}$$

From (40) and (41) we now conclude that all time derivatives of arbitrary (finite) order converge strongly toward zero if (34) is satisfied.

Furthermore the s_n are bounded by

$$\|\mathrm{s}_n\| \leqq E^{n/2}\ \forall\, \Omega, \quad E = \max_p \varepsilon_p\,. \tag{44}$$

similarly we see from (40), (41) that the n'th time derivatives are bounded by

$$\left\| \frac{d^n}{dt^n} \mathrm{s} \right\| \leqq c^n$$

where c is independent of Ω. Thus

$$\mathrm{s}(t) = \sum_{n=0}^{N} \frac{d^n}{dt^n} \mathrm{s}(t)\big|_{t=0} \frac{t^n}{n!} + R_N$$

$$\|R_N\| = \left\| \sum_{n=N}^{\infty} \left(\frac{d^n}{dt^n} \mathrm{s} \right) \cdot \frac{t^n}{n!} \right\| \leqq \frac{c^N}{N!} e^{ct}\,. \tag{45}$$

By chosing N sufficiently large one can always make R_N arbitrarily small and since all finite time derivatives converge strongly towards zero if (34) holds we have shown that in these representations s converges strongly to a constant c-number.

Similarly for the time dependence of the $\boldsymbol{\sigma}_p$ we obtain f.i.

$$i \frac{d}{dt} \sigma_p^+ = 2\,\varepsilon_p \sigma_p^+ = 4\,T_c s^+ \sigma_p^{(z)} = [\sigma_p^+, H_\Omega] \to [\sigma_p^+, H_B] \tag{46}$$

since H_B is constructed such that it has, in the limit, the same commutators with the σ_p's as H. If (34) holds so that the commutators of $\mathbf{s}$ with H_Ω converge to zero we see that the n'th time derivative $\sigma_p^{(n)}$ converges strongly towards the corresponding commutator with H_B. Furthermore we have again $\|\sigma_p^{(n)}\| \leqq c^n$ so that by the estimate used in (45) we conclude

$$\boldsymbol{\sigma}_p(t) = e^{itH_\Omega} \boldsymbol{\sigma}_p e^{-itH_\Omega} \to e^{itH_B} \boldsymbol{\sigma}_p e^{-itH_B} \tag{47}$$

if (34) holds. However in a general representation where (34) does not hold, H_B will give a totally incorrect time dependence. This is exhibited immediately in the strong coupling limit where one finds

$$s_z = \text{const.}, \quad s^\pm(t) = s^\pm(0)\, e^{\pm it(2\varepsilon - 4T_c s_z)}$$

i.e. a rotation around the z-axis with the frequency 2μ of (5). H_B now gives a rotation of the $\boldsymbol{\sigma}$'s around a fixed $\mathbf{s}$ with a frequency Δ. The correct time dependence given by H_Ω is this rotation around $\mathbf{s}$ which, however, rotates itself around the z-axis. Only if $\mu = 0$ and therefore states with different N have the same energy, H_B gives the correct motion.

It should be pointed out that the convergence of the higher time derivatives and their uniform boundedness is essential for obtaining the correct time dependence. This is not guaranted by the fact that $[H_\Omega, b] \to$ $\to [H_B, b]\ \forall\, b \in B(l_2)$ as can be seen by the following counterexample from l_2: Define

$$\sigma_\Omega = \begin{array}{c|cccc} 1 & 0 & 0 & 0 & \cdots \\ 2 & 0 & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ \cdot & 0 & & & \\ \Omega & 1 & & & \\ \cdot & 0 & & & \end{array}, \quad \Sigma_\Omega = \begin{array}{|ccccccc} 1 & & & & & & \\ & 2 & & & & & \\ & & 3 & & & & \\ & & & \ddots & & & \\ & & & & \Omega & & \\ & & & & & 0 & \\ & & & & & & 0 \end{array} \tag{48}$$

and $A_\Omega = \frac{1}{\Omega} \sigma_\Omega^+$. The σ, σ^+ generate the $B(l_2)$ algebra and their commutator with A_Ω goes uniformly to zero as $\Omega \to \infty$. Hence A_∞ should be a c-number in an irreducible representation and we see that actually $A_\Omega \Rightarrow 0$. Take as Hamiltonian

$$H_\Omega = \Sigma_\Omega A_\Omega + A_\Omega^+ \Sigma_\Omega = \sigma_\Omega^+ + \sigma_\Omega \rightharpoonup 0\,. \tag{49}$$

Thus again H_Ω converges weakly towards the limit which we obtain if we insert for A_Ω its c-number limit. Furthermore the commutator of H_Ω with all σ, σ^+ goes to zero. However $e^{iH_\Omega t} \Sigma_1 e^{-iH_\Omega t} \rightharpoonup \Sigma_1 \cdot \cos^2 t$, because $[H_\Omega [H_\Omega, \Sigma_1]] \neq 0$.

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