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Uniqueness and Symmetry Breaking in S-Matrix Theory

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Abstract. Assuming that the physical world is a solution of the S matrix equations, nonlinear functional analysis enables its uniqueness to be tested experimentally. As a first step, we develop such tests within the limits of partial wave dispersion relations, *with crossing symmetry included.* They are closely related to Levinson's theorem. We show that they give conditions for the validity of the bootstrap hypothesis, of the dynamical generation of symmetries, and of Dashen-Frautschi perturbation theory. They do not appear to be satisfied experimentally.

1. Introduction

Of the various theoretical frameworks for elementary particle physics, S matrix theory is especially remarkable for its mathematical inadequacy and its experimental success. In part the former is due to the incompleteness of its axioms, about which I have nothing to say. However, even in the case where equations exist and where all the experimental successes have been obtained, most calculations employ very crude approximations. There has been considerable doubt as to whether the deeper predictions of unique sets of self-consistent particles possessing dynamically determined symmetries, do in fact follow from the S matrix equations, and are not just self-consistency conditions on the rough approximations used. In the present work, I want to investigate this question using nonlinear functional analysis.

The equations of S matrix theory are nonlinear integral equations. They define nonlinear operators in a Banach space of physically acceptable scattering amplitudes. The best known results of nonlinear functional analysis are fixed point theorems, which decide whether solutions of such equations exist. However they involve intricate topological questions (and would lead to solipsist titles), so at present I will only consider the easier problem of whether a given solution is unique. Unless S matrix theory is completely wrong, the observed physical universe must be a solution of the S matrix equations. Some people hope it is the only solution. Nonlinear functional analysis lets us test this hope *experimentally.* This is done by Banach space implicit function theorems which tell us, using only experimentally observable quantities, whether the observed universe is an isolated solution of the S matrix equations or whether it belongs to a continuum of solutions, assuming of course that it is a solution at all.

This is obviously a matter of considerable philosophic interest. It turns out to be also an essential one for S matrix perturbation theory, and indeed for the whole idea of bootstrap generation of symmetries. These theories require that, even if the original solution is nonunique, it shall at least be so in a controlled and limited way.

In the present paper, I shall only consider a finite number of partial wave dispersion relations with unitarity and crossing, the many-particle contributions being assumed given. Although such a framework is not as general as might be wished, it nevertheless includes all the bootstrap models which have been claimed as evidence for dynamical symmetries. My reason for this limitation is that the proofs then go through rather easily. This means the techniques are not being used at anything like full stretch, so there is every hope that extensive generalization will be possible.

A short summary of this work has been given elsewhere [1] for the less mathematically minded. In the present paper, I shall assume knowledge of linear functional analysis. Standard theorems will be quoted from DUNFORD and SCHWARTZ [2], referred to as *DS*.

Section 2 defines the appropriate Banach space, and considers unitarity and dispersion relations as operations in it. This Banach space consists of functions with certain continuity properties in the physical region. In accord with the general principles of S matrix theory, the definition of the Banach space only refers to directly observable quantities. This is a big difference from potential scattering. Before any application of functional analysis, we must verify as in Section 2 that our problem satisfies the axioms. Even if this Section read by itself seems just to be giving long names to the physically obvious, the reader must remember that it is the essential foundation for all that follows.

Section 3 formulates the mathematical problems involved in the uniqueness of S matrix solutions, in spontaneous symmetry breaking, and in the validity of Dashen-Frautschi perturbation theory [3—27]. A rigorous formulation of any of them leads to the concept of the Fréchet differential, which is explained here.

Section 4 gives the fundamental uniqueness theorem which relates properties of the first term of the perturbation expansion to those of the exact solution. All numerical calculations involve replacing the integral equations of S matrix theory by finite-dimensional algebraic ones. Bootstrappers have usually assumed, not only that this can be done, but also that if the original integral equations were free of arbitrary constants, then the algebraic equations also will not contain any. All the unique results claimed from various bootstraps depend on this assumption. However, the "implicit variable" theorem of Section 4 shows that it is not necessarily true — the process of reducing the integral equations to algebraic ones may itself introduce or remove a certain number of arbitrary parameters. This number is called the index, and can be calculated.

When higher spin particles are exchanged, a left-hand cutoff is needed to make the partial wave dispersion relations consistent. Section 5 studies crossing symmetry, supposing such a cutoff to be given. We show that the index is independent of the left-hand cut contribution. This is a rather special feature of the present problem, and depends on the fact that perturbations arising from two-particle channels die out at infinity. For the strip approximation, the index would almost certainly depend on the leading term in the asymptotic behaviour.

Having eliminated the left-hand cut, we calculate the index explicitly in Section 6. It turns out to be very closely related to Levinson's theorem, and to the conditions for CDD poles. If the index is positive, the Dashen-Frautschi equation requires subtractions, leading to ambiguities. After these have been included, the compactness proof of Section 5 justifies the algebraic approximation. We also discuss the external mass perturbations — these can be reduced to a very simple form, but it shows that first order perturbation theory will not hold near moving thresholds.

Section 7 discusses our solution of the uniqueness problem. Unlike similar work on the N/D equations [28—31], crossing has been included, so that we can apply it to bootstraps. For the reciprocal $N - N^*$ bootstrap the experimental situation is rather clear — the uniqueness condition is not satisfied. This means that S matrix theory neither sustains the $SU(2)$ symmetry in the absence of an electromagnetic interaction, nor gives a definite answer for the effects of the latter. Any unique number obtained for the $n - p$ mass difference must be a consequence of special dynamical assumptions made, knowingly or unknowingly, during the calculation and not of S matrix theory itself.

In Appendix A, we verify the uniqueness theorem for certain relativistic soluble models. The result for coupled two-particle channels is contained in Appendix B, the main body of the paper having been restricted to the uncoupled case for expositional clarity.

2. *S* matrix theory in a normed ring

a. Continuity assumptions

To apply functional analysis, one must always start by defining an appropriate Banach space. The first restriction on our choice is that the right-hand cut dispersion integral shall represent a bounded operator. There are many Banach spaces in which this is true. However, the space of Hölder-continuous functions is particularly suitable, since such properties can be inferred from unitarity. Before defining it, it is convenient to transform the right-hand cut into a finite interval by the change of variable

$$u = \frac{s - s_1}{s - s_0} \tag{2.1}$$

where s_1 is the threshold of the right-hand cut, and s_0 is some real subtraction point between the two cuts (which we assume not to overlap). For convenience, we shall write $\delta(u)$ for $\delta(s(u))$, etc. In this variable u, the right-hand cut is $[0, 1]$ and the left-hand cut is $[1, u_2]$, where u_2 is finite. $s = \infty$ becomes $u = 1$, and $u = \infty$ corresponds to the subtraction point $s = s_0$. We now define the Banach space $H(R; \mu)$ to be the set of all real-valued functions which satisfy a Hölder condition in u on the interval $[0, 1]$. The norm is

$$|f| = \sup_{0 \leqq u \leqq 1} |f(u)| + \sup_{\substack{0 \leqq u \leqq 1 \\ 0 \leqq u' \leqq 1}} \left| \frac{f(u) - f(u')}{(u - u')^{\mu}} \right| \tag{2.2}$$

with $0 < \mu < 1$. We need not completely specify μ. For a proof that this is a Banach space, see MUSKHELISHVILI [32], p. 132. The reader not familiar with Hölder-continuous functions may also find an account of their elementary properties in the first chapter of this book. Note that we only assume Hölder continuity in the physical region, and that the functions in our Banach space need only be defined on the physical region. This is a rather nice feature of the application of functional analysis to S matrix theory, in which it differs from potential scattering.

It is essential that the Banach space be defined over the field of *real* numbers, in order that unitarity, which involves a modulus, shall be a differentiable operation. For each partial wave, we must therefore take two copies of the Banach space $H(R; \mu)$, one for the real part and one for the imaginary part. Alternatively, we may use the phase and elasticity as our basic quantities.

In defining the Banach space to which the physical scattering amplitudes are to belong, we are making three assumptions about them:

i) they are Hölder-continuous in s. This is known to follow from unitarity plus a minimal amount of analyticity [33]. At two-particle thresholds $\mu \leqq \frac{1}{2}$, while elsewhere $\mu \leqq 1$;

ii) in order that continuity hold in the bounded variable u, we must assume that the phase and elasticity both achieve limiting values as $s \to +\infty$, which is physically reasonable. (This is required in particular by the use of the Arzelà-Ascoli theorem in Section 5.);

iii) to get Hölder continuity in u at $u = 1$, we must further assume that they attain their limiting value like some fractional power. Thus for the phase shifts

$$\delta(s) = \delta(\infty) + O(s^{-\mu}), \quad \mu > 0 . \tag{2.3}$$

This third assumption is much less justifiable. However, it is probably unnecessarily restrictive. With some extra work it should, I think, be possible to generalize everything to

$$\delta(s) = \delta(\infty) + O(1/\ln s) \tag{2.4}$$

(see also Appendix A).

b. Unitarity

For the present we consider only uncoupled two-particle channels. The generalization to coupled channels is in Appendix B. We take our basic quantities to be the real and imaginary parts, $x_j(u)$, $y_j(u)$, normalized to their unitarity limits (i.e., without the kinematic factor). The subscript j indexes the different partial waves, and we shall omit it when not needed. Unitarity is then

$$y(u) = [x(u)]^2 + [y(u)]^2 + \frac{1}{4}\{1 - [\eta(u)]^2\}, \tag{2.5}$$

the last term being the contribution of many-particle states.

Now it is known that the product of two Hölder-continuous functions is also Hölder-continuous, and in fact satisfies

$$|fg| \leqq |f| \, |g| \tag{2.6}$$

where $|f|$ is the norm defined by (2.2). Therefore the Banach space $H(R;\mu)$ is a commutative normed ring, and the unitarity formula (2.5) has a meaning in it.

The real and imaginary parts are given in terms of the phase shift $\delta(u)$ and elasticity $\eta(u)$ by

$$x(u) = \frac{1}{2}\eta(u)\sin[2\delta(u)], \tag{2.7}$$

$$y(u) = \frac{1}{2}\{1 - \eta(u)\cos[2\delta(u)]\} \tag{2.8}$$

where unitarity requires

$$0 \leqq \eta(u) \leqq 1 . \tag{2.9}$$

By means of power series convergent in the strong topology, we can define any entire function in the normed ring. Therefore, if $\delta(u)$ and $\eta(u)$ are in $H(R;\mu)$, (2.7)—(2.8) are nonlinear equations in this normed ring, and give $x(u)$, $y(u)$ as elements of it.

We shall eventually require $\eta_j(u) \neq 0$, thus excluding total absorption. However, it is instructive to see why (end of Section 6.b.), and therefore we suppose at present only that

$$\eta_j(s) \sim s^{-H(j)} \sim (1-u)^{H(j)}, \quad \text{with} \quad H(j) \geqq 0 \tag{2.10}$$

at high energies, and that the phase goes to a well-defined and Hölder-continuous limit. The exclusion of total absorption does not seem to be a serious physical limitation, since experimentally

$$\sigma_{el}(\pi p)/\sigma_{\text{tot}}(\pi p) \sim 0.25$$

at high energies, showing that the average value of η_j for partial waves inside the diffraction region is 0.5.

c. Boundedness of the dispersion integral

To get a quantity satisfying a dispersion relation free of kinematic singularities, we must divide $x_j(s) + i y_j(s)$ by a kinematic factor $\varrho_j(s)$. We shall *not* include the centrifugal barrier factor k^{2l} in this. The specific form of $\varrho_j(s)$ is, for pion-pion scattering

$$\varrho(s) = [(s - 4\mu^2)/4s]^{1/2}, \tag{2.11}$$
$$s_1 = 4\mu^2, \quad \mu = \text{pion mass},$$

and for pion-nucleon

$$\varrho(s) = k = \{[s - (M+\mu)^2]\,[s - (M-\mu)^2]/4s\}^{1/2}, \tag{2.12}$$
$$s_1 = (M+\mu)^2, \quad M = \text{nucleon mass}.$$

All we shall assume about $\varrho_j(s)$ is that it is bounded, nonvanishing and Hölder-continuous for $s_1 \leqq s < \infty$, with

$$\varrho_j(s) \sim (s - s_1)^{1/2} \sim u^{1/2} \tag{2.13}$$

at threshold, and

$$\varrho_j(s) \sim s^{R(j)} \sim (1-u)^{-R(j)}, \tag{2.14}$$
$$0 \leqq R(j) < 1,$$

at infinity in such a way that $(1-u)^{R(j)}\varrho_j(u)$ belongs to $H(R;\mu)$. Unitarity shows that the partial wave dispersion relation will require one subtraction if $R(j) = 0$, and no subtraction if $R(j) > 0$[1]. We shall leave out the arguments from $R(j)$ and $H(j)$ when only one partial wave is being considered.

The formulae for transforming the dispersion relation to the variable u are

$$\frac{ds'}{s'-s} = \left(\frac{1-u}{1-u'}\right)\frac{du'}{u'-u}, \tag{2.15}$$

$$\frac{s-s_0}{s'-s_0} = \frac{1-u'}{1-u}. \tag{2.16}$$

We therefore get in the case with one subtraction (e.g., pion-pion)

$$x_j(u) = a_j\varrho_j(u) + \frac{\varrho_j(u)}{\pi}\int_0^1 \frac{du'\,y_j(u')}{\varrho_j(u')\,(u'-u)} + \frac{\varrho_j(u)}{\pi}\int_1^{u_2}\frac{du'\,g_j(u')}{u'-u} \tag{2.17}$$

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[1] That the left-hand cut does not require more will be shown from crossing symmetry in Section 5 (see especially Lemma 5B).

where $g_j(u)$ is the imaginary part on the left-hand cut, and for no subtraction (e.g., pion-nucleon)

$$x_j(u) = \frac{(1-u)\,\varrho_j(u)}{\pi} \int_0^1 \frac{du'\, y_j(u')}{(1-u')\,\varrho_j(u')\,(u'-u)} + \\ + \frac{(1-u)\,\varrho_j(u)}{\pi} \int_1^{u_2} \frac{du'\, g_j(u')}{(1-u')\,(u'-u)}\,. \tag{2.18}$$

Possible bound state terms will be considered later (Section 2.d.).

The boundedness of the right-hand dispersion integral is based on the following theorem, which is obtained by putting together various results proved in MUSKHELISHVILI's book [32], pp. 46, 53, 75.

Theorem 2 A.

$$\mathrm{E}_{\beta\gamma}\mathrm{f} = \frac{u^\beta (1-u)^\gamma}{\pi} \int_0^1 \frac{du'\, f(u')}{(u')^\beta\,(1-u')^\gamma\,(u'-u)} \tag{2.19}$$

for

$$0 \leqq \beta < 1, \quad 0 \leqq \gamma < 1 \tag{2.20}$$

defines a bounded operator $E_{\beta\gamma}$ in the Banach space $H(R;\mu)$, for $0 < \mu < 1$.

Note that there is a difference between a bounded operator in a normed ring, and an element of it. Multiplication by any element of the normed ring defines a bounded operator in it, but not conversely. Note also that $\mu = 1$ is excluded.

By (2.13) and (2.14), the theorem proves that the right-hand dispersion integrals in (2.17) or (2.18) are bounded operators in $H(R;\mu)$. For the unsubtracted case (2.17) we have of course $R(j) = 0$, while for the subtracted case (2.18), we consider the factors $(1-u)\,\varrho_j(u)$ together, and remember that $R(j) > 0$ or there would be no subtraction.

Note that in this result we do not need all physical information available, since for short-range interactions the threshold singularity of $1/\varrho_j(s)$ will actually be cancelled by $y_j(s) \lesssim (s-s_1)^{1/2}$. The fact that only unitarity limits are assumed means that the results will hold for electromagnetic interactions, apart from the question of infrared divergences.

In Section 5 we will show that the left-hand cut term in (2.17) or (2.18), including the crossing relation which expresses $g_j(u)$ as a linear function of $y_k(u')$, is also bounded.

d. Bound-state poles and subtraction constants

We call the rings of Hölder-continuous functions, which contain the real and imaginary parts of the various partial waves, the *continuous* rings. The dispersion relation may also contain explicitly various real parameters — subtraction constants, bound state positions and residues. For each of these, we form a one-dimensional ring of real numbers, which we call the *discrete* rings.

In the case with no subtraction $R(j) = 0$, so by (2.13) and (2.14) $\varrho_j(u)$ will be an element of the continuous ring $H(R; \mu)$. The first term on the right of (2.17) is then a bounded operator from the discrete ring containing a_j to the continuous ring containing $x_j(u)$.

Bound-state poles contribute

$$\varrho_j(u)\,\Gamma_B^j/(u_B^j - u) \tag{2.21}$$

to (2.17) or

$$(1-u)\,\varrho_j(u)\,\Gamma_B^j/\{(1-u_B)\,(u_B - u)\} \tag{2.22}$$

to (2.18). Provided $u_B^j < 0$, these contributions are easily seen to be Hölder-continuous in u. They are therefore bounded operators from the product of the discrete rings containing Γ_B^j and u_B^j to the continuous ring containing $x_j(u)$. In the case of u_B^j, the operator is nonlinear.

We shall also require the real part to vanish at $s = +\infty$, which is physically reasonable in view of the diffraction picture. However, we do not include it in the definition of our Banach space, as this would greatly complicate the boundedness proofs. Instead, we add to our equations $x(u = 1) = 0$. By the norm (2.2), the operator $x(u) \to x(1)$ is a bounded linear operator from the ring $H(R; \mu)$ to the ring of real numbers (i.e., an element of the adjoint space of $H(R; \mu)$). The same is true of

$$\delta(s = \infty) = n\pi/2\,, \tag{2.23}$$

which is equivalent to the vanishing of the real part, by (2.7). In the case with a subtraction, this condition cancels the resulting arbitrariness.

e. Sum rules

The centrifugal barrier requires the threshold behaviour

$$x_j(s)/\varrho_j(s) \sim (s - s_1)^{l(j)} \tag{2.24}$$

where $l(j)$ is the orbital angular momentum. If $l(j) \geqq 1$, this leads to sum rules of the form

$$0 = \frac{1}{\pi}\int\limits_{s_1}^{\infty} \frac{ds'\,y_j(s')}{\varrho_j(s')\,(s'-s_1)^n} + \frac{1}{\pi}\int\limits_{-\infty}^{s_2} \frac{ds'\,g_j(s')}{(s'-s_1)^n}\,, \quad \text{for} \quad n = 1, \ldots, l(j)\,, \tag{2.25}$$

which have to be imposed as additional conditions. These can be regarded as operators from the ring $H(R; \mu)$ to the ring of real numbers. It is easy to see that they are closed operators (DS.II.2.3) because convergence of a series of functions in the norm (2.2) implies uniform convergence, which commutes with the Lebesgue integral. However, unlike those we have dealt with so far, they will not be bounded operators, since the integral need not converge for all $y_j(s)$ in $H(R;\mu)$. Their domains will plainly be the set for which it does converge, namely

$$y_j(s) \lesssim (s-s_1)^{l(j)-1/2}/\ln^2(s-s_1)\,. \tag{2.26}$$

It follows from (2.24) and unitarity (2.5) that

$$y_j(s) \sim (s-s_1)^{2l(j)+1} \tag{2.27}$$

so that physical scattering amplitudes will always be in the domain of the centrifugal sum rules.

f. Summary of the normed ring structure

We suppose that we have a finite number of partial waves, indexed by j. For each of them, we take two copies of the normed ring $H(R;\mu)$ of real-valued Hölder-continuous functions, the norm being given by (2.2). These are for the real and imaginary parts. Besides these continuous rings, we also take one copy each of the ring of real numbers for the various subtraction constants, and bound state masses and residues appearing in the equations. The complete Banach space is the outer product of all these normed rings, with addition and multiplication being defined in each ring separately.

Unitarity is a nonlinear equation in the normed rings containing the real and imaginary parts. Alternatively it can be considered as a nonlinear equation from similar normed rings containing the phase and elasticity to those containing the real and imaginary parts. These nonlinear functions are defined on the whole of the normed ring. The right-hand dispersion integral is a bounded linear operator from the ring containing the imaginary part to that containing the real part. We shall show later that the left-hand dispersion integral plus crossing is also a bounded linear operator between these rings (except that in this case it mixes different partial waves). The bound-state pole terms and subtraction constants in the dispersion relation are bounded operators from the ring of real numbers to the ring of Hölder-continuous functions containing the real parts $x_j(u)$. The condition that the real part vanish at infinity gives a bounded operator from the continuous function real part ring to one of real numbers. The centrifugal sum rules are unbounded closed linear operators from the ring of Hölder-continuous functions containing the imaginary parts to a ring of real numbers.

This completes our Banach space foundation for one-variable S matrix theory, and we are now ready to start building. Our assumptions regarding crossing will be given in Section 5, since we want to avoid writing down the complicated formulae twice. Briefly they are that either Shirkov equations, or Chew-Mandelstam equations with a left-hand cutoff vanishing at least like $s^{1-l'-\varepsilon}$ for $l' \geqq 1$, are satisfied (l' being the exchanged spin).

3. Perturbations and Fréchet differentials

We can summarize all these equations as a single nonlinear equation $f(x) = 0$, where f and x are both Banach space vectors. To investigate uniqueness we must consider the possibility of perturbed solutions

$$f(x + \delta x) = 0 \tag{3.1}$$

and for Dashen-Frautschi perturbation theory, we are also interested in

$$f(x + \delta x) = \delta f \tag{3.2}$$

where δf is the "driving term". We thus need the concept of the derivative of one Banach space vector with respect to another. Consider first the finite-dimensional case. We then have a set of nonlinear functions $f_j(\mathbf{x})$ depending on a set of variables x_i, and the derivative is given by the Jacobian matrix of partial derivatives

$$\|\partial f_j(\mathbf{x})/\partial x_i\| \, .$$

It is a *linear* operator from the vector space of variables x_i to the vector space of functions f_j, which can be used as a local approximation to the nonlinear operator f:

$$f_j(\mathbf{x} + \delta\mathbf{x}) = f_j(\mathbf{x}) + \sum_i \frac{\partial f_j(\mathbf{x})}{\partial x_i}\, \delta x_i + o(\delta x_i) \, .$$

In infinite-dimensional Banach spaces, the derivative of one vector with respect to another is similarly a linear operator. The main difference is that this may now be bounded or unbounded. The bounded case is called the Fréchet differential, and the unbounded the Gâteaux differential. Only the former will interest us. To define it, we use the strong derivative of a vector with respect to a real number

$$f'(x; \delta x) = \text{strong} \lim_{\alpha \to 0} \alpha^{-1} [f(x + \alpha\, \delta x) - f(x)] \, . \tag{3.3}$$

If such a derivative is defined for each sufficiently small (but finite) vector δx, and if it is a linear bounded function of δx, then this is called the Fréchet differential.

An alternative definition uses the concept of a *polynomial form*. This is a vector-valued Banach space function satisfying

$$P(\alpha x + \beta y) = \sum_{m=0}^{N} \sum_{n=0}^{N-m} \alpha^n \beta^m P_{n,m}(x, y) \, . \tag{3.4}$$

Here x, y are Banach-space vectors, P and $P_{n,m}$ are vector-valued functions, α, β are arbitrary real numbers, and the point of the definition is that $P_{n,m}$ must be independent of α, β. The polynomial form is *homogeneous* of order N if it only contains terms with $n + m = N$. A homogeneous polynomial form of order N is *bounded* if

$$|P_N(x)| \leqq A\, |x|^N, \quad A < \infty \, , \tag{3.5}$$

for all x. It is plain that a bounded homogeneous form of order 1 defines a bounded linear operator T

$$P_1(x) = Tx\,. \tag{3.6}$$

If we consider a bounded linear operator $g(x, y, \ldots)$ from the product space $(X, X, \ldots)$ (N times) to the space F containing f, and then take its diagonal part $g(x, x, \ldots)$, this will give us a bounded homogeneous polynomial form of order N, and all such can be obtained in this way [34].

If, for all sufficiently small δx, the given nonlinear operator can be approximated (in the strong topology) by a series of bounded homogeneous polynomials of orders $n \leqq N$

$$f(x + \delta x) = f(x) + \sum_{n=1}^{N} P_n(\delta x) + o(|\delta x|^N)\,, \tag{3.7}$$

then it is N-times Fréchet-differentiable, and the linear polynomial $P_1(\delta x)$ gives the first Fréchet differential. As in (3.6), we may then write

$$P_1(\delta x) = f'(x)\,\delta x \tag{3.8}$$

where $f'(x)$ will be a bounded linear operator, which of course depends on the point x at which the derivative is taken. Its dependence on x may be continuous — continuity is defined using the norm topology for the operator $f'(x)$ and the strong topology for the vector x — in this case $f(x)$ is said to be *continuously Fréchet-differentiable.* The higher terms in the polynomial expansion (3.7) define higher Fréchet differentials. If the second Fréchet differential exists in some neighbourhood, then the first will certainly be continuous there.

In the case of strongly convergent power series in a normed ring, Fréchet differentiability to all orders follows immediately and gives the expected formulae. Thus the Fréchet differentials of the real and imaginary parts with respect to the phase shift will be, by (2.7)—(2.8),

$$\begin{aligned} \delta x(u)/\delta\delta(u) &= \eta(u)\cos[2\delta(u)]\,, \\ \delta y(u)/\delta\delta(u) &= \eta(u)\sin[2\delta(u)]\,. \end{aligned} \tag{3.9}$$

However, though the S matrix case involves a normed ring, the applications of nonlinear functional analysis to offshell theory do not [35]. It is therefore important to note that neither the concept of the Fréchet differential, nor that of a polynomial form, nor the implicit function theorem, in any way assume the Banach space to be a normed ring. Multiplication of a vector by a real number is all that is needed [see Eqs. (3.3) and (3.4)].

If we have a function $f(x, y)$ (in the Banach space F), depending on two Banach space arguments in the spaces X and Y, then we can consider it as a function of one vector in the product space (X, Y). If this is

Fréchet-differentiable, then it is easy to see that the partial Fréchet differentials $f'_x(x, y)$ and $f'_y(x, y)$, in which one vector is fixed, will also be bounded linear operators from X to F, and Y to F, respectively.

If we now reread the DASHEN-FRAUTSCHI papers [3—7], we will recognize $1 - A$, where A is their A-matrix, as the Fréchet differential of (3.2). Thus to calculate the effect of a given driving term (to first approximation), we must invert the Fréchet differential, while spontaneous symmetry breaking will occur if the Fréchet differential has zero as an eigenvalue. The concept of the Fréchet differential thus enables us to eliminate the restriction to algebraic perturbations (coupling shifts and mass splittings) assumed by DASHEN and FRAUTSCHI and by all their followers. However, it does much more than this. For there exist implicit function theorems, to be described in the next Section, which relate qualitative uniqueness properties of the *exact* perturbed solution to those of the Fréchet differential, and also approximation schemes of guaranteed convergence under suitable conditions [36] which give the exact solution itself by successive solutions of the linear approximation. When spontaneous symmetry breaking occurs, the linear approximation obviously has a continuum of solutions. It has been suggested [37—38] (as usual on the basis of finite-dimensional algebraic models) that this continuum ought to be eliminated when higher order perturbations are considered, giving just a discrete perturbed solution, slightly removed from the unperturbed one. As we shall see, this is not necessarily so in the realistic infinite-dimensional case.

Now let us consider the form of the Fréchet differential, and whether it is in fact bounded. If the many-particle contributions are fixed, then we have to consider changes in the phase shifts $\delta\delta_j(u)$, in the subtraction constants δa_j, in the external masses, and in the bound-state parameters. The last two are finite dimensional and therefore essentially trivial from the mathematical point of view. We shall forget them for the time being, so as not to clutter the equations, and put them back in Sections 6.b, c. Because the right-hand dispersion integral is a bounded linear operator, we can then insert (3.9) into (2.17) to get

$$\begin{aligned}\delta f_j(u) = {} & \eta_j(u)\cos[2\delta_j(u)]\,\delta\delta_j(u) - \varrho_j(u)\,\delta a_j - \\ & - \frac{\varrho_j(u)}{\pi}\int_0^1 \frac{du'\,\eta_j(u')\sin[2\delta_j(u')]\,\delta\delta_j(u')}{\varrho_j(u')\,(u'-u)} \\ & - \frac{\varrho_j(u)}{\pi}\int_1^{u_2} \frac{du'}{u'-u}\sum_k\int_0^1 du''\,\frac{\delta g_j(u')}{\delta y_k(u'')}\,\times \\ & \times\,\eta_k(u'')\sin[2\delta_k(u'')]\,\delta\delta_k(u'')\,,\end{aligned} \tag{3.10}$$

with a similar equation resulting from (2.18) in the unsubtracted case. The imaginary part on the left-hand cut $g_j(u)$ will depend linearly on the right-hand imaginary parts $y_k(u)$ through crossing, thus giving rise to the $\delta g_j(u')/\delta y_k(u'')$ factor. We shall consider the precise form of this in Section 5. The linear operator which gives the driving term $\delta f_j(u)$ of (3.10) in terms of the change in the phase shift $\delta\delta_j(u)$ is the Fréchet differential, provided we can show it is bounded. However, we know already that the right-hand dispersion integral is a bounded operator, while the quantities $\cos[2\delta_j(u)]$, $\sin[2\delta_j(u)]$ belong to the normed ring, so that multiplication by them also defines bounded operators. Accepting, as will be proved in Section 5, that the left-hand dispersion integral plus crossing is bounded as well, it follows that (3.10) is bounded and is therefore the Fréchet differential. In fact, using the strong convergence of the power series for $\cos[2\delta_j(u)]$ and $\sin[2\delta_j(u)]$, it is easy to see that (2.17) is Fréchet-differentiable to all orders everywhere. Thus in particular the first differential will be continuous.

We saw in Section 2.e that the centrifugal sum rules in higher partial waves were unbounded operators. If we differentiate them with respect to $\delta_j(s)$, we find nevertheless that the first differential is bounded, because the threshold behaviour (2.27) of the unperturbed solution cancels out the singular denominator. However, the second differential becomes unbounded again. Thus they are Fréchet-differentiable, but not continuously so.

For further information on Fréchet and Gâteaux differentials see Hille and Phillips [34]. However, it should be noted that some of their theorems use complex variable results (e.g., the maximum modulus principle) in an essential way, and therefore would not hold in a Banach space over the field of real numbers, such as we require for unitarity.

4. Uniqueness theorems

The way in which the Fréchet differential determines the uniqueness properties of the exact solution is through the Banach space implicit function theorem. This very closely resemble the implicit function theorem for two real variables, but with the partial derivatives replaced by Fréchet differentials. It says:

Theorem 4 A (Hildebrandt-Graves theorem). Let X, Y, F be three *Banach spaces. Let $f(x, y)$ be a function from the product space (X, Y) to F, which is continuously Fréchet-differentiable in some open set. Let (x_0, y_0) be a point in this open set satisfying the equation $f(x_0, y_0) = 0$, and let the partial differential $f'_y(x_0, y_0)$ have a bounded inverse there. Then there exists a continuously Fréchet-differentiable function $y(x)$, with $y(x_0) = y_0$, which is the unique solution of the equation $f(x, y) = 0$ in a certain neighbourhood* *of the original solution. This neighbourhood is bounded only by points at which either $f(x, y)$ ceases to be continuously Fréchet-differentiable, or $f'_y(x, y)$ ceases to have a bounded inverse. If $f(x, y)$ is n-times continuously Fréchet-differentiable, then so will be $y(x)$.*

Apart from the original paper of Hildebrandt and Graves [35a], a proof of most of this theorem may be found in the book of Kantorovich and Akilov [36], and what is not there follows easily by similar methods. It is essentially an application of the contraction mapping principle.

Our present interest in this theorem is mainly as a stepping stone to a more general one. This concerns cases when the Fréchet differential does not have a bounded inverse, and when the vector x may be only implicit in the equation. First we must define a class of operators to which the Fréchet differential will belong. It is well known that integral equations with compact kernels ($T = 1 - C$, C compact) have a property called the Fredholm alternative. This says that the number of linearly independent solutions of the homogeneous equation $Tx = 0$ is finite and equal to the number of linearly independent vectors y (driving terms) for which the inhomogeneous equation $Tx = y$ has no solution. The class of operators we are interested in have these two numbers finite but unequal.

Definition 4 B. Let T be a linear operator acting from the Banach space Y to the Banach space F, with T^* the adjoint operator acting from F^* to Y^*. The *null space* of T is the linear manifold in Y of all solutions of the equation $Ty = 0$. We define α_T to be its dimension, and β_T to be the dimension of the null space of its adjoint T^*. $\varkappa_T = \alpha_T - \beta_T$ is called the *index* of T.

Definition 4 C. A linear operator T is called a *Φ-operator* if it is closed, has a closed range, and if the two numbers α_T, β_T are both finite.

For a Φ-operator, β_T has a simpler interpretation. It is the dimension of its *defect space*, i.e., the set of all vectors y which cannot be represented in the form $y = Tx$ for any x. The index is thus the excess of the number of solutions of the homogeneous equation, over the number of forbidden driving terms for the inhomogeneous equation. The Fredholm alternative corresponds to the index being zero. Φ-operators and the concept of the index arose in connection with singular integral equations, and a special case of them will be familiar to readers of Muskhelishvili's book [32]. A general theory has been given by Gokhberg and Krein [39], from which we shall be quoting some results, and to which we refer the reader for background information.

We now quote an extension of the implicit function theorem to Φ-operators, due to Vainberg and Trenogin [40]:

Theorem 4 D ("Implicit variable" theorem). *Let $f(y)$ be a (non-linear) function from the Banach space Y to the Banach space F, with $f(y_0) = 0$. Let it be n-times continuously Fréchet-differentiable in some* *neighbourhood of y_0, and let $f'(y_0) = T$ be a Φ-operator. Then there exists a neighbourhood of y_0 within which the Banach space equation $f(y) = 0$ is exactly equivalent to β_T nonlinear scalar equations in α_T real variables:*

$$\varphi_j(\eta_1, \ldots, \eta_{\alpha_T}) = 0, \quad j = 1, \ldots, \beta_T. \tag{4.1}$$

Each of these scalar functions φ_j is n-times continuously differentiable in all its arguments.

A full proof may be found in the paper of VAINBERG and TRENOGIN [40]. However, the idea is quite simple: the restriction of the Fréchet differential acting between the factor space of the null space and the factor space of the defect space has a bounded inverse, so that Theorem 4.A can be applied to solve it uniquely. Left over are α_T variables coming from the null space, and β_T equations in them coming from the defect space.

Thus, after the nonlinear integral equation has been reduced to algebraic ones, the index $\varkappa_T$ of the Fréchet differential gives the excess of the number of variables over the number of equations available to determine them (assuming that both matched in the integral equation, otherwise it gives the increase in this number). In general we can expect it to be the dimensionality of the manifold of solutions of the equation $f(x) = 0$, though there is the possibility of this dimension being reduced if x_0 corresponds to a singular point of the equations (4.1). Essentially this means that accidental degeneracy may occur. However, in the case $\beta_T = 0$ even this if forbidden and there will then certainly be α_T solutions. For further limitation on possible degeneracies see the end of Section 6.

We have called 4.D the "implicit variable" theorem, because it shows how, even though the original equation appears to contain no arbitrary parameters, it may still have a continuum of solutions if its Fréchet differential has a positive index. To avoid possible misunderstandings, we repeat again that this ambiguity refers to solutions of the *exact* equation.

We will apply this theorem to (3.10), the next two Sections containing the proof that the Fréchet differential is a Φ-operator and the calculation of its index. The centrifugal sum rules (2.24) are not continuously Fréchet-differentiable, but are only finite dimensional anyway. We can therefore leave them out of the main argument, and add them to the φ-equations afterwards. They will only differ from the others in not being necessarily continuously differentiable.

As we shall see, normal stable symmetries correspond to $\alpha_T = 0$, $\beta_T = 0$. Spontaneous symmetry breaking will occur if $\alpha_T = 1$, $\beta_T = 1$. (4.1) will then give one nonlinear equation for one unknown, so that we can expect discrete solutions as argued by previous authors [37—38]. The Fréchet differential then has a zero eigenvalue, but it still satisfies the Fredholm alternative. For both these situations, the index is zero. The possibility of a nonzero index implies new phenomena, not considered by previous authors because they only worked with finite-dimensional algebraic models. These will be discussed in Sections 6, 7.

5. Compactness of the crossed term

The index of a Φ-operator has a very important stability property.

Theorem 5 A. *Let T be a Φ-operator, and U a compact operator, then $T + U$ is also a Φ-operator with the same index as T.*

For the proof see GOKHBERG and KREIN [39].

Our strategy can now be revealed. Having used the implicit variable theorem to reduce the uniqueness question to properties of the Fréchet differential, we are now going to use the stability theorem to eliminate the left-hand cut contribution, which we shall prove compact. The right-hand cut part can then be solved in closed form, and the index calculated explicitly.

The imaginary part on the left-hand cut is related to that on the right by a linear integral transform of the general form

$$g_j(s) = \sum_k \int_{s_1}^{\infty} ds' \, K_{jk}(s, s') \, y_k(s')/\varrho_k(s') \,. \tag{5.1}$$

This leads to $\delta g_j(u')/\delta y_k(u'')$ factor in the left-hand term of (3.10).

To analyze its structure, we shall decompose the whole left-hand cut contribution to (3.10) into a sequence of maps between several different Banach spaces. The first and last will be $H(R; \mu)$, the space of functions Hölder-continuous in u on the right-hand cut with the norm (2.2) as before. We also use $C(R; \mu)$ which is the space of functions $f(u)$ for which

$$f(u)/|1 - u|^{\mu} \tag{5.2}$$

is continuous, but not Hölder-continuous, in u on the right-hand cut, the norm being

$$|f|_{C(R;\mu)} = \sup_{0 \leq u \leq 1} |f(u)/(1 - u)^{\mu}| \,. \tag{5.3}$$

Continuity in u means not only continuity in s, but also the existence of a limit as $s \to +\infty$. Thus

$$\lim_{s \to \infty} s^{\mu} f(s) \tag{5.4}$$

must exist for a function to be in $C(R; \mu)$. A special case of $C(R; \mu)$ is $C(R; 0)$ — the space of continuous functions with the usual norm. For the imaginary part on the left-hand cut we shall use $C(L; \mu)$, which is the space of functions for which (5.2) is continuous in u on the left-hand cut, the norm being

$$|f|_{C(L;\mu)} = \sup_{1 \leq u \leq u_2} |f(u)/(u - 1)^{\mu}| \,. \tag{5.5}$$

Again, a function will belong to this space only if it is both continuous in s, and the limit (5.4) exists at $s = -\infty$.

First consider the relation of the spaces $H(R;\mu)$ and $C(R;0)$. The latter is just the space of all continuous functions on $0 \leqq u \leqq 1$. Obviously, every Hölder-continuous function is *a fortiori* continuous, so there exists an embedding of $H(R;\mu)$ into $C(R;0)$, by which each function $f(u)$ in the former becomes the same function considered as an element of the latter. This embedding plainly defines a linear operator from $H(R;\mu)$ into $C(R;0)$, and it follows immediately from the Arzelà-Ascoli theorem (DS.IV.6.7) that it is compact — any bounded set of $H(R;\mu)$ becomes a compact set of $C(R;0)$ under the embedding.

The next stage is the multiplication by

$$\eta_k(u)\sin[2\delta_k(u)]\,. \tag{5.6}$$

We now use the additional equation mentioned in Section 2.d, that the real part $x(s)$ should vanish at $s = +\infty$. This implies that the unperturbed phase shift satisfies

$$\begin{aligned}\delta_k(s) = n\pi/2 + O(s^{-\mu}) = n\pi/2 + O([1-u]^{\mu}),&\\ \text{as}\quad s \to +\infty,\quad \text{i.e.}\quad u \to 1&\end{aligned} \tag{5.7}$$

[by (2.23) and (2.3)]. So (5.6) must vanish at least like $(1-u)^{\mu}$ as $u \to 1$. Multiplication by (5.6) is then obviously a bounded linear operator from $C(R;0)$ to $C(R;\mu)$.

Now we consider the crossing equation itself, which takes the imaginary part on the right-hand cut into that on the left. This may take different forms according to whether we use the CHEW-MANDELSTAM, Ref. [41], or SHIRKOV [42] equations. We consider the simpler Shirkov case first and use the variable s. The crossing relation is then (Λ_{jk} being a numerical crossing matrix)

$$g_j(s_1 + s_2 - s) = \sum_k \Lambda_{jk} y_k(s)/\varrho_k(s) \tag{5.8}$$

which we decompose into two parts: an obviously bounded map from $C(R;\mu)$ to $C(L;\mu)$

$$\bar{g}_k(s_1 + s_2 - s) = y_k(s) \tag{5.9}$$

and a multiplicative matrix factor

$$g_j(s) = \sum_k \Lambda_{jk}\bar{g}_k(s)/\varrho_k(s_1 + s_2 - s)\,. \tag{5.10}$$

The denominators in (5.10) will give inverse square root singularities at the threshold of the left-hand cut. We will therefore consider them as part of the left-hand dispersion integral (see below).

The case of the Chew-Mandelstam crossing relations is more complicated — for higher partial waves the crossing operator is certainly not bounded. However, it is known that the Chew-Mandelstam equations are not consistent for higher partial waves without a left-hand cutoff, and it turns out that the minimum modification needed to make them consistent is precisely that which will make (5.1) into a bounded operator.

Consider the typical case of pion-pion scattering (mass = 1), and put

$$\nu = \frac{s}{4} - 1 \,. \tag{5.11}$$

We shall decompose the Chew-Mandelstam crossing relations [41] into three parts

i) an integral transform in the space $C(R;\mu)$

$$\begin{aligned}\bar{y}_l^T(\nu) = \varrho(-\nu-1) \sum_{T'l'} A_{TT'}\, c_{ll'}^{TT'}(\nu) \,\times \\ \times \frac{1}{\nu+1} \int_0^\nu d\nu'\, P_l\left(1 - 2\,\frac{\nu'+1}{\nu+1}\right) P_{l'}\left(1 - 2\,\frac{\nu}{\nu'}\right) y_{l'}^{T'}(\nu')/\varrho(\nu')\,,\end{aligned} \tag{5.12}$$

where

$$\varrho(\nu) = [\nu/(\nu+1)]^{1/2} \,. \tag{5.13}$$

Here l, l' are the orbital angular momenta in the direct and crossed channels, T, T' the isospins, $A_{TT'}$ are numerical coefficients, and $c_{ll'}^{TT'}(\nu)$ is the cutoff function;

ii) a mapping of the space $C(R;\mu)$ into the space $C(L;\mu)$

$$\bar{g}_l^T(-\nu-1) = \bar{y}_l^T(\nu) \,. \tag{5.14}$$

iii) a multiplicative factor

$$g_l^T(\nu) = \bar{g}_l^T(\nu)/\varrho(-\nu-1) \,. \tag{5.15}$$

This cancels the first factor on the right of (5.12).

To determine the boundedness properties of (5.12), we use the following lemma.

Lemma 5 B. The integral transform

$$X f(\nu) = \frac{1}{t(\nu)} \int_0^\nu d\nu'\, r(\nu')\, f(\nu') \tag{5.16}$$

will be a bounded operator in $C(R;\mu)$ if the following conditions are satisfied[2]:

(a) $[t(\nu)]^{-1}$, $r(\nu)$ are both continuous for $0 < \nu < \infty$,

(b) $t(\nu) \sim \nu^{\alpha}$, $r(\nu) \lesssim \nu^{\alpha-1}$, as $\nu \to \infty$, with $\alpha > \mu$,

(c) $t(\nu) \sim \nu^{\beta}$, $r(\nu) \lesssim \nu^{\beta-1}$, as $\nu \to 0$, with $\beta \neq -\mu$.

Proof. Boundedness in the norm (5.3) follows easily from the mean value theorem and a little algebra. However, we also have to prove the

[2] Here $f(\nu) \lesssim \nu^{\alpha}$ means $\lim \nu^{-\alpha} f(\nu) = A$ exists and is finite, whereas $f(\nu) \sim \nu^{\alpha}$ means also $A \neq 0$.

continuity of

$$(1-u)^{-\mu} Xf(u) \sim (\nu-\nu_0)^{\mu} Xf(\nu) . \tag{5.17}$$

This is obvious except at the end points $\nu = 0$ and $\nu = \infty$. To show continuity at $\nu = \infty$, we apply L'Hospital's rule to get

$$\lim_{\nu\to\infty} \nu^{\mu} Xf(\nu) = (\alpha-\mu)^{-1} \lim_{\nu\to\infty} \{[r(\nu)/\nu^{\alpha-1}]\,[\nu^{\mu} f(\nu)]\} \tag{5.18}$$

where both factors in square brackets are continuous at $\nu = \infty$ by assumption. The argument at $\nu = 0$ is similar.

If we put $c(\nu) = 1$ for the cutoff in (5.12), then it is obvious that the terms containing

$$\nu^{n-1}/(\nu')^{n}, \quad n = 1, \ldots, l' , \tag{5.19}$$

arising from the expansion of the second Legendre polynomial will be the only ones violating the conditions of this lemma. Therefore, the crossing operator (5.12) will be bounded provided the cutoff is at least as strong as

$$c_{ll'}^{TT'}(\nu) \sim \nu^{1-l'-\mu}, \quad l' \geqq 1 \tag{5.20}$$

(for $l' = 0$, of course no cutoff is needed). If there were no cutoff, the imaginary part on the left-hand cut would go like $(\nu)^{l'}$. Since we have assumed the existence of a limit on the right-hand cut, it then follows from the Phragmén-Lindelöf theorem [43] that the real part at $\nu = +\infty$ would diverge, in contradiction to unitarity. For $l' = 1$, the need for a cutoff is more subtle [44][3] the convergence as $\nu \to \infty$ of the integral in the term

$$\sim \int_0^{\nu} d\nu'\, y_1^1(\nu')/[\nu' \varrho(\nu')] \tag{5.21}$$

then imposes additional restrictions on $y_1^1(+\infty)$, which turn out to be incompatible with the equality of the imaginary parts at $\mp\infty$ required by the Phragmén-Lindelöf theorem [43]. This difficulty will be eliminated if we multiply (5.21) by a cutoff $\sim \nu^{-\mu}$.

Equation (5.14) is obviously a bounded operator from $C(R;\mu)$ to $C(L;\mu)$. We are left then, in both the Shirkov and Chew-Mandelstam cases, with the left-hand dispersion integral combined with some kinematic factors

$$L\bar{g}(u) = \frac{\varrho(u)}{\pi} \int_1^{u_2} \frac{du'\,\bar{g}(u')}{\varrho([u_2-u']/[1-2u'+u'u_2])\,(u'-u)} \tag{5.22}$$

[3] At first sight the inconsistency of vector-meson exchange without a cutoff seems to be contradicted by various papers on singular N/D equations [45—47]. However, these all approximate the *two*-particle exchange cut by a *one*-particle exchange cut, which is a very strong cutoff. No such approximation was made in [44].

if the original dispersion relation had one subtraction, and

$$L\bar{g}(u) = \frac{(1-u)\,\varrho(u)}{\pi} \int_1^{u_2} \frac{du'\,\bar{g}(u')}{(u'-u)\,(1-u')\,\varrho([u_2-u']/[1-2u'+u'u_2])} \tag{5.23}$$

if it had none. We want to show that L defined by these equations is a bounded operator from $C(L;\mu)$ back into the space $H(R;\mu)$ containing the real part in the physical region $0 \leqq u \leqq 1$. It is important to note that only the end of this region $u = 1$ is a singularity of the integral. Therefore, we use the following lemma:

Lemma 5 C. *If $f(u)$ satisfies*

$$|f(u)| < A\,|1-u|^{\mu}, \quad \text{for} \quad 0 \leqq u \leqq 1\,, \tag{5.24}$$

$$|f'(u)| < B\,|1-u|^{\mu-1}, \quad \text{for} \quad 0 \leqq u < 1\,, \tag{5.25}$$

then it belongs to $H(R;\mu)$, and the norm (2.2) *is bounded by*

$$|f| \leqq 2A + \mathrm{Max}[A, \mu A + B]\,. \tag{5.26}$$

Proof. We use a result of MUSKHELISHVILI [32] (bottom of p. 16), which says that, if $\varphi(u)$ is a function in $H(R;\mu)$, and $\omega(u)$ is bounded on $[0, 1]$ and its derivative satisfies

$$|\omega'(u)| < C/[1-u], \quad 0 \leqq u < 1\,, \tag{5.27}$$

then $[\varphi(u) - \varphi(1)]\,\omega(u)$ is in $H(R;\mu)$. We put $\varphi(u) = (1-u)^{\mu}$. The bound (5.26) on the norm comes from a detailed inspection of MUSKHELISHVILI's proof.

To apply the lemma, we omit the $\varrho(u)$ from (5.22), differentiate under the integral sign, which we can certainly do for $u < 1$, and get

$$\left|\frac{d}{du}\left[\frac{L\bar{g}(u)}{\varrho(u)}\right]\right| \leqq C_1 \int_1^{1+\varepsilon} \frac{du'\,(u'-1)^{\mu}\,|\bar{g}|}{(u'-u)^2} + \\ + C_2 \int_{1+\varepsilon}^{u_2} \frac{du'\,|\bar{g}|\,(u'-1)^{\mu}}{(u_2-u')^{1/2}\,(u'-u)^2}\, 0 \leqq u < 1 \tag{5.28}$$

where $|\bar{g}|$ is the $C(L;\mu)$ norm defined by (5.5), and we have used the assumptions (2.13)—(2.14) about the behaviour of the kinematic factor $\varrho(\bar{u})$ in the denominator, remembering that $R = 0$ for the case with one subtraction. The second term in (5.28) is obviously uniformly bounded, and the first has a bound like (5.25). To prove (5.24) is even easier. Thus (5.22) without the $\varrho(u)$ factor is a bounded operator from $C(L;\mu)$ to $H(R;\mu)$ by (5.26), and the $\varrho(u)$ factor will be an element of $H(R;\mu)$ in this case, because $R = 0$ in (2.14).

In the case of (5.23), the absorptive part of the integral will behave near $u = 1$ as $\sim (u'-1)^{\mu+R-1}|\bar{g}|$. We therefore instead divide (5.23) by the factor $(1-u)^R\,\varrho(u)$, which belongs to $H(R;\mu)$ by (2.14), and obtain bounds (5.24) and (5.25) for the remaining factors $(1-u)^{1-R}$ times the integral. Thus again Lemma 5.C. shows L is bounded.

Putting all the pieces together, we see that the left-hand cut contribution consists of (i) a compact map from $H(R;\mu)$ to $C(R;0)$ (the embedding), (ii) a bounded map from $C(R;0)$ to $C(R;\mu)$ [multiplication by (5.6)], (iii) a bounded map from $C(R;\mu)$ to $C(R;\mu)$ [the integral transform (5.12) with cutoff (5.20)], (iv) a bounded map from $C(R;\mu)$ to $C(L;\mu)$ [the substitution (5.14)], (v) a bounded map from $C(L;\mu)$ back into $H(R;\mu)$ [the left-hand dispersion integral (5.22) or (5.23)]. The product of a bounded operator with a compact operator is itself compact (DS.VI.5.4), and therefore the whole sequence defines a compact operator from $H(R;\mu)$ to $H(R;\mu)$. QED.

The conclusion is therefore that the index of the Fréchet differential, and the question of whether it is a Φ-operator, will not be affected by the omission of the left-hand cut contribution from (3.10).

The crucial rôle played in this proof by the factor (5.6) should be noted. If it were not present, then we would have to embed $H(R;\mu)$ directly in $C(R;\mu)$, and though this is a bounded operator, it is not a compact one. Therefore, the left-hand cut term in the original unperturbed equation is merely bounded — what makes its contribution to the Fréchet differential compact is that the perturbation has to die out at $s=\infty$, because of our subsidiary condition $\delta(\infty)=n\pi/2$, and the fact that $\eta(s)$ is not perturbed. However, the left-hand cut contribution could always be made compact by strengthening the cutoff sufficiently. The contribution from all singularities which do not touch the physical region anywhere including ∞ will necessarily be compact. Thus the proof is not affected by unequal masses, and the kinematic factors for cases with spin need only be considered at infinity.

6. Calculation of the index

a. Construction of the D function

We are now left with (3.10) minus its left-hand cut term, and propose to solve it in closed form. This can be done by the Muskhelishvili-Omnès method [32], but we must be rather careful about boundary conditions, especially at infinity.

We define the amplitude (the partial waves are now independent so we drop j)

$$\Phi(u)=\delta a+\frac{1}{\pi}\int_0^1\frac{du'\,\sin[2\delta(u')]\,\eta(u')\,\delta\delta(u')}{\varrho(u')\,(u'-u)} \tag{6.1}$$

in terms of which, (3.10) without its left-hand cut becomes

$$\cos[2\delta(u)]\,\eta(u)\,\delta\delta(u)/\varrho(u) - \delta f(u)/\varrho(u) = \frac{1}{2}[\Phi(u+i\varepsilon)+\Phi(u-i\varepsilon)]\,, \tag{6.2}$$

$$i\sin[2\delta(u)]\,\eta(u)\,\delta\delta(u)/\varrho(u) = \frac{1}{2}[\Phi(u+i\varepsilon)-\Phi(u-i\varepsilon)] \tag{6.3}$$

leading immediately to the Hilbert problem

$$\Phi(u+i\varepsilon) = e^{4i\delta(u)}\Phi(u-i\varepsilon) + [e^{4i\delta(u)}-1]\,\delta f(u)/\varrho(u)\,. \tag{6.4}$$

According to the Muskhelishvili method [32], we have to start by constructing one particular solution of the homogeneous Hilbert problem, (6.4) with $\delta f(u) = 0$. The Dashen-Frautschi method [3] involves cancelling out the right-hand cut by multiplying by $D^2(s)$, where $D(s)$ is the denominator function of the N/D decomposition [41], and then writing a dispersion relation for the result. In the case with no left-hand cut and no driving term, the Dashen-Frautschi and Muskhelishvili methods are in fact entirely equivalent, $[D(s)]^{-2}$ being a solution of the homogeneous Hilbert problem. In other cases, the Dashen-Frautschi method is more special, since it assumes that the inhomogeneous and compact terms have known analytic properties, but gives a neater answer (see end of Section 6).

We must therefore start by constructing the D function belonging to the unperturbed solution. Let us work in the variable u of (2.1). The first requirement is that $D(u)$ shall have no cut but the right-hand cut $0 \leqq u \leqq 1$, and shall have the phase there

$$\arg D(u+i\varepsilon) = -\delta(u)\,. \tag{6.5}$$

As is well known [48—52], one such function is given by

$$\bar{D}(u) = e^{-\Gamma(u)} \tag{6.6}$$

where

$$\Gamma(u) = \frac{1}{\pi}\int_0^1 \frac{du'\,\delta(u')}{u'-u} \tag{6.7}$$

and the most general solution of this part of the problem is

$$D(u) = R(u)\,\bar{D}(u) \tag{6.8}$$

where $R(u)$ is any meromorphic function. Because of the analyticity of $\Gamma(u)$, $\bar{D}(u)$ will be finite and nonvanishing in the u plane, excluding $[0, 1]$, but including $u = \infty$. As for the cut itself, a result of MUSKHELISHVILI [32] (p. 46) and the assumption that the unperturbed $\delta(u)$ is in $H(R;\mu)$ show that $\Gamma(u)$ will also be Hölder-continuous on (0,1) excluding the end points. Another result of MUSKHELISHVILI (p. 74) shows that the values of $\Gamma(u)$ near these ends will be

$$\Gamma(u) \approx \frac{1}{\pi}\,\delta(\infty)\ln(1-u)\,, \quad \text{as} \quad u \to 1\,, \tag{6.9}$$

$$\Gamma(u) \approx -\frac{1}{\pi}\,\delta(0)\ln(u)\,, \quad \text{as} \quad u \to 0\,, \tag{6.10}$$

where $\delta(\infty)$ means the phase shift at $s = \infty$ ($u = 1$). Without loss of generality, we can define the phase shift to vanish at threshold. Transforming to the variable s, we then see that $\overline{D}(s)$ will be finite, non-vanishing and Hölder-continuous in the entire s plane with the possible exception of $s = \infty$, where it behaves like

$$\overline{D}(s) \sim s^{\delta(\infty)/\pi}\,. \tag{6.11}$$

A further requirement on $D(s)$ is that it have zeros at the bound-state energies s_B, and nowhere else in the finite s plane. Also, since the Dashen-Frautschi method involves multiplying by $D^2(s)$ and then writing a dispersion relation, $D(s)$ should have no poles and be of finite degree at infinity. These conditions determine the meromorphic function $R(s)$ of (6.8) uniquely (apart from a multiplicative constant, which we can fix without subsequent loss of generality). The only possible $D(s)$ is

$$D(s) = \prod_B (s_B - s)\,\overline{D}(s) \tag{6.12}$$

and its behaviour at $s = \infty$ is

$$D(s) \sim s^{\varkappa/2} \tag{6.13}$$

where

$$\varkappa = 2n_B + 2\delta(\infty)/\pi\,, \tag{6.14}$$

n_B being the number of bound states.

b. Solution of the right-hand equation

If the original dispersion relation (2.17) had bound-state poles, then it is easy to see that the change to (3.10) will be such that we must add to $\Phi(s)$ as given by (6.1)

$$\sum_B \left\{\frac{\delta\Gamma_B}{s_B - s} - \frac{\Gamma_B\,\delta s_B}{(s_B - s)^2}\right\} \tag{6.15}$$

(6.2) and (6.3) will then hold unchanged. We now return to the variable s and consider the dispersion relation satisfied by $D^2(s)\,\Phi(s)$ with $\Phi(s)$ given by (6.1) plus (6.15). The bound-state poles of (6.15) will be cancelled by the zeros of (6.12). The only cut is the right-hand cut, and (6.2), (6.3), (6.5) show that the imaginary part there is given by

$$\operatorname{Im}\{D^2(s + i\varepsilon)\,\Phi(s + i\varepsilon)\} = |D(s)|^2 \sin[2\delta(s)]\,\delta f(s)/\varrho(s)\,. \tag{6.16}$$

By (6.13), (2.14) and (5.7), this will have asymptotic behaviour

$$\sim s^{\varkappa - \mu - R}\,\delta f(\infty) \tag{6.17}$$

where $\varkappa$ is given by (6.14). We therefore obtain a dispersion relation with $\varkappa$ subtractions if $\varkappa \geqq 0$

$$D^2(s)\,\Phi(s) = \sum_{n=0}^{\varkappa-1} A_n (s-s_0)^n + \frac{(s-s_0)^\varkappa}{\pi} \int_{s_0}^{\infty} \frac{ds' |D(s')|^2 \sin[2\delta(s')]\,\delta f(s')}{(s'-s_0)^\varkappa \varrho(s')(s'-s)} \tag{6.18}$$

leading to

$$\delta\delta(s) = \cos[2\delta(s)]\delta f(s)/\eta(s) + \frac{\varrho(s)}{\eta(s)|D(s)|^2} \left\{ \sum_{n=0}^{\varkappa-1} A_n (s-s_0)^n + \right. \\ \left. + \frac{(s-s_0)^\varkappa}{\pi} \int_{s_1}^{\infty} \frac{ds' |D(s')|^2 \sin[2\delta(s')]\,\delta f(s')}{(s'-s_0)^\varkappa \varrho(s')(s'-s)} \right\} \tag{6.19}$$

Using Theorem 2.A, it is straightforward to show that the last term of (6.19), like the first one, gives the asymptotic behaviour

$$\delta\delta(s) \sim s^H\,\delta f(s), \quad \text{if} \quad \varkappa \geqq 0\,, \tag{6.20}$$

where H is given by (2.10). (The other terms will be smaller.) We shall see shortly that it is necessary to assume $H = 0$, thus excluding pure absorption. Our supplementary condition on the Banach space — that the real part shall vanish at infinity — implies that the driving term $\delta f(s)$ shall also vanish there. Thus (6.20) with $H = 0$ makes the change in the phase vanish at $s = \infty$, again in accord with the supplementary condition. It is then easy to see that (6.19) is in $H(R;\mu)$.

For $\varkappa < 0$, there will be no subtraction, but the last term of (6.19) will then increase like $s^{|\varkappa|}$ even for $H = 0$. Thus supplementary conditions have to be put on the driving term in order that the perturbed phase shift be bounded. To find them, we write a dispersion relation for $(s-s_0)^{|\varkappa|} D^2(s)\,\Phi(s)$, which will require no subtractions

$$D^2(s)\,\Phi(s) = \frac{1}{\pi}(s-s_0)^{-|\varkappa|} \int_{s_1}^{\infty} \frac{ds'(s'-s_0)^{|\varkappa|} |D(s')|^2 \sin[2\delta(s')]\,\delta f(s')}{\varrho(s')(s'-s)} \tag{6.21}$$

Now s_0 is just the subtraction point, and by (6.1) $\Phi(s)$ should not have any poles there. So $\delta f(s)$ must satisfy

$$\frac{1}{\pi}\int_{s_1}^{\infty} ds'(s'-s_0)^n |D(s')|^2 \sin[2\delta(s')]\,\delta f(s')/\varrho(s') = 0\,, \\ n = 0, 1, \ldots, |\varkappa|-1\,. \tag{6.22}$$

Provided these supplementary conditions on the driving term are satisfied [the integrals in them will always converge by (6.13)], then the perturbed shift is given by

$$\delta\delta(s) = \cos[2\delta(s)]\,\delta f(s)/\eta(s) + \\ + \frac{\varrho(s)(s-s_0)^\varkappa}{\pi\eta(s)|D(s)|^2} \int_{s_0}^{\infty} \frac{ds' |D(s')|^2 \sin[2\delta(s')]\,\delta f(s')}{(s'-s_0)^\varkappa \varrho(s')(s'-s)}\,, \text{ for } \varkappa < 0\,, \tag{6.23}$$

and will have the asymptotic behaviour $\delta\delta(s) \sim s^H \delta f(s)$. If $H = 0$ (6.23) is easily seen to lie in $H(R; \mu)$.

The subtraction terms in (6.19) with the coefficients A_n will be solutions of the homogeneous equation with $\delta f(s) = 0$. It follows from the construction that they are the only ones in $H(R; \mu)$. Thus the dimension of the null space is $\alpha = \mathrm{Max}(\varkappa, 0)$. If $\varkappa \geqq 0$, the inhomogeneous equation will have a solution in $H(R; \mu)$ for any driving term $\delta f(s)$ in $H(R; \mu)$, according to (6.19), while for $\varkappa < 0$, this requires $-\varkappa$ supplementary conditions (6.22) on the driving term. Thus the dimension of the defect space is $\beta = \mathrm{Max}(-\varkappa, 0)$. The index is therefore $\alpha - \beta = \varkappa$, as given by (6.14).

Putting all the partial waves together, and using the compactness theorem of Section 5, we find that the index of the Fréchet differential (3.10) is

$$\varkappa = \sum_j \varkappa(j) = \sum_j \left[2 n_B^j + 2\delta_j(\infty)/\pi\right]. \tag{6.24}$$

This result is independent of whether or not (3.10) is subtracted (as can be seen by examining the previous argument), and includes the supplementary condition that the real part shall vanish at infinity. It does not yet include the centrifugal sum rules (2.24) however.

To show that the Fréchet differential is a Φ-operator (Definition 4.C), we must therefore prove it to be closed and have a closed range. By Theorem 5.A it is sufficient to show this with the left-hand cut omitted. However, it follows from Theorem 2.A that the right-hand part of (3.10) is bounded and therefore *a fortiori* closed. The range consists of the set of all possible driving terms, which will be the whole space for $\varkappa \geqq 0$, and the subspace satisfying the conditions (6.22) for $\varkappa < 0$. It is easy to see from Theorem 2.A that the inverse, as defined by (6.19) or (6.23), will be a bounded (but not necessarily single-valued) operator mapping $\delta f(s)$ into $\delta\delta(s)$ for all $\delta f(s)$ in the range. Therefore, by DS.VI.9.15 (ii), the range will be closed.

Finally, we give the reason why complete absorption $H > 0$ has to be excluded for the proof to work. For in this case, (6.20) shows that we must assume $\delta f(s) = o(s^{-H})$ to get any solution in the Banach space $H(R; \mu)$. Also, in order that $\eta(s)$ as given by (2.10) shall belong to the Banach space $H(R; \mu)$ (as assumed at numerous places in the proof), we must have $H \geqq \mu$. However, the functional

$$0 = \lim_{u \to 1} (1 - u)^H \delta f(u) \tag{6.25}$$

which defines the defect space, is unbounded for $H > \mu$ by (2.2), so that it does not correspond to a projection, and by DS.VI.3.1 the range will not be closed. Therefore, if total absorption occurs, the Fréchet differential will not be a Φ-operator (Definition 4.C), except in the special case

$H = \mu$. This could presumably be overcome by suitable redefinition of the Banach space.

c. External mass perturbations

Besides the perturbations of phase shifts and bound state pole parameters already discussed, the Dashen-Frautschi perturbation theory, [3—27], also considers changes in the external masses which define the kinematics, e.g., M and μ in (2.12). In terms of the amplitude which satisfies a dispersion relation

$$A_j(s) = [x_j(s) + y_j(s)]/\varrho_j(s) \tag{6.26}$$

these give

$$\delta A_j(s) = \frac{\delta A_j(s)}{\delta \delta_j(s)}\, \delta \delta_j(s) - \frac{A_j(s)}{\varrho_j(s)} \frac{d\varrho_j(s)}{dm}\, dm\,. \tag{6.27}$$

Now the kinematic factor always has the general form

$$\varrho_j(s) = [(s - r_a)(s - r_b) \ldots / (s - r_e)(s - r_f) \ldots]^{1/2} \tag{6.28}$$

where $r_a, r_b, \ldots, r_e, r_f, \ldots$ are the thresholds of various cuts (physical or kinematic). Examples are (2.11) and (2.12). Therefore it will satisfy a differential equation

$$\frac{d\varrho_i(s)}{dm} = \sum_n \frac{c_n}{r_n - s}\, \varrho_j(s) \tag{6.29}$$

where the c_n are certain known constants depending only on the unperturbed masses. From the dispersion relation satisfied by the unperturbed solution can be derived, by algebraic manipulation,

$$\sum_n \frac{c_n \operatorname{Re} A_j(s)}{r_n - s} = \sum_n \frac{c_n}{\pi} \int_{-\infty}^{\infty} \frac{ds' \operatorname{Im} A_j(s')}{(r_n - s')(s' - s)} + \sum_n \frac{c_n \operatorname{Re} A_j(r_n)}{r_n - s} \tag{6.30}$$

(we have omitted subtractions and bound-state poles for convenience of writing). The perturbed amplitude must also satisfy a dispersion relation

$$\operatorname{Re} \delta A_j(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Im} \delta A_j(s')\, ds'}{s' - s}\,. \tag{6.31}$$

By substituting (6.29) and (6.30), the contribution of the dm term in (6.27) to the right-hand cut cancels and we get

$$\begin{aligned} &\frac{\delta \operatorname{Re} A_j(s)}{\delta \delta_j(s)}\, \delta \delta_j(s) \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{ds'}{s' - s} \left(\frac{\delta \operatorname{Im} A_j(s')}{\delta \delta_j(s')}\right) \delta \delta_j(s') - \sum_n \frac{c_n \operatorname{Re} A_j(r_n)}{r_n - s}\, dm\,. \end{aligned} \tag{6.32}$$

Thus the only effect of the external mass perturbations is to add pole terms at the thresholds of the various cuts. Their residues will be proportional to the external mass perturbation dm, and the constant of proportionality depends only on the unperturbed solution. They are not cancelled by anything in $D^2(s)$ and will therefore appear also in the Dashen-Frautschi equation.

These poles in $\delta A_j(s)$ will lead to terms in the perturbed phase shift

$$\delta \delta_j(s) \sim (s - s_1)^{-1/2} \tag{6.33}$$

which do not belong to the Banach space $H(R;\mu)$. Thus the equations of S matrix theory are not Fréchet-differentiable with respect to the external masses. That this is a real difficulty, and not just due to the way we have formulated the problem, may be seen by considering the behaviour of an S wave phase shift when the threshold is changed by variation of the external mass. We will get, say

$$\begin{aligned} \delta(s) &\approx a(s - s_1)^{1/2}, \\ \delta(s) + \Delta\delta(s) &\approx [a + \Delta a](s - s_1 - \Delta s_1)^{1/2}, \end{aligned} \tag{6.34}$$

which are both physically permissible, but differentiation with respect to Δs_1 will give an inverse square root. Thus the equation

$$\Delta\delta(s) = \frac{d\delta(s)}{dm}\Delta m \tag{6.35}$$

can never be true near threshold, no matter how small the perturbation. This shows that first order perturbation theory will not be valid for changes in the external masses, whenever scattering lengths or other quantities close to threshold are being calculated.

Nevertheless, the external masses will only be finite in number. If we generalize the definition of the index to be the excess of variables over equations available to determine them, as in Theorem 4.D, then this will have a meaning even in the absence of Fréchet differentiability with respect to a finite number of the variables, for we can simply introduce them as new variables into the φ-equations of Theorem 4.D. In the case of the external masses, either the change in them will be given, as in the rho bootstrap, or it will be determined by a relation between the external mass and a bound-state pole position, as in the $N - N^*$ reciprocal bootstrap. In either case there will be as many extra equations as extra parameters, and the "index" will be unchanged.

d. Total number of implicit variables

So far the centrifugal sum rules (2.25) have not been included, since their Fréchet differentials are not continuous. However, there are only a finite number of them. Therefore, we may add them to the φ-equations of Theorem 4.D, without affecting the essential structure of the latter. Using our generalized notion of the index as the excess of variables over equations, it will be reduced by $-\sum_j l(j)$ by the centrifugal sum rules.

Changes in the original subtraction constants, δa_j of (3.10), are already included in the index. By (6.1), their value is got by evaluating (6.18) at $s = s_0$ corresponding to $u = \infty$, in the absence of a left-hand cut, and in a similar way from the subtracted Dashen-Frautschi equation when the left-hand cut is included.

The bound-state terms are also compact, and could therefore have been dropped without changing the index. Tho asymptotic behaviour of $D(s)$ [(6.13)—(6.14)] will then improve, thus at first sight decreasing the index, but the bound-state parameters will be left hanging undetermined in the air, thus increasing it again. This answers a question which may have puzzled some people in connection with the static model where there are poles in the crossed term — the reason bound-state poles contribute to Levinson's theorem is not that they are poles, but that they have independent variable parameters associated with them.

Putting this all together, the final value of the index is

$$\varkappa = \sum_j [2 n_B^j + 2\delta_j(\infty)/\pi - l(j)] . \tag{6.36}$$

This assumes that the left-hand cutoffs are given. Some bootstrap calculations adjust these to get better agreement with experiment — in such cases the value of the index must be increased. The corresponding index formula for N_j coupled channels is (B.30) of Appendix B.

We have shown that the index is not affected by the left-hand cut contributions, nor by the external mass terms. However, these are just the two places where group-theoretical factors enter the Dashen-Frautschi equations [5]. Therefore, the index must be the same for all types of symmetry breaking, since they are all coupled by unitarity to the same unperturbed state. There is no question of getting dynamical symmetries, such as octet dominance, from changes in the index. In fact all previous papers on dynamical symmetries have implicitly assumed that the index was zero.

There is an easy and enlightening way in which this formula for the index can be checked. If the left-hand cut contribution to the perturbed dispersion relation is omitted, then the problem is identical to that for a *fixed* left-hand cut. As is well known, this can be solved by the N/D method [41], which has been extensively investigated by FRYE and WARNOCK [28]. Their work shows that CDD poles [53] may occur, and will change Levinsons' theorem to [54—55]

$$\delta(\infty) - \delta(s_1) = (n_C - n_B)\pi \tag{6.37}$$

where n_C is the number of CDD poles. Each CDD pole contributes two parameters — its position and residues, so that the index $\varkappa$ as given by (6.24) is simply the number of CDD parameters. What we have shown beyond the N/D results is (a) that these CDD parameters are not determined by crossing symmetry, but lead to ambiguities in the exact solution (this was known previously in some soluble models, but never generally), and (b) that when symmetry breaking occurs, there will be *different* arbitrary CDD parameters for each mode of symmetry breaking, if the unperturbed solution had CDD poles. The ambiguity is to some extent reduced by the centrifugal sum rules in higher partial waves, again in agreement with the N/D work [28—31].

The stability Theorem 5.A only applies to the index $\varkappa = \alpha - \beta$. The values of α and β (the dimensions of the null and defect spaces) may individually be affected by the left-hand cut, and thus by the group-theoretical factors. Indeed this is the way in which spontaneous symmetry breaking (in the usual sense) occurs. However, the possibilities of this are limited by the following theorem, proved in GOKHBERG and KREIN [39].

Theorem 6 A. *Let $T(\lambda)$ be an operator-valued analytic function of the complex variable λ, whose values are Φ-operators for λ in a certain domain D. Then the index $\varkappa_{T(\lambda)}$ is constant throughout D, and $\alpha_{T(\lambda)}$, $\beta_{T(\lambda)}$ are also constant with the possible exception of some isolated points at which their values are larger than elsewhere.*

To apply this theorem, we embed the real Banach space $H(R;\mu)$ in a complex one, in which both the "real part" and "imaginary part" of the scattering amplitude may become complex. We then consider the elements of the crossing matrix [e.g., $\Lambda_{TT'}$ of (5.12)] as complex variables. The Fréchet differential is linear in them and therefore obviously analytic. When the crossing matrix is zero, there is no crossed term, and α, β are then known from Section 6.b to be

$$\alpha = \mathrm{Max}(\varkappa, 0), \quad \beta = \mathrm{Max}(-\varkappa, 0) \tag{6.38}$$

and in practice we always have

$$\alpha = 2n_C, \quad \beta = 0\,, \tag{6.39}$$

n_C being the number of CDD poles. As we vary a particular element of the crossing matrix, Theorem 6.A tells us that (6.39) will remain true except at some isolated points, where α and β may both increase by the same number. At these isolated values, the Fréchet differential will have another zero eigenvector, and spontaneous symmetry breaking will occur. However, any additional ambiguity thus created, though it will appear to be continuous in the linear approximation, can only be discrete in the exact solution by Theorem 4.D. This corresponds to the sort of spontaneous symmetry breaking discussed in algebraic models by previous authors [37—38]. By contrast, the ambiguities due to CDD poles go through into the exact solution.

Theorem 6.A also has some bearing on the possibility of the CDD ambiguities being reduced by accidental degeneracies, i.e., singular points of the φ-equations of Theorem 4.D. According to Theorem 4.D, there are no φ-equations unless $\beta > 0$, and according to Theorem 6.A and (6.39) this only occurs for isolated values of any single element of the crossing matrix. Thus for accidental degeneracies to stabilize symmetries, we must (a) be at one of these isolated crossing matrices, (b) the φ-equations must just happen to have a singular point at the unperturbed solution there, (c) this double coincidence must occur for every possible mode of symmetry breaking. This seems beyond belief. However, the centrifugal sum rules and the equations determining the external mass variations are more difficult to analyze since they are not continuously differentiable, and accidental degeneracies in them are perhaps slightly more plausible.

We have shown that the left-hand cut contribution to the Fréchet differential is compact, while the right-hand cut part can be inverted in closed form. It is then possible to reduce the inversion of the complete Fréchet differential to that of an operator $1 - C$ with C compact, which can then be solved by standard methods. Several such reduction techniques are given in MUSKHELISHVILI [32]. However, for the case when the inhomogeneous term and the compact part of the kernel have known analytic properties, the Dashen-Frautschi method [3, 5] is neater than any of them. In order to prove that it gives a compact kernel, we note that from the results of Section 6.a

$$(s - s_0)^{-\varkappa} D^2(s) \tag{6.40}$$

and its inverse, will both belong to the normed ring $H(R; \mu)$. We therefore write a dispersion relation for the perturbed amplitude multiplied by this quantity, and as argued by DASHEN and FRAUTSCHI it will have no right-hand cut. The results of Section 5, together with these properties of (6.40), then show that the left-hand cut contribution is compact. If $\varkappa \neq 0$ we will get either arbitrary subtraction constants in the equation ($\varkappa > 0$), or subsidiary conditions to be added to it ($\varkappa < 0$), just as in Section 6.b.

There exists a method for finding the exact perturbed solution by iterative solution of the linear approximation. It is shown in Chapter 18 of KANTOROVICH and AKILOV [36] that this will converge under certain conditions on the second Fréchet differential. It would be interesting to try to verify them, but this would lead us too far out of our path.

7. Discussion

We have shown that a quantity $\varkappa$ called the index determines the uniqueness of a given S matrix. $\mathrm{Max}(\varkappa, 0)$ is the number of arbitrary parameters on which it depends. $\varkappa$ has been calculated for partial wave dispersion relations (see (6.36) for the one-channel answer, and (B.30) of Appendix B for many channels), and is independent of the left-hand cut contribution, and of all group-theoretical factors. Normal dynamical symmetries, and also spontaneous symmetry breaking if it is to be well defined, both require the index to be zero. A positive index means that the symmetry is nondynamical — the S matrix equations by themselves allow symmetry breaking to occur in all directions without restriction, even with no driving term. This is like the situation in Lagrangian field theory, where $SU(3)$ violating Lagrangians are not excluded or restricted by any general principle. A negative index, on the other hand, would give a super-stable symmetry, highly resistent to any perturbation, and imposing constraints on the weak interactions.

There is no known example of a model with a negative index, but positive indices can occur if there are CDD poles. In this case the Dashen-Frautschi perturbation theory [3—37] will break down, due to their equation acquiring arbitrary subtraction constants different for each mode of symmetry breaking. (A statement in one of the Dashen-Frautschi papers [77] that CDD poles would not affect their argument is wrong.)

The failure of unique dynamical generation of symmetries has been demonstrated by a number of authors in static models [56—63], and Huang and Low have suggested that Levinson's theorem (without CDD poles) should be added to S matrix theory as a formulation of the bootstrap hypothesis. Our work confirms this, in so far as we show that quite dreadful things will happen to bootstraps if Levinson's theorem is not satisfied. However another question ought to be asked: is Levinson's theorem true in nature?

The best place to look is obviously the pion-nucleon P_{33} state. This is known experimentally up to 1311 MeV [64—73], with good agreement between all the phase-shift analyses[4]. It is elastic up to 700 MeV. The dynamics is believed to be well understood, and does not depend on inelasticity [74]. If Levinson's theorem is valid at all, then it is hard to imagine a more favourable opportunity for verifying it. Unfortunately, all the analyses show $\delta(P_{33})$ going to 180° at high energies, which means that Levinson's theorem is not satisfied, and a CDD pole must be present. Of course, the experiments do not extend to infinity. However, in the Chew-Low model where Levinson's theorem is valid, the phase levels off quickly after the resonance [75], and precisely this feature of the model causes drastic disagreement with experiment above 200 MeV. In fact it has been suggested before [76] that this could be due to a CDD pole.

The situation in the $\pi N P_{11}$ state is similar, though here the inelasticity is large. According to Levinson's theorem, the phase should go to $-180°$, because of the nucleon bound state, and all bootstrap models show it going strongly negative. However, again it is just this prediction

[4] The agreement is even better than might appear from the published papers, because (a) P_{33} was accidentally misdrawn in the earliest Chilton paper, (b) the Livermore P_{33} has changed since their last publication, due to new polarization experiments. In both cases the changers are towards 180°.

of the models which disagrees with experiment. The P_{11} phase unquestionably goes positive above 200 MeV, and while there is some dispute between the different analyses about whether it ends up at 0° or 180t, none give any indication that it goes to $-180°$. The D_{13} resonance appears to behave like the P_{33}.

If we then consider the uniqueness of the $N - N^*$ coupled bootstrap using the experimental phase shifts, the CDD poles will give four parameters, two of which are used up in imposing the P wave centrifugal barriers. Thus, even assuming that the left-hand cutoff is fixed, the final index is $+2$, indicating breakdown of the bootstrap hypothesis, and invalidity of Dashen-Frautschi perturbation theory. On S matrix theory alone, there is nothing stopping the isospin symmetry from spontaneous and complete disintegration. The long-range nature of electromagnetism should, if anything, make things worse by removing the centrifugal barrier. We may, if we wish, declare an interest only in those perturbations which die out at high energies fast enough to allow the Dashen-Frautschi equation to converge without subtractions, meaning $\delta\delta(s) = o(s^{-2})$ in the $N - N^*$ case, but this is very arbitrary and would give the super-stable index -2. To get the normal zero index situation, we would have to assume just one subtraction (instead of the two allowed by the unitarity limit), which is more arbitrary still.

The experimental evidence for CDD poles has been noticed independently by ATKINSON and HALPERN [77], who suggest they could be due to the coupled channels required by $SU(6)$, though their arguments are obviously inconclusive. If so, then Dashen-Frautschi perturbation theory would be applicable to $SU(6)$ breaking, but not to $SU(2)$ breaking. In any case, the index has a definite form for coupled channels (see Appendix B), and it should be possible to test this experimentally before very long. If tempted to assume the answer, let us recall that not so long ago people were confidently declaring all CDD poles to be theologically impossible.

Levinson's theorem is closely related to the requirement that Regge trajectories return to the left-half plane at high energies [78]. The experimental situation is similar: whereas in potential scattering the trajectories turn back rather quickly, the physical ones just seem to go up and up. It is obviously tempting to link the two phenomena. This is an argument against the coupled channel explanation, since it would not prevent Regge trajectories turning back.

The next possibility that will be noted by convinced bootstrappers is that the index might be changed by the Mandelstam representation, or the inclusion of many-particle states. Indeed, if we were to include a sufficient number of high angular states without CDD poles (assuming we can find a sufficient number experimentally, which is not certain in view of what has just been said about Regge trajectories) then the centrifugal barrier restrictions would counterbalance the CDD ambiguities. The simple answer is that this is not the way in which bootstrap calculations have always been done. To get through this loophole we must jettison all existing bootstrap models, and all the evidence for dynamical symmetries along with them.

Nevertheless, the extension to the Mandelstam representation and many-particle states should certainly be tried. The present work probably only scratches the surface of what can be done with these techniques. In particular, the fact that the compactness proof of Section 5 did not require any detailed knowledge of the location of the unphysical singularities seems very promising for many-particle states. Only their behaviour at points where they touch the physical region, and in particular the point at infinity, is required.

The present approach is based on the direct dispersion relations, and not on the N/D method used in almost all previous analyses. The former have the obvious advantage of automatically excluding ghost states, and also of giving crossing symmetry a simple form. The N/D equations are linear, so that classical techniques can be used, but are nevertheless a dead end, since there is no hope of ever including crossing. The really interesting questions of bootstraps and dynamical generation of symmetries are thus permanently closed to it. By contrast, the nonlinearity of the direct equations requires unfamiliar analysis, but once this has been learned, the way is open.

Appendix A

Some soluble examples

To check that nothing has been overlooked in the mathematical proofs, the reader would no doubt like to see some soluble models in which the theorem gives the right answer. There is a very large literature on static models for meson-baryon scattering. However, the static limit leads to divergences at high energies not present in relativistic theories, to cancel which a right-hand cutoff is introduced. This causes the kinematic factor $\varrho(s)$ of Section 2.b to vanish strongly at high energies, so that the condition (2.14) needed for the right-hand integral to be a bounded operator is not satisfied. This means that the formulation in which the real and imaginary parts normalized to their unitarity limits are taken as the fundamental Hölder-continuous quantities, is not suited to static models. No doubt this could be overcome by suitable reformulation, and the general techniques applied to such cases. However, since this feature of the static model is clearly unphysical and due merely to the use of nonrelativistic kinematics at high energies, we have not thought such changes worth while. Instead, we shall consider some relativistic models.

We note however that the CDD ambiguity and its relation to Levinson's theorem are well verified in static models, so that the additive constant in the index formula is the only point at issue.

The first model we consider is the scattering of two neutral pions, according to the Shirkov equations. EFREMOV et al. [79] showed that this is exactly soluble by the Castillejo-Dalitz-Dyson technique of generalized R functions [53]. The general solution is $\left(\text{with } \nu = \frac{s}{4} - 1\right)$

$$\begin{aligned}\sqrt{\frac{\nu}{\nu+1}}\cot\delta(\nu) = \lambda^{-1} - c(2\nu+1)^2 - \sum_n \frac{R_n(2\nu+1)^2}{\omega_n[\omega_n - (2\nu+1)^2]} - \\ -1 + \frac{1}{\pi}\sqrt{\frac{\nu}{\nu+1}}\ln\left[\frac{\sqrt{\nu}+\sqrt{\nu+1}}{\sqrt{\nu}-\sqrt{\nu+1}}\right] + \frac{1}{\pi}\sqrt{\frac{\nu+1}{\nu}}\ln\left[\frac{\sqrt{\nu+1}+\sqrt{\nu}}{\sqrt{\nu+1}-\sqrt{\nu}}\right],\end{aligned} \tag{A.1}$$

where

$$\lambda \geqq 0, \quad c \geqq 0, \quad R_n \geqq 0, \quad \omega_n > 1, \tag{A.2}$$

but are otherwise arbitrary. The uniqueness theorem says that any solution satisfying

$$\delta(\nu) = \delta(\infty) + O(\nu^{-\mu}), \quad \mu > 0, \quad \text{as} \quad \nu \to +\infty \tag{A.3}$$

should belong to a continuum of dimension

$$2n_B + 2[\delta(\infty) - \delta(0)]/\pi. \tag{A.4}$$

First consider (A.1) when all $R_n = 0$. $\cot\delta(\nu)$ must then be finite in between threshold and infinity, so that $\delta(\infty) - \delta(0)$ will be either 0 or $\pm\pi$. To find which, we must examine whether $\cot\delta(\nu)$ goes to $+\infty$ or $-\infty$, at $\nu = 0$ and $\nu = \infty$. At the symmetry point $\nu = -\frac{1}{2}$, (A.1) will be positive by (A.2). By making λ sufficiently small we can certainly ensure that there will be no bound state, so that it will still be positive at $\nu = 0$. It will go to $-\infty$ as $\nu \to \infty$ unless c vanishes. This implies $\delta(\infty) = \pi$, so by (A.4) we should have a two-parameter family of solutions, which we do — the parameters being λ and c. For the case with bound states, it is easily seen, by studying what happens to (A.1) as a bound state crosses threshold, that (A.4) is unchanged. As for the solutions with $R_n \neq 0$, each will give a pole in $\cot\delta(\nu)$ with negative residue, implying that $\delta(\nu)$ must *increase* through a multiple of π. Thus each CDD pole adds $+2$ to the index (A.4), and two parameters R_n and ω_n to the manifold of solutions. These different families of solutions cannot perturb into each other, because they correspond to different values of $\delta(\infty)$, and will therefore never approach each other in the Banach space norm (2.2).

Thus we have shown that this model satisfies the uniqueness theorem, except in the special cases $c = 0$ and $\lambda = 0$. For $c = 0$ the asymptotic behaviour of the unperturbed solution is

$$\delta(\nu) = \delta(\infty) + O(1/\ln\nu), \tag{A.5}$$

so that the assumption (2.3) is not satisfied. In fact, it is known that this case corresponds to a one-parameter family of solutions satisfying Levinson's theorem. No doubt, with a bit of extra work, the theorem could be extended to these weaker asymptotic behaviours. In particular, we would have to prove Theorem 2.A in a Banach space for which the imaginary part merely satisfied

$$y(s) = y(\infty) + O(\ln^{-2} s) \tag{A.6}$$

instead of the Hölder condition at infinity, and also take logarithmic factors into account in calculating the asymptotic behaviour of $\Gamma(s)$ in Section 6.a.

The case $\lambda = 0$ is exceptional because the unperturbed solution vanishes identically. Therefore the Fréchet differential (3.10) reduces to multiplication by $\eta(\nu) = 1$ and has index zero. The zero solution is thus isolated, in accord with the fact that there is a discontinuous change in the high-energy behaviour as the interaction is switched off. Thus the boundaries $c = 0$ and $\lambda = 0$ of the manifolds of solutions, required by (A.2), correspond to the solution moving out of the Banach space, and the equation ceasing to be continuously Fréchet-differentiable, respectively. This checks with Theorem 4.A.

In the case of pions with isotopic spin 1, the Shirkov equations have two S waves and one P wave. They cannot then be solved exactly, but have been extensively investigated numerically [80] and the dimensionality of some of the manifolds of solutions are known with reasonable certainty. The P wave contributes an extra -1 to the index, because of the centrifugal barrier, so it is

$$\varkappa = \sum_{T=0}^{2} \{n_B^T + [\delta_T(\infty) - \delta_T(0)]/\pi\} - 1 \,. \tag{A.7}$$

The subtraction constants will be determined by the requirement that the real part vanish at infinity. This will relate them to integrals over the imaginary parts, and will therefore ensure that the symmetry relation at the subtraction point

$$2A_0^0\left(-\frac{1}{2}\right) + 9A_1^1\left(-\frac{1}{2}\right) - 5A_0^2\left(-\frac{1}{2}\right) = 0 \tag{A.8}$$

is automatically satisfied. There are known to be three asymptotic behaviours [81]

$$A_l^T(\nu) \sim \frac{1}{\ln \nu}\,, \quad \sim \frac{1}{\nu}\,, \quad \text{or} \quad \sim \frac{1}{\nu^2}\,. \tag{A.9}$$

As in the neutral case, our theorem does not apply to the first, because it violates (2.3). In the absence of CDD poles, this one is known to lead to the one-parameter S dominant solutions. Serebryakov and Shirkov [80] have obtained numerically a set of solutions with the second asymptotic behaviour, giving resonances in A_0^0 and A_1^1 but none in A_2^2 (in qualitative agreement with experiment). The resonant phase shifts go to 180° at infinity, so we should expect them to depend on three parameters, according to (A.7), which indeed is the case. These solutions have CDD poles at infinity.

Appendix B

Coupled channels

We now consider the case of N coupled two-particle channels. The unperturbed S matrix is then an $N \times N$ matrix related to the quantities $A_{\alpha\beta}(s)$ satisfying a dispersion relation by

$$S_{\alpha\beta}(s) = \delta_{\alpha\beta} + 2i\,[\varrho_\alpha(s)]^{1/2}\,A_{\alpha\beta}(s)\,[\varrho_\beta(s)]^{1/2}\,. \tag{B.1}$$

Here $\varrho_\alpha(s)$ is a kinematic factor with square-root behaviour at the appropriate threshold, and asymptotic behaviour $\sim s^R$. It follows from time-reversal invariance that A and S are symmetric matrices, and therefore, because of the dispersion relation, each element satisfies

$$[A_{\alpha\beta}(s)]^* = A_{\alpha\beta}(s^*) \tag{B.2}$$

(* when applied to *elements* of a matrix means complex conjugate only, but Hermitian adjoint when there are no subscripts). Our Banach space will consists of normed rings of matrices whose elements are real-valued functions Hölder-continuous in the variable u. To avoid difficulty with thresholds, we take

$$\begin{aligned} x(s) &= (s - s_0)^R\,\mathrm{Re}\,A(s)\\ y(s) &= (s - s_0)^R\,\mathrm{Im}\,A(s) \end{aligned} \tag{B.3}$$

as our basic Banach-space vectors this time, where $\sim s^R$ is the common asymptotic behaviour of all $\varrho_\alpha(s)$. We are thus assuming short-range interaction threshold behaviour. We also exclude total absorption by imposing the condition.

$$\det[S(s)] \neq 0\,, \quad s_1 \leqq s \leqq \infty\,. \tag{B.4}$$

We do not exclude many-particle contributions, but suppose they are not perturbed, so that

$$\delta S(s)\,S(s^*) + S(s)\,\delta S(s^*) = 0\,. \tag{B.5}$$

The right-hand cut as used in the definition of u [see Eq. (2.1)] is taken down to the lowest threshold s_1. We assume that the right and left-hand cuts do not overlap, though this could probably be relaxed by doing more work on the Hölder continuity of the left-hand imaginary part. The proofs that the right-hand dispersion integral is a bounded operator, and that the left-hand term of the unperturbed equation is *bounded*, then go through with only trivial modifications. However, to show the *compactness* of the perturbed left-hand term, we require a convergence factor analogous to (5.6), and this needs a little consideration. Equations (B.3) and (B.1) imply

$$\begin{aligned} \delta \operatorname{Re} S(s) &= -2(s-s_0)^{-R} \varrho^{1/2} \delta y(s) \varrho^{1/2}, \\ \delta \operatorname{Im} S(s) &= 2(s-s_0)^{-R} \varrho^{1/2} \delta x(s) \varrho^{1/2}, \end{aligned} \tag{B.6}$$

while from (B.5) we get

$$\{\delta \operatorname{Re} S(s), \operatorname{Re} S(s)\} + \{\delta \operatorname{Im} S(s), \operatorname{Im}(Ss)\} = 0 . \tag{B.7}$$

The supplementary condition that the real parts $x(s)$ shall vanish at infinity implies by (B.1) that $\operatorname{Im} S(\infty) = 0$, and hence by (2.2), $\operatorname{Im} S(s) \sim \sim s^{-\mu}$. Equation (B.4) then shows that the $\operatorname{Re} S(\infty) \neq 0$. Therefore, the consistency of (B.7), together with (B.6), implies a bound

$$\delta y(s) \sim s^{-\mu} |\delta x(s)|, \quad \text{as} \quad s \to \infty , \tag{B.8}$$

which can be used to get the requisite convergence factor in the compactness proof.

The main difficulty of course comes in the solution of the unitarity equation, required in the index calculation of Section 6. For this we proceed as follows.

Corresponding to (6.1), we define a matrix

$$\Phi(u) = \delta a + \frac{1}{\pi} \int_0^1 \frac{du' \, \delta y(u')}{u' - u} \tag{B.9}$$

(taking the case $R = 0$ for simplicity), and the analogue of (6.2)—(6.3) is

$$\Phi(u) = \frac{1}{2i} \varrho^{1/2} \delta S(u) \varrho^{1/2} - \delta f(u) \tag{B.10}$$

where $\delta f(u)$ is a matrix of driving terms. Like $S(u)$, $\Phi(u)$ will be a symmetric matrix, so that each element will satisfy

$$\Phi_{\alpha\beta}(u)^* = \Phi_{\alpha\beta}(u^*) \tag{B.11}$$

and we have

$$\Phi(u^*) = -\frac{1}{2i} \varrho^{1/2} \, \delta S(u^*) \, \varrho^{1/2} - \delta f(u^*) . \tag{B.12}$$

Therefore, by (B.5) and (B.4), we get

$$\begin{aligned} \varrho^{-1/2} \Phi(u) \varrho^{-1/2} = S(u) \{\varrho^{-1/2} \Phi(u^*) \varrho^{-1/2}\} [S(u^*)]^{-1} - \\ - \varrho^{-1/2} \, \delta f(u) \, \varrho^{-1/2} + S(u) \, \varrho^{-1/2} \, \delta f(u^*) \, \varrho^{-1/2} [S(u^*)]^{-1} \end{aligned} \tag{B.13}$$

We thus find ourselves with a matrix Hilbert problem of the form

$$X(u) = A(u) \, X(u^*) \, B(u) , \tag{B.14}$$

and the corresponding inhomogeneous problem. (Here u has a small positive imaginary part.) Consider first the *vector* Hilbert problems

$$U_\alpha(u) = \sum_\gamma A_{\alpha\gamma}(u)\, U_\gamma(u^*)\,, \tag{B.15}$$

$$V_\beta(u) = \sum_\delta V_\delta(u^*)\, B_{\delta\beta}(u)\,. \tag{B.16}$$

It is shown in Chapter 18 of MUSKHELISHVILI's book [32] that each of these will possess N independent fundamental solutions $U_\alpha^\varrho(u)$ and $V_\beta^\sigma(u)$, respectively, satisfying the conditions [e.g., for $U_\alpha^\varrho(u)$]

$$\det\|U_\alpha^\varrho(u)\| \neq 0\,, \quad \text{in the whole finite } u \text{ plane,} \tag{B.17}$$

$$0 < |\det\|u^{-\varkappa_\varrho}\, U_\alpha^\varrho(u)\| < \infty\,, \quad \text{as} \quad u \to \infty\,. \tag{B.18}$$

Here $\varkappa_\varrho = \varkappa_\varrho(A)$ are integers, known as the *partial indices* of the Hilbert problem (B.15), and satisfy

$$[\arg\det\|A_{\alpha\gamma}(u)\|]_{u=0}^{u=1} = \sum_{\varrho=1}^{N} \varkappa_\varrho(A)\,. \tag{B.19}$$

It is obvious that

$$X(u) = U_\alpha^\varrho(u)\, V_\beta^\sigma(u) \tag{B.20}$$

will give a particular solution of the homogeneous *matrix* Hilbert problem (B.14). The trick now is to write the $N \times N$ matrix X(u) as a vector with N^2 components. Equation (B.14) is then a vector Hilbert problem of the same type as (B.16), but in N^2 dimensions. Its matrix is an outer product of two N-dimensional matrices A and B^T. Using the formula for the determinant of the *outer* product of two matrices

$$\det(A \otimes B) = (\det A\, \det B)^N\,, \tag{B.21}$$

we see that $N^2 \times N^2$ matrix with components (B.20), where α, β label the row and ϱ, σ label the column, satisfy equations like (B.17) and (B.18) and are therefore a matrix of fundamental solutions of the N^2 dimensional Hilbert problem (B.14). According to MUSKHELISHVILI's results there can only be N^2 fundamental solutions for a vector Hilbert problem of dimension N^2, and all solutions of the homogeneous equation (B.14) are given by linear combinations of them with polynomial coefficients. The partial indices will be, by (B.21) and (B.18),

$$\varkappa_{\varrho\sigma} = N[\varkappa_\varrho(A) + \varkappa_\sigma(B)] \tag{B.22}$$

and the total index

$$N\left[\sum_{\varrho=1}^{N} \varkappa_\varrho(A) + \sum_{\sigma=1}^{N} \varkappa_\sigma(B)\right] = N[\arg\det A(u) + \arg\det B(u)]_{u=0}^{u=1}\,. \tag{B.23}$$

In our particular case, the vector Hilbert problems corresponding to (B.15) and (B.16) will be

$$\begin{aligned} U_\alpha(u) &= \sum_\gamma S_{\alpha\gamma}(u)\, U_\gamma(u^*) \\ V_\beta(u) &= \sum_\delta V_\delta(u^*)\, \{[S(u^*)]^{-1}\}_{\delta\beta}\,. \end{aligned} \tag{B.24}$$

Using the symmetry of the S matrix, and (B.4), and conjugating, it is easily seen that any solution of the one will also determine a solution of the other. Therefore, we can identify

$$V^{\varrho}_{\alpha}(u) = [U^{\varrho}_{\alpha}(u^*)]^* = U^{\varrho}_{\alpha}(u) \tag{B.25}$$

by analytic continuation to $u < 0$. First let us consider the solution of (B.13) with the same boundary conditions as Muskhelishvili — that is $\varrho^{-1/2}\Phi(u)\varrho^{-1/2}$ finite everywhere in the u plane, Hölder-continuous on the cut, and vanishing at $u = \infty$. The general solution of the homogeneous Hilbert problem is then

$$\varrho_{\alpha}^{-1/2}\Phi_{\alpha\beta}(u)\varrho_{\beta}^{-1/2} = \sum_{\varrho,\sigma=1}^{N} U^{\varrho}_{\alpha}(u)\, P_{\varrho\sigma}(u)\, U^{\sigma}_{\beta}(u) \tag{B.26}$$

where $P_{\varrho\sigma}(u)$ is an arbitrary polynomial in u of order $\leqq \varkappa_{\varrho} + \varkappa_{\sigma}$, if this is $\geqq 0$. If $\varkappa_{\varrho} + \varkappa_{\sigma} < 0$, then this term is absent from the homogeneous solution, and instead we get $-\varkappa_{\varrho} - \varkappa_{\sigma}$ conditions on the driving term, in order that the inhomogeneous solution shall vanish at $u = \infty$. It would appear therefore that the index in the sense of Definition 4.B is given by (B.23). However, we have forgotten that $\Phi_{\alpha\beta}(u)$ must be a symmetric matrix, in order that the perturbed S matrix given by (B.10) be symmetric. This means that the polynomials $P_{\varrho\sigma}(u)$ of (B.26) must be symmetric in ϱ and σ. Also the conditions on the driving terms when the partial index is negative will be partly dependent on each other, since $\delta f_{\alpha\beta}(u)$ will be symmetric. It is easily seen that the effect of this symmetry on the index is to replace it by

$$\begin{aligned}\frac{1}{2}\sum_{\varrho\neq\sigma}[\varkappa_{\varrho}+\varkappa_{\sigma}] + \sum_{\varrho}[\varkappa_{\varrho}+\varkappa_{\varrho}] &\\ &= (N+1)\sum_{\varrho=1}^{N}\varkappa_{\varrho} \\ &= (N+1)\,[\arg\det\|S_{\alpha\beta}(s)\|]^{\infty}_{s_1}\,.\end{aligned} \tag{B.27}$$

Our boundary conditions differ from MUSKHELISHVILI's in the following respect: (a) the solution must vanish at $u = 1$ corresponding to $s = \infty$, (b) the solution must go to a constant *at* $u = \infty$, corresponding to the subtraction point $s = s_0$, but need not vanish there, (c) the solution for the perturbed S matrix may have a double pole at each of the n_B bound-state positions [compare (6.15)]. These bound-state poles will occur in each element of the S matrix. (a) and (b) are easily seen to cancel in the index, since multiplication of a Muskhelishvili type solution by $(u - 1)$ will always satisfy (a) without violating (b). The bound states mean that each of the fundamental solutions (B.20) can be multiplied by $\prod_B (u_B - u)^{-2}$. This will decrease by $u^{-2n}B$ the behaviour at $u = \infty$, and therefore allow each of the polynomials of (B.26) to have order higher by $2n_B$. The index will then be, instead of (B.27)

$$\frac{1}{2}\sum_{\varrho\neq\sigma}[\varkappa_\varrho+\varkappa_\sigma+2n_B]+\sum_{\varrho}[\varkappa_\varrho+\varkappa_\varrho+2n_B] = \varkappa = (N+1)\{n_B+[\arg\det\|S_{\alpha\beta}(s)\|]_{s_1}^{\infty}\}\,. \tag{B.28}$$

This has a simple physical interpretation, for

$$n_C = n_B + [\arg\det\|S^j_{\alpha\beta}(s)\|]_{s_2}^{\infty} \tag{B.29}$$

will be the number of CDD poles, and vanishes for the many-channel form of Levinson's theorem [82—88]. Each CDD pole corresponds to an elementary particle, and therefore has $N+1$ arbitrary parameters, namely its mass, and its coupling to each of the N channels.

To determine the number of centrifugal sum rules without undue complication, we assume the orbital parities to be equal in all coupled channels. When these are included, and the independent partial waves put together, the index becomes finally

$$\tilde{\varkappa} = \sum_j (N_j+1)\{n_B^j+[\arg\det\|S^j_{\alpha\beta}(s)\|]_{s_1}^{\infty}\} - \sum_j (N_j+1)\left[\sum_{\alpha=1}^{Nj}\frac{1}{2}l_\alpha(j)\right] \tag{B.30}$$

$l_\alpha(j)$ being the orbital angular momentum of the α^{th} channel of the j^{th} partial wave.

The reader will notice that we have not assumed any commutation properties among the matrices S, S^*, δS and δS^*.

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