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# Non-Existence of Trouser-Worlds

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**Abstract.** Simple restrictions are derived on the global structure of normal hyperbolic Riemannian spaces. As a consequence, cosmological models of the trouser type are ruled out irrespective of field equations.

A well-known phenomenon in astronomy is the appearance of new visible objects in the sky. This led JORDAN to the conjecture that gravity theory might offer cosmological models in which separate *spaces unite* at some finite time, yielding a discrete growth of the observable matter in the universe. In order to have an imaginable model in mind, think of a 2-dimensional pair of trousers in Minkowskian 3-space such that the vertical direction is timelike: horizontal sections of this "*trouser world*" are two circles near the bottom, and one circle at hip level. In other words: space sections are disconnected at some time, and connected at another. However, closer inspection shows that this model does not carry a regular metric: near the saddle point its signature changes from normal hyperbolic to (positive) definite. Is this failure unavoidable, i.e. independent of imbedding, and dimension? We are going to prove a simple lemma which indeed denies the existence of trouser worlds. (Further relevant global properties will be presented by SEIFERT in a later article.)

**Lemma.** *Consider a g-complete* [1] *Riemannian space of continuity class* $C_2$ *containing a continuous surface* $\Sigma$, *and a continuous unit vector field* $u^a$ *over* $\Sigma$. *Follow the geodesics which start on* $\Sigma$ *with direction* $u^a$ *up to (eigen) distance* $s$ *(from* $\Sigma$*). The "geodesic map"* $\Sigma \to \Sigma(s)$ *of* $\Sigma$ *onto the points* $\Sigma(s)$ *thus obtained is continuous for every* $s$.

*Proof.* It suffices to show that the image points $x_k(s)$ of a converging point sequence $x_k$ in $\Sigma$ converge towards the image point $x(s)$ of the limit $x$ of $x_k$ (with respect to the coordinate topology). Note that the geodesics of a $C_2$-space cannot split because their differential equation satisfies a Lipschitz condition. Suppose now that for a certain $s$, $x_k(s)$ does not converge towards $x(s)$, and call $s_0$ the infimum of all such $s$. Consequently, the geodetic segments $g_k$ converge pointwise towards $g$ for all $s$ smaller than $s_0$; and so do their tangent vectors

$\dot{x}^a := dx^a/ds$, and coordinate acceleration vectors $\ddot{x}^a := -\Gamma^a_{bc}\dot{x}^b\dot{x}^c$. For if not, one could find a subsequence whose $\dot{x}^a_k(s_1)$ converge towards a vector differing from $\dot{x}^a(s_1)$; which conflicts with the convergence of $g_k$ around $s_1$. Next observe that $x_k(s_0)$ converges towards $x(s_0)$. In order to see this we introduce a normal coordinate system $S$ in $x(s_0)$, and assume that convergence is violated. It is then possible to choose a subsequence of $g_k$ whose intersections with a small sphere around $x(s_0)$ converge towards a point not contained in $g$. But this contradicts the fact that $\dot{x}^a(s)$ converges for all $s < s_0$, and that $\ddot{x}^a$ is a first order quantity at the origin of $S$. Finally we see that $x_k(s)$ converges towards $x(s)$ for an open interval of distances $s$ containing $s_0$. This follows from the continuous dependence of solutions of differential equations on their initial values in sufficiently small domains: $x_k(s) \to x(s)$ holds for $x_k(s_0) \to x(s_0)$, $\dot{x}^a_k(s_0) \to \dot{x}^a(s_0)$. The existence of an $s_0$ has thereby been refuted.

The lemma has interesting *consequences*. Recall for instance that the continuous image of a connected (compact) manifold is connected (compact), and that a simply connected image implies a simply connected inverse image. If in cosmology, $\Sigma$ is taken to be a spacelike hypersurface intersecting every matter worldline, and if the matter 4-velocity is chosen as $u^a$, then the lemma implies that matter can *never separate* into disjoint pieces.

In conclusion, let us define a *trouser world* as a spacetime model which a) can be divided, by means of two space sections $\Sigma_\pm$, into three disjoint parts: the future of $\Sigma_+$, the past of $\Sigma_-$, and the rest, and such that b) $\Sigma_+$ is connected whereas $\Sigma_-$ consists of at least two connectivity components $\Sigma_j$ with disjoint past, c) timelike geodesics meeting $\Sigma_+$ meet $\Sigma_-$ within a uniform interval of eigen time, and d) for each $\Sigma_j$ there exists a timelike geodesic meeting both $\Sigma_j$, and $\Sigma_+$. This definition expresses what we understand by "uniting spaces". The lemma implies that such a world cannot exist: Choose $\Sigma_+$ as the surface $\Sigma$ from the lemma, and take for $u^a$ a past directed timelike vector field such that in some points $x_j$ of $\Sigma_+$, $u^a$ agrees with the tangent of a geodesic meeting $\Sigma_j$. For sufficiently large $s$, the geodesic image $\Sigma(s)$ must have swept past $\Sigma_-$, i.e. must have torn if the past of $\Sigma_-$ was disconnected; which contradicts the lemma.

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## Literatur

[1] KUNDT, W.: Z. Physik **172**, 488 (1963).