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Degenerate Representations of Non-Compact Unitary Groups. II. Continuous Series

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Abstract. Three degenerate principal series of irreducible unitary representations of an arbitrary non-compact unitary group $U(p, q)$ are derived. These series are determined by the eigenvalues of the first and second-order invariant operators, the former having a discrete spectrum and the latter a continuous one. The explicit form of the corresponding harmonic functions is derived and the properties of the continuous representations are discussed.

1. Introduction

In our previous paper [1] we obtained two degenerate principal series, $D^L_M(X^{p,q}_+)$ and $D^L_M(X^{p,q}_-)$, of irreducible unitary representations of an arbitrary non-compact unitary group $U(p, q)$. These series have been realized in the Hilbert spaces of functions defined in the domains

$$X^{p,q}_+ = U(p, q)/U(p-1, q) \quad \text{and} \quad X^{p,q}_- = U(p, q)/U(p, q-1) \tag{1.1}$$

respectively, which are homogeneous with respect to the action of the $U(p, q)$ group (see [2]). The representation labels M and L determine the eigenvalues, M and λ, of the first and second-order invariant operators $\hat{M}$ and $\Delta(X^{p,q}_\pm)$ respectively and both possess a discrete spectrum.

In the present paper we investigate the properties of the continuous series of degenerate representations of the $U(p, q)$ groups which are characterized by continuous values of λ and discrete values of M. We derive three such series of representations, the first two being related to the manifolds $X^{p,q}_+$ and $X^{p,q}_-$ given by (1.1) and the third being related to the manifold

$$X^{p,q}_0 = U(p, q)/T^{p+q-2} \boxed{s}\, U(p-1, q-1)\,. \tag{1.2}$$

Here, T^{p+q-2} is the group of translations in the $(p+q-2)$-dimensional complex space C^{p+q-2} and $\boxed{s}$ means the semidirect product. As will be shown later, the homogeneous spaces $X^{p,q}_\pm$ and $X^{p,q}_0$ can be represented as certain hypersurfaces in the $2(p+q)$-dimensional Minkowski space $M^{2p,2q}$.

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The method is analogous to that used in our first paper [1]. We choose the biharmonic coordinate system on the manifold considered and then we solve the eigenvalue problem for the invariant operators $\hat{M}$ and Δ

$$\begin{aligned} \hat{M}\Psi^{\lambda}_{M} &= M\Psi^{\lambda}_{M} \\ \Delta(X^{p,q}_{\pm})\Psi^{\lambda}_{M} &= \lambda\Psi^{\lambda}_{M} \end{aligned} \tag{1.3}$$

where

$$\hat{M} = \sum_{i=1}^{p+q} Z^{c}_{i} \tag{1.4}$$

and $\Delta(X^{p,q}_{\pm})$ is the Laplace-Beltrami operator on the manifold $X^{p,q}_{\pm}$:

$$\Delta(X^{p,q}_{\pm}) = \frac{1}{\sqrt{|\bar{g}|}}\,\partial_{\alpha} g^{\alpha\beta}(X^{p,q}_{\pm})\sqrt{|\bar{g}|}\,\partial_{\beta}\,. \tag{1.5}$$

Here, Z^{c}_{i} are the generators of a Cartan subgroup of the $U(p, q)$ group, $g_{\alpha\beta}(X^{p,q}_{\pm})$ is the metric tensor on $X^{p,q}_{\pm}$ and $\bar{g} = \det\{g_{\alpha\beta}(X^{p,q}_{\pm})\}$. In the case of the manifold $X^{p,q}_{0}$ the second-order invariant operator is defined in a different way (see Section 3).

In Section 2 we derive two sets of harmonic functions, one related to the manifold $X^{p,q}_{+}$ and the other related to the manifold $X^{p,q}_{-}$. The harmonic functions having the manifold $X^{p,q}_{0}$ as their domain are derived in Section 3. Section 4 is devoted to the construction of three continuous degenerate series of representations related to these three sets of harmonic functions. In Section 5 we discuss the structure of the Hilbert spaces corresponding to the series of representations obtained. Finally, in the Appendix, we show that the series of representations obtained are irreducible.

2. Harmonic functions for the non-compact unitary groups defined on the manifolds $X^{p,q}_{+}$ and $X^{p,q}_{-}$

As it was explained in [1], the homogeneous spaces $X^{p,q}_{+}$ and $X^{p,q}_{-}$ defined by (1.1) can be represented with the help of certain manifolds having the same dimension and the same stability group and embedded into the $(p+q)$-dimensional complex space C^{p+q}:

$$\begin{gathered} z^1\bar{z}^1 + \cdots + z^p\bar{z}^p - z^{p+1}\bar{z}^{p+1} - \cdots - z^{p+q}\bar{z}^{p+q} = \pm 1 \\ z^k \in C^{p+q},\ k = 1, 2, \ldots, p+q, \end{gathered} \tag{2.1}$$

where the right-hand side is ± 1 for $X^{p,q}_{\pm}$. These manifolds, which in the following will be denoted also by $X^{p,q}_{\pm}$, can be considered as certain hypersurfaces (namely, hyperboloids) in the $2(p+q)$-dimensional Minkowski space $M^{2p,2q}$.

To obtain diagonal metric tensors on $X^{p,q}_{+}$ and $X^{p,q}_{-}$ we introduce the biharmonic coordinate system (see [1]) by choosing a set of $2(p+q)$

parameters r, θ, ω, $\tilde{\omega}$,

$$\begin{aligned}\omega &\equiv \{\varphi^1, \ldots, \varphi^p, \vartheta^2, \ldots, \vartheta^p\}\\ \tilde{\omega} &\equiv \{\tilde{\varphi}^1, \ldots, \tilde{\varphi}^q, \tilde{\vartheta}^2, \ldots, \tilde{\vartheta}^q\}\end{aligned} \tag{2.2}$$

where

$$0 \leqq r < \infty, \quad 0 \leqq \varphi^k \leqq 2\pi, \quad 0 \leqq \tilde{\varphi}^l \leqq 2\pi, \quad 0 \leqq \vartheta^i \leqq \frac{\pi}{2},$$

$$0 \leqq \tilde{\vartheta}^j \leqq \frac{\pi}{2}, \; k = 1, 2, \ldots p; \quad l = 1, 2, \ldots q; \quad i = 2, 3, \ldots p;$$

$$j = 2, 3, \ldots q$$

and by parametrizing (in a recursive manner) the C^{p+q} space in the following way:

$$\begin{gathered} z^k = r e^{i\varphi^k} \sigma(\theta) \sin\vartheta^p \ldots \sin\vartheta^{k+1} \cos\vartheta^k \\ \tilde{z}^l \equiv z^{p+l} = r e^{i\tilde{\varphi}^l} \tau(\theta) \sin\tilde{\vartheta}^q \ldots \sin\tilde{\vartheta}^{l+1} \cos\tilde{\vartheta}^l \\ k = 1, 2, \ldots p, \quad l = 1, 2, \ldots q, \quad \vartheta^1 \equiv 0, \quad \tilde{\vartheta}^1 \equiv 0\,. \end{gathered} \tag{2.3}$$

The manifolds $X_+^{p,q}$ and $X_-^{p,q}$ are then obtained simply by putting

$$0 \leqq \theta < \infty, \quad \sigma(\theta) = \operatorname{ch}\theta, \quad \tau(\theta) = \operatorname{sh}\theta, \quad r = 1 \tag{2.4}$$

$$0 \leqq \theta < \infty, \quad \sigma(\theta) = \operatorname{sh}\theta, \quad \tau(\theta) = \operatorname{ch}\theta, \quad r = 1 \tag{2.5}$$

respectively.

Using now the same procedure as in [1], we obtain the following expression for the Laplace-Beltrami operator on the manifold $X_+^{p,q}$:

$$\begin{aligned}\Delta(X_+^{p,q}) = -\frac{1}{\operatorname{ch}^{2p-1}\theta \operatorname{sh}^{2q-1}\theta} \frac{\partial}{\partial\theta} \operatorname{ch}^{2p-1}\theta \operatorname{sh}^{2q-1}\theta \frac{\partial}{\partial\theta} + \\ + \frac{\Delta(X^p)}{\operatorname{ch}^2\theta} - \frac{\tilde{\Delta}(X^q)}{\operatorname{sh}^2\theta}\end{aligned} \tag{2.6}$$

where $\Delta(X^p)$ and $\tilde{\Delta}(X^q)$ are the Laplace-Beltrami operators of the compact unitary groups $U(p)$ and $U(q)$ respectively, X^a being an abbreviation for $X_+^{a,0}$.

The invariant first-order operator $\hat{M} \equiv \hat{M}(X_+^{p,q})$ has the form

$$\hat{M} = \hat{M}_p + \hat{\tilde{M}}_q \tag{2.7}$$

where $\hat{M}_p = -i \sum_{k=1}^{p} \frac{\partial}{\partial\varphi^k}$ and $\hat{\tilde{M}}_q = -i \sum_{l=1}^{q} \frac{\partial}{\partial\tilde{\varphi}^l}$ are the invariant first-order operators of the $U(p)$ and $U(q)$ group respectively. The explicit form of the eigenfunctions of the operators $\Delta(X^p)$ and $\hat{M}_p$ is given in the Appendix of [1].

Representing the simultaneous eigenfunctions of the operators $\Delta(X^{p,q}_{+})$ and $\hat{M}(X^{p,q}_{-})$ in the form of a product of eigenfunctions of $\Delta(X^p)$ and $\hat{M}_p$ times eigenfunctions of $\tilde{\Delta}(X^q)$ and $\hat{\tilde{M}}_q$ times an unknown function $\psi^{\lambda}_{J_p,\tilde{J}_q}(\theta)$ we obtain the following differential equation for $\psi^{\lambda}_{J_p,\tilde{J}_q}(\theta)$:

$$\Big[-\frac{1}{\operatorname{ch}^{2p-1}\theta\operatorname{sh}^{2q-1}\theta}\frac{d}{d\theta}\operatorname{ch}^{2p-1}\theta\operatorname{sh}^{2q-1}\theta\frac{d}{d\theta}- \\ -\frac{J_p(J_p+2p-2)}{\operatorname{ch}^2\theta}+\frac{J_q(J_q+2q-2)}{\operatorname{sh}^2\theta}-\lambda\Big]\psi^{\lambda}_{J_p,\tilde{J}_q}(\theta)=0 \tag{2.8}$$

where $-J_p(J_p+2p-2)$ and $-\tilde{J}_q(\tilde{J}_q+2q-2)$ are eigenvalues of the operators $\Delta(X^p)$ and $\tilde{\Delta}(X^q)$ for $p>1$ and $q>1$ respectively, J_p and $\tilde{J}_q$ being non-negative integers. For $p=1$ or $q=1$ the eigenvalues of $\Delta(X^1)$ and $\tilde{\Delta}(X^1)$ turn out to be equal to $-J_1^2$ or $-\tilde{J}_1^2$ respectively, J_1 and $\tilde{J}_1$ being arbitrary integers.

The solution of (2.8) which is regular at $\theta=0$ is given by (see [1])

$$\psi^{\lambda(\alpha)}_{J_p\tilde{J}_q}(\theta)=\operatorname{th}^{|\tilde{J}_q|}\theta\operatorname{ch}^{-\alpha}\theta\times \\ \times{}_2F_1\left(\frac{|\tilde{J}_q|-|J_p|+\alpha}{2}-p+1,\frac{|\tilde{J}_q|+|J_p|+\alpha}{2};|\tilde{J}_q|+q;\operatorname{th}^2\theta\right) \tag{2.9}$$

where $\alpha=p+q-1+\sqrt{(p+q-1)^2-\lambda}$. Whereas in [1] we were interested in the square-integrable solutions of (2.8), now we are considering the continuous part of the spectrum of λ. As it will be shown in Part III of our work, the continuous spectrum of λ is given by

$$\lambda\geqq(p+q-1)^2\,. \tag{2.10}$$

Thus, we represent α in the form

$$\alpha\equiv\alpha(\Lambda)=p+q-1+i\Lambda \tag{2.11}$$

where Λ is an arbitrary non-negative number.

The set of orthogonal functions related to definite values of invariant numbers Λ and M (M being an arbitrary integer) is given by the following formula

$$\mathbf{Y}^{-\bar{\alpha}(\Lambda),J_2,\dots,J_p,\tilde{J}_2,\dots,\tilde{J}_q}_{M,M_1,\dots,M_p,\tilde{M}_1,\dots,\tilde{M}_q}(\theta,\Omega,\tilde{\Omega}) \\ =V^{-\bar{\alpha}(\Lambda)}_{J_p,\tilde{J}_q}(\theta)\,\mathbf{Y}^{J_2,\dots J_p}_{M_1,\dots M_p}(\Omega)\,\mathbf{Y}^{\tilde{J}_2,\dots\tilde{J}_q}_{\tilde{M}_1,\dots\tilde{M}_q}(\tilde{\Omega}) \tag{2.12}$$

where $V^{-\bar{\alpha}(\Lambda)}_{J_p,\tilde{J}_q}(\theta)$ is defined by

$$V^{-\bar{\alpha}(\Lambda)}_{J_p,\tilde{J}_q}(\theta)=\frac{1}{\sqrt{N}}\psi^{\lambda(\bar{\alpha})}_{J_p,\tilde{J}_q}(\theta)\,. \tag{2.13}$$

The normalization constant N turns out to be (see [3])

$$N=2\pi\left|\frac{\Gamma(|\tilde{J}_q|+q)\,\Gamma(\alpha-p-q+1)}{\Gamma\left[\frac{1}{2}(|\tilde{J}_q|-|J_p|+\alpha)-p+1\right]\Gamma\left[\frac{1}{2}(|\tilde{J}_q|+|J_p|+\alpha)\right]}\right|^2. \tag{2.14}$$

The functions $Y^{J_2,\ldots,J_p}_{M_1,\ldots,M_p}(\Omega)$ are simultaneous eigenfunctions of the invariant operators $\Delta(X^p)$ and $\hat{M}_p$ and their form is given in the Appendix of [1], where, also, properties of the corresponding unitary representations of the $U(p)$ group are discussed in detail. We recall here only the conditions which are imposed on the numbers labelling the eigenfunctions (2.12):

$$
\begin{aligned}
&M_1 \equiv J_1 \\
&|M_2 - M_1| + |M_1| = J_2 - 2n_2 \quad n_2 = 0, 1, \ldots, \tfrac{1}{2}(J_2 - |M_2|) \\
&|M_3 - M_2| + J_2 = J_3 - 2n_3 \qquad n_3 = 0, 1, \ldots, \tfrac{1}{2}(J_3 - |M_3|) \\
&\ldots\ldots\ldots\ldots \qquad\qquad \ldots\ldots\ldots\ldots \\
&|M_p - M_{p-1}| + J_{p-1} = J_p - 2n_p \quad n_p = 0, 1, \ldots, \tfrac{1}{2}(J_p - |M_p|) .
\end{aligned}
\tag{2.15}
$$

Analogous relations hold also among $\tilde{M}_1, \ldots, \tilde{M}_q, \tilde{J}_2, \ldots, \tilde{J}_q$. Finally,

$$M_p + \tilde{M}_q = M . \tag{2.16}$$

Note that the numbers $M_2, \ldots, M_p, \tilde{M}_2, \ldots, \tilde{M}_q$ and M have the same parity as the numbers $J_2, \ldots J_p, \tilde{J}_2, \ldots \tilde{J}_q$ and $J_p + \tilde{J}_q$ respectively.

For reasons explained in [1] we have replaced in (2.12) the sets ω, $\tilde{\omega}$ of variables by new sets

$$
\begin{aligned}
\Omega &\equiv \{\phi^1, \ldots \phi^p, \vartheta^2, \ldots \vartheta^p\} \\
\tilde{\Omega} &\equiv \{\tilde{\phi}^1, \ldots \tilde{\phi}^q, \tilde{\vartheta}^2, \ldots \tilde{\vartheta}^q\}
\end{aligned}
\tag{2.17}
$$

respectively, where $\phi^1, \ldots \phi^p, \tilde{\phi}^1, \ldots \tilde{\phi}^q$ are connected with $\varphi^1, \ldots \varphi^p$ $\tilde{\varphi}^1, \ldots \tilde{\varphi}^q$ by a certain linear transformation (see equation (2.25) of [1]).

The set of harmonic functions (2.12) is orthogonal with respect to the left-invariant Riemannian measure $d\mu(X^{p,q}_+)$ given by the metric tensor expressed in the biharmonic coordinates $\theta, \Omega, \tilde{\Omega}$:

$$d\mu(X^{p,q}_+) = d\mu(X^p)\, d\tilde{\mu}(X^q)\, \mathrm{ch}^{2p-1}\theta\, \mathrm{sh}^{2q-1}\theta\, d\theta \tag{2.18}$$

with

$$
\begin{aligned}
d\mu(X^p) &= \prod_{j=2}^{p} \sin^{2j-3}\vartheta^j \cos\vartheta^j\, d\vartheta^j \prod_{k=1}^{p} d\phi^k \\
d\tilde{\mu}(X^q) &= \prod_{j=2}^{q} \sin^{2j-3}\tilde{\vartheta}^j \cos\tilde{\vartheta}^j\, d\tilde{\vartheta}^j \prod_{k=1}^{q} d\tilde{\phi}^k .
\end{aligned}
$$

In a similar way, the set of harmonic functions related to the homogeneous space $X^{p,q}_-$ defined by (1.1) is obtained. The expression for the harmonic functions differs from (2.12) only by replacing $V^{-\tilde{\alpha}(\Lambda)}_{J_p,\tilde{J}_q}(\theta)$ by

$V_{\tilde{J}_q, J_p}^{-\bar{\alpha}(\Lambda)}(\theta)$. Correspondingly, the role of the numbers J_p, p and $\tilde{J}_q, q$ is interchanged in the formulae (2.9), (2.14) and (2.18).

The two sets of harmonic functions related to the homogeneous spaces $X_+^{p,q}$ and $X_-^{p,q}$ can be used for construction of degenerate irreducible unitary representations of the $U(p, q)$ groups. These representations will be constructed in Section 4.

3. Harmonic functions for the non-compact unitary groups defined on the manifold $X_0^{p,q}$

The set of harmonic functions defined on the manifold $X_0^{p,q}$ (see (1.2)) can be obtained in a similar way. As a model of this manifold we choose the subspace of the C^{p+q} space defined by the equation

$$z^1\bar{z}^1 + \cdots + z^p\bar{z}^p - z^{p+1}\bar{z}^{p+1} - \cdots - z^{p+q}\bar{z}^{p+q} = 0\,. \tag{3.1}$$

As we see from equation (3.1), the model of $X_0^{p,q}$ can also be considered as a $(2(p+q)-1)$-dimensional cone embedded in the $2(p+q)$-dimensional Minkowski space $M^{2p,2q}$.

Equation (3.1) can be automatically satisfied if we parametrize the coordinates z^i, $i = 1, \ldots p+q$, according to the prescription (2.3) and then put

$$\sigma(\theta) = \cos\theta, \quad \tau(\theta) = \sin\theta, \quad \theta = \frac{\pi}{4}, \quad 0 \leqq r < \infty\,. \tag{3.2}$$

The first-order invariant operator $\hat{M}$ has on the manifold $X_0^{p,q}$ the same form as on $X_\pm^{p,q}$, i.e., it is given by (2.7). On the other hand, the construction of the second-order invariant operator meets here two difficulties. First, as the metric tensor $g_{\alpha\beta}(X_0^{p,q})$ on $X_0^{p,q}$ is singular, (see [3]), the second-order invariant operator cannot be calculated by using formula (1.5). Second, due to the fact that the group $U(p,q)$ is not semi-simple, the Cartan metric tensor g_{ik} is singular so that we cannot use the standard formula $Q_2 = \sum g^{ik} Z_i Z_k$ to determine the second-order invariant operator. To construct it, we note that the operator

$$\begin{aligned}\hat{I}_2 = \left(\sum_{i<j=1}^{p} + \sum_{p<i<j=p+1}^{p+q}\right)(L_{ij}^{+2} + L_{ij}^{-2}) + \\ + \sum_{i=1}^{p+q} L_{ii}^{-2} - \sum_{i=1}^{p}\sum_{j=p+1}^{p+q}(B_{ij}^{+2} + B_{ij}^{-2})\end{aligned} \tag{3.3}$$

commutes with the whole algebra $R_{p,q}$[1]. Representing the basis of the $R_{p,q}$ algebra by the Lie algebra of operators of differentiation with

[1] The basis of the $R^{p,q}$ algebra is chosen in the same form as in [1], where the commutation relations are explicitly given.

respect to $z^1, \ldots z^{p+q}, \bar{z}^1, \ldots \bar{z}^{p+q}$ [2],

$$\begin{aligned}
L^+_{k,l} &= z^k \frac{\partial}{\partial z^l} - z^l \frac{\partial}{\partial z^k} + \bar{z}^k \frac{\partial}{\partial \bar{z}^l} - \bar{z}^l \frac{\partial}{\partial \bar{z}^k} \\
L^-_{kl} &= i \left(z^k \frac{\partial}{\partial z^l} + z^l \frac{\partial}{\partial z^k} - \bar{z}^k \frac{\partial}{\partial \bar{z}^l} - \bar{z}^l \frac{\partial}{\partial \bar{z}^k} \right) \\
B^+_{kl} &= z^k \frac{\partial}{\partial z^l} + z^l \frac{\partial}{\partial z^k} + \bar{z}^k \frac{\partial}{\partial \bar{z}^l} + \bar{z}^l \frac{\partial}{\partial \bar{z}^k} \\
B^-_{kl} &= i \left(z^k \frac{\partial}{\partial z^l} - z^l \frac{\partial}{\partial z^k} - \bar{z}^k \frac{\partial}{\partial \bar{z}^l} + \bar{z}^l \frac{\partial}{\partial \bar{z}^k} \right)
\end{aligned} \tag{3.4}$$

and expressing them through r, ω and $\tilde{\omega}$ by using (2.3) and (3.2), we obtain the following expression for the operator $\hat{I}_2$ on the manifold $X^{p,q}_0$:

$$\begin{gathered}
\hat{I}_2 = -r^2 \frac{\partial^2}{\partial r^2} - (2p + 2q - 1)\, r \frac{\partial}{\partial r} + \left(\sum_{k=1}^{p+q} \frac{\partial}{\partial \varphi^k} \right)^2, \\
\varphi^{p+l} \equiv \tilde{\varphi}^l, \quad l = 1, 2, \ldots q
\end{gathered} \tag{3.5}$$

where the last term is obviously equal to $-\hat{M}^2$ on $X^{p,q}_0$.

The set of differential equations

$$\begin{aligned}
\hat{I}_2 \Psi^\lambda_M &= (\lambda - M^2) \Psi^\lambda_M \\
\hat{M} \Psi^\lambda_M &= M \Psi^\lambda_M
\end{aligned} \tag{3.6}$$

can now be solved, again by the method of separation of variables. We obtain

$$\begin{gathered}
\left[r^2 \frac{d^2}{dr^2} + (2p + 2q - 1)\, r \frac{\partial}{\partial r} + \Lambda^2 + (p + q - 1)^2 \right] R(r) = 0 \\
\left(-i \sum_{k=1}^{p+q} \frac{\partial}{\partial \varphi^k} - M \right) \Phi(\varphi^1, \ldots \varphi^{p+q}) = 0
\end{gathered} \tag{3.7}$$

where the form $\Lambda^2 + (p + q - 1)$ of the spectrum is chosen in analogy with the spectrum of λ in the preceding section. In Part III of this work we shall show that the spectrum is complete in this form.

The solution of the system (3.6) of equations can be written in the form

$$r^{-\alpha} \exp \left(i \sum_{k=1}^{p+q} m_k \varphi^k \right) F(\vartheta^2, \ldots, \vartheta^p, \tilde{\vartheta}^2, \ldots, \tilde{\vartheta}^q) \tag{3.8}$$

[2] The harmonic functions are not analytic in the variables $z^1, z^2, \ldots, z^{p+q}$ but they can be considered as differentiable functions of $2(p+q)$ independent real variables $x^k \equiv \operatorname{Re} z^k$, $y^k \equiv \operatorname{Im} z^k$, $k = 1, 2, \ldots, p+q$. The set of relations

$$z^k = x^k + i y^k \quad \bar{z}^k = x^k - i y^k$$

has the meaning of a regular linear transformation, implying

$$\frac{\partial}{\partial z^k} = \frac{1}{2} \left(\frac{\partial}{\partial x^k} - i \frac{\partial}{\partial y^k} \right), \quad \frac{\partial}{\partial \bar{z}^k} = \frac{1}{2} \left(\frac{\partial}{\partial x^k} + i \frac{\partial}{\partial y^k} \right).$$

Note that, according to this definition, the derivatives $\frac{\partial \bar{z}^k}{\partial z^k}$ exist in every point and are equal to zero.

where $\alpha = p + q - 1 + \sqrt{(p + q - 1)^2 - \lambda}$. As the invariant operators $\hat{I}_2$ and $\hat{M}$ are independent of $\vartheta^2, \ldots, \vartheta^p, \tilde{\vartheta}^2, \ldots, \tilde{\vartheta}^q$, the function F in (3.8) can be chosen in an arbitrary way. We specify the form of the function F by requiring it to be the common eigenfunction of the maximal set of commuting operators in the enveloping algebra of the $U(p, q)$ group (see [1] Section 5). We obtain (by the index 0 we denote the eigenfunctions defined on the manifold $X_0^{p,q}$)

$$ {}_0\mathrm{Y}^{-\alpha(\Lambda), J_2, \ldots, J_p, \tilde{J}_2, \ldots, \tilde{J}_q}_{M, M_1, \ldots M_p, \tilde{M}_1 \ldots \tilde{M}_q}(r, \Omega, \tilde{\Omega}) = r^{-\alpha}\, \mathrm{Y}^{J_2, \ldots J_p}_{M_1, \ldots, M_p}(\Omega)\, \mathrm{Y}^{\tilde{J}_2, \ldots \tilde{J}_q}_{\tilde{M}_1, \ldots \tilde{M}_q}(\tilde{\Omega}) \tag{3.9} $$

where the Y functions on the right-hand side are identical with those occurring in (2.12) and are defined in the Appendix of [1].

The harmonic functions ${}_0\mathrm{Y}^{-\alpha(\Lambda), J_2, \ldots, J_p, \tilde{J}_2, \ldots, \tilde{J}_q}_{M, M_1, \ldots, M_p, \tilde{M}_1, \ldots, \tilde{M}_q}(r, \Omega, \tilde{\Omega})$ constitute an orthogonal set of functions with respect to the left-invariant Riemannian measure $d\mu(X_0^{p,q})$ given by

$$ d\mu(X_0^{p,q}) = d\mu(X^p)\, d\tilde{\mu}(X^p)\, r^{2p+2q-3}\, dr \tag{3.10} $$

where $d\mu$ and $d\tilde{\mu}$ are defined by (2.18).

4. Continuous degenerate representations of non-compact unitary groups

In the two preceding sections we derived three sets of harmonic functions defined on three different manifolds, $X_+^{p,q}$, $X_-^{p,q}$ and $X_0^{p,q}$, and orthogonal on these manifolds with respect to the left-invariant measures $d\mu(X_+^{p,q})$, $d\mu(X_-^{p,q})$ and $d\mu(X_0^{p,q})$ respectively. In this section we shall construct Hilbert spaces in which the continuous degenerate representations of arbitrary non-compact unitary groups corresponding to these sets of harmonic functions are realized.

Let us start with the representations related to the homogeneous space $X_+^{p,q}$. We define a set of functions $f(\theta, \Omega, \tilde{\Omega})$ having the form

$$ f(\theta, \Omega, \tilde{\Omega}) = \exp\left(-\sum_{i=1}^{p+q} z^i \bar{z}^i\right) P(z^1, \ldots, z^{p+q}, \bar{z}^1, \ldots \bar{z}^{p+q}) \tag{4.1} $$

where $P(z^1, \ldots, z^{p+q}, \bar{z}^1, \ldots, \bar{z}^{p+q})$ is an arbitrary polynomial in the variables $z^1, \ldots, z^{p+q}, \bar{z}^1, \ldots, \bar{z}^{p+q}$ restricted by the condition (2.1) to the manifold X_+^{p+q} and expressed in terms of the biharmonic coordinates (2.3), (2.4)[3].

As the harmonic functions (2.12) are not square integrable, they cannot create a Hilbert space. Nevertheless, we can use them to construct the Hilbert space in an indirect way, by defining the generalized Fourier transforms χ of the functions $f(\theta, \Omega, \tilde{\Omega})$ (see [3] and [5]). We define

[3] As is shown in [4], the considered set of functions $f(\theta, \Omega, \tilde{\Omega})$ creates a dense set of functions in $L^2(X_+^{p,q}, \mu)$.

them as follows:

$$\chi^{A,J_2,\ldots,J_p,\tilde J_2,\ldots,\tilde J_q}_{M,M_1,\ldots,M_p,\tilde M_1,\ldots,\tilde M_q} = \langle \mathrm{Y}^{-\bar\alpha(A),J_2,\ldots,J_p,\tilde J_2,\ldots,\tilde J_q}_{M,M_1,\ldots,M_p,\tilde M_1,\ldots,\tilde M_q}, f\rangle$$
$$= \int_{X^{p,q}_+} \overline{\mathrm{Y}^{-\bar\alpha(A),J_2,\ldots,J_p,\tilde J_2,\ldots,\tilde J_q}_{M,M_1,\ldots,M_p,\tilde M_1,\ldots\tilde M_q}(\theta,\Omega,\tilde\Omega)}\, f(\theta,\Omega,\tilde\Omega)\, d\mu(X^{p,q}_+)\,. \tag{4.2}$$

The set of all such generalized Fourier transforms

$$\chi^A_M \equiv \{\chi^{A,J_2,\ldots J_p,\tilde J_2,\ldots \tilde J_q}_{M,M_1,\ldots,M_p,\tilde M_1,\ldots,\tilde M_q}\}$$

fulfilling the condition

$$\|\chi^A_M\|^2 = \sum_{\substack{J_2,\ldots,J_p,\tilde J_2,\ldots,\tilde J_q\\ M_1,\ldots,M_p,\tilde M_1,\ldots,\tilde M_q}} |\chi^{A,J_2,\ldots,J_p,\tilde J_2,\ldots,\tilde J_q}_{M,M_1,\ldots,M_p,\tilde M_1,\ldots,\tilde M_q}|^2 < \infty \tag{4.3}$$

spans the Hilbert space $\mathscr{H}^A_M(X^{p,q}_+)$ of vectors, the scalar product being defined by

$$(\chi^A_M, \eta^A_M) = \sum_{\substack{J_2,\ldots,J_p,\tilde J_2,\ldots,\tilde J_q\\ M_1,\ldots,M_p,\tilde M_1,\ldots,\tilde M_q}} \overline{\chi^{A,J_2,\ldots,J_p,\tilde J_2,\ldots,\tilde J_q}_{M,M_1,\ldots,M_p,\tilde M_1,\ldots\tilde M_q}}\, \eta^{A,J_2,\ldots,J_p,\tilde J_2,\ldots,\tilde J_q}_{M,M_1,\ldots,M_p,\tilde M_1,\ldots,\tilde M_q}\,. \tag{4.4}$$

The sum in (4.3) and in (4.4) is taken over all possible values of the labels admitted by relations (2.15) and (2.16).

Each Hilbert space $\mathscr{H}^A_M(X^{p,q}_+)$ can be represented as a direct sum of the form

$$\mathscr{H}^A_M(X^{p,q}_+) = \sum_{\substack{\tilde J_q=0\\ (J_p+\tilde J_q+M\ \text{even})}}^{\infty} \sum_{J_p=0}^{\infty} \sum_{\substack{\tilde M_q=-\tilde J_q\\ (\tilde M_q+\tilde J_q\ \text{even})}}^{\tilde J_q} \sum_{\substack{M_p=-J_p\\ (M_p+J_p\ \text{even})}}^{J_p} \oplus$$
$$\oplus\, \mathscr{H}^{A\,J_p,\tilde J_q}_{M,M_p,\tilde M_q}(X^{p,q}_+)\,\delta_{M,M_p+\tilde M_q} \tag{4.5}$$

where $\mathscr{H}^{A\,J_p,\tilde J_q}_{M,M_p,\tilde M_q}(X^{p,q}_+)$ is a finite-dimensional subspace of $\mathscr{H}^A_M(X^{p,q}_+)$ in which the irreducible unitary representation of the maximal compact subgroup $U(p)\times U(q)$ determined by the invariants J_p, M_p, $\tilde J_q$ and $\tilde M_q$ is realized. Note that in the special cases $p > q = 1$ and $p = q = 1$ the three-dimensional sum in (4.5) reduces to a two-dimensional and to a one-dimensional sum respectively.

It is easy now to determine the action of the generators of the $U(p,q)$ group on the vectors $\{\chi^{A,J_2,\ldots,J_p,\tilde J_2,\ldots,\tilde J_q}_{M,M_1,\ldots,M_p,\tilde M_1,\ldots,\tilde M_q}\} \equiv \chi^A_M$ of the Hilbert space $\mathscr{H}^A_M(X^{p,q}_+)$. Using the representation (3.4) of the algebra of $U(p,q)$ and expressing z^k, $\bar z^k$, $\frac{\partial}{\partial z^k}$ and $\frac{\partial}{\partial \bar z^k}$ in terms of the biharmonic coordinates (2.3), (2.4) we determine the form of the generators on $X^{p,q}_+$. Then the action of an arbitrary generator Z_i on a vector χ^A_M of $\mathscr{H}^A_M(X^{p,q}_+)$ is defined as follows:

$$Z_i\{\chi^{A,J_2,\ldots,J_p,\tilde J_2,\ldots,\tilde J_q}_{M,M_1,\ldots,M_p,\tilde M_1,\ldots,\tilde M_q}\} = \{\langle \mathrm{Y}^{-\bar\alpha(A),J_2,\ldots,J_p,\tilde J_2,\ldots,\tilde J_q}_{M,M_1,\ldots,M_p,\ldots,\tilde M_1,\ldots,\tilde M_q} Z_i f\rangle\}\,. \tag{4.6}$$

In this way, we have obtained a series of continuous degenerate representations for an arbitrary non-compact unitary group $U(p, q)$. These representations are determined by two numbers Λ and M, Λ being an arbitrary non-negative number (see (2.11)) and M being an arbitrary integer. The irreducibility of these representations is proved in the Appendix and the proof of unitarity will be given in Part III of our work. We shall denote this series of representations by the symbol $C_M^\Lambda(X_+^{p,q})$.

In an analogous way, also the series $C_M^\Lambda(X_-^{p,q})$ and $C_M^\Lambda(X_0^{p,q})$ of continuous degenerate representations related to the homogeneous spaces $X_-^{p,q}$ and $X_0^{p,q}$ are obtained. We can repeat everything that has been said in this section, replacing only $X_+^{p,q}$ by $X_-^{p,q}$ or $X_0^{p,q}$ respectively and using the corresponding set of harmonic functions.

5. Discussion

Let us discuss briefly the structure of the Hilbert spaces corresponding to the different representation series obtained. The structure of the Hilbert spaces $\mathscr{H}_M^\Lambda(X_+^{p,q})$ is given by (4.5) and can be represented with the help of "net" diagrams each point of which represents a subspace $\mathscr{H}_{M, M_p, \tilde{M}_q}^{\Lambda, J_p, \tilde{J}_q}(X_+^{p,q})$ of $\mathscr{H}_M^\Lambda(X_+^{p,q})$. These net diagrams are constructed in the same way as the analogous diagrams for the representations $D_M^L(X_+^{p,q})$ discussed in [1], the only difference being that the numbers Λ, J_p and $\tilde{J}_q$ are not restricted by any condition.

If $p \geqq q > 1$, we have a three-dimensional net of points with the coordinates J_p, $\tilde{J}_q$ and M_p, say $(\tilde{M}_q = M - M_p)$. Figures 1 and 2 represent sections through this net with $\tilde{J}_q = \text{const}$ and $J_p = \text{const}$, respectively. For different values of Λ the structure is the same.

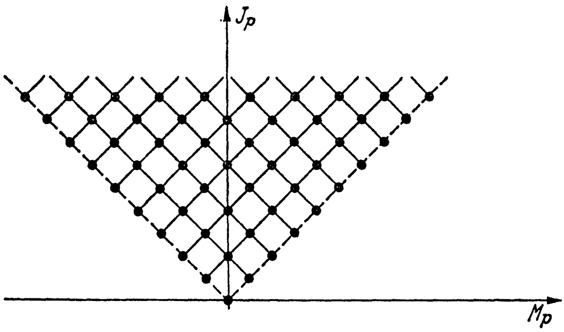

Fig. 1. A $\tilde{J}_q$ = const section through the three-dimensional net representing admissible values of J_p and M_p in a given representation $C_M^\Lambda(X_+^{p,q})$, $p \geqq q > 1$. The same diagram represents the full net in the case $p > q = 1$

In the case $p > q = 1$, the net is two-dimensional and can be represented by Fig. 1. (Note that $\tilde{J}_1 \equiv \tilde{M}_1 = M - M_p$ in this case.) Finally, for $p = q = 1$ the structure of the Hilbert spaces can be represented by an infinite string of points $M_1 + \tilde{M}_1 = M$ $(J_1 \equiv M_1, \tilde{J}_1 \equiv \tilde{M}_1)$.

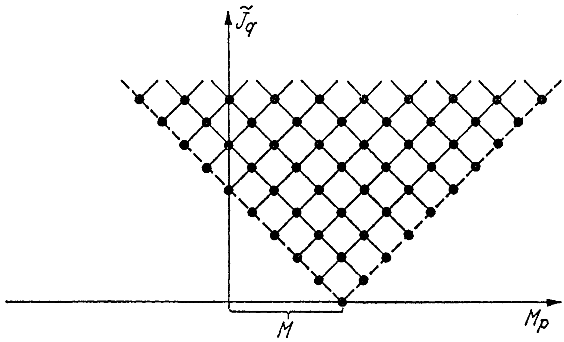

Fig. 2. A J_p = const section through the three-dimensional net representing admissible values of $\tilde{J}_q$ and M_p in a given representation $C_M^{\Lambda}(X_+^{p,q})$, $p \geqq q > 1$

Let us remark that for the definite Λ the number M can be an arbitrary integer.

The Hilbert spaces $\mathscr{H}_M^{\Lambda}(X_-^{p,q})$ and $\mathscr{H}_M^{\Lambda}(X_0^{p,q})$ have the same structure as the Hilbert spaces $\mathscr{H}_M^{\Lambda}(X_+^{p,q})$.

The parity operator P, defined by

$$P z^k = -z^k, \quad k = 1, 2, \ldots, p + q$$

commutes with all generators of the $U(p, q)$ group. Thus, for a given irreducible representation the parity is determined and is equal to $(-1)^M$.

The maximal set of commuting operators in the given representation space is

$$\Delta(X_\pm^{p,q}), \hat{M}, C_p, \tilde{C}_q \tag{5.1}$$

in the case of the manifolds $X_\pm^{p,q}$ and

$$\hat{I}_2, \hat{M}, C_p, \tilde{C}_q \tag{5.2}$$

in the case of the manifold $X_0^{p,q}$ (see (3.5)). The symbols C_p and $\tilde{C}_q$ denote the maximal sets of commuting operators of the compact subgroups respectively:

$$C_p \equiv \{\Delta(X^p), \hat{M}_p, \Delta(X^{p-1}), \hat{M}_{p-1}, \ldots \Delta(X^2), \hat{M}_2, \hat{M}_1\}$$
$$\tilde{C}_q \equiv \{\tilde{\Delta}(X^q), \hat{\tilde{M}}_q, \tilde{\Delta}(X^{q-1}), \hat{\tilde{M}}_{q-1}, \ldots \tilde{\Delta}(X^2), \hat{M}_2, \hat{M}_1\}\,.$$

The number of operators of the set (5.1) or (5.2) is strongly reduced in comparison with the case of the principal non-degenerate series, being equal to $2(p + q) - 1$.

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Appendix

In this Appendix we shall prove the irreducibility of the representations $C_M^A(X_+^{p,q})$, $C_M^A(X_-^{p,q})$ and $C_M^A(X_0^{p,q})$ which we have derived in Section 4. The proof can be performed by the method used in [1] for proving irreducibility of the representations $D_M^L(X_+^{p,q})$ and $D_M^L(X_-^{p,q})$ (see also [3]).

We shall denote by $R^{p,q}$ the Lie algebra of the $U(p, q)$ group. By the symbols $R_+^{p,q}$, $R_-^{p,q}$ and $R_0^{p,q}$ we shall denote the representations of $R^{p,q}$ given by (3.4), (2.3) with the condition that the coordinate parameters are chosen according to (2.4), (2.5) and (3.2) respectively.

Let us start with the $C_M^A(X_+^{p,q})$ representation. We have to show that the Hilbert space $\mathscr{H}_M^A(X_+^{p,q})$ contains no subspace invariant with respect to the action of the Lie algebra $R_+^{p,q}$. The structure of the space $\mathscr{H}_M^A(X_+^{p,q})$ is given by (4.5).

In the Appendix of [1] it has been shown that our representation of the algebra of the maximal compact subgroup $U(p) \times U(q)$ is irreducible in the space $\mathscr{H}_{M, M_p, \tilde{M}_q}^{A, J_p, \tilde{J}_q}$. Thus, in complete analogy with the proof given in Section 4 of [1], it is now sufficient to find some operators $B_i \in R_+^{p,q}$ and one vector of the type (4.2) such that the operators B_i can perform the transition into all nearest neighbouring Hilbert spaces $\mathscr{H}_{M M_p' \tilde{M}_q'}^{A J_p' \tilde{J}_q'}$ with $J_p' = J_p \pm 1$, $J_q' = \tilde{J}_q \pm 1$ and $M_p' = M_p \pm 1$. (Let us mention that the maximal number of the nearest neighbours is 8, 4 and 2 in the cases $p \geqq q > 1$, $p > q = 1$ and $p = q = 1$ respectively).

By an explicit calculation we can see that the operators

$$B_\pm \equiv B_{p,p+q}^+ \pm i B_{p,p+q}^- = e^{\pm i(\varphi^p - \tilde{\varphi}^q)} \Big[\cos\vartheta^p \cos\tilde{\vartheta}^q \frac{\partial}{\partial\theta} - \\ - \operatorname{cth}\theta \cos\vartheta^p \left(\sin\tilde{\vartheta}^q \frac{\partial}{\partial\tilde{\vartheta}^q} \pm \frac{i}{\cos\tilde{\vartheta}^q} \frac{\partial}{\partial\tilde{\varphi}^q}\right) - \\ - \operatorname{th}\theta \cos\tilde{\vartheta}^q \left(\sin\vartheta^p \frac{\partial}{\partial\vartheta^p} \mp \frac{i}{\cos\vartheta^p} \frac{\partial}{\partial\varphi^p}\right)\Big] \tag{A.1}$$

have the desired properties if acting on

$$\chi_{M, M_p, \tilde{M}_q}^{A, J_p, \tilde{J}_q} \equiv \{\chi_{M, 0, \ldots, 0, M_p, 0, \ldots, 0, \tilde{M}_q}^{A, 0, \ldots, 0, J_p, 0, \ldots, 0, \tilde{J}_q}\}. \tag{A.2}$$

The calculation in the case $p \geqq q > 1$ leads to the following result:

$$\begin{aligned} B_+ \chi^{\Lambda, J_p, \tilde{J}_q}_{M, M_p, \tilde{M}_q} = & - |J + \tilde{J} + 1 + i\Lambda|\, a^+_+ \tilde{a}^+_- \chi^{\Lambda, J_p+1, \tilde{J}_q+1}_{M, M_p+1, \tilde{M}_q-1} + \\ & + |J - \tilde{J} + 1 + i\Lambda|\, a^+_+ \tilde{a}^-_- \chi^{\Lambda, J_p+1, \tilde{J}_q-1}_{M, M_p+1, \tilde{M}_q-1} - \\ & - |J - \tilde{J} - 1 + i\Lambda|\, a^-_+ \tilde{a}^+_- \chi^{\Lambda, J_p-1, \tilde{J}_q+1}_{M, M_p+1, \tilde{M}_q-1} + \\ & + |J + \tilde{J} - 1 + i\Lambda|\, a^-_+ \tilde{a}^-_- \chi^{\Lambda, J_p-1, \tilde{J}_q-1}_{M, M_p+1, \tilde{M}_q-1} \end{aligned} \tag{A.3}$$

where

$$\begin{aligned} a^\pm_+ \equiv a^\pm_+ (J, M_+, M_-) &= \frac{\sqrt{(J \pm M_+)(J \pm M_- \pm 2)}}{2\sqrt{J(J \pm 1)}} \\ a^\pm_- (J, M_+, M_-) &= a^\pm_+ (J, -M_-, -M_+) \\ \tilde{a}^\pm_\pm &\equiv a^\pm_\pm (\tilde{J}, \tilde{M}_+, \tilde{M}_-) \end{aligned} \tag{A.4}$$

and

$$\begin{aligned} J = J_p + p - 1, \quad & M_\pm = M_p \pm (p - 1) \\ \tilde{J} = \tilde{J}_q + q - 1, \quad & \tilde{M}_\pm = \tilde{M}_q \pm (q - 1)\,. \end{aligned} \tag{A.5}$$

The action of the B_- operator can be obtained from (A.3) by performing the following changes:

$$\begin{aligned} \chi^{\Lambda, J'_p, \tilde{J}'_q}_{M, M_p+1, \tilde{M}_q-1} \to \chi^{\Lambda, J'_p, \tilde{J}'_q}_{M, M_p-1, \tilde{M}_q+1} \quad & \text{for} \quad J'_p = J_p \pm 1 \\ & \qquad\; \tilde{J}'_q = \tilde{J}_q \pm 1 \\ a^\pm_+ \to a^\pm_- \qquad & \\ \tilde{a}^\pm_- \to \tilde{a}^\pm_+ \,. \qquad & \end{aligned} \tag{A.6}$$

If $p > q = 1$, the formula (A.3) has to be replaced by

$$\begin{aligned} B_+ \chi^{\Lambda, J_p}_{M, M_p, \tilde{M}_1} = & \frac{\tilde{M}_1}{|\tilde{M}_1|} |J - \tilde{M}_1 + 1 + i\Lambda|\, a^+_+ \chi^{\Lambda, J_p+1}_{M, M_p+1, \tilde{M}_1-1} + \\ & + \frac{\tilde{M}_1}{|\tilde{M}_1|} |J + \tilde{M}_1 - 1 + i\Lambda|\, a^-_+ \chi^{\Lambda, J_p-1}_{M, M_p+1, \tilde{M}_1-1}\,, \end{aligned} \tag{A.7}$$

the action of B_- being obtained, again, by changing the symbols according to (A.6) and $\tilde{M}_1 \to -\tilde{M}_1$.

Finally, for $p = q = 1$ we obtain:

$$B_\pm \chi^{\Lambda}_{M, M_1, \tilde{M}_1} = \pm \frac{\tilde{M}_1}{|\tilde{M}_1|} |\pm (M_1 - \tilde{M}_1) + 1 + i\Lambda|\, \chi^{\Lambda}_{M, M_1 \pm 1, \tilde{M}_1 \mp 1}\,. \tag{A.8}$$

By a successive application of the operators $B^+_{p,p+q} \pm i B^-_{p,p+q}$, each admissible value of the numbers J_p, $\tilde{J}_q$, M_p and $\tilde{M}_q$ can be reached starting from any other admissible value. Moreover, we see from (A.3), (A.4) and (A.7) that the factors $J \mp M_\pm$, $\tilde{J} \mp \tilde{M}_\pm$ stop the raising or the lowering process at the points lying on the boundary of the corresponding diagram (see Figs. 1 and 2). For instance, $J + M_-$ vanishes at the point $M_p = -J_p$, in which J_p and M_p cannot be simultaneously lowered.

The irreducibility of the representation $C_M^\Lambda(X_-^{p,q})$ is proved by the same method. The corresponding formulae are obtained by exchanging the twiddled and "untwiddled" quantities.

In the same way, the proof of irreducibility of the $C_M^\Lambda(X_0^{p,q})$ representation is also performed. The formulae (A.1), (A.3), (A.7) and (A.8) are to be replaced, respectively, by

$$\begin{aligned} B_\pm \equiv B^+_{p,p+q} \pm i B^-_{p,p+q} = e^{\pm i(\varphi^p - \tilde{\varphi}^q)} \Big[\cos\vartheta^p \cos\tilde{\vartheta}^q \, r \frac{\partial}{\partial r} - \\ - \cos\vartheta^p \left(\sin\tilde{\vartheta}^q \frac{\partial}{\partial \tilde{\vartheta}^q} \pm \frac{i}{\cos\tilde{\vartheta}^q} \frac{\partial}{\partial \tilde{\varphi}^q} \right) - \cos\tilde{\vartheta}^q \left(\sin\vartheta^p \frac{\partial}{\partial \vartheta^p} \mp \frac{i}{\cos\vartheta^p} \frac{\partial}{\partial \varphi^p} \right) \Big] \end{aligned} \tag{A.1'}$$

$$\begin{aligned} B_+ \, {}_0\chi^{\Lambda, J_p, \tilde{J}_q}_{M, M_p, \tilde{M}_q} = & - (J + \tilde{J} + 1 - i\Lambda)\, a^+_+ \tilde{a}^+_- \, {}_0\chi^{\Lambda, J_p+1, \tilde{J}_q+1}_{M, M_p+1, \tilde{M}_q-1} - \\ & - (J - \tilde{J} + 1 - i\Lambda)\, a^+_+ \tilde{a}^-_- \, {}_0\chi^{\Lambda, J_p+1, \tilde{J}_q-1}_{M, M_p+1, \tilde{M}_q-1} + \\ & + (J - \tilde{J} - 1 + i\Lambda)\, a^-_+ \tilde{a}^+_- \, {}_0\chi^{\Lambda, J_p-1, \tilde{J}_q+1}_{M, M_p+1, \tilde{M}_q-1} + \\ & + (J + J - 1 + i\Lambda)\, a^-_+ \tilde{a}^-_- \, {}_0\chi^{\Lambda, J_p-1, \tilde{J}_q-1}_{M, M_p+1, \tilde{M}_q-1} \end{aligned} \tag{A.3'}$$

$$\begin{aligned} B_+ \, {}_0\chi^{\Lambda, J_p}_{M, M_p, \tilde{M}_1} = & - (J - \tilde{M}_1 + 1 - i\Lambda)\, a^+_+ \, {}_0\chi^{\Lambda, J_p+1}_{M, M_p+1, \tilde{M}_q-1} + \\ & + (J + \tilde{M}_1 - 1 + i\Lambda)\, a^-_+ \, {}_0\chi^{\Lambda, J_p-1}_{M, M_p+1, \tilde{M}_q-1} \end{aligned} \tag{A.7'}$$

$$B_\pm \, {}_0\chi^{\Lambda}_{M, M_1, \tilde{M}_1} = (\pm(\tilde{M}_1 - M_1) - 1 + i\Lambda)\, {}_0\chi^{\Lambda}_{M, M_1\pm 1, \tilde{M}_1 \mp 1} \, . \tag{A.8'}$$

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