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Current Commutation Relations in the Framework of General Quantum Field Theory

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Abstract. In this paper we give a rigorous formulation of Gell-Mann's equal time commutation relations in the framework of general quantum field theory. We show that this can be achieved despite the nonexistence of charge operators for non-conserved currents. Starting from the properly formulated equal time commutation relations of "generalized charges", we justify the application of the Gauss-Theorem and we discuss the limits for large times of time dependent "generalized charges". The Jost-Lehmann-Dyson representation is used in order to show that the equal time commutation relations always lead to exactly one, frame independent, sum rule. We discuss the connection between properties of the Jost-Lehmann-Dyson spectral function and the convergence of Adler-Weisberger type sum rules.

1. Introductory remarks

The most convincing success of the equal time commutation relations between vector and axial vector currents originally proposed by GELL-MANN [1] is the derivation of sum rules of the Adler-Weisberger type [2], [3]. In the original presentation of ADLER and WEISBERGER this derivation was very involved and the independence on the frame of reference was not shown. FUBINI and FURLAN [4] proposed subsequently a simpler and aesthetically more appealing covariant method based on the commutation relations of "generalized charges"; however, the weak point of their derivation was the discussion of the boundary terms for large times and the ambiguity due to the bad large distance behaviour of the retarded matrix-element. Furthermore, it was not realized, that retardation i.e. the multiplication with the step function $\Theta(x_0)$ does not introduce any ambiguous subtraction constant. This follows from a more careful formulation of the commutation relations which takes into account the distribution theoretical aspects. The use of Gauss's theorem for field operators and a careful computation of time limits will resolve the ambiguities for low energies (i.e. the intermediate one particle ambiguity).

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By starting from the stronger and more questionable assumptions of equal time commutation relations for current densities and unsubtracted dispersion relations for different amplitudes, FUBINI by-passed this low energy ambiguity in a recent paper [5].

As regards the problems of high energy behaviour and covariance of sum rules, we show that for a certain class of Jost-Lehmann-Dyson spectral functions $\psi(u, s)$ the covariant sum rules are identical with the Adler-Weisberger type sum rules. We discuss preliminary results on the general connection between the behaviour of $\psi(u, s)$ for large s and the form of the sum rule.

We will discuss the mentioned statements in the framework of general quantum field theory[1]. However, the mathematical rigour of our presentation is modest and more on the level of the LSZ formulation than present day axiomatic field theory. Some of the results of the next chapter, especially the first two statements are implicit in the work of KASTLER, ROBINSON and SWIECA [10]. We avoid however the algebraic framework used by these authors.

2. Definition of charges

The first problem we investigate is the question in what sense a "charge" operator Q can be connected with a conserved current $j_\mu(x)$:

$$\partial^\mu j_\mu(x) = 0 \, . \tag{1}$$

We observe first, that irrespective of the conservation law (1) the matrix element

$$\langle \Phi | \, j_\mu(\mathbf{x}, t) \, | \Psi \rangle \tag{2}$$

is a smooth and fast decreasing function in $\mathbf{x}$, whenever $|\Psi\rangle$ and $|\Phi\rangle$ are quasilocal states, i.e. states of the form

$$|\Phi\rangle = \sum_{m=1}^{n} \int g_m(x_1, \ldots, x_m) \, A_1(x_1), \ldots, A_m(x_m) \, |0\rangle \tag{3}$$

where the A's are from the basic set of local fields (resp. currents) in terms of which the theory is defined, and g_m are fast decreasing smooth functions. Here we assumed $\langle j_\mu(x)\rangle_0 = 0$ and the restricted spectrum condition, i.e. the non-occurrence of zero rest-mass states.

The smoothness property of (2) comes (due to translational invariance of the vacuum expectation values) directly from the smoothness of the g's, whereas the fall-off property for large $\mathbf{x}$ uses in addition locality and is a special case of the so-called linked cluster property [7], [8], [9]. Hence the spatial integral

$$\int d^3x \langle \Phi | \, j_\mu(\mathbf{x}, t) \, | \Psi \rangle \tag{4}$$

always exists and defines a bilinear form.

[1] For a short exposition of general quantum field theory see [6].

In the conserved case we expect, however, to be able to define an operator Q called "charge", and we try for the connection with the charge density the following formula

$$Q = \lim_{R\to\infty} j_0(f_R, f_T) \tag{5}$$

i.e. we ask in what sense the sequence of unbounded operators $j_0(f_R, f_T)$ has an operator limit.

In choosing our space and time smearing functions f_R, f_T we followed the suggestion of KASTLER, ROBINSON and SWIECA [10]:

$$f_T \geqq 0, \quad \operatorname{supp} f_T \subset [-T, T], \quad \int dt\, f_T(t) = 1 \tag{6a}$$

$$f_R(\mathbf{x}) = f_R(|\mathbf{x}|) = \begin{cases} 1 & |\mathbf{x}| < R \\ 0 & |\mathbf{x}| > R + L\,. \end{cases} \tag{6b}$$

Hopefully as we expect from analogy to the classical case, the limit (5) turns out to be independent of T, so that $T \to 0$ is superfluous.

We first want to show that (5) cannot exist in the sense of strong convergence:

Statement I.

$$\langle 0|\, j_0(f_R, f_T)\, j_0(f_R, f_T)\, |0\rangle \xrightarrow[R\to\infty]{} C R^2$$

with $C \neq 0$ *unless* $j_\mu(x) \equiv 0$. ‖

Proof. For a conserved current we have the following KÄLLÈN-LEHMANN [11], [12] representation:

$$\langle j_\mu(x)\, j_\nu(y)\rangle_0 = -i \int d\varkappa^2 \left(g_{\mu\nu} + \frac{\partial_\mu \partial_\nu}{\varkappa^2}\right) \Delta_\varkappa^{(+)}(x-y)\, \varrho(\varkappa^2) \tag{7}$$

hence

$$\langle j_0(f_R, f_T)\, j_0(f_R, f_T)\rangle_0 = \int d^3p\, |p \tilde{f}_R(\mathbf{p})|^2\, g(p) \tag{8}$$

with

$$g(p) = \frac{1}{2\pi} \int d\varkappa^2 \frac{\varrho(\varkappa^2)}{2\varkappa^2 \sqrt{p^2+\varkappa^2}}\, |\tilde{f}_T(\sqrt{p^2+\varkappa^2})|^2\,.$$

We have

$$\begin{aligned} p\tilde{f}_R(p) &= 4\pi \int dr\, r \left(\frac{d}{dr} f_R(r)\right) \frac{1}{p} \left[\cos pr - \frac{\sin pr}{pr}\right] \\ &= \frac{4\pi}{p} \int r f'(r-R) \left[\cos pr - \frac{\sin pr}{pr}\right] dr \end{aligned} \tag{9}$$

where $f'(r)$ is the derivative of (6b) for $R = 0$.

By change of variable:

$$p\tilde{f}_R(p) = \frac{4\pi}{p} \int f'(\varrho) \left[\cos p(\varrho + R) - \frac{\sin p(\varrho+R)}{p(\varrho+R)}\right] (\varrho + R)\, d\varrho\,. \tag{10}$$

Inserting (10) into (8), using the addition theorem for sin and cos and taking only the leading term in R we obtain:

$$\langle j_0(f_R, f_T)\, j_0(f_R, f_T)\rangle_0 = 32\pi^2 R^2 \int dp\, g(p)\, (|\tilde{f}_1'(p)|^2 + |\tilde{f}_2'(p)|^2) \tag{11}$$

with

$$\tilde{f}'_{\binom{1}{2}}(p) = \int f'(\varrho) \left(\frac{\sin p\varrho}{\cos p\varrho}\right) d\varrho\,. \tag{12}$$

The coefficient of the leading term vanishes if and only if $\varrho(\varkappa^2) \equiv 0$. According to a well known theorem [13], [14][2] this is equivalent to $j_\mu(x) \equiv 0$.

It is easy to see that with our choice of an infinitely smooth test function in time, the leading term is approached faster than any inverse power in R.

Next we want to show that the limit (5) exists in the weak sense on a dense set of states. First we show for this purpose that the vacuum is annihilated weakly.

Statement II.

$$\lim_{R\to\infty} \langle\Phi|\, j_0(f_R, f_T)\, |0\rangle = 0 \tag{13}$$

for states Φ of the form $|\Phi\rangle = \int d^3x\, h(\mathrm{x})\, U(\mathrm{x})\, B\, |0\rangle$ where B is quasilocal, i.e. of the form (3). *$U(\mathrm{x})$ is the translation operator and $h(\mathrm{x})$ is a smooth function which decreases for large r such that*

$$\lim_{r\to\infty} r^2 h(\mathrm{x}) = 0\,. \quad \rfloor\!| \tag{14a}$$

This statement is the transcription of a Lemma by KASTLER, ROBINSON and SWIECA [10] from their algebraic framework to the field theoretical framework.

Proof. As in the paper of KASTLER, ROBINSON and SWIECA, we "divide" $|\Phi\rangle$ by the energy operator. Here we use the fact that if $B|0\rangle$ is a quasilocal state of the form (3) with $\langle 0|\, B\, |0\rangle = 0$ then $|\Psi\rangle = \frac{1}{H} B|0\rangle$ is again quasilocal. This is so since the smearing function $\tilde{g}_m(p_1, \ldots, p_m)$ in (3), which according to the finite restmass spectrum condition can be chosen such that

$$\tilde{g}_m(p_1, \ldots, p_m) = 0 \quad \text{for} \quad \sum_{i=1}^{m} p_{i0} < \frac{M}{2} \quad (M = \text{smallest rest-mass})\,,$$

allow division by Σp_{i0} and yield again smooth and fast decreasing test functions, and hence $|\Psi\rangle$ is again quasilocal. Therefore we have:

$$\begin{aligned} &\langle\Phi|\, \frac{i}{H}\, [H, j_0(f_R, f_T)]\, |0\rangle \\ &\qquad = \int d^3x\, h(\mathrm{x})\, \langle\Psi|\, U^\dagger(\mathrm{x})\, j_r(f'_R, f_T)\, |0\rangle \end{aligned} \tag{14b}$$

where j_r = component along the radius vector x.

Now we consider the left hand state as the sum of two states

$$\int d^3x\, h(\mathrm{x})\, U(\mathrm{x})\, |\Psi\rangle = \int_0^{R/2} d^3x\, h(\mathrm{x})\, U(\mathrm{x})\, |\Psi\rangle + \int_{R/2}^{\infty} d^3x\, h(\mathrm{x})\, U(\mathrm{x})\, |\Psi\rangle\,. \tag{15}$$

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[2] This theorem, which is nowadays called the Federbush-Johnson Theorem, was rediscovered by P. G. FEDERBUSH, K. A. JOHNSON [15].

The first state is effectively localized in the sphere with radius $R/2$ and the second one behaves in norm as

$$\begin{aligned}&\left\|\int_{R/2}^{\infty} h(\mathrm{x})\, U(\mathrm{x})\, |\Psi\rangle\, d^3x\right\| \\ &= \left\{\int_{R/2}^{\infty} d^3x \int_{R/2}^{\infty} d^3y\, h(\mathrm{x})\, h(\mathrm{y})\, \langle\Psi|\, U(\mathrm{y}-\mathrm{x})\, |\Psi\rangle\right\}^{1/2} \sim 0(R^{-1/2})\,.\end{aligned} \tag{16}$$

This estimate holds because of RUELLE's result [8]

$$\lim_{(\mathrm{x}-\mathrm{y})^2\to\infty} (\mathrm{x}-\mathrm{y})^{2N}\, \langle\Psi|\, U(\mathrm{y}-\mathrm{x})\, |\Psi\rangle = 0$$

for all $N > 0$, and the assumed fall-off properties of $h(\mathrm{x})$.

The contribution to (14b) from the effectively localized first state is

$$\int_0^{R+L} d^3x \int d^3y\, h(\mathrm{x})\, f_R'(\mathrm{y})\, \langle\Psi|\, U(\mathrm{y}-\mathrm{x})\, j_r(0, f_T)\, |0\rangle \tag{17}$$

and hence because the matrix element vanishes again faster than any inverse power of $(\mathrm{x}-\mathrm{y})^2$, the integration (17) leads to a function of R which vanishes rapidly for $R \to \infty$. For the second state in (15) we use the Schwarz inequality and obtain

$$\begin{aligned}&\left|\int_{R/2}^{\infty} d^3x\, h(\mathrm{x})\, \langle\Psi|\, U^\dagger(\mathrm{x})\, j_r(f_R', f_T)\, |0\rangle\right| \leqq \\ &\qquad \leqq \left\|\int_{R/2}^{\infty} d^3x\, h(\mathrm{x})\, U(\mathrm{x})\, |\Psi\rangle\right\| \, \|j_r(f_R', f_T)\, |0\rangle\| \,.\end{aligned} \tag{18}$$

If we use for f_R space-smearing functions of type (6b) with $L =$ constant, we obtain for the second norm

$$\lim_{R\to\infty} \|j_r(f_R', f_T)\, |0\rangle\| < CR\,.$$

However, by using instead of (6b) a sequence of "stretched" functions

$$f_R(\mathrm{x}) = f\left(\frac{r}{R}\right)$$

where $f(r)$ is a smooth function which is one inside a certain fixed radius and vanishes outside a larger radius, we obtain for the derivative

$$\frac{d}{dr} f\left(\frac{r}{R}\right) \leqq \frac{1}{R} \max f'\,,$$

and hence for the norm

$$\lim_{R\to\infty} \int d^3x \int d^3y\, f_R'(\mathrm{x})\, f_R'(\mathrm{y})\, \langle 0|\, j_r(\mathrm{x}, f_T)\, j_r(\mathrm{y}, f_T)\, |0\rangle < C R^{1/2}\,. \tag{19}$$

Together with (16) we obtain a vanishing right hand side in (18) for $R \to \infty$.

We would like to mention that our estimates are optimal in any conserved current theory. This can easily be seen by taking a state $|\Phi\rangle = \int d^3x\, h(\mathrm{x})\, j_0(\mathrm{x}, f_T)\, |0\rangle$ with $\lim_{r\to\infty} \mathrm{x}^2\, h(\mathrm{x}) \neq 0$. Such a state is still normalizable; however, a consideration which is similar to the statement I shows that $L(\Phi) = \lim_{R\to\infty} \langle\Phi| j_0(f_R, f_T) |0\rangle$ vanishes if and only if $j_\mu \equiv 0$.

Hence we have learned that in any theory the formula

$$Q = w\text{-}\lim_{R\to\infty} j_0(f_R, f_T) \tag{20}$$

breaks down if one of the states in which the weak limit is taken has a "long range". The linear form $L(\Phi)$ vanishes (and therefore is bounded) on the dense set of quasilocal states fulfilling eq. (14a), and hence the operator Q (if it exists) must annihilate the vacuum.

If the connection between the $j_0(f_R, f_T)$ and a charge operator (20) makes any sense, both operators should have a dense domain, which are independent of R. The "natural" domain of $j_0(f_R, f_T)$ are the quasilocal states and hence one would expect that Q has to have the vacuum in its domain. But then we can show that a nonconserved current cannot give rise to an operator Q. This was first conjectured and made plausible by COLEMAN [16] (see also [17]).

Statement III (COLEMAN). *For a nonconserved current* $\partial^\mu j_\mu(x) \equiv A(x) \neq 0$ *the linear form*

$$L(\Phi) = \lim_{R\to\infty} \langle\Phi| j_0(f_R, f_T) |0\rangle \tag{21}$$

is unbounded in $|\Phi\rangle$. $\rfloor\!\!|$

Here $|\Phi\rangle$ runs through the same set of quasilocal states as in the previous statement. It should be stressed that $L(\Phi)$ is unbounded in both the conserved and nonconserved case. Only on the subset of quasilocal states do we have boundedness (since it vanishes) in the conserved case.

Proof. Again dividing the state $|\Phi\rangle$ by the energy operator one obtains:

$$iL(\Phi) = \lim_{R\to\infty} \langle\Psi| j_r(f'_R, f_T) |0\rangle + \lim_{R\to\infty} \langle\Psi| A(f_R, f_T) |0\rangle \tag{22}$$

with

$$|\Psi\rangle = \frac{1}{H} |\Phi\rangle .$$

The first term vanishes according to the previous consideration. We want to show that the second term is unbounded in $|\Psi\rangle$. For this purpose we choose a sequence of quasilocal states as

$$|\Psi_\varrho\rangle = \|A(f_\varrho, f_T) |0\rangle\|^{-1} A(f_\varrho, f_T) |0\rangle . \tag{23}$$

The norm behaves for large ϱ like

$$\|A(f_\varrho, f_T)\,|0\rangle\| = \{\int d^3x \int d^3y\, f_\varrho(\mathrm{x})\, f_\varrho(\mathrm{y})\, \langle A(\mathrm{x}, f_T)\, A(\mathrm{y}, f_T)\rangle_0\}^{1/2}$$

$$= \left\{\frac{1}{2\pi} \int d^3p\, |\tilde{f}_\varrho(p)|^2 \int d\varkappa^2 \frac{\sigma(\varkappa^2)}{2\sqrt{p^2+\varkappa^2}}\, |\tilde{f}_T(\sqrt{p^2+\varkappa^2})|^2\right\}^{1/2} \tag{24}$$

$$\xrightarrow[\varrho\to\infty]{} C\varrho^{3/2}$$

with $C = 0 \Leftrightarrow A(x) = 0$.

Therefore

$$\lim_{R\to\infty} \langle\Psi_\varrho|\, A(f_R, f_T)\,|0\rangle \xrightarrow[\varrho\to\infty]{} \varrho^{-3/2} \lim_{R\to\infty} \int d^3p\, \tilde{f}_\varrho^*(p)\, \tilde{f}_R(p)\, \times$$

$$\times \int d\varkappa^2 \frac{\sigma(\varkappa^2)}{2\sqrt{p^2+\varkappa^2}}\, |\tilde{f}_T(\sqrt{p^2+\varkappa^2})|^2 = \varrho^{-3/2} \int d^3x\, f_\varrho(\mathrm{x}) \int d\varkappa^2 \frac{\sigma(\varkappa^2)}{2\varkappa}\, |\tilde{f}_T(\varkappa)|^2$$

$$\sim \varrho^{3/2}$$

which can be made arbitrarily large by choice of $|\Psi_\varrho\rangle$. Since $|\Phi_\varrho\rangle = H|\Psi_\varrho\rangle$ and hence

$$\||\Phi_\varrho\rangle\|^2 \xrightarrow[p\to\infty]{} \frac{1}{C^2\varrho^3}\,\varrho^3 \int d\varkappa^2\, \sigma(\varkappa^2)\, \varkappa\, |\tilde{f}_T(\varkappa)|^2 < \infty\,,$$

$L(\Phi)$ is unbounded in $|\Phi\rangle$.

Let us now come back to the formula (20) in the conserved case. We consider the action of $j_0(f_R, f_T)$ on the dense set of states $B|0\rangle$ as in formula (3), but with compact support test functions $g_m(x, \ldots x_m)$:

$$j_0(f_R, f_T)\, B|0\rangle = [j_0(f_R, f_T), B]\,|0\rangle + B j_0(f_R, f_T)\,|0\rangle\,. \tag{25}$$

According to locality the first term is independent of R for large R and again has the form (3) with compact support test functions. The last term converges weakly to zero as $R \to \infty$. Hence the formula (20) defines an operator Q which has in its domain all states (3) with compact support test functions and furthermore the operator Q can be applied repeatedly on this domain. It is just slightly more complicated to see that also quasilocal states; i.e. states with concompact (but decreasing) test function and multiparticle in- (out) states with non-overlapping wave functions belong to the domain of Q.

Finally it is worthwhile to mention that all our considerations go through if the current has other tensorial indices in addition to the index in which the conservative law holds i.e. for currents

$$j_{\mu,\mu_1,\ldots\mu_n}(x)\,.$$

In this case the decomposition of the two point function into standard covariants is more involved; however, due to the requirement that all relations hold for arbitrary $\mu_1, \ldots \mu_n$ we obtain the analog result.

3. Formulation of equal time commutation relations

The main point of this section is to show that despite the impossibility of defining a charge operator in the nonconserved case, the commutation relations of "generalized charges" can be given meaning since in the commutator the infinities of the norm cancel each other. Let $f_T(t)$ be any symmetric time smearing function with compact support in $[-T, T]$.

Statement IV.

$$\langle \Phi | \, [j_0^i(f_R, f_T), j_0^K(f_R, f_T)] \, | \Phi \rangle = 0 \tag{26}$$

between TCP invariant states.

Proof. If Θ is the TCP operator we have

$$\begin{aligned}\langle \Phi | \, \Theta \Theta j_0^i(f_R, f_N) \, j_0^K(f_R, f_T) \Theta \Theta \, | \Phi \rangle &= \langle \Phi | \, j_0^i(f_R, f_T) \, j_0^K(f_R, f_T) \, | \Phi \rangle^* \\ &= \langle \Phi | \, j_0^K(f_R, f_T) \, j_0^i(f_R, f_T) \, | \Phi \rangle\end{aligned}$$

because of the choice of symmetric test functions f_R and f_T. Hence we have (26). Here the index i, k designates any vector or axial vector current. If the state $|\Phi\rangle$ is the vacuum we can omit the smearing in space due to the fact that $|0\rangle$ is rotational invariant.

In the literature one finds very often the statement that the vacuum expectation value of the equal time commutator vanishes. This is wrong because there is no equal time meaning to this quantity. Even for free field currents the two-point function, although perfectly well defined as a Wightmann distribution, can hardly be given meaning for equal times. However, our symmetric time smearing process takes care of this problem, i.e. it truncates the matrix element automatically.

In order to avoid a lengthy discussion due to generalities we take as a model the axial vector commutation relations of Adler and Weisberger. The currents $j_\mu(x)$ lead after smearing in time to one particle truncated expectation values

$$\langle \Psi | \, j_0^{(+)5}(\mathrm{x}, f_T) \, j_0^{(-)5}(\mathrm{y}, f_T) \, | \Phi \rangle - \langle \Psi | \Phi \rangle \, \langle j_0^{(+)5}(\mathrm{x}, f_T) \, j_0^{(-)5}(\mathrm{y}, f_T) \rangle_0 \tag{27}$$

which are infinitely smooth functions in x and y and decrease in these variables faster than any inverse power. This statement is a direct consequence of Ruelle's results' [8] since the one particle (wave packet) states are quasilocal. Hence the integration with $f_R(\mathrm{x}) \, f_R(\mathrm{y})$ and the limit $R \to \infty$ causes no difficulties[3].

In our nonconserved current case the result will, however, depend on the time smearing function $f_T(t)$. Let f_T be a sequence of symmetric compact support test functions which for $T \to 0$ approaches the δ-function. The statement of equal time commutation relation in the case of

[3] Here and in the following a smearing in space is superfluous since the spatial integrals converge in the ordinary sense. We will, however, keep the f_R's because they serve as a convenient reminder that the spatial integration in general cannot be interchanged with other limits.

our special expectation value is now the assertion that

$$\lim_{T\to 0}\lim_{R\to\infty}\langle\Psi|\,[j_0^{(+)5}(f_R, f_T), j_0^{(-)5}(f_R, f_T)]\,|\Phi\rangle = 2\langle\Psi|\,I_3\,|\Phi\rangle \tag{28}$$

where I_3 is the 3rd component of the isospin operator. Such an assertion does not run into any obvious difficulties with the principles of quantum field theory. However, an explicit perturbation-theoretical check in some renormalizable models would certainly add a lot to the credibility of relation (28). We will discuss this problem in a future paper.

Symbolically we could write

$$\lim_{T\to 0}\lim_{R\to\infty}[j_0^{(+)5}(f_R, f_T), j_0^{(-)5}(f_R, f_T)] = 2I_3 \tag{29a}$$

if we only consider the left hand side between states which lead to fall off properties in x and y and hence to the existence of $\lim R\to\infty$. All so-called quasilocal states certainly belong to the set of admissable states, but a more detailed investigation shows that (29a) can also be taken between multiparticle in- (or out) states with nonoverlapping wave packets.

In this section we have studied the commutation relation between space integrals. Often one also formulates commutation relations between densities, for example

$$\lim_{T\to 0}[j_0^{(+)5}(\mathrm{x}, f_T), j_0^{(-)5}(\mathrm{y}, f_T)] = 2j_0^{(3)}(\mathrm{x}, 0)\,\delta(\mathrm{x}-\mathrm{y}) \tag{29b}$$

or equivalently

$$[j_0^{(+)5}(\mathrm{x}, t), j_0^{(-)5}(\mathrm{y}, t)]_{tr} = 2j_0^{(3)}(\mathrm{x}, t)\,\delta(\mathrm{x}-\mathrm{y})\,.$$

The validity of such commutation relations is doubtful, even if the current is conserved. The derivation of sum rules is, however, much less complicated for that case.

4. Derivation of sum rules

Consider now (omitting the index 5)

$$j_0^{(+)}(f_R, f_T) = -\int_0^t \frac{d}{d\tau} j_0^{(+)}(f_R, f_T^\tau)\,d\tau + j_0^{(+)}(f_R, f_T^t) \tag{30a}$$

and

$$j_0^{(-)}(f_R, f_T) = \int_{-t}^0 \frac{d}{d\tau} j_0^{(-)}(f_R, f_T^\tau)\,d\tau + j_0^{(-)}(f_R, f_T^{-t}) \tag{30b}$$

where

$$U(\tau)\,j_0^{(\pm)}(f_R, f_T)\,U^\dagger(\tau) = j_0^{(\pm)}(f_R, f_T^\tau);\quad f_T^\tau(t) = f_T(t-\tau)\,.$$

Lemma 1.

$$\int_0^t\int_{-t}^0 d\tau\, d\tau' \lim_{R\to\infty} \langle\Psi| \left\{ \left[\frac{d}{d\tau} j_0^{(+)}(f_R, f_T^\tau), \frac{d}{d\tau'} j_0^{(-)}(f_R, f_T^{\tau'})\right] - \right. \\ \left. - \left[D^{(+)}(f_R, f_T^\tau), D^{(-)}(f_R, f_T^{\tau'})\right]\right\} |\Phi\rangle = 0 \tag{31a}$$

$$\int_{-t}^0 d\tau \lim_{R\to\infty} \langle\Psi| \left[j_0^{(+)}(f_R, f_T^t), \frac{d}{d\tau} j_0^{(-)}(f_R, f_T^\tau) - D^{(-)}(f_R, f_T^\tau)\right] |\Phi\rangle - \\ - \int_0^t d\tau \lim_{R\to\infty} \langle\Psi| \left[\frac{d}{d\tau} j_0^{(+)}(f_R, f_T^\tau) - D^{(+)}(f_R, f_T^\tau), j_0^{(-)}(f_R, f_T^{-t})\right] |\Phi\rangle \\ = 0\,. \tag{31b}$$

Here $|\Phi\rangle$ and $|\Psi\rangle$ are quasilocal states and

$$D^{(\pm)}(x) = \partial^\mu j_\mu^{(\pm)}(x)\,.$$

Proof. Since

$$\frac{d}{d\tau} j_0^{(\pm)}(f_R, f_T^\tau) = j_r^{(\pm)}(f_R', f_T^\tau) + D^{(\pm)}(f_R, f_T^\tau)$$

in order to prove (31 a) we have to show that:

$$\langle\Psi| [j_r^{(+)}(f_R', f_T^\tau), D^{(-)}(f_R, f_T^{\tau'})] |\Phi\rangle_{tr} + \\ + \langle\Psi| [D^{(+)}(f_R, f_T^\tau), j_r^{(-)}(f_R', f_T^{\tau'})] |\Phi\rangle_{tr} \\ + \langle\Psi| [j_r^{(+)}(f_R', f_T^\tau), j_r^{(-)}(f_R', f_T^{\tau'})] |\Phi\rangle_{tr} \xrightarrow[R\to\infty]{} 0 \tag{32}$$

where tr (truncation) indicates subtraction of the vacuum expectation values[4].

We prove that every single term in (32) goes to zero. Consider for example the first term explicitly

$$\int_R^{R+L} d^3x \int_0^{R+L} d^3y\, f_R'(\mathrm{x})\, f_R(\mathrm{y})\, \langle\Psi| [j_r^{(+)}(\mathrm{x}, f_T^\tau), D^{(-)}(\mathrm{y}, f_T^{\tau'})] |\Phi\rangle_{tr}\,.$$

According to Ruelle [8] the truncated matrix element

$$\langle\Psi| j_r^{(+)}(\mathrm{x}, f_T^\tau)\, D^{(-)}(\mathrm{y}, f_T^{\tau'}) |\Phi\rangle_{tr} \in \mathfrak{S}_{\mathrm{x},\mathrm{y}} \tag{33}$$

i.e. is a smooth function which decreases rapidly in x and y. Hence after integration with $f_R(\mathrm{y})$ the remaining expression decreases rapidly in x and hence the integral over the ring $R \leqq |\mathrm{x}| \leqq R + L$ gives a decreasing function in R. Therefore the first commutator decreases rapidly as $R \to \infty$. The argument for the decrease of the other terms as well as for (31 b) is the same.

[4] If the integration over the τ's is performed as in the Lemma the vacuum expectation value of the commutators vanishes; hence the truncation would be superfluous. However, working with the integrands only, the truncation is necessary for the existence of the limit $R \to \infty$.

Specializing $|\Phi\rangle = |\Psi\rangle = \int \frac{d^3p}{2p_0}\,\psi(p)\,|p\rangle$ to a one particle state (proton state) with a smooth decreasing wave packet (such states are quasilocal [7], [8], [9]) and using lemma 1 we obtain:

$$\begin{aligned}
\lim_{R\to\infty} \langle\Psi|\,[j_0^{(+)}(f_R, f_T), j_0^{(-)}(f_R, f_T)]\,|\Psi\rangle \\
&= -\int_0^t d\tau \int_{-t}^0 d\tau' \lim_{R\to\infty} \langle\Psi|\,[D^{(+)}(f_R, f_T^{\tau}), D^{(-)}(f_R, f_T^{\tau'})]\,|\Psi\rangle + \\
&\quad + \int_{-t}^0 d\tau \lim_{R\to\infty} \langle\Psi|\,[j_0^{(+)}(f_R, f_T^{t}), D^{(-)}(f_R, f_T^{\tau})]\,|\Psi\rangle - \\
&\quad - \int_0^t d\tau \lim_{R\to\infty} \langle\Psi|\,[D^{(+)}(f_R, f_T^{\tau}), j_0^{(-)}(f_R, f_T^{-t})]\,|\Psi\rangle + \\
&\quad + \lim_{R\to\infty} \langle\Psi|\,[j_0^{(+)}(f_R, f_T^{t}), j_0^{(-)}(f_R, f_T^{-t})]\,|\Psi\rangle\,.
\end{aligned} \tag{34}$$

We now want to show that the contribution for large t of the 2nd and 3rd term vanishes.

For this purpose, we use translational invariance and obtain for the second term:

$$\int_{-2t}^{-t} d\tau \lim_{R\to\infty} \langle\Psi|\,[j_0^{(+)}(f_R, f_T^{0}), D^{(-)}(f_R, f_T^{\tau})]\,|\Psi\rangle\,. \tag{35}$$

For $D^{(-)}(x)$ one has the formal development into incoming fields:

$$\begin{aligned}
D^{(-)}(x) = \int \frac{d^3k}{2k_0} \{\langle 0|\,D^{(-)}(x)\,|k\pi^+\rangle\, A^{\text{in}}_{\pi^+}(k) + \\
+ \langle\pi^- k|\,D^{(-)}(x)\,|0\rangle\, A^{\text{in}\,\dagger}_{\pi^-}(k)\} + \\
+ \sum_{s_3, s_3} \int \frac{d^3k}{2k_0} \int \frac{d^3k'}{2k'_0} \langle k', s'_3; n|\,D^{(-)}(x)\,|k, s_3; p\rangle\, \Psi^{\text{in}\,\dagger}_n(k', s'_3)\, \Psi^{\text{in}}_p(k, s_3) + R.
\end{aligned} \tag{36a}$$

The first term in this formal decomposition is clearly similar to the LSZ term. The only modification is due to the fact that the matrixelement $\langle 0|\,D^{(-)}(x)\,|k\pi^+\rangle$ is different from the wave function of the pion by the π-decay constant. The bilinear term in the neutron creation and proton annihilation operator is the next term in this series. Terms with two particle creation- resp. annihilation operators and all the higher multilinear terms have not been written down in (36a) because of the following theorem due to H. ARAKI and R. HAAG[5].

Theorem. *Let $|\Psi\rangle$ and $|\Phi\rangle$ be in- resp. out-states with nonoverlapping wave packets. Then formula* (36a) *is correct for the matrixelements $\langle\Phi|\,D^{(-)}(x)\,|\Psi\rangle$. The matrixelement of the rest $\langle\Phi|\,R\,|\Psi\rangle$ decreases to zero faster than any inverse power of t for $t \to -\infty$ uniformly in* $\mathbf{x}$. ‖

[5] "Collision cross sections in Terms of Local Observables." Unpublished manuscript. We thank Professor HAAG for communicating to us some results contained in this manuscript.

The LSZ term evidently decreases like $t^{-3/2}$ and gives after wave packet integration the time independent creation resp. annihilation operator of an incoming particle. The second term goes like t^{-3} for $t \to -\infty$ and hence vanishes after wave packet integration (as well as the rest). However, integrating (36a) over x space leads to a result in which only the second term survives[6].

$$\int d^3x\, D^{(-)}(\mathbf{x}, t) \xrightarrow[t\to-\infty]{} (2\pi)^3 \sum_{s_3, s_3'} \int \frac{d^3k}{(2k_0)^2} \times \\ \times \langle k, s_3'; n|\, D^{(-)}(0)\, |k, s_3; p\rangle\, \Psi_n^{\mathrm{in}\dagger}(k, s_3')\, \Psi_p^{\mathrm{in}}(k, s_3)\,. \tag{36b}$$

The validity of this formula taken between states $|\Phi\rangle$ and $|\Psi\rangle$ as defined in the theorem is a immediate consequence of the theorem. However, in formula (35) only one state $|\Psi\rangle$ has the form required in the theorem:

$$\langle\Psi|\, j_0^{(+)}(f_R, f_T) \int d^3x\, D^{(-)}(x)\, |\Psi\rangle = \langle X| \int d^3x\, D^{(-)}(x)\, |\Psi\rangle$$

whereas the other state $|X\rangle$ has no simple interpretation in terms of incoming or outgoing states. Fortunately, this does not matter because of the following statement also due to ARAKI and HAAG[5].

Statement. For the matrixelement of space integrals of local (or quasilocal) operators taken between states of which at least one consists of incoming or outgoing nonoverlapping wave packets, only the bilinear "density" term remains in the limit $t \to \mp\infty$ and this term is approached faster than any inverse power in t. Applying this statement to the two operators D and j_0 and taking into account that $\langle k|\, D\, |k\rangle = 0$ we obtain:

$$\langle X| \int d^3x \int D^{(-)}(\mathbf{x}, f_T^t)\, |\Psi\rangle \xrightarrow[t\to\pm\infty]{} 0 + R \tag{37a}$$

$$\langle X| \int d^3x\, j_0^{(-)}(\mathbf{x}, f_T^t)\, |\Psi\rangle \xrightarrow[t\to\pm\infty]{} (2\pi)^3 \times \\ \times \sum_{s_3, s_3'} \int \frac{d^3k}{(2k_0)^2} \langle k, s_3', n|\, j_0^{(-)}(0, f_T^0)\, |k, s_3; p\rangle \times \\ \times \langle X|\, \psi_n^{\overset{\mathrm{out}\dagger}{\mathrm{in}}}(k, s_3')\, \psi_p^{\overset{\mathrm{out}}{\mathrm{in}}}(k, s_3)\, |\Psi\rangle + R \tag{37b}$$

where R decreases faster than any inverse power of t. The first relation yields immediately:

$$\lim_{t\to+\infty} \int_{-2t}^{-t} \langle X| \int d^3x\, D^{(-)}(\mathbf{x}, f_T^\tau)\, |\Psi\rangle\, d\tau = 0$$

and hence leads to the vanishing of the second term in (34). In an analogous fashion one shows that the third term in (34) vanishes in the limit.

---

[6] For our special D field this contribution vanishes since $\langle k|\, D^{(\pm)} |k\rangle = 0$ but this would not be the case if we take $j_0^{(\pm)}(x)$ instead of $D^{(\pm)}(x)$.

With the help of (37b) the fourth term in (34) yields:

$$(2\pi)^6 \sum_{s_3} \int \frac{d^3 k}{(2k_0)^2} |\psi(k)|^2 \langle k; p | j_0^{(+)}(0, f_T) | k, s_3; n \rangle \times \\ \times \langle k, s_3; n | j_0^{(-)}(0, f_T) | k; p \rangle \tag{38}$$

where the summation goes over the two spin states of the neutron. Since $\int f_T(t)\, dt = 1$, the f_T is dropping out and we obtain:

$$G_A^2 \int \frac{d^3 k}{2k_0} |\psi(k)|^2 \left(1 - \left(\frac{M}{k_0}\right)^2\right) \tag{39}$$

where we used the definition of the axialvector renormalization constant:

$$\langle k; p | j_0^{(+)}(0) | k; n \rangle = \frac{2MG_A}{(2\pi)^3} \bar{u}_p(k)\, \gamma_0 \gamma_5 u_n(k)\,. \tag{40}$$

Now we want to discuss the remaining first term on the r.h.s. in eq. (34). After integration over one τ and evaluation of $\lim R \to \infty$ one obtains

$$-(2\pi)^3 \int dx_0\, \Theta(x_0)\, x_0 \int d^3 x \times \\ \times \langle p | [D^{(+)}(\mathrm{x}/2, f_T^{x_0/2}), D^{(-)}(-\mathrm{x}/2, f_T^{-x_0/2})] | p \rangle \tag{41}$$

where we have omitted the wave-packet integration.

From the form of eq. (41), one would expect, after taking the limit $T \to 0$ that eq. (41) transforms with respect to Lorentz transformations like the zero component of a four vector, i.e. is proportional to p_0. Suppose the retarded commutator appearing in (41) is Lorentz-invariant[7] and the integration over the whole Minkowski space is independent of the order of integrations, then this proportionality to p_0 would be the case indeed. But this independence of the order of integration is not fulfilled for the one-particle contribution, as will become evident later on. The definite prescription given for the order of integrations in (41) is an immediate consequence of our distribution-theoretic definition of equal time commutators. It agrees with the limiting procedure given by OKUBO [18] in momentum space.

Eq. (41) may be written as

$$\lim_{\omega \to 0} \frac{\partial}{\partial \omega} R_{f_T}(\omega)_p \tag{42}$$

with

$$R_{f_T}(\omega)_p = i(2\pi)^3 \int dx_0\, e^{i\omega x_0}\, \Theta(x_0) \times \\ \times \int d^3 x \langle p | [D^{(+)}(\mathrm{x}/2, f_T^{x_0/2}), D^{(-)}(-\mathrm{x}/2, f_T^{-x_0/2})] | p \rangle$$

if $R_{f_T}(\omega)_p$ is an analytic function in ω whose first derivative approaches a continuous boundary value at $\omega = 0$. That this is indeed the case will be shown in the subsequent discussion.

[7] That this is generally not true even for local commutators will be discussed in the next section.

Due to the smearing in time the product of the step function with the commutator in eq. (42) is a well defined quantity. Therefore, we may evaluate the x-integration which results in the Hilbert relation

$$R_{f_T}(\omega)_p = (2\pi)^2 \int_{-\infty}^{+\infty} d\omega' \frac{M(\omega', 0)_p}{\omega' - \omega - i\varepsilon} \tilde{f}_T^2(\omega') \tag{43}$$

where M is the Fourier transform of the matrixelement of the commutator

$$M(q)_p = \int d^4x\, e^{iq\cdot x} \langle p|\, [D^{(+)}(x/2), D^{(-)}(-x/2)]\, |p\rangle\,. \tag{44}$$

According to the spectral conditions $M(q)_p$ may be decomposed as follows

$$M(q)_p = -(2\pi)^{-2}\, \varepsilon(q_0 + p_0)\, q^2 K^2(q^2)\, \delta((q+p)^2 - M^2) + M_c(q)_p \tag{45}$$

where the support of the continuous part M_c is given by

$$q_0 \geqq -p_0 + \sqrt{(M+\mu)^2 + (\mathfrak{q}+\mathfrak{p})^2} \tag{46a}$$

$$q_0 \leqq p_0 - \sqrt{(M+\mu)^2 + (\mathfrak{q}-\mathfrak{p})^2} \tag{46b}$$

and the vertex function K is defined as follows

$$\langle p|\, D(0)\, |p'\rangle = (2\pi)^{-3}\, 2M \bar{u}(p)\, \gamma_5 u(p')\, K((p-p')^2)\,.$$

$K(0)$ is connected to the axial-vector coupling constant renormalization G_A by

$$K^2(0) = (2MG_A)^2\,. \tag{47}$$

The first term in eq. (45) results from the one-particle intermediate state in eq. (44) and its "crossed counterpart", i.e. the partially disconnected contribution from the 3-particle intermediate state (one antineutron, two protons).

We require that the operators $D^{(\pm)}(x)$ are local relative to each other i.e.

$$[D^{(+)}(x), D^{(-)}(y)] = 0 \quad \text{if} \quad (x-y)^2 < 0\,. \tag{48}$$

For the further development, it is advantageous not to work directly with the decomposition eq. (45) but with the closely connected "causal decomposition"

$$M(q)_p = M^{(1)}(q)_p + M^{(2)}(q)_p \tag{49}$$

where $M^{(1)}$ is defined by

$$M^{(1)}(q)_p = -(2\pi)^{-2}\, q^2 K^2(0)\, \varepsilon(q_0 + p_0)\, \delta((q+p)^2 - M^2)\,. \tag{50}$$

The Fourier transforms $\tilde{M}^{(i)}(x)_p$ are causal functions (i.e. vanish for $x^2 < 0$) because this is true for $\tilde{M}^{(1)}$ by construction and for $\tilde{M}$ it follows from the requirement eq. (48).

By insertion of eq. (49) into eq. (43) and evaluation of the $M^{(1)}$ contribution we obtain

$$R_{f_T}(\omega) = -\frac{2p_0}{\omega + 2p_0 + i\varepsilon} K^2(0) \tilde{f}_T^2(4p_0^2) + \\ + (2\pi)^2 \int_{-\infty}^{+\infty} d\omega' \frac{M^{(2)}(\omega', \mathfrak{o})_p}{\omega' - \omega - i\varepsilon} \tilde{f}_T^2(\omega') . \tag{51}$$

With the aid of eq. (45) and (46), we get the support of $M^{(2)}(\omega', 0)_p$:

$$M^{(2)}(\omega', \mathfrak{o})_p = 0 \quad \text{if} \quad |\omega'| < -p_0 + \sqrt{(M+\mu)^2 + \mathfrak{p}^2} . \tag{52}$$

Therefore $R_{f_T}(\omega)$ is an analytic function of ω within a circle around $\omega = 0$ with radius $-p_0 + \sqrt{(M+\mu)^2 + \mathfrak{p}^2}$. The differentiation of $R_{f_T}(\omega)$ at $\omega = 0$ may be carried out now, leading to

$$\frac{K^2(0)\, \tilde{f}_T^2(4p_0^2)}{2p_0} + (2\pi)^2 \int_{-\infty}^{+\infty} d\omega' \frac{M^{(2)}(\omega', \mathfrak{o})_p}{\omega'^2} \tilde{f}_T^2(\omega') . \tag{53}$$

On the premises of the existence of equal-time commutators for axial-vector charges the limit $T \to 0$ of the r.h.s. of eq. (53) exists.

We now make the technical assumption that $\lim T \to 0$ may be taken under the integral, which excludes certain oscillatory behaviour of $M^{(2)}(\omega', \mathfrak{o})_p$ for $\omega' \to \infty$, i.e. we obtain in this limit for eq. (53) (resp. eq. (41))[8]:

$$\frac{K^2(0)}{2p_0} + (2\pi)^2 \int_{-\infty}^{+\infty} d\omega' \frac{M^{(2)}(\omega', \mathfrak{o})_p}{\omega'^2} . \tag{54}$$

We mention that the prescription given for the order of integrations in eq. (41) led to the absence of a one-particle contribution, but to the presence of its "crossed counterpart" in eq. (54)[9].

To exploit locality for $M^{(2)}$ we use the Jost-Lehmann-Dyson (JLD)-representation [19], [20] in its Lorentz invariant non-unique form [20]:

$$M^{(2)}(q)_p = \int d^4u \int ds\, \varepsilon(q_0 - u_0)\, \delta((q-u)^2 - s)\, \boldsymbol{\psi}^{(2)}(u, s)_p \tag{55}$$

---

[8] In the usual treatment of equal time commutation relations without testing functions for example in S. ADLER's treatment [2] this formula follows directly from intermediate state insertion.

[9] If we would disregard this proved prescription and exchange the order of space and time integration in eq. (41) by considering

$$\lim_{T\to 0} \lim_{\mathfrak{q}\to 0} \lim_{\omega\to 0} (2\pi)^2 \frac{\partial}{\partial\omega} \int d\omega' \frac{M(\omega', \mathfrak{q})_p}{\omega' - \omega - i\varepsilon} \tilde{f}_T^2(\omega')$$

the result would differ from eq. (54) by the one-particle contribution

$$\frac{p_0}{2} K^2(0) \lim_{\mathfrak{q}\to 0} \mathfrak{q}^2/(\mathfrak{q}\cdot\mathfrak{p})^2$$

which may take any desired value.

with

$$\operatorname{supp} \boldsymbol{\psi}^{(2)} = \{(p \pm u) \in V_+, \sqrt{s} \geqq$$
$$\geqq \operatorname{Max}(0, M - \sqrt{(p+u)^2}, M + \mu - \sqrt{(p-u)^2})\} .$$

As is well known, this support of $\boldsymbol{\psi}^{(2)}$ leads to support properties for $M^{(2)}(q)_p$ as given by eq. (45) and (46). Because we have split off $M^{(1)}$ from M in defining $M^{(2)}$ we obtain a further restriction on $\operatorname{supp} \boldsymbol{\psi}^{(2)}$:

$$\boldsymbol{\psi}^{(2)}(u, s)_p = 0 \quad \text{if} \quad s \leqq u^2 . \tag{56}$$

The proof of eq. (56) is given in the appendix.

The high-energy problem

Next we would like to remember that the existence of the integral in eq. (54) which requires a good high-energy behaviour for the even (with respect to ω) part $M_e^{(2)}$ of $M^{(2)}$, i.e.

$$M_e^{(2)}(\omega, \mathbf{o})_p \underset{\omega \to \infty}{\simeq} \omega^{1-\varepsilon}, \quad \varepsilon > 0 \tag{57}$$

is an immediate consequence of the existence of the equal time commutation relation eq. (28). Therefore eq. (54) already leads to a well defined sum rule without containing any unknown subtraction constant.

In the following, we want to discuss sufficient conditions for the spectral function $\boldsymbol{\psi}^{(2)}(u, s)_p$ to obtain from eq. (54) the sum rule in its dispersion-theoretic like form (i.e. the Adler-Weisberger relation [2], [3]) whereas more general cases are discussed in the next section.

Necessary and sufficient for the validity of eq. (57) is the following behaviour of $\boldsymbol{\psi}^{(2)}$

$$\int d^4u \; \boldsymbol{\psi}^{(2)}(u, (\omega - u_0)^2 - \mathbf{u}^2)_p \big|_{\text{even}} \underset{\omega \to \infty}{\sim} \omega^{1-\varepsilon}, \; \varepsilon > 0 . \tag{58}$$

We now consider the *sufficient requirement*, that the good large s behaviour of $\boldsymbol{\psi}^{(2)}(u, s)_p$ is valid uniformly[10] in u:

$$|\boldsymbol{\psi}^{(2)}(u, s)_p| \underset{s \to \infty}{\leqq} C s^{1/2(1-\varepsilon)}, \; \varepsilon > 0 \tag{59}$$

for every $u \in \operatorname{supp} \boldsymbol{\psi}^{(2)}$.

This requirement is not necessary, as will be shown by counter examples in the next section[11].

Inserting the JLD-representation eq. (55) into eq. (54), we may interchange the order of integrations by considering eq. (59) and (56) and evaluate the ω'-integration with the aid of the δ-function. In this

[10] In eq. (58) and (59) we treated $\boldsymbol{\psi}^{(2)}$ as a function, however, the distribution-theoretical modification can easily be formulated.

[11] But it turns out, that these counter examples are more or less of a pathological nature.

way we obtain

$$\int_{-\infty}^{+\infty} d\omega' \frac{M^2(\omega', \mathbf{o})_p}{\omega'^2} = - \int d^4 u \int ds \frac{2 u_0 \psi^{(2)}(u, s)_p}{(u^2 - s)^2}. \tag{60}$$

Because of Lorentz-invariance $\psi^{(2)}(u, s)_p$ is only a function of the invariants u^2, $u \cdot p$ and s. Therefore, the r.h.s of eq. (60) transforms like the 0-component of a four vector, i.e. is proportional to p_0, as expected. The non-covariant term $K^2(0)/2p_0$ in eq. (54) just cancels the non-covariant contribution from the boundary term in eq. (39). Therefore, with the requirement eq. (59), we always get the sum rule in a frame independent form.

For reasons of simplicity, we now work in the laborsystem $\mathbf{p} = 0$ (this will be indecated in the following simply by suppressing the index p).

Theorem.

$$\int_{-\infty}^{+\infty} d\omega' \frac{M^{(2)}(\omega', \mathbf{o})}{\omega'^2} = \int_{-\infty}^{+\infty} d\omega' \frac{M(\omega', \mathbf{e}\,\omega')}{\omega'^2} \tag{61}$$

where $\mathbf{e}$ is an unit vector.

The r.h.s. of eq. (61) has exactly the unsubtracted dispersion-theoretic form. To prove this theorem, we first observe that $M^{(1)}(\omega', \mathbf{e}\,\omega') = 0$ because $M^{(1)}(q)_p \sim q^2$. Inserting now the JLD-representation for $M^{(2)}$ into the r.h.s. of eq. (61), we may again interchange the order of integrations and get by a simple and straightforward calculation, thereby using the rotational invariance of $\psi^{(2)}(u, s)$ and the support properties eq. (55)

$$- \int d^4 u \int ds \frac{2 u_0 \psi^{(2)}(u, s)}{(u^2 - s)^2}$$

which is due to eq. (60) the desired result.

The integral on the right hand side of (61) may now be rewritten with the help of the formula

$$D^{(\pm)}(x) = \frac{G_A\, 2M\, \mu^2}{\sqrt{2}\, g_{NN\pi}\, K_{NN\pi}(0)}\, \varphi_\pi^{(\pm)}(x) \tag{62}$$

and one obtains:

$$2 p_0 \left(\frac{G_A\, 2M}{g_{NN\pi}\, K_{NN\pi}(0)} \right)^2 \frac{1}{(2\pi)^3} \int_{(M+\mu)^2}^{\infty} \frac{ds}{s - M^2} \left[\sigma_{p\pi^-}^{\text{tot}}(s, 0) - \sigma_{p\pi^+}^{\text{tot}}(s, 0) \right] \tag{63}$$

where $\sigma_{p\pi^+}^{\text{tot}}(s, \zeta) \frac{k(s, \zeta)}{k(s, \mu^2)}$ are analytic up to $\zeta = \mu^2$ [21] and $\sigma_{p\pi^+}^{\text{tot}}(s, \mu^2)$ is the physical total cross section of $p\,\pi^\pm$ scattering ($k(s, \zeta)$ is the CMS momentum for pions of mass $\sqrt{\zeta}$).

Collecting all terms for the l.h.s. of the equal time commutation relation eq. (28) we finally arrive at the Adler-Weisberger relation [2], [3]:

$$1 = G_A^2 \left\{ 1 + \frac{2M^2}{g_{NN\pi}^2 K_{NN\pi}^2(0)} \times \right. \\ \left. \times \frac{1}{\pi} \int_{(M+\mu)^2}^{\infty} \frac{ds}{s - M^2} \left[\sigma_{p\pi^-}^{\text{tot}}(s, 0) - \sigma_{p\pi^+}^{\text{tot}}(s, 0) \right] \right\}. \tag{64}$$

We would like to emphasize that formula (62) (often called the PCAC-hypothesis) is not an assumption but a result of quantum field theory. It follows from general collision theory [22], [23], [24] that if a local field has a nonvanishing matrixelement between the vacuum and the one $\pi^\pm$ state

$$\langle 0 | D^{(\pm)}(0) | k, \pi^\mp \rangle = (2\pi)^{-3/2} c \neq 0 \tag{65}$$

then the field[12] $\varphi_\pi^{(\pm)}(x) = \frac{1}{c} D^{(\pm)}(x)$ is a possible candidate of an interpolating pion field[12]. In other words, this field inserted into the LSZ reduction formula gives the correct on-mass shell scattering amplitude. The application of this observation to the problem of a "universal" ϱ-meson coupling was pointed out by HAAG, NISHIJIMA and SCHROER (unpublished) and used extensively by GELL-MANN [1]. This result of general collision theory seems to have been overlooked, however, in the recent literature on PCAC.

Since the vertex function of the field φ_π taken at the momentum transfer μ^2 is also an S-matrix observable (residue of the pole term)

$$\langle k; n | (\Box + \mu^2) \varphi_\pi^{(-)}(0) | k'; p \rangle |_{(k-k')^2 = \mu^2} = g_{NN\pi} \frac{2M}{(2\pi)^3} \bar{u}_n(k) \gamma_5 u_p(k') \tag{66}$$

we can, by taking in addition the definition of the axial vector coupling constant (40), in a well-known manner compute the normalization c and obtain formula (62). Of course, in order to obtain the right on-mass shell πN-scattering amplitude, we could have taken any element of the Borchers class [25], [26] of $D(x)$. However, the equal time commutation relation for the "generalized charges" of the axial vector current force us to study the off-mass shell extrapolation with the help of the field $\frac{D(x)}{C}$. The "smoothness" of the extrapolation from $\zeta = \mu^2$ to $\zeta = 0$ is, in contrast to the analyticity in ζ, not a property of the whole Borchers-class of D.

[12] Anybody who has a distrust of general quantum field theory may check this insensitivity of the in and out fields with respect to the particular form of the interpolating field in perturbation theory.

The nontrivial part of the PCAC hypothesis is the claim that matrix elements of the preferred (by weak interactions) physical field $(\Box + \mu^2)\, D^{\pm}(x)$ have this smoothness property in ζ. It is well known that from observed π-lifetime the "smoothness" $K_{NN\pi}(0) \cong K_{NN\pi}(\mu^2)$ i.e. the Goldberger Treimann relation follows [27], [28]. A sufficient condition for this smoothness would be according to GELL-MANN [1] an unsubtracted dispersion relation with fast decreasing absorptive part. However, for matrixelements entering into the Adler-Weisberger relation such dispersion relations in ζ have not been formulated.

5. A more general discussion of the high-energy problem

In the last section, we only discussed JLD-spectral functions $\psi(u, s)_p$ which are bounded for $s \to \infty$ by $C s^{1/2(1-\varepsilon)}$ $(\varepsilon > 0)$ uniformly in u. Now we want to relax this restrictive condition by considering a more general class of spectral functions bounded only by a polynom in s but leading to the existence of the equal-time commutation relation, i.e. to the existence of the integral

$$\int d\omega' \frac{M_e(\omega', \mathfrak{o})_p}{\omega'^2} \tag{67}$$

To avoid unnecessary complications with one-particle states, which do not influence the high-energy behaviour, we consider the example of the commutator of axial vector charges transforming with respect to $SU(3)$ like the $\pm$-components of a V-spin vector, taken between one-pion states.

The support of our $M(q)_p$ is then given by

$$\begin{aligned} q_0 &\geqq -p_0 + \sqrt{(m+\mu)^2 + (\mathfrak{q}+\mathfrak{p})^2} \\ q_0 &\leqq p_0 - \sqrt{(m+\mu)^2 + (\mathfrak{q}-\mathfrak{p})^2} \end{aligned} \tag{68}$$

where μ resp. m is the mass of the pion resp kaon i.e. we have in the system $\mathfrak{p} = 0$ a symmetrical spectrum. This allows the application of the unique Jost-Lehmann (JL)-representation [19] for M_e. Due to Lorentz-invariance of M we have

$$M(\omega', \mathfrak{o})_p = M\left(\omega' \frac{p_0}{\mu}, -\mathfrak{p}\frac{\omega'}{\mu}\right) \tag{69}$$

Using the JL-representation for the even (with respect to ω') part of eq. (69) we obtain

$$M_e(\omega', \mathfrak{o})_p = \varepsilon(\omega')\, \omega' \frac{p_0}{\mu} \int d^3u \int ds\, \varphi(u,s)\, \delta\left(\omega'^2 - \frac{\omega'}{\mu} 2\mathfrak{p}\cdot\mathfrak{u} - u^2 - s\right) \tag{70}$$

with $u = |\mathfrak{u}|$ and

$$\operatorname{supp} \varphi(u, s) = \{u \leqq \mu, \sqrt{s} \geqq \sqrt{s_0} = m + \mu - \sqrt{\mu^2 - u^2}\}\,.$$

Necessary and sufficient for the existence of the integral eq. (67) is the condition

$$\int d^3u \, \boldsymbol{\varphi}\left(u, \omega'^2 - \frac{\omega'}{\mu} 2\mathfrak{p} \cdot \mathfrak{u} - u^2\right) \underset{\omega' \to \infty}{\sim} (\omega')^{-\varepsilon}, \quad \varepsilon > 0 . \tag{71}$$

Next we consider the special case $\mathfrak{p} = 0$. Then a very general class of JL-spectral functions satisfying eq. (71) but only bounded by a polynom in s for large s is given by the ansatz

$$u^2 \boldsymbol{\varphi}(u, s) = u^2 \mathring{\boldsymbol{\varphi}}(u, s) + \sum_{n=0}^{N} a_n'(u) \, (s + u^2)^n \, \Theta(s - s_0(u)) \tag{72}$$

with $|\mathring{\boldsymbol{\varphi}}(u, s)| \underset{s \to \infty}{\leqq} C s^{1/2\varepsilon}$, $\varepsilon > 0$ for every $u \in \operatorname{supp} \boldsymbol{\varphi}$ and $\operatorname{supp} a_n(u) = (0, M]$.

Because the $\mathring{\boldsymbol{\varphi}}$-part of (72) has been considered already in the last section, we restrict ourselves in the following discussion to the polynomial part (which we call $\hat{\boldsymbol{\varphi}}$). By partial integration in u we obtain from our ansatz

$$\begin{aligned} &\int d^3u \, \hat{\boldsymbol{\varphi}}(u, \omega'^2 - u^2) \\ &\quad = 4\pi \int du \sum_{n=0}^{N} a_n(u) \, \omega'^{2n} (2u + s_0'(u)) \, \delta(\omega'^2 - u^2 - s_0(u)) \end{aligned} \tag{73}$$

i.e. we have a $\hat{M}_e(\omega', \mathfrak{o})$ with compact support.

But this ansatz fulfills eq. (71) for arbitrary $\mathfrak{p}$ only after imposing some supplementary conditions for the $a_n(u)$: Calculating the l.h.s. of eq. (71) by using our $\hat{\boldsymbol{\varphi}}$ we get

$$f(\omega') + P_{2(N-1)}(\omega', |\mathfrak{p}|) \tag{74}$$

with

$$P_{2(N-1)}(\omega', (\mathfrak{p})) = \begin{cases} 0 \quad \text{if} \quad N \leqq 1 \\ \displaystyle\sum_{\substack{m=0 \\ m \text{ even}}}^{2(N-1)} \omega'^m b_m(|\mathfrak{p}|) \end{cases}$$

and

$$b_m(|\mathfrak{p}|) = \sum_{k \text{ odd}}^{m-1} \int du \, a_{\frac{m+k+1}{2}} \, u^k |\mathfrak{p}|^{k+1} \, g_{m,k}$$

where $f(\omega')$ has compact support.

The r.h.s. of eq. (71) requires

$$b_m(|\mathfrak{p}|) = 0 \quad \text{for every} \quad |\mathfrak{p}|$$

i.e.

$$\int du \, a_n(u) \, u^k = 0 \tag{75}$$

if $n > 1$, k odd and $k \leqq n - 1$.

Further integral conditions for the $a_n(u)$ we obtain from the requirement of Lorentz-invariance of the "naive defined" retarded commutator:

$$H(q, p) = \int dq'_0 \frac{\hat{M}_e(q'_0, \mathfrak{q})_p}{q'_0 - q_0 - i\varepsilon} . \tag{76}$$

The r.h.s. of eq. (76) always exists, because due to conditions (75) $\hat{M}_e(q'_0, \mathfrak{q})_p$ has compact support with respect to q'_0.

Because $H(q, p)$ is already rotational invariant by construction it is sufficient to consider an infinitesimal special Lorentz transformation along the i-th coordinate axis. Invariance of (76) then requires

$$\int dq'_0 \frac{\partial}{\partial q_i} \hat{M}_e(q'_0, \mathfrak{q})_p = 0. \tag{77}$$

In coordinate space eq. (77) is a condition for the equal time commutator of the divergences of our currents

$$\tilde{M}_e((0, \mathrm{x}))_p \sim \delta(\mathrm{x}) . \tag{78}$$

We have not the intention to write down the complicated conditions for the $a_n(u)$ following from eq. (77) resp. (78) for a general N. We only give the results for the two simplest cases:

$$\begin{aligned} N &= 0: \text{ no condition} \\ N &= 1: \int du\, a_1(u) u = 0 . \end{aligned} \tag{79}$$

It is immediately clear, that from the Lorentz invariance of $H(q, p)$ the desired proportionality of the expression eq. (67) to p_0 follows, provided that $H(q \cdot p, q^2)$ has a continuous first derivative in q^2 at $q^2 = 0$ (which is the case for our $\hat{M}_e$). Since the r.h.s. of our equal time commutator is proportional to p_0, we are forced to require Lorentz-invariance of $H(q, p)$.

After these considerations, the important question arises whether the unsubtracted dispersion-theoretic like sum rule i.e. the expression

$$\int d\omega' \frac{\hat{M}_e(\omega', \mathrm{e}\,\omega')}{\omega'^2} \tag{80}$$

exists. Unfortunately, this is not always the case. A simple calculation shows

$$\int d^3u\, \hat{\varphi}(u, 2\,\mathrm{e} \cdot \mathfrak{u}\,\omega' - u^2) \xrightarrow[\omega' \to \infty]{} \sum_{n=1}^{N} c_n \omega'^n + o(\omega'^{-1}) \tag{81}$$

with $c_n \sim \int du\, a_n(u)\, u^{n-1}$.

According to eq. (75), we have already

$$c_n = 0 \quad \text{if} \quad n \text{ even}$$

from the existence of the equal-time commutator, but no condition for c_n with n odd is onhand.

Suppose only terms with n even are present in $\hat{\varphi}$ so that the expression eq. (80) exists. Even in this case the equal-time commutator, i.e. the expression $\int \frac{\hat{M}_e(\omega', 0)}{\omega'^2}$ is different from the dispersion-theoretic (i.e. Adler-Weisberger) expression $\int \frac{\hat{M}_e(\omega', e\,\omega')}{\omega'^2}$. This corresponds to the situation in dispersion theory where an analytic function $\left(\text{in our case} \int \frac{\hat{M}_e(\omega', e\,\omega)}{\omega' - \omega}\right)$ may be represented in the cut ω-plane by the unsubtracted dispersion integral plus a real polynom in ω.

It is our hope, that one may exclude the considered $\hat{\varphi}$ by imposing more conditions following from general quantum field theory on our commutator matrix element. From the point of view of perturbation theory, these $\hat{\varphi}$ are of a pathological nature, because in this case unsubtracted dispersion relations and naive multiplication with the Θ function are always synonymous.

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Appendix

A theorem on the support of the JLD-spectral function for $M^{(2)}(q)_p$

The purpose of this appendix is to prove eq. (56). Next we consider the JLD-spectral function $\psi(u, s)_p$ for $M(q)_p$ which has support in $D(u, s)$:

$$D(u, s) = \{(P \pm u) \in V_+, \quad \sqrt{s} \geqq \mathrm{Max}(0, M - \sqrt{(P+u)^2}, M + \mu - \sqrt{(p-u)^2})\} . \tag{A1}$$

This ψ may be decomposed into two parts

$$\psi = \psi_0 + \psi_1 \tag{A2}$$

where $\underset{1}{\psi_0}$ have support in $\underset{1}{D_0}$:

$$\begin{aligned} D_0 &= \bigcap_{\varepsilon} (D \cap U_\varepsilon) \quad \text{with} \quad U_\varepsilon = \{s = u^2 + \delta, |\delta| < \varepsilon\} \\ D_1 &= D - D_0 . \end{aligned} \tag{A3}$$

Theorem.

$$1.\ \psi_0 = 0 \quad \text{if} \quad u \neq -p$$

$$2.\ \psi_1 = 0 \quad \text{if} \quad s \leqq u^2 \qquad \rule{0pt}{0pt}\rfloor\!\!\|$$

Proof. The domain D_0 consists of all points u, s for which the hyperboloids $(q-u)^2 - s = 0$ are admissable in the sense of DYSON [20] with the subsidary condition $(u, s) \in U_\varepsilon$ i.e. we have for $u \in D_0$ in the Lorentz-frame $p = (M, \mathbf{o})$ for arbitrary $\mathbf{q}$:

$$u_0 + \sqrt{\mathbf{q}^2 - 2\,\mathbf{u}\cdot\mathbf{q} + u_0^2 + \delta} \geqq -M + \sqrt{M^2 + \mathbf{q}^2} \tag{A5}$$

$$u_0 - \sqrt{\mathbf{q}^2 - 2\,\mathbf{u}\cdot\mathbf{q} + u_0^2 + \delta} \leqq M - \sqrt{(M+\mu)^2 + \mathbf{q}^2} \tag{A6}$$

with $|\delta| < \varepsilon > 0$, $\varepsilon \to 0$.

Consider now (A5) and (A6) for one particular $\mathbf{q} : \mathbf{q} = 0$. Then we obtain

$$\left.\begin{aligned} &u_0 + |u_0| + \frac{\delta}{2\,|u_0|} \geqq 0; &&\text{(A7)}\\ &u_0 - |u_0| - \frac{\delta}{2\,|u_0|} \leqq -\mu\,. &&\text{(A8)}\end{aligned}\right. \quad ; \quad |\delta| < \varepsilon > 0,\ \varepsilon \to 0$$

From (A8) we infer $u_0 < 0$. With that the limit case $\varepsilon = 0$ only allows the equality sign in (A7). Then the same is true for (A5) because $\mathbf{q}$ is an arbitrary vector. But with the equality sign and $\delta = 0$ (A5) has the unique solution $u_0 = -M$, $\mathbf{u} = 0$. It may easily be seen that this solution fulfills (A6). This proves the first part of our theorem.

Corollary. *The hyperboloid* $(q - u)^2 - s = 0$ *is not admissible for* $s < u^2$ *because it is a monotonic function of* s *and* (A7) *and* (A8) *are inconsistent for* $\delta < 0$.

With that we have proved the second part of our theorem.

$$\psi_1(u, s) = 0 \quad \text{if} \quad s \leqq u^2$$

From the support of ψ_0 and the JLD-representation for $M(q)_p$

$$M(q)_p = \int d^4u \int ds\ \psi(u, s)_p\ \delta((q - u)^2 - s)\ \varepsilon(q_0 - u_0) \tag{A9}$$

it is immediately clear that ψ_0 only gives a nonvanishing contribution to M on the one-particle mass shell $(p + q)^2 = M^2$. In order to fix the contribution of ψ at $s = u^2$ we consider the derivative of $M(\omega, \mathbf{o})$ at $\omega \to 0$

$$\begin{aligned}\frac{\partial}{\partial\omega} M(\omega, \mathbf{o}) \xrightarrow[\omega\to 0]{} \frac{K^2(0)}{(2\pi)^2}\,[2M\,\delta(\omega^2 + 2\omega M) + {}\\ + 4\omega M^2\,\delta'(\omega^2 + 2\omega M)]\,\Theta(\omega + M)\,.\end{aligned} \tag{A10}$$

On the other hand we get, according to eq. (A9), and our theorem applied to $\frac{\partial}{\partial s}\psi(u, s)$

$$\begin{aligned}\frac{\partial}{\partial\omega} M(\omega, \mathbf{o}) \xrightarrow[\omega\to 0]{} -2\int d^4u \int ds\ \varepsilon(\omega - u_0)\,u_0 \times{}\\ \times\delta((\omega - u_0)^2 - \mathbf{u}^2 - s)\left(\frac{\partial}{\partial s}\psi(u, s)\right)_0\,.\end{aligned} \tag{A11}$$

The following ansatz satisfies eq. (A10)

$$\left(\frac{\partial}{\partial s}\psi(u, s)_p\right)_0 = -\frac{K^2(0)}{(2\pi)^2}\,\delta(s - u^2)\,\frac{\partial}{\partial u^\mu}\,u^\mu\,\delta(u + p)\,. \tag{A12}$$

Therefore $\psi(u, s)_p$ may be decomposed as follows

$$\psi(u, s)_p = \psi^{(1)}(u, s)_p + \psi^{(2)}(u, s)_p$$

with

$$\psi^{\binom{1}{2}}(u, s)_p = \int\limits_0^s ds' \left(\frac{\partial}{\partial s'} \psi(u, s')_p\right)_{\substack{0\\1}} . \tag{A13}$$

With the aid of eq. (A12), (A13) and the definition of $M^{(1)}(q)_p$ it may easily be seen that $\psi^{(1)}(u, s)_p$ is the JLD-spectral function generating $M^{(1)}$, therefore $\psi^{(2)}$ generates $M^{(2)}$ and has according to eq. (A13) and our theorem the desired property

$$\psi^{(2)}(u, s) = 0 \quad \text{if} \quad s \leqq u^2 . \tag{A14}$$

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