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Divergence of Perturbation Theory for Bosons

By

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Abstract. Perturbation theory is studied in two dimensional space-time. There all non-derivative boson self-interactions are renormalizable and in each order of perturbation theory there are no divergences, that is all renormalizations are finite in perturbation theory. Thus the unrenormalized perturbation series may be studied and it is shown that any interaction of the general form $H_I(x) = \lambda \sum_{j=3}^{\infty} a_j \times$ $\times : \varphi(x)^j :, a_j \geqq 0$ leads to Green's functions which are not analytic in λ at $\lambda = 0$. This result holds in momentum space at a large set of points, enough to show that the Green's functions are not distributions in the momenta which are analytic in λ at $\lambda = 0$. Furthermore the proper self energy and the two-particle scattering amplitude are shown not to be analytic in λ at $\lambda = 0$ for certain momenta on or below the bare mass shell. In the course of this analysis we use the integral representations for Feynman graphs to derive a minorization of the form $|I(p_1, \ldots, p_e)| > A\, B^n$ for the contribution from all n^{th} order connected graphs in a theory with an interaction of the form $H_I(x) = \lambda \sum_{j=3}^{Q} a_j : \varphi(x)^j :$. Then the constants A and B depend only on the momenta p_i, and not on the structure of a particular graph.

I. Introduction

It is interesting to study perturbation theory for self interacting bosons to discover whether it can be used as a tool to prove the existence of solutions to the field equations. The problem is to expand as a power series in the coupling constant either the vacuum expectation values of the (time-ordered) Heisenberg fields, the S-matrix elements, or the kernels which occur in the numerator and denominator of the Green's functions or S-matrix elements, and then to determine whether the expansion in question defines an analytic function of the coupling constant at zero coupling. If the answer is no, then some approximation scheme more sophisticated than perturbation theory must be used to investigate solutions of the field equations.

In order to work in the interaction picture and remain completely within a Hilbert space formalism, Haag's theorem tells us that it is necessary to study an approximate theory. This follows from the fact that we want to write the interaction Hamiltonian H_I as the integral of an energy density $H_I(x)$ so that

$$H_I = \int_{x_0 = 0} H_I(x)\, d\vec{x}\,,$$

where $H_I(x)$ is a local function of a free field. It then follows that H_I cannot be an operator in the Fock representation unless it annihilates the no-particle state Φ_0. Since H_I given above is invariant under space translations, and since Φ_0 is also translation invariant, we see that

$$\|H_I\,\Phi_0\|^2 = \int d\vec{x}\,d\vec{y}\,(\Phi_0, H_I(x)\,H_I(y)\,\Phi_0) = \begin{cases} 0 & \text{if } H_I\,\Phi_0 = 0 \\ \infty & \text{otherwise}\,. \end{cases}$$

But $H_I\,\Phi_0 \neq 0$ for all interactions which are normally studied, for instance $H_I(x) = \lambda : \varphi(x)^4 :$, so we see that H_I cannot be applied to the Fock vacuum. Hence we can use the interaction representation legitimately only if we study an approximate H_I which is not translation invariant. By the same argument, each term in the sum

$$\sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int dy_1, \ldots, dy_n (\Phi_0, T H_I(y_1), \ldots, H_I(y_n)\,\Phi_0)$$

is infinite, and this is the so-called "self energy of the bare vacuum".

On the other hand, if we try to use the above H_I and make a formal expansion in λ, then the vacuum self energy occurs as a multiplicative factor in both the numerator and the denominator of the formal expansion for the time-ordered vacuum expectation values of the Heisenberg fields. (It also factors from the S-matrix elements which are related to the expectation values of the fields by L. S. Z.) In fact the formal Gell-Mann-Low expansions [1]

$$(\psi_0, T\,\varphi(x_1), \ldots, \varphi(x_e)\,\psi_0) = \frac{\sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int (\Phi_0, T\,\varphi_I(x_1), \ldots, \varphi_I(x_e)\,H_I(y_1), \ldots, H_I(y_n)\,\Phi_0)\,dy_1, \ldots, dy_n}{\sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int (\Phi_0, T H_I(y_1), \ldots, H_I(y_n)\,\Phi)\,dy_1, \ldots, dy_n} \tag{1}$$

can be expressed by a formal division of the denominator and then by truncation as

$$(\psi_0, T\,\varphi(x_1), \ldots, \varphi(x_e)\,\psi_0)^T = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \times \\ \times \int (\Phi_0, T\,\varphi_I(x_1), \ldots, \varphi_I(x_e)\,H_I(y_1), \ldots, H_I(y_n)\,\Phi_0)^T\,dy_1, \ldots, dy_n\,. \tag{2}$$

We will take the expansion (2) as a starting point and ask whether it defines a function of λ which is analytic at $\lambda = 0$.

Even though expansion (2) was heuristically derived, if the sum does converge, it is a good candidate for a field theory. In fact in every order of perturbation theory, it satisfies Lorentz invariance and locality, and the time-ordered two point function satisfies the spectral condition. The last statement means that the Fourier transform is an analytic function in momentum space for all $p^2 < m^2$. (This follows as an immediate consequence of the fact that in (A.5), $V > 0$ for $p^2 < m^2$.) The corresponding result for the n-point Green's function cannot be proved since "anomalous thresholds" occur.

This fact that the two point function satisfies the spectral condition in each order of perturbation theory can be turned round to make plausible the fact that the perturbation expansion does not converge. This argument is reminiscent of, but different from, an argument of Dyson [2]. We use the fact that Baym [3] has shown that the $\lambda\,\varphi^3$ theory can have no state of lowest energy, and hence does not satisfy the spectral condition. In two dimensional space-time it is straightforward to extend Baym's proof to any self interaction of degree three, or higher, which formally is unbounded below. That is, no theory with an interaction $\lambda\,\varphi^{2k+3}$ for $k \geqq 0$, or $\lambda\,\varphi^{2k}$ for $k \geqq 2$ and $\lambda < 0$ will satisfy the spectral condition. Thus in general no theory with an interaction of degree three or more and with a negative coupling constant will satisfy the spectral condition. However, we noted that the two point function satisfies the spectral condition in every order of perturbation theory. Hence, if we use the perturbation expansion for an arbitrarily small negative value of λ, we are trying to approximate a solution which will not obey the spectral condition by an approximate solution which does obey the spectral condition in every order. This makes it reasonable that the perturbation expansion (2) will never converge for $\lambda < 0$ and thus will never be analytic at zero coupling for an interaction of degree three or higher*.

In this note we will show that this plausibility argument is in fact borne out. We will see how to generalize the classic results of Hurst, Thirring and Petermann [4—6] to deal with an extremely large class of interactions in two-dimensional space-time. Furthermore, we will see how to analyze the complete set of Green's functions off the mass shell, and it will turn out that none of the Green's functions define distributions in the momenta which are analytic in the coupling constant λ at $\lambda = 0$. In the case of the two point function for the $\lambda\,\varphi^3$ theory, this proof will essentially reduce to the proof in Reference [5] which capitalizes on the fact that for certain momenta the contributions from all graphs of a given order add coherently. In order to deal with the general case, we

* Indeed a quadratic Lagrangian will or will not have a lowest energy state depending on whether or not the mass matrix (including the bare mass terms) is positive. It is just at the value of the coupling constant which changes positivity of the mass matrix that the exact solution is singular as a function of the coupling constant.

use the integral representations for FEYNMAN graphs [22—30] which were not mechanized at the time when HURST and THIRRING undertook their original investigations.

II. Basic approach

In order to study the perturbation theory in a concrete case, we must further specify the interaction H_I and the mass of the interaction picture field φ_I which appear in expansion (2). Aside from the particular choice of a theory (degree of interaction) one must choose between using the *unrenormalized* or the *renormalized* perturbation theory. Since in two dimensional space-time all renormalizations are finite in every order of perturbation theory, either of these two methods might be considered.

1. Unrenormalized theory

If we are interested, for example, in an interaction of the form $\lambda : \varphi^k :$, then we would set

$$H_I(x) = \lambda : \varphi_I^k(x) :$$

$$H_0(x) = \frac{1}{2} : \varphi_I^2(x)\, m^2 + (\nabla\, \varphi_I(x))^2 + (\partial_0\, \varphi_I)^2(x) : ,$$

where φ_I is a free field of mass m, the bare mass, and λ is the bare coupling constant. Then each Feynman graph will occur with the mass m on each line and the power series expansion is made in λ. The two point function will have a delta function at the mass $\mu(\lambda)$ which defines the renormalized (physical) mass. Then we set

$$\mu^2(\lambda) \equiv m^2 + \delta\, m^2(\lambda)$$

and denote $\mu^2(\lambda)$ and $\delta\, m^2(\lambda)$ functions of λ. If the time ordered two point function of φ is denoted $G(p^2)$ and the free field time ordered two point function of φ_I is denoted $G_0(p^2)$, then we define the "proper self energy" Π by the equation

$$G(p^2) = G_0(p^2) + G_0(p^2)\, \Pi(p^2)\, G(p^2)\,. \tag{3}$$

It then follows that on the physical mass shell where $p^2 = \mu^2(\lambda)$, we have

$$\Pi(\mu^2(\lambda); \lambda) = \frac{(2\pi)^2}{i}(\mu^2 - m^2) = \frac{(2\pi)^2}{i}\,\delta\, m^2(\lambda)\,. \tag{4}$$

2. Renormalized theory

In the renormalized theory of the same interaction we take the physical mass μ as a given constant, independent of λ. That is we require that the two point function have a delta function at $p^2 = \mu^2$. The Hamiltonian is then taken to be [7]

$$H_I(x) = g(1-L) : \varphi_I^k : -\frac{1}{2} A : \varphi_I^2 : + \frac{1}{2} B : (\nabla \varphi_I)^2 + (\partial_0 \varphi_I)^2 + + \mu^2 (\varphi_I)^2 : \qquad H_0(x) = \frac{1}{2} : \mu^2 \varphi_I^2 + (\nabla \varphi_I)^2 + (\partial_0 \varphi_I)^2 : ,$$

where φ_I is a free field of mass μ. Then each Feynman graph will occur with a mass μ on every line, and the power series expansion is made in g. In order to ensure that the delta function occurs at $p^2 = \mu^2$, A is chosen as a function of g so that it just compensates the mass shift made by the other interaction terms. Two other conditions are given to determine L and B order by order as functions of g.

Hence if we denote by Π_R the proper self energy which is calculated in the renormalized theory, we have by definition that on the physical mass shell $p^2 = \mu^2$, in place of (4)

$$\Pi_R(\mu^2) = 0 \, . \tag{5}$$

In this case the bare mass is regarded as depending on the coupling constant, and we have the relation

$$\mu^2 = m^2(g) + \delta m^2(g) \, . \tag{6}$$

If δm^2 is infinite, then only the renormalized perturbation theory can be used for calculations. Even though δm^2 might be finite, if it is infinite in any order of perturbation theory, we must also use the renormalized expansion. However, in two dimensional space-time, δm^2 is finite in every order of perturbation theory. Thus both methods may be tried. On the other hand, there is no closed expression known at present for the n^{th} order term in the renormalized expansion which includes the development of the renormalization constants in g. Hence in most of what follows we will discuss the unrenormalized expansion. A few comments about the renormalized expansion appear in Section VI.

III. Review of previous papers

A number of authors have studied the convergence of perturbation series in approximate field theories of various forms.

1. The classical equation $(\Box + m^2)\,\varphi = -\lambda\,\varphi^k$ has been treated by Jörgens, Browder and Segal [8—11]. In two and three dimensional space-time, the perturbation theory converges and global existence theorems are given for $\lambda \geqq 0$ and $k = 2n$. In four dimensional space-time, this result follows for $\lambda \geqq 0$, $k = 4$. In two and four dimensions these just correspond to what in the quantum case would be the renormalizable theories with positive energy. However, in three dimensional space-time,

Dyson's criteria for renormalizability [12, 13] imply that only for $k \leqq 6$ is $\lambda\, \varphi^k$ renormalizable. On the other hand Segal's proof holds for all $\lambda\, \varphi^{2k}$.

2. Other authors have considered approximate, but quantized theories [14—17]. The approximations are introduced either to make the Hamiltonian an operator in the Fock representation [14—17] or to make integrations over time converge [15—17]. After the introduction of these approximations (including time switching of the interaction) it is not possible to show that expansion (1) factors to give (2). This factorization is assumed in the non-local interaction [14]. EDWARDS [15] worked in the Schwinger functional formulation and introduced the Gaussian transform to analyze the analyticity of the two point function at $\lambda = 0$. While much of this argument is heuristic, this method was utilized by FRANK [16] in his study of the vacuum-vacuum matrix element of the S-matrix, $S_{00}(\lambda)$. He showed that for an approximate trilinear interaction, $S_{00}(\lambda)$ is not analytic at $\lambda = 0$. CAIANIELLO et al. [17] study the functions which appear in both the numerator and denominator of (1), again in an approximate $\lambda\, \varphi^4$ theory. However, once the approximation is introduced, it is not possible to show that the non-analytic behavior of the expansion does not factor out of the physical ratio which appears in (1). Moreover, it is possible that when the approximations are removed, the higher order graphs decrease in size fast enough to restore analyticity.

3. That this is not the case for the two point function in the fully relativistic expansion of the $\lambda\, \varphi^3$ theory was shown in the classical work of HURST, THIRRING and PETERMANN [4—6]. They demonstrated that the two point function was not analytic in the coupling constant at zero coupling. However, since they worked in four dimensional space-time it was necessary to carry out the field strength and mass renormalizations. All the counter-terms were not included in THIRRING's work, and an explicit form for the n^{th} order term in the fully renormalized expansion has not yet been given.

IV. Unrenormalized theory

Let us introduce the following notation:

$$G(p_1, \ldots, p_e; \lambda)\, \delta(p_1 + \cdots + p_e)$$
$$= \int \exp\left(i \sum_{j=1}^{e} p_j\, x_j\right) (\psi_0, T\, \varphi(x_1), \ldots, \varphi(x_e)\, \psi_0)^T\, dx_1, \ldots, dx_e\,,$$

where we take for $(\psi_0, T\, \varphi(x_1), \ldots, \varphi(x_e)\, \psi_0)^T$ the Gell-Mann-Low expansion (2) for the truncated vacuum expectation values of a product of time-ordered Heisenberg picture fields. Note that in perturbation theory *truncation* just corresponds to taking the connected part of an amplitude, that is to summing connected graphs.

In momentum space this gives

$$G(p_1, \ldots, p_e; \lambda) = \left[\prod_{j=1}^{e} \frac{i}{(2\pi)^{(s+1)}} \frac{1}{p_j^2 - m^2 + i\varepsilon}\right] \sum_{n=0}^{\infty} \sum_{\substack{n^{\text{th}}\text{ order}\\ \text{connected}\\ \text{graphs}}} \frac{\lambda^n}{n!} I(p_1, \ldots, p_e), \tag{7}$$

where I is normalized as in Appendix A.

Note that $I(p_1, \ldots, p_e)$ will always stand for a contribution which arises from a Feynman *graph* as opposed to the more common Feynman *diagrams*. The difference between a graph and a diagram is that a *graph* has internal vertices which are labeled in a particular way. A *diagram* is the sum of all graphs with different internal labelings which correspond to the same physical process. The contribution from a diagram is related to that from a corresponding graph by a numerical factor. For n^{th} order *connected diagrams* this factor is just $n!$ A direct expansion of the integrand in (2) gives rise to a sum of contributions from *graphs*. Expressed in terms of diagrams, this would just cancel the $\frac{1}{n!}$ factor in front of the integral in (2). We will, however, stick with graphs throughout.

We will consider Hamiltonians in which all the terms enter with the same sign. Thus

$$H_I(x) = \lambda \sum_{j=3}^{\infty} a_j : \varphi(x)^j :, \quad a_j \geqq 0 . \tag{8}$$

We start the sum at $j = 3$ to exclude any mass renormalization ($j = 2$ term) from the equation. Thus any such term is put in the bare mass. The absence of a $j = 1$ term insures us that the current $J(x)$ derived from $H_I(x)$ has zero vacuum expectation value. In certain cases we will want to insist that $H_I(x)$ include some non-zero term for $j \geqq 4$. In this case we say the Hamiltonian is of the form (8′).

In three or more dimensional space-time, $H_I(x)$ will define a local field only if the sum over j is finite [18]. However, in two dimensional space-time infinite sums are allowed, and we can consider any $H_I(x)$ which corresponds to a set of a_j such that

$$f(z) = \sum_{j=3}^{\infty} a_j z^j$$

defines an entire function of exponential type. Thus the interaction can be an entire function of the free field [19].

Definition: A set of e vectors $\{p_1, \ldots, p_e\}$ has property S if

1) $\sum_{i=1}^{e} p_i = 0,$

2) *The sum of any proper subset of the p_i is a space-like vector.*

Remarks:

1) By general arguments, if $G(p_1, \ldots, p_e; \lambda)$ exists at a momentum point with property S, then for fixed λ it is analytic there in the momenta.

2) There are many points with property S, since if $p_1, \ldots, p_{e-1}$ is a Jost point, then $\{p_1, \ldots, p_{e-1}, -\sum_{j=1}^{e-1} p_j\}$ has property S. $\{p_1, \ldots, p_{e-1}\}$ is a Jost point if for all $a_i \geqq 0$, $\sum_{i=1}^{e-1} a_i \neq 0$, $\left(\sum_{i=1}^{e-1} a_i\, p_i\right)^2 < 0$. In other words, the convex cone spanned by the vectors $p_1, \ldots, p_{e-1}$ is a space-like region.

Theorem 1: *Let $\{p_1, \ldots, p_e\}$ have property S and consider $G(p_1, \ldots, p_e; \lambda)$ in a theory with an interaction of the form* (8). *Then $G(p_i, \ldots, p_e; \lambda)$ defined by expansion* (7) *is not analytic in λ at $\lambda = 0$.*

Remark: In the unrenormalized series, we recall that all renormalizations are finite in every order of perturbation theory.

Definition: Let $\mathfrak{D}(O)$ be infinitely differentiable functions with support in O.

Theorem 2: *Let S denote the set of points with property S. Let $f > 0$, $f \in \mathfrak{D}(S)$. Then*

$$(G\,\delta)(f, \lambda) = \int G(p_1, \ldots, p_e; \lambda)\, f(p_1, \ldots, p_e)\, \delta(p_1 + \cdots + p_e)\, dp_1 \ldots dp_e$$

is not analytic in λ at $\lambda = 0$. Hence $G\,\delta$ is not a distribution in $\mathfrak{D}(S)'$ analytic in λ at $\lambda = 0$.

Outline of Proof of Theorems 1 and 2: We study $G(p_1, \ldots, p_e; \lambda)$ at-momenta $(p_1, \ldots, p_e)$ which have property S. Then we find,

1. In a given order of perturbation theory, that is for a fixed number n of internal vertices, and for an interaction of type (8), all graphs contribute to G with the same phase. Hence it is sufficient to consider graphs from only one term in the interaction, $\lambda\, \varphi^k$ for $k \geqq 3$.

2. We give a generous lower bound on the number of connected (in fact one particle irreducible) graphs in n^{th} order. For n greater than e this number is greater than

$$n!\,(n - e + 1)!!\,.$$

3. We give a lower bound on the magnitude of the contribution from any connected graph, in the form of the following lemma.

Critical Lemma: *Let $I(p_1, \ldots, p_e)$ be the integral corresponding to a connected Feynman graph in a theory given by*

$$H_I(x) = \sum_{j=1}^{Q} a_j : \varphi(x)^j : .$$

Let $p_1, \ldots, p_e$ have property S. Then there exist strictly positive constants A and B which depend only on $p_1, \ldots, p_e$ and such that

$$|I(p_1, \ldots, p_e)| > A\, B^n\,,$$

where n is the order of the graph (number of internal vertices). As the momenta vary over a compact set of points with property S, A and B vary over compact intervals on the positive real line.

Note that this is a stronger statement than what is needed to prove the theorems, where it is sufficient to know the critical lemma in the case that only one $a_j \neq 0$, $j \geqq 3$. However, we get the same type of uniform minorization of Q non-zero interaction terms.

Proof of Theorem 1.

We now combine steps 1, 2 and 3 to show that for momenta with property S the Gell-Mann-Low expansion (2) does not define an analytic function of λ at $\lambda = 0$. We can write (7) as

$$G(p_1, \ldots, p_e; \lambda) = D \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} a_n \,,$$

where

$$D = D(p_1, \ldots, p_e) = \prod_{j=1}^{e} \frac{i}{(2\pi)^{s+1}} \; \frac{1}{p_j^2 - m^2 + i\varepsilon}$$

and

$$a_n = \sum_{\substack{n^{\text{th}}\text{ order}\\ \text{connected}\\ \text{graphs}}} I(p_1, \ldots, p_e) \,.$$

Then for momenta with property S and $n > e$ we know that

$$|a_n| > n!\,(n - e + 1)!!\, A B^n$$

from 1, 2 and 3 above. If G defines a function analytic in λ at $\lambda = 0$, then in some neighborhood of $\lambda = 0$ the series converges absolutely, that is $\sum_{n=0}^{\infty} \left|\frac{\lambda^n}{n!} a_n\right| < \infty$. However

$$\sum_{n=0}^{\infty} \left|\frac{\lambda^n}{n!} a_n\right| > \sum_{n=e+1}^{\infty} \frac{|\lambda|^n}{n!} n!\,(n - e + 1)!!\, A B^n = A \sum_{n=e+1}^{\infty} |\lambda B|^n (n - e + 1)!! \,.$$

which diverges for all $\lambda \neq 0$. Thus G is not analytic at $\lambda = 0$.

We now proceed to prove 1—3.

1. This follows from the specific form of the Feynman integrals. Note that the contribution $I(p_1, \ldots, p_e)$ from a particular graph has been normalized in (7) so that all phase factors are included in $I(p_1, \ldots, p_e)$. From the integral representation (A.5) for I and the fact that $D > 0$, $V > 0$ for momenta $p_1, \ldots, p_e$ with property S we see that I has a phase

$$-i(-1)^{n-1}(i)^e \prod_{\substack{\text{internal}\\ \text{vertices}}} (\operatorname{sgn} \lambda_j) \,,$$

where λ_j is the coupling constant at vertex j. Since for all interactions of type (8), $\lambda_j = \lambda\, a_j$ and $a_j \geqq 0$, we see that $(\prod \operatorname{sgn} \lambda_j) = (\operatorname{sgn} \lambda)^n$.

Hence this is constant in a given order. Thus for fixed n and e, interactions of type 8 yield contributions to G from I which have constant phase.

2. We want to give a lower bound on the number of terms in the Hafnian expansion of

$$(\Phi_0, T\, \varphi_I(x_1) \ldots \varphi_I(x_e)\, H_I(y_1) \ldots H_I(y_n)\, \Phi_0)^T\,,$$

where $H_I = \lambda : \varphi_I^k :$ and $k \geqq 3$. Such a lower bound then clearly gives a lower bound on the number of graphs from any H_I of type (8). In fact we will not only give a bound on the number of connected graphs, but we will give an estimate on the number of graphs without one particle singularities. Each graph can be represented by n internal vertices to each of which $k \geqq 3$ lines are attached, and e external vertices to each of which exactly one line is attached. No lines start and end at the same vertex since the interaction is Wick ordered.

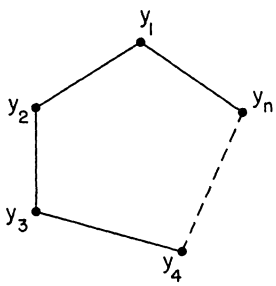

Fig. 1. Arrangement of internal vertices

Let $n > e$ and arrange the internal vertices on a circle. There are $(n-1)!$ ways to permute the points on the circle. Use two lines from each interaction vertex to form the ring (see Fig. 1).

Attach the external lines to e distinct vertices. These can be chosen in $\binom{n}{e}$ ways. This leaves $(n-e)$ internal vertices with at least one more unconnected line. These can be connected in $(n-e+1)!!$ ways. Thus there are more than $(n-1)!\ \binom{n}{e}\ (n-e+1)!! > n!\,(n-e+1)!!$ one particle irreducible graphs in n^{th} order.

3. *Proof of the Critical Lemma*: The proof of the lemma is based on three estimates:

3.1. We take the external momenta to have specially chosen Euclidean values, so we can use the *Nakanishi path theorem* (see Appendix A.8). This gives an absolute upper bound on the terms $\sum\limits_h \frac{W_h}{U}$ which occur in V of the Nakanishi form of the integral representation (A.6).

3.2. We take the external momenta to have property S. Then it is possible to use the estimate 3.1 to show that there are positive constants C_1 and D_1 such that

$$\left|\frac{1}{[V - i\,\varepsilon]^{n-1}}\right| > C_1\, D_1^n,$$

so this factor can be taken outside the integral. Furthermore, C_1 and D_1 remain strictly positive as $p_1, \ldots, p_e$ varies over a compact set of momenta with property S.

3.3. Now by going back to the Symanzik form of the representation and using the connection of $D(\alpha)$ with the skeleton of the graph, we can estimate the remaining integral by the lower bound

$$\int_0^\infty \left(\Pi \frac{d\alpha}{\alpha}\right) \frac{\delta(1 - \Sigma 1/\alpha)}{D(\alpha)} > \frac{C_2 D_2^n}{n!},$$

for positive constants C_2, D_2.

3.4. Combining estimates 3.2 and 3.3 with formula (A.6) gives the minorization

$$\begin{aligned} |I(p_1, \ldots, p_e)| &> (n-2)!\, C_3 D_3^n \left|\frac{1}{V - i\varepsilon}\right|_{\min}^{(n-1)} \int_0^\infty \left(\Pi \frac{d\alpha}{\alpha}\right) \frac{\delta(1 - \Sigma 1/\alpha)}{D(\alpha)} \\ &> (C_1 C_2 C_3)(D_1 D_2 D_3)^n \frac{(n-2)!}{n!} \\ &> \frac{A \tilde{B}^n}{n^2} > A B^n . \end{aligned}$$

which is the desired result. Thus we need only derive 3.1—3.3.

3.1. Let $p_i = \frac{mp}{e-1}$, where $p^2 = 1, i = 1, 2, \ldots, e-1$ and let $p_e = -mp$. Then p_i are Euclidean, parallel, and satisfy $\frac{m^2}{(e-1)^2} \leqq \Big(\sum_{\substack{\text{proper}\\ \text{subset}}} p_i\Big)^2 \leqq m^2$.
Hence by the Nakanishi path theorem (Appendix A.8) we deduce that $V \geqq 0$ and hence

$$\sum_h \frac{W_h}{U} \leqq (e-1)^2,$$

since $W_h \geqq 0$, and $U \geqq 0$*.

3.2. If the momenta p_i have property S, then

$$V = m^2 - \frac{1}{U}\sum_h W_h \Big(\sum_{i \in h} p_i\Big)^2 < m^2 + (e-1)^2 \sup_h \Big(\sum_{i \in h} p_i\Big)^2 .$$

It then follows that

$$m^2 < V < m^2 + |L(p_i)|,$$

and L varies over a compact set as p_i varies over a compact set of points with property S.

Thus

$$\left|\frac{1}{V - i\varepsilon}\right|^{n-1} > C_1 D_1^n,$$

where C_1, D_1 depend on p_i but vary over a compact interval as p_i varies over a compact set with property S.

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* It is possible to avoid using the Nakanishi path theorem at this point by giving a direct estimate on $\left|\frac{1}{V - i\varepsilon}\right|_{\min}$ when the parameters α are restricted to the subregion $2N < \alpha_i < 3N$, $i = 2, 3, \ldots, N$. Then this estimate would be combined with 3.3. However, it seems necessary to employ the theorem later to pass to the mass shell. Hence to avoid making these separate types of estimates we will now introduce the path theorem.

3.3. We wish to estimate

$$\int_0^\infty \left(\prod_{i=1}^N \frac{d\alpha_i}{\alpha_i}\right) \frac{\delta\left(1-\sum_{i=1}^N 1/\alpha_i\right)}{D(\alpha)} = \int_0^\infty \alpha_1 \left(\prod_{i=2}^N \frac{d\alpha_i}{\alpha_i}\right) \frac{1}{D(\alpha)}\Bigg|_{\left(\sum_{i=1}^N \frac{1}{\alpha_i}\right)=1} >$$

$$> \int_{2N}^{3N} \alpha_1 \left(\prod_{i=2}^N \frac{d\alpha_i}{\alpha_i}\right) \frac{1}{D(\alpha)}\Bigg|_{\left(\sum_{i=1}^N \frac{1}{\alpha_i}\right)=1}.$$

Now we use the fact that $D(\alpha)$ can be expressed as the sum of the α-products corresponding to all skeletons of the graph, each counted once (see Appendix A). There are at most Q^{n-1} skeletons in an n^{th} order connected graph, and each skeleton contains exactly $(n-1)$ lines. Thus we have an upper bound on $D(\alpha)$ in the region

$$2N < \alpha_i < 3N \quad i = 2, 3, \ldots, N\,.$$

In this region $1 < \alpha_1 < 2$ and

$$D(\alpha) < (3N)^{n-1}\, Q^{n-1}\,,$$

or

$$\frac{1}{D(\alpha)} > \frac{1}{(3NQ)^{n-1}}\,.$$

Hence

$$\int_0^\infty \left(\prod_{i=1}^N \frac{d\alpha_i}{\alpha_i}\right) \frac{\delta\left(1-\sum_{i=1}^N 1/\alpha_i\right)}{D(\alpha)} > \frac{(\log 3/2)^{N-1}}{(3NQ)^{n-1}}\,.$$

Since $N \leqq \frac{Q\,n-e}{2}$ and $n^{n-1} < c\, e^n\, n!$, we have

$$\int_0^\infty \left(\prod_{i=1}^N \frac{d\alpha_i}{\alpha_i}\right) \frac{\delta\left(1-\sum_{i=1}^N 1/\alpha_i\right)}{D(\alpha)} > \frac{C_2 D_2^n}{n!}\,.$$

This completes the proof of the Critical Lemma and Theorem 1.

Proof of Theorem 2. In order to prove Theorem 2, we note that in the proof of the Critical Lemma, only the constants C_1 and D_1 depend on the external momenta. Furthermore, as p_i vary over the compact set $\{\operatorname{supp} f\}$ of points with property S, then C_1 and D_1 vary over a compact interval of the positive real line. Hence take for C_1 and D_1 their minimum values on this interval so that

$$|I(p_1, \ldots, p_e)| > A\,B^n$$

for all $\{p_i\} \in \operatorname{supp} f$, and all n.

Hence

$$G(f, \lambda) = \int f(p_1, \ldots, p_e)\, \delta(p_1 + \cdots + p_e) \prod_{j=1}^{e} \left[\frac{i}{(2\pi)^2} \frac{1}{p_j^2 - m^2 + i\varepsilon} \right] \times$$

$$\times \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \sum_{\substack{\text{conn. } n^{\text{th}} \\ \text{order graphs}}} I(p_1, \ldots, p_e)\, dp_1 \ldots dp_e\,.$$

Since $f \geqq 0$ and $p_j^2 < m^2$ in $\operatorname{supp} f$, we have for

$$a_n = \int f(p_1, \ldots, p_e)\, \delta(p_1 + \cdots + p_e) \prod_{j=1}^{e} \frac{i}{(2\pi)^2} \frac{1}{p_j^2 - m^2 + i\varepsilon} \times$$

$$\times \sum_{\text{conn. } n^{\text{th}} \text{ order}} I(p_1, \ldots, p_e)\, dp_1 \ldots dp_e\,,$$

that

$$|a_n| > \left[\inf_{\substack{1 \leqq j \leqq e \\ \{p_i\} \in \operatorname{supp} f}} \left| \frac{1}{p_j^2 - m^2 + i\varepsilon} \right| \right]^e \left[\int f(p_1, \ldots, p_e)\, \delta(p_1 + \cdots + p_e) \times \right.$$

$$\left. dp_1 \ldots dp_e \right] \times A\, B^n (n - e + 1)!!\, n!\,, \quad \text{if} \quad n > e\,.$$

Hence $\sum_{n=0}^{\infty} \left| \frac{\lambda^n a_n}{n!} \right| \geqq \sum_{n=e+1}^{\infty} \left| \frac{\lambda^n a_n}{n!} \right|$ diverges for all $\lambda \neq 0$.

This completes the proof of Theorem 2.

V. Passage to the mass shell and the scattering amplitude

The quantities of interest on the mass shell are the *amputated* vacuum expectation values of time-ordered products of field operators. That is, they are the Green's functions with the external lines removed. The momenta we considered in the last section were all points $\{p_i\}$ with property S. This implies that each momentum variable p_i is space-like and so all these points lie off the mass shell defined by $\sum_{i=1}^{e} p_i = 0$, $p_i^2 = m^2$.

The difficulty in generalizing the above results comes from the possibility that thresholds can occur in the momentum variables. These singularities correspond to the production of real intermediate states. Such thresholds can destroy the coherence of terms which contribute to the n^{th} order amplitude and thus rule out the possibility of summing the minorizations of individual graphs to find a total minorization of the n^{th} order amplitude. In order to pass to the mass shell and preserve the essential features in the proof of Theorems 1 and 2, we consider partial amplitudes in which the threshold singularities do not occur on the mass shell.

1. *Proper self energy part*

The proper self energy part $\Pi(p)$ is just the one particle irreducible contribution to the two-point function. It is related to the propagator G and the free-particle propagator G_0 by the unrenormalized equation (3). Since intermediate states in Π have at least two particles, we will be able to apply the path theorem to these graphs with threshold $(2m)^2$ to derive

Theorem 3: *Let $\Pi(p_1, p_2, \lambda)$ be the proper self energy part in any theory of type 8. Let the external momenta satisfy*

$$p_1 + p_2 = 0$$
$$p_1^2 < 4m^2 . \tag{9}$$

Then $\Pi(p_1, p_2, \lambda)$ is not analytic in λ at $\lambda = 0$.

If $f(p_1, p_2)$ is a positive test function with compact support in the set of momenta satisfying (9), *then*

$$F(f, \lambda) = \int \Pi(p_1, p_2, \lambda)\, \delta(p_1 + p_2)\, dp_1\, dp_2$$

is not analytic at $\lambda = 0$. In other words, $\Pi\,\delta$ is not a distribution in any region below the two particle threshold, which is analytic at $\lambda = 0$.

Proof: We just extend to proof of Theorems 1 and 2. First we note that we will see that for momenta satisfying $p_1 + p_2 = 0$ and $p_1^2 < 4m^2$ we have $V > 0$. Hence in a given order all graphs contribute coherently. Secondly, we note that the number of graphs estimated in the proof of Theorem 1 actually gave a lower bound on the number of one particle irreducible graphs. Hence this estimate can be used for the proper self energy part to show that there are more than

$$n!\,(n-1)!!$$

graphs in n^{th} order. Thirdly, we now extend the Critical Lemma to hold for the external momenta in question in the case of proper self energy graphs. Let $p_1 = 2p$, $p_2 = -2p$, $p^2 = m^2$. Then $p_1^2 = p_2^2 = (2m)^2$, $p_1 + p_2 = 0$. Since the momenta are parallel, Euclidean and satisfy the threshold condition for two particle intermediate states, we can apply the Nakanishi path theorem (Appendix A.8) to deduce that

$$V > 0 .$$

Hence $\frac{W}{U} < \frac{1}{4}$ independently of the momenta. There is no sum over h since only one partition of proper self energy graphs can be made. From this we see that

$$V = m^2 - \frac{W}{U}(p_1)^2 > m^2 - \frac{1}{4} p_1^2 ,$$

or $V > 0$ if $p_1^2 < 4m^2$. Furthermore if $p_1^2 < 0$ we know the theorem is true as a special case of the Critical Lemma. For $p_1^2 > 0$ we have $V < m^2$ or

$$\left|\frac{1}{V}\right|^{n-1} > m^2 \left|\frac{1}{m^2}\right|^{n} .$$

This estimate on $\left|\frac{1}{V}\right|^{n-1}$ will be sufficient to complete the proof of the Critical Lemma. The proof of the theorem now follows exactly the proofs of Theorems 1 and 2.

2. Two particle scattering amplitude

The four momenta $p_1, \ldots, p_4$ involved in the two particle scattering amplitude are restricted on the mass shell by

$$\sum_{i=1}^{4} p_i = 0 \qquad p_i^2 = m^2 . \tag{10}$$

The second condition implies $|(p_i, p_j)| \geqq m^2$ which in turn leads to either

$$(p_i + p_j)^2 \geqq 4m^2 \quad \text{or} \quad (p_i + p_j)^2 \leqq 0 . \tag{11}$$

We say that a point lies *on or below* the mass shell if in (10) the relation $p_i^2 = m^2$ is replaced by $p_i^2 \leqq m^2$.

Let us consider the subset $\mathfrak{N}$ of points which satisfy:

$$\sum_{i=1}^{4} p_i = 0 \qquad p_i^2 < \frac{9}{2} m^2 \qquad (p_i + p_j)^2 \leqq \frac{9}{2} m^2 . \tag{$\mathfrak{N}$}$$

This includes a certain subset of the mass shell. The two particle scattering amplitude M will be *defined* as

$$M(p_1, \ldots, p_4; \lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \sum_{\substack{n^{\text{th}}\text{ order}\\ \text{connected}\\ \text{graphs}}} I(p_1, p_2, p_3, p_4) , \tag{12}$$

and let $M^{(2)}(p_1, \ldots, p_4; \lambda)$ be the two particle irreducible contribution to $M(p_1, \ldots, p_4; \lambda)$. One would then expect that $M^{(2)}(p; \lambda)$ for fixed λ is analytic in the momenta at points with property $\mathfrak{N}$. This in fact does follow in each order of perturbation theory so we can prove

Theorem 4: *Consider $M^{(2)}(p_1, p_2, p_3, p_4; \lambda)$ in any theory of type (8′) with the momenta p_i satisfying property $\mathfrak{N}$. Then $M^{(2)}$ is not analytic in λ at $\lambda = 0$. If $f(p_1, \ldots, p_4)$ is a positive test function with compact in $\mathfrak{D}(\mathfrak{N})'$ support in the set of points with property $\mathfrak{N}$, then*

$$(M^{(2)} \delta)(f, \lambda) = \int M^{(2)}(p_n, \ldots, p_4)\, \delta(p_1 + \cdots + p_4)\, f(p_1, \ldots, p_4)\, dp_1 \ldots dp_4$$

is not analytic at $\lambda = 0$. Hence $(M^{(2)} \delta)$ does not define a distribution in $\mathfrak{D}(\mathfrak{N})'$ which is analytic in λ at $\lambda = 0$.

Proof: We wish to extend the proofs of Theorems 1—2 to cover the case of the external momenta with property $\mathfrak{N}$. To do this we must give a lower bound on the number of two particle irreducible graphs in the n^{th} oredr. Secondly, we must get bounds for V to show that it does not vanish in the range of integration of the Feynman parameters. Hence all graphs in a given order will be coherent. Lastly we will show how to get a bound on V to prove the Critical Lemma for this case.

Take any graph of the form used in the proof of Theorem 1, but now with four external lines and assume at least a four point interaction at each vertex (Fig. 2a).

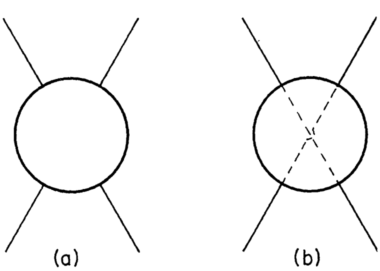

Fig. 2a and b. Certain two particle irreducible contributions to the two particle scattering amplitude

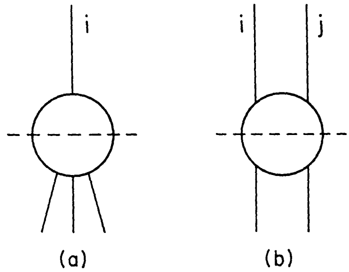

Fig. 3 a and b. Partitions of the two particl scattering amplitude

By making the connections shown in Fig. 2b, we end up with a two particle irreducible graph. This does not involve any of the remaining $(n-4)$ vertices so we still have more than

$$n!\,(n-3)!!$$

two particle irreducible graphs in n^{th} order if $n > 4$. In order to apply the path theorem, we choose

$$p_i = 3m\,p,\ p_2 = -3m\,p,\ p_3 = p_4 = 0\,,\quad p^2 = 1\,.$$

Then if W_i corresponds to a division of the graph in two parts as shown in Fig. 3a, and W_{ij} corresponds to the division in Fig. 3b, we have for this choice of momenta

$$V = m^2 - \sum_h \frac{W_h}{U}\Big(\sum_{i\in h} p_i\Big)^2 = m^2 - 9m^2\left(\frac{W_1 + W_2 + W_{13} + W_{14}}{U}\right) \geqq 0\,.$$

This gives

$$\frac{W_1 + W_2 + W_{13} + W_{14}}{U} \leqq \frac{1}{9}\,.$$

With all other choices of pairs of momenta we get similar bounds. Adding these gives

$$\frac{3(W_1 + W_2 + W_3 + W_4) + 4(W_{12} + W_{13} + W_{14})}{U} \leqq \frac{6}{9}\,.$$

or

$$\frac{W_1 + W_2 + W_3 + W_4 + W_{12} + W_{13} + W_{14}}{U} \leqq \frac{2}{9}\,.$$

Hence for $\{p_i\}$ with property $\mathfrak{A}$,

$$V = m^2 - \sum_h \frac{W_h}{U} \left(\sum_{i \in h} p_i\right)^2 > m^2 \left(1 - \frac{2}{9} \cdot \frac{9}{2}\right) = 0 \, .$$

Hence V does not vanish in the range of integration and so the denominator has constant sign. Furthermore $V \leqq m^2 + |L(p_i)|$ so that

$$\frac{1}{V} \geqq \frac{1}{m^2 + |L(p_i)|} \, ,$$

and the rest of the proof of the Critical Lemma follows for all two particle irreducible graphs contributing to the two particle scattering amplitude evaluated on the mass shell.

This completes the proof of Theorem 4, since the remainder of the proof just follows the proofs of Theorems 1 and 2.

VI. Mass renormalization

1. In the preceding sections we considered the unrenormalized theory. That is we assumed that the field satisfies an equation of the form

$$(\Box + m^2)\, \varphi(x) = \lambda\, J(x) \, ,$$

where the bare mass m is a given constant. Equivalently the current $J(x)$ contains no terms linear in φ. We then take an interaction Hamiltonian with interaction representation fields of mass m and expand φ in perturbation theory. For $\lambda = 0$, the two point function of φ will have a delta function in the spectral weight at $p_0 = (\vec{p}^{\,2} + m^2)^{1/2}$. However for $\lambda \neq 0$ the interaction $J(x)$ will shift the position of the delta function to $p_0 = (\vec{p}^{\,2} + \mu^2(\lambda))^{1/2}$. Hence the interacting field will give rise to one-particle states of mass $\mu(\lambda)$, and this defines the physical (renormalized) mass μ. We can then consider the mass shift $\delta\, m^2$ caused by the interaction

$$\mu^2(\lambda) = m^2 + \delta\, m^2(\lambda) \, .$$

Furthermore, we saw in Section II that $\delta\, m^2$ is related to the proper self energy Π by

$$\delta\, m^2(\lambda) = C\, \Pi(\mu^2(\lambda), \lambda) \, , \tag{4}$$

where

$$\Pi(\mu^2, \lambda) = \Pi(p_1, p_2, \lambda)\,|_{p_1^2 = p_2^2 = \mu^2(\lambda)} \, , \quad \text{and}$$

C is a constant. Thus we can ask if it is possible to use Theorem 3 to show that $\delta\, m^2(\lambda)$ is not an analytic function of λ at $\lambda = 0$. This question has not been resolved.

2. We can ask how the above discussion of non-analyticity in λ could be affected by renormalization. In order to discuss the $\lambda\,\varphi^3$ theory in four dimensional space-time, HURST, THIRRING, and PETERMANN had to consider a renormalized theory. What THIRRING calculates [5] (neglecting the field strength renormalization) is a perturbation expansion for the proper self energy in which the mass μ is put on each line and the expansion made in g, and yet it resembles an unrenormalized expansion in the $g\,\varphi^3$ theory. Then if this defines the function $\Sigma(p^2)$, he constructs "the renormalized" Σ,

$$\Sigma_R(p^2) \equiv \Sigma(p^2) - \Sigma(\mu^2)\,,$$

which has the desired property that each contribution to $\Sigma_R(p^2)$ vanishes for $p^2 = \mu^2$. However, as THIRRING points out, it is not necessarily the correct $\Pi_R(p^2)$. If, however, we take this definition of the renormalized expansion, Theorem 3 can be generalized to the statement that in any theory of type (8), $\Sigma_R(p^2)$ is not analytic in λ at $\lambda = 0$ if $p^2 < \mu^2$.

The proof is immediate since we need only extend the Critical Lemma to these subtracted integrals. Since

$$V(p^2) = \mu^2 - p^2\frac{W}{U} > \mu^2\left(1 - \frac{W}{U}\right) \geqq \frac{3\,\mu^2}{4}\,,$$

we have that

$$\left|\frac{1}{V(p^2)}\right|^{n-1} \leqq \left|\frac{1}{\mu^2\left(1-\frac{W}{U}\right)}\right|^{n-1} = \left|\frac{1}{V(\mu^2)}\right|^{n-1}.$$

Hence for $p^2 < \mu^2$, $\left|\frac{1}{V(p^2)}\right|^{n-1} < \left|\frac{1}{V(\mu^2)}\right|^{n-1}$,

and the Critical Lemma holds.

The problem of including all the graphs which arise from mass and charge renormalization counter terms has been studied by PETERMANN [6]. He claims that an arbitrary finite mass renormalization will not affect the divergence of the perturbation series, while on the other hand some particular choice of charge renormalization gauge might do this. It has not been possible to extend the methods of this paper to cover the cases in Petermann's discussion.

3. *Sign of the mass renormalization.* We note that under the assumptions that

a) the two point function of φ has a delta function at mass μ^2 and a continuum starting above this value,

b) the interacting field satisfies equal time commutation relations (with possibibly an infinite field strength renormalization),

c) $J(x)$ contains no terms linear in φ and has zero vacuum expectation value, it follows that $\delta\,m^2 < 0$ [20, 21]. In unrenormalized perturbation theory we see from (A.6) that for interactions of even degree and positive coupling constant every contribution to $i\Pi(p^2)$ is positive if $p^2 < 4m^2$. Hence this is true in every order of perturbation theory for interactions which formally satisfy the spectral condition.

Hence the physical mass shell lies below the bare mass shell for such interactions, and the scattering amplitude on the physical mass shell is evaluated at momenta covered by Theorem 4. However the implicit mass shell dependence on λ has not been accounted for. It is open whether $M^{(2)}$ is analytic when evaluated on the physical mass shell (with amputation now of external lines of physical mass).

Another open question is the extension of the above results to an arbitrary interaction which is an entire function of the free field. This general class of interactions does not seem accessible by the method discussed above.

VII. Conclusion

We conclude that for a large class of interactions, $H_I(x) = \lambda \sum_{j=3}^{\infty} a_j : \varphi(x)^j :$, $a_j \geqq 0$, the unrenormalized perturbation theory in two dimensional space-time is finite in each order, but does not define Green's functions which are analytic in λ at $\lambda = 0$. The explicit statements are made in Theorems 1—4. In particular this is true for external momenta with property S.

The basic idea of the proof is to use the trick of Hurst and Thirring [4, 5], who show that the contributions from all graphs of a given order add coherently.

However, we here derive a simple and general lower bound for the contribution from an *arbitrary* connected Feynman graph of order n,

$$|I(p_1, \ldots, p_e)| > A\,B^n\,,$$

where A, B are constants depending only on $p_1, \ldots, p_e$. Such an estimate holds in two-dimensional space-time for interactions

$$H_I(x) = \lambda \sum_{j=1}^{Q} a_j : \varphi(x)^j :$$

1. At points $(p_1, p_2, \ldots, p_e)$ with property S,
2. For the proper self energy parts of $p^2 < 4m^2$,
3. For the two particle irreducible contribution to the two particle scattering amplitude for points with property $\mathfrak{R}$.

This set of points is large enough to show that perturbation series *does not* define Green's functions which are distributions in $(p_1, \ldots, p_e)$ and which are analytic in the parameter λ at $\lambda = 0$.

We have not been able to include in this analysis the effects of renormalization as discussed by Petermann [6].

Appendix A

Integral representations [22—30]

We use the following notation:

e = number of external lines
n = number of internal vertices
k = order of interaction
N = number of internal lines
l = number of loop variables (non-trivial integrations)
s = number of space dimensions
h = partition of the set of external momenta into two proper subsets. It also may refer to the collection of momenta in one of these two sets of momenta.
$s(h)$ denotes the set of intermediate states which induce the division h of the external momenta into two sets. An intermediate state is a set of internal lines which when cut divide the graph in two (Notation of NAKANISHI [27]).

Note that for a fixed degree k of interaction, $H_I(x) = \lambda : \varphi^k :$,

$$\begin{aligned} N &= \frac{k\,n - 2}{2}\,, \\ l &= N - (n-1) = \frac{(k-2)\,n - e + 2}{2}\,, \\ 2l - 1 &= (k-2)\,n - e + 1\,. \end{aligned} \tag{A.1}$$

For convenience we collect here some forms of the Chisholm-Nambu-Symanzik-Nakanishi integral representation for amputated, connected graphs in $(s+1)$ dimensional space time. The normalization of phase is chosen to give formula (7).

Symanzik form [23]

$$I(p_1, p_2, \ldots, p_e) = -i\left(N - \frac{s+1}{2}\,l - 1\right)!\,(-1)^{(n-1)}\,A\,B^l \left(\prod_{\substack{\text{vertices}\\ \text{in graph}}} a_j\right) \times$$

$$\times \int_0^\infty \left(\prod_{i=1}^N \frac{d\alpha_i}{\alpha_i^{(3-s)2}}\right) \frac{\delta\left(1 - \sum_{i=1}^N 1/\alpha_i\right)}{D(\alpha)^{\left(\frac{s+1}{2}\right)}\left[M(\alpha) + \dfrac{D(p,\alpha)}{D(\alpha)} - i\,\varepsilon\right]^{\left[N - \left(\frac{s+1}{2}\right)l\right]}}\,. \tag{A.2}$$

Here A and B are positive constants which depend on s. The α's are inverse Feynman parameters, and will be enumerated in two ways. Since each internal line has a corresponding α, they may be indexed α_i where $1 < i < N$. On the other hand, if α corresponds to the l^{th} line connecting vertex j of a graph to vertex k, then α might be denoted α_{jk}^l where

$1 \leqq j, k \leqq n$ and $l = 1, 2, \ldots$. Use $\alpha_{jk}^l = \alpha_{kj}^l$, $\alpha_{kk}^l \equiv 0$, $B_j = \sum_{k=1}^{n} \beta_{jk}$, $\beta_{jk} = \sum_l \alpha_{jk}^l$, and

$$k_i = \sum_{\substack{\text{all external momenta } p_j \\ \text{entering vertex } i}} p_j \,.$$

Then $M(\alpha) = \sum_{i=1}^{N} \frac{m_i^2}{\alpha_i} = m^2$ in the equal mass case.

$$C(\alpha) = \begin{pmatrix} B_1 & -\beta_{12} & -\beta_{13} & \cdots & -\beta_{1n} \\ -\beta_{12} & B_2 & -\beta_{23} & \cdots & -\beta_{2n} \\ -\beta_{13} & & & & \\ \vdots & & & & \\ -\beta_{1n} & & & & B_n \end{pmatrix}.$$

$$C(p, \alpha) = \begin{pmatrix} 0 & k_1 & k_2 & \cdots & k_n \\ k_1 & & & & \\ \vdots & & C(\alpha) & & \\ k_n & & & & \end{pmatrix}. \tag{A.3}$$

$D(\alpha)$ is the minor of $C(\alpha)$ corresponding to omitting some column and the corresponding row. $D(p, \alpha)$ is the minor of $C(p, \alpha)$ corresponding to omitting some column and the corresponding row, but not the first. The derivation of these formulae is in TODOROV [29].

In two dimensional space time we have

$$I(p_1, \ldots, p_e) = -i(-1)^{n-1}(n-2)!\, 4\pi^2 \left(\frac{1}{4\pi}\right)^l \Bigl(\prod_{\substack{\text{vertices}\\ \text{in graph}}} a_i\Bigr) \times$$

$$\times \int_0^\infty \left(\prod_{i=1}^{N} \frac{d\alpha_i}{\alpha_i}\right) \frac{\delta\left(1 - \sum_{i=1}^{N} 1/\alpha_i\right)}{D(\alpha)\left[M(\alpha) + \frac{D(p, \alpha)}{D(\alpha)} - i\varepsilon\right]^{n-1}} \tag{A.4}$$

and in four dimensional space-time

$$I(p_1, \ldots, p_e) = -i(-1)^{n-1}(n-l-2)!\,(2\pi)^4 \left(\frac{1}{4\pi}\right)^{2l} \Bigl(\prod_{\substack{\text{vertices}\\ \text{in graph}}} a_j\Bigr) \times$$

$$\times \int_0^\infty (\prod d\alpha_i) \frac{\delta\left(1 - \sum_{i=1}^{N} 1/\alpha_i\right)}{D(\alpha)^2\left[M(\alpha) + \frac{D(p, \alpha)}{D(\alpha)} - i\varepsilon\right]^{n-l-1}}.$$

SYMANZIK [23] remarks that $D(\alpha)$ is the sum of all α-products corresponding to all skeletons of the graph, each counted once. A skeleton is a simply connected subgraph of the original graph which contains all the original vertices. The proof of this statement is in BOTT and MAYBERRY [26].

Nakanishi form [27—28]

NAKANISHI uses the usual Feynman parameters $x_i = \frac{1}{\alpha_i}$. Then the integral representation becomes

$$I(p_1, \ldots, p_e) = -i\left(N - \frac{s+1}{2}\, l - 1\right)! \, (-1)^{(n-1)} A\, B^l \left(\prod_{\substack{\text{vertices}\\ \text{in graph}}} a_j\right) \times$$

$$\times \int\limits_0^\infty \prod_{i=1}^N \frac{dx_i}{x_i^{(s+1)2}} \frac{\delta\left(1 - \sum_{i=1}^N x_i\right)}{D\left(\frac{1}{x_i}\right)^{\frac{s+1}{2}} [V - i\,\varepsilon]^{\left[N - \left(\frac{s+1}{2}\right) l\right]}}, \tag{A.5}$$

which reduced in two dimensional space-time to

$$I(p_1, \ldots, p_e) = -i(n-2)! \, (-1)^{n-1} (4\pi^2) \left(\frac{1}{4\pi}\right)^l \left(\prod_{\substack{\text{vertices}\\ \text{in graph}}} a_j\right) \times$$

$$\times \int\limits_0^\infty \left(\prod_{i=1}^N \frac{dx_i}{x_i}\right) \frac{\delta\left(1 - \sum_{i=1}^N x_i\right)}{D\left(\frac{1}{x_i}\right) [V - i\,\varepsilon]^{n-1}}, \tag{A.6}$$

where $D \geqq 0$ and, $U = \left(\prod_{i=1}^N x_i\right) D\left(\frac{1}{x}\right) \geqq 0$,

$$V = \sum_{i=1}^N x_i m_i^2 - \sum_h \frac{W_h}{U} \left(\sum_{i \in h} k_i\right)^2. \tag{A.7}$$

W_h is a positive function of the Feynman parameters x_i, depending on h.

Nakanishi path theorem

If (1) k_i *are parallel*; (2) k_i *are Euclidean*; *and* (3) *for all intermediate states* S *in* $S(h)$

$$0 < \left(\sum_{i \in h} k_i\right)^2 < \left(\sum_{i \in S} m_i\right)^2, \tag{A.8}$$

Then $V \geqq 0$.

A set of vectors k_i is *Euclidean* if for all real a_i, $(\sum a_i k_i)^2 \geqq 0$.

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