Communications in **M**athematical **A**nalysis
Volume 22, Number 2, pp. 35–60 (2019) www.math-res-pub.org/cma
ISSN 1938-9787

Two Results Relating an L^p Regularity Condition and the L^q Dirichlet Problem for Parabolic Equations

Luis San Martín*
Instituto de Matemáticas - Unidad Cuernavaca
Universidad Nacional Autónoma de México
Cuernavaca CP 62210, México

Jorge Rivera-Noriega†
Departamento de Matemáticas
Instituto Tecnológico Autónomo de México
Rio Hondo 1, Col. Progreso Tizapán
Ciudad de México CP 01080, México

(Communicated by Palle Jorgensen)

Abstract

We consider variations and generalizations of the initial Dirichlet problem for linear second order divergence form equations of parabolic type, with vanishing initial values and non-continuous lateral data, in the setting of Lipschitz cylinders. More precisely, lateral data in adequations of the Lebesgue classes L^p, and a family of Sobolev-type classes are considered. We also establish some basic connections between estimates related to solvability of each of these problems. This generalizes some of the well-known works for Laplace's equation, heat equation and some linear elliptic-type equations of second order.

AMS Subject Classification: Primary: 35K10, 35K20; Secondary: 35B09, 35B65

Keywords: Initial L^p Dirichlet problem, initial L^q regularity problem, Lipschitz cylinder

1 Introduction and preliminary definitions

Some historical background and context

Questions of solvability of boundary value problems associated to equations of elliptic and parabolic type have always been a central topic in the theory of partial differential equations.

*E-mail address: luisr@matcuer.unam.mx
†E-mail address: rnoriega@itam.mx

Of particular interest are the generalizations where either the datum or the boundary of the underlying domain are assumed less regular than in classical results.

In the present work we take up an issue related to this circle of ideas, focusing on estimates associated to initial-boundary problems for second order linear equations of parabolic type and divergence form over a Lipschitz cylider. These equations have the form

$$\operatorname{div}(A(x)\nabla u(x,t)) - \frac{\partial u(x,t)}{\partial t} = 0, \quad x \in \mathbb{R}^n,\, t \in \mathbb{R}, \tag{1.1}$$

where both the divergence and the gradient are taken with respect to x variables only. The $n \times n$ matrix $A(x)$ consists of smooth bounded functions, and is assumed to be symmetric, and to satisfy the ellipticity condition (1.2) below.

It has been a common ocurrence that some techniques originally designed for problems associated to boundary value problems associated to the Laplace operator are later adapted to the heat equation. And the same can be said about more general equations of elliptic and parabolic type. However, the adaptation of techniques that work in the elliptic setting is not always trivial or straightforward, as the references we are about to mention have shown.

The Dirichlet-type problems that motivated questions of this kind were originally studied for Laplace equation on C^1 and Lipschitz domains (see e.g. [19, 12, 7, 8, 20, 30]) and for second order, divergence form linear equations of elliptic type (see e.g. [16, 27]). Actually, this last reference is the main motivation for studying the questions we solve herein. For similar equations of parabolic type, see e.g. [21, 11, 14, 10], and in the setting of non-cyilindrical domains see e.g. the fundamental works [17, 18] and some more recent work in [24, 25, 26].

For the initial-Dirichlet problem, one may attach to the parabolic equation (1.1) a *vanishing initial condition*, along with a condition on the *lateral boundary* of a cylinder $\Omega = D \times [0,T)$, where D is a bounded Lipschitz domain in $\mathbb{R}^n$ and $T > 0$:

$$u(x,0) = 0 \quad \text{for } x \in D, \qquad u(y,s) = f(y,s) \quad \text{for } (y,s) \in \partial D \times [0,T).$$

A special feature of the non-classical problem we want to describe, is that f is not necessarily continuous, and so this boundary datum is attained almost everywhere. Hence one imposes an extra condition in the form of the boundedness of certain boundary *non-tangential maximal operator* associated to u.

Having said this, we emphasize that our argumentations are focused on the maximal estimates associated to two variations of the previously described problem, described in the next paragraphs.

We should point out a more recent work [6] that deals with related problems, although with a different approach than ours.

Initial description of results

Having described some of the historical background, we now start the technical description of the problems we have been mentioned before.

The main results in this paper describe a couple of particular connections between estimates related to the solvability of the initial L^p Dirichlet problem and of the initial L^q

regularity problem on Lipschitz cylinders. This type of domains have the form $\Omega = D \times \mathbb{R}$, where D is a bounded Lipschitz domain in $\mathbb{R}^n$, $1 < p, q < \infty$.

In these initial-Dirichlet problems, the datum provided in the lateral boundary S belongs to an adequate Lebesgue class $L^p(S)$, $1 < p < \infty$, and a Sobolev-type class $W_p^{1,1/2}(S)$ defined through (1.7) below, respectively.

For our results we also require the main coefficients of the parabolic operator to be independent of the time variable (as for instance in [6]). This feature will guarantee that one may obtain solutions to the adjoint equation from solutions to the original equation via a *reflection in the time variable*.

An estimate for a non-tangential maximal function (from now on refered to as the $(D)_p$ condition) turns out to be the right estimate to pose the initial L^p Dirichlet problem (see (1.5) below). We propose here the condition $(R)_q$ as the adequate maximal condition (see (1.8) below) for a initial L^q regularity problem.

A rough description of the main results in this work are as follows:

- In our first main result (Theorem 1.3 below), we prove that the condition $(R)_p$ implies the $(D)_{p'}$ condition, where $1 < p < \infty$, and $p' = p/(p-1)$. This result resembles [16, Theorem 5.4], only that we do not define a *regularity boundary value problem* subsumed in the L^q regularity condition that we define herein.

- Our second main theorem establishes a partial reciprocal result, which as far as we know is the first adaptation to parabolic equations of the result in [27]. On way to describe the result is by saying that if the $(D)_{p'}$ condition holds along with the $(R)_q$ for certain $1 < q < p$, then the condition $(R)_p$ holds. See the Theorem 1.4 for the precise statement.

The regularity condition $(R)_p$ adopted in our work contains a definition of a Sobolev-type space on the boundary of the Lipchitz Ω_T, and is aimed to generalize for $p \neq 2$ the one introduced in [4].

The adaptations we provide are not straightforward consequences from the situation for elliptic equations. For instance, unlike the elliptic operator in [16, 27], the parabolic operator is not self adjoint, and by the evolutionary nature of the parabolic equations, certain basic estimates for solutions, parabolic measure and Green's function have a "shift in the time variable" which requires different argumentations than those for elliptic equations. To describe more properly the results in this work we introduce some notation and definitions.

Notations and definitions

Now that we have presented the basic description and background for our results, we are ready to provide more technical material in order to give precise statements of what we described in the previous paragraphs.

Points in $\mathbb{R}^{n+1}$ will be denoted by $X = (x,t) = (x', x_n, t)$, where the variables $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \equiv \mathbb{R}^n$ are referred to as the *space variables* and $t \in \mathbb{R}$ is conceived as the *time variable*.

The operator of parabolic type has the divergence form $\mathcal{L} = \operatorname{div}(A(x)\nabla) - \partial_t$, where ∇ denotes the gradient with respect to space variables only, $\partial_t = \dfrac{\partial}{\partial t}$, and where the coefficients form a symmetric matrix of functions $A(x) = (a_{i,j}(x))$ which are assumed to be smooth, and to satisfy the ellipticity condition

$$\lambda|\xi|^2 \le \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}(x)\xi_j\xi_i \le \frac{1}{\lambda}|\xi|^2 \quad \text{for all } x \in \mathbb{R}^n \text{ and } \xi = (\xi_1,\xi_2,\dots,\xi_n) \in \mathbb{R}^n, \tag{1.2}$$

and where $|\xi|$ denotes the Euclidean norm of ξ.

As usual, the smoothness assumption for the coefficients of $\mathcal{L}$ is adopted to initially consider smooth solutions to the equation $\mathcal{L}u = 0$, that is $u \in C^{2,1}$ (two continuous derivatives in space variables and one continuous derivative in t variable). Well known limiting arguments allows us to extend the results to weaker notions of solutions (see e.g. [1, Theorem 1, p. 634]). The reason is that the only quantitative information that will arise in the constants of the results and estimates we invoke, comes from the ellipticity constant λ, the dimension n, and geometric constants of the domain D we are about to specify.

We emphasize that we have assumed that the main coefficients of the operator $\mathcal{L}$ are independent of t. To justify this assumption recall the definition of adjoint solutions associated to $\mathcal{L}$. These are functions $v \in C^{2,1}$ which are solutions to the equation $\mathcal{L}^*v = 0$, where $\mathcal{L}^* = \operatorname{div}(A(x)\nabla) + \partial_t$. Hence, if $\mathcal{L}u = 0$ on a domain Ω then $v(x,t) = u(x,-t)$ is solution to $\mathcal{L}^*v = 0$ on $\tilde{\Omega}$, the reflection in t variable $t \mapsto -t$ of the domain Ω.

And even though $\mathcal{L}^*$ has the same ellipticity coefficients than $\mathcal{L}$, they may be very different operators. A couple of times in our argumentations we employ auxiliar adjoint solutions arising from the application of this *reflection mapping*, and in particular, for Green's function for $\mathcal{L}$ in Ω, we compare a couple of different values in the *adjoint variable*. This is also the reason why our proofs work on cylindrical domains.

An open and bounded domain $D \subset \mathbb{R}^n$ is a *Lipschitz domain* if its boundary is given locally by Lipschitz functions. This means that for every $P = (p_1,\dots,p_{n-1},p_n) \in \partial D$, there is a new local coordinate system (x',x_n), $x' = (x_1,\dots,x_{n-1}) \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$, and with respect to these coordinates, one can find

- A *rectangle of radius* $r > 0$ of the form

$$R = R(P,r) = \Big\{(x',x_n) : |x_i - p_i| < r,\ \ |x_n - p_n| < 2nm_p r, i = 1,\dots,n-1\Big\};$$

- A function $\phi = \phi_P : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying $|\phi(x') - \phi(y')| \le m_P|x' - y'|$ for certain $m_P > 0$,

with the following significance. In this new local coordinate system (x',x_n), one has

(i) $2R \cap \partial D = 2R \cap \{(x',x_n) : x_n = \phi(x')\}$

(ii) $2R \cap D = 2R \cap \{(x',x_n) : x_n > \phi(x')\}$

where $2R$ is the rectangle concentric to R with twice its radius.

By compactness of ∂D we can choose a finite number of rectangles $R_1,\cdots,R_N$ with the same radius r_0 covering ∂D and a finite number of Lipschitz functions $\phi_1,\cdots,\phi_N$ satisfying the conditions above with an absolute and unique Lipschitz constant m. In fact one can always take $0 < r_0 < 1$. Once this constant is fixed, one can define local geometric objects within Lipschitz cylinders, whose definition we recall shortly.

An *infinite Lipschitz cylinder with constants m and* r_0 is an open set of the form $\Omega = D\times\mathbb{R}$ where D is a Lipschitz domain with constants m and r_0 as described above. We denote by S the *lateral boundary of* Ω, defined as $S = \partial D\times\mathbb{R}$. On S we can consider the surface measure σ given by the product measure $d\sigma = d\widetilde{\sigma}\times dt$, where $\widetilde{\sigma}$ denotes the surface measure on the Lipschitz domain D and dt is the Lebesgue measure on $\mathbb{R}$.

For $r < r_0/10$ and $Q = (q,s)\in S$ we define the *Carleson boxes*, *surface balls* and *right and left corkscrew points* (in that order) as

$$\begin{aligned}\Psi_r(Q) = \{X = (x',x_n,t)\in\Omega : |x_i - q_i| < r,\ i = 1,\cdots,n-1,\\ \psi(x',t) < x_n < \psi(x',t)+4nmr,\ |s-t| < r^2\};\end{aligned}$$

$$\Delta_r(Q) = S\cap\overline{\Psi_r(Q)};$$

$$\overline{\mathcal{A}}_r(Q) = (q',\psi(q',s)+6nmr, s+2r^2), \qquad \underline{\mathcal{A}}_r(Q) = (q',\psi(q',s)+6nmr, s-2r^2).$$

The *parabolic cubes in* $\mathbb{R}^{n+1}$ are defined by

$$Q_r(X) = \left\{Y = (y,s)\in\mathbb{R}^{n+1} : |x-y| < r,\ |t-s| < r^2\right\}, \quad 0 < r < r_0.$$

In order to define the conditions $(R)_p$ and $(D)_p$, we introduce the *non tangential approach regions* $\Gamma_\alpha(Q) = \{X\in\Omega : \delta(X,Q)\le(1+\alpha)\delta(X)\}\cap\Psi_{r_0}(Q)$. Here, $\delta(X) = \delta(X;S)$ is the parabolic distance from X to the lateral boundary S and is given by $\delta(X;S) := \inf_{Q\in S}\delta(X;Q)$, where the *parabolic distance* between $X = (x,t)\in\mathbb{R}^{n+1}$ and $Y = (y,s)\in\mathbb{R}^{n+1}$ is $\delta(X;Y) = |x-y| + |t-s|^{1/2}$.

Our main results are stated in the setting of a finite Lipschitz cylinder of the form $\Omega_T = D\times(0,T]$ with lateral boundary $S_T = S\times(0,T]$, with $T > 0$ fixed, and where a fixed parabolic center has been defined as $\Xi = (0, T+1)$. This makes it easier to pose as an adequate definition of an L^q regularity problem adapted to Ω_T.

Conditions $(D)_p$ and $(R)_p$ and statement of the main results

It is well known that a Lipschitz cylinder Ω is a regular domain for Dirichlet-type problems associated to any parabolic operator that satisfies condition (1.2) (for instance using parabolic capacity, see e.g. [9]). This implies, for instance, that for every continuous function f defined and compactly supported on S_T, there exists a unique solution $u\in C^{2,1}(\Omega)\cap C(\overline{\Omega})$ such that

$$\begin{cases}\mathcal{L}u(X) = 0 & X\in\Omega\\ \lim\limits_{\substack{X\to Q\\ X\in\Omega}} u(X) = f(Q) & Q\in S_T.\end{cases} \tag{1.3}$$

From this, through the Riesz Representation Theorem, we can define the parabolic measure. The *$\mathcal{L}$-parabolic measure at* $X\in\Omega$ is the unique Borel measure $\omega^X = \omega^X_{\mathcal{L}}$ such that the solution to (1.3) is represented at $X\in\Omega$ by

$$u(X) = \int_{S_T} f(Q)\,d\omega^X(Q). \tag{1.4}$$

We note that by the maximum principle this measure is actually supported on S_T.

To define solvability of the L^p *Dirichlet problem* for $\mathcal{L}$ in Ω, one requires that the solution to (1.3) also must satisfy

$$\|(u)^*\|_{L^p(S_T)} \le c\|f\|_{L^p(S_T)}, \quad \text{where the constant } c > 0 \text{ is independent of } f. \tag{1.5}$$

Here $(u)^*$ is the *non tangential maximal function* of u, defined as

$$(u)^*(Q) = \sup_{X \in \Gamma(Q)} |u(X)|. \tag{1.6}$$

When (1.5) holds, we say that *the* $(D)_p$ *condition holds for* $\mathcal{L}$ *in* Ω_T.

The L^p Dirichlet problem associated to parabolic operators as $\mathcal{L}$, even with coefficients depending on time variable, is by now a well understood topic, see [24]. For instance, following ideas from related problems associated to elliptic equations, it has been noted that in order to obtain the estimate (1.5), because of comparability of u^* and Hardy-Littlewood's maximal operator (see e.g. [24, p. 224-226]), it is equivalent to obtain a reverse Hölder property for the Radon-Nikodým property of the parabolic measure with respect to the surface measure.

In other words, obtaining the estimate (1.5) becomes a local matter, in the sense that one must prove an estimate for every surace cube in S_T. For this reason, when proving the estimate (1.5) we work in the setting of a domain above a graph, as we mention later in this section.

Based on the definitions of similar problems for elliptic equations from [16], and the heat equation in [3], we consider an additional regularity condition for this Dirichlet-type problem associated to $\mathcal{L}u = 0$ on a finite Lipschitz cylinder Ω_T. Given the nature of a parabolic equation we use a Sobolev-type space over S with the usual derivatives in the (space) tangent directions, and a half order derivative in time direction.

The mixed norm space $W_p^{1,\frac{1}{2}}(S)$ is defined as the closure of the set

$$\left\{g = f|_S : f \in C_0^\infty(\mathbb{R}^n \times (0,\infty))\right\}$$

with respect to the seminorm

$$\|f\|_{W_p^{1,\frac{1}{2}}(S)} = \left(\int_S (|\nabla_{tan} f|^p) d\sigma\, dt + \int_S |\partial_t^{\frac{1}{2}} f|^p\, d\sigma\right)^{\frac{1}{p}} \tag{1.7}$$

where $\nabla_{tan} f = \nabla f - \nu(\nabla f \cdot \nu)$ is the *tangential gradient* of f, ∇f is the spatial gradient of f, ν is the exterior normal unit vector to ∂D and

$$\partial_t^{\frac{1}{2}} f(x,s) = \left(\int_{-\infty}^T \frac{|f(x,s) - f(x,t)|^2}{|s-t|^2}\, dt\right)^{\frac{1}{2}}.$$

This definition is taken from [28, p. 1034].

Remark 1.1. The definition of the mixed norm space $W_p^{1,\frac{1}{2}}(S_T)$ is meant to generalize to $p \neq 2$ the space adopted in [3, p. 352-353] when solving an initial L^2 regularity problem for the heat equation. In [4] it is adopted another extension, using parabolic Riesz potentials, following a definition from [11]. This extension is also adopted by [17, 18] in the setting of non-cylindrical domains. In any case, for $p = 2$ the definitions coincide, by Plancherel's theorem, as observed for instance in [17, p. 353].

We say that the $(R)_p$ *condition holds for* $\mathcal{L}$ *on* Ω_T whenever the following estimate holds

$$\|\mathcal{N}(\nabla u)\|_{L^p(S_T)} \leq c\|f\|_{W_p^{1,\frac{1}{2}}(S_T)} \tag{1.8}$$

for each $f \in C_0^\infty(\mathbb{R}^n \times (0,\infty))$ and u the corresponding solution to (1.3). Here, the *modified non-tangential maximal function* of a continuous function v defined on Ω is defined as

$$\mathcal{N}v(Q) = \mathcal{N}_\alpha v(Q) = \sup_{X\in\Gamma_\alpha(Q)} \left(\fint_{Q(X)} |v|^2 \, dY \right)^{\frac{1}{2}}, \tag{1.9}$$

where $Q(X) \equiv Q_{c^{-1}\delta(X)}(X)$ is such that $Q(X) \subset \Omega$ and $\alpha = \alpha(m) > 0$ is fixed. We are adopting the notation $\fint_{Q(X)} v \, dY$ for the *integral average* $\frac{1}{\delta(X)^{n+1}} \int_{Q(X)} v \, dY$. Integral averages with measure different to the Lebesgue measure on $\mathbb{R}^{n+1}$ will be used later on in this paper, and its meaning should be clear from the context. Also, in later arguments $\mathcal{N}(\nabla u)$ will be used instead of $\mathcal{N}(|\nabla u|)$.

Remark 1.2. Given $\alpha, \beta > 0$, using standard arguments (see e.g. [29, §6.2-6.4] or the original argumentation in [15, p. 166]) one may prove that for any function h defined on Ω

$$\|\mathcal{N}_\alpha(h)\|_{L^p(S_T)} \approx \|\mathcal{N}_\beta(h)\|_{L^p(S_T)}.$$

This will become useful when proving the Theorem 1.3 that we state below.

With all the previous definitions and remarks, we are now in position to make precise description of our goal in this work. We prove the following two results relating the conditions $(R)_q$ and $(D)_p$ under the assumptions stated above.

Theorem 1.3. *Let* Ω *be a Lipschitz cylinder and* $\mathcal{L}$ *an operator satisfying* (1.2)*. If condition* $(R)_p$ *holds for* $\mathcal{L}$ *in* Ω_{T+1}*, then condition* $(D)_{p'}$ *holds for* $\mathcal{L}$ *in* Ω_T*,* $1/p + 1/p' = 1$.

Theorem 1.4. *Let* Ω *be a Lipschitz cylinder and* $\mathcal{L}$ *an operator satisfying* (1.2)*. Let* $1 < p < \infty$*, and suppose that the condition* $(R)_q$ *for* $\mathcal{L}$ *in* Ω_T *for some* $1 < q < p$*. If* $1/p + 1/p' = 1$ *then the condition* $(D)_{p'}$ *for* $\mathcal{L}$ *in* Ω_T *implies that the condition* $(R)_p$ *for* $\mathcal{L}$ *in* Ω_T *holds.*

The next result is an immediate consequence of these theorems, a well-known property of the condition $(D)_p$ and the classical theory of Muckenhoupt weights and reverse Hölder inequalities (see e.g. [24, Theorem 6.1]).

Corollary 1.5. *Let* $1 < q < \infty$ *and assume that the* $(R)_q$ *condition is satisfied on* Ω_T*. Then there exists* $\epsilon > 0$ *such that the* $(R)_s$ *condition is satisfied on* Ω *for every* $q < s \leq q + \epsilon$.

Proof. If $(R)_q$ holds by Theorem 1.3 we know $(D)_{q'}$ holds, with $1/q + 1/q' = 1$. But then thete exists $\epsilon > 0$ such that $(D)_s$ holds for $s \in (q' - \epsilon, q')$. Noticing that $s' \in (q, q + \epsilon)$, where $s' = s/(s-1)$, by Theorem 1.4 we now know $(R)_s$ for $s \in (q, q + \epsilon)$. □

Sections 3 and 4 are devoted to prove Theorem 1.3, while the remaining sections deal with the proof of Theorem 1.4. The next section gathers some preliminary results.

2 Some known results

The constants playing a role in each of the following results depend only on the ellipticity constant, the dimension and the geometric features of the Lipschitz cylinder Ω, such as m, the Lipschitz constant of D.

Theorem 2.1 (Harnack's Inequality)**.** *[23, Theorem 2] Let u be a nonnegative solution of $\mathcal{L}u = 0$ in Ω_T. Let D' be a convex subdomain of D such that $\delta = dist(D', \partial D) > 0$. Then for all $x, y \in D'$ and $0 < s < t \leq T$ we have*

$$u(y,s) \leq u(x,t) \exp\left[c\left(\frac{|x-y|^2}{t-s} + \frac{t-s}{R} + 1 \right) \right]$$

where $c = c(n, \lambda)$ and $R = min\{1, s, \delta^2\}$.

Theorem 2.2 (Carleson-type estimate)**.** *[10, Theorem 0.3] Let $Q = (q,s) \in S$ and $0 < r < \min\left\{r_0, \sqrt{s}\right\}$. Then for any nonnegative solution of $\mathcal{L}u = 0$ in Ω vanishing continuously on $\Delta(Q, 2r)$, we have*

$$\sup_{\Psi_r(Q)} u \leq c\, u(\overline{\mathcal{A}}_r(Q))$$

where the constant $c = c(n, \lambda, m, r_0) > 0$.

Theorem 2.3. *[21, Lemma 1.1] Let $Q = (q,s) \in S_T$ and $0 < r < \frac{1}{2}\min\left\{r_0, \sqrt{s}, \sqrt{T-s}\right\}$. Then, for each $X \in \Psi_{r/2}(Q)$ we have*

$$\omega^X(\Delta_r(Q)) \geq c$$

where $c = c(n, \lambda, m) > 0$.

The *Green's function of $\mathcal{L}$ on Ω* with pole at $X = (x,t) \in \Omega$ is denoted by $G(X;Y)$ and defined as

$$G(X;Y) = \Gamma(X;Y) - \int_{\partial_p \Omega} \Gamma(Z;Y)\, d\omega_X(Z) \tag{2.1}$$

where $\Gamma(X;Y)$ is the fundamental solution of $\mathcal{L}$. In the next two Theorems, $G(X,Y)$ denotes the Green function, for $X, Y \in D \times \{-1 < t < T+2\}$.

Theorem 2.4. *[10, Theorem 1.4] Let $Q = (q,s) \in S_T$ and $0 < r < \frac{1}{2}\min\left\{r_0, \sqrt{s}, \sqrt{T-s}\right\}$. Then, for each $X = (x,t) \in \Omega_T$ with $s + 4r^2 \leq t \leq T$ we have*

$$c^{-1} r^n G(X; \overline{\mathcal{A}}_r(Q)) \leq \omega^X(\Delta_r(Q)) \leq c\, r^n G(X; \underline{\mathcal{A}}_r(Q))$$

where $c = c(n, \lambda, m, r_0, T) > 0$.

Theorem 2.5. *[10, Corollary 2.3] Let $Q = (q,s) \in S_T$ and $0 < r < \frac{1}{2}\min\left\{r_0, \sqrt{s}, \sqrt{T-s}\right\}$. Then we have*

$$c^{-1} \leq \frac{G(\Xi; \underline{\mathcal{A}}_r(Q))}{G(\Xi; \overline{\mathcal{A}}_r(Q))} \leq c$$

where $c = c(n, \lambda, m, r_0, T) > 0$.

Theorem 2.6 (Local comparison principle)**.** *[10, Theorem 1.6] Let $Q \in S_T$ and u, v be two positive solutions of $\mathcal{L}u = 0$ in $\Psi_{2r}(Q)$ vanishing continuously on $\Delta_{2r}(Q)$. Then, for each $X \in \Psi_{\frac{r}{8}}(Q)$ and $0 < r < \frac{1}{2}\min\left\{r_0, \sqrt{s}, \sqrt{T-s}\right\}$, we have*

$$\frac{u(X)}{v(X)} \le c\frac{u(\overline{\mathcal{A}}_r(Q))}{v(\underline{\mathcal{A}}_r(Q))}$$

where the constant $c = c(n, \lambda, m, r_0) > 0$.

Theorem 2.7 (Hölder continuity)**.** *[24, Theorem 1.3] Let $\mathcal{L}u = 0$ in Ω. Then there exists $\alpha = \alpha(n, \lambda) > 0$ such that $u \in C^\alpha(\Omega)$. Furthermore, if $Q_{2r}(x,t) \subset \Omega$ and $(z,w), (y,s) \in Q_r(x,t)$, then*

$$|u(z,w) - u(y,s)| \le c\left(\frac{|z-y|}{r} + \frac{|w-s|}{r^2}\right)^\alpha r\left(\fint_{Q_r(x,t)} |\nabla u|^2 dy\right)^{\frac{1}{2}}$$

where the constant $c = c(n, \lambda) > 0$.

There is a version of this result adapted for solutions vanishing on a portion of the boundary. The form of this result that we now present may be found in [22, Lemma 5], and the ideas for its proof are the same as sketched therein.

Theorem 2.8. *Let $\mathcal{L}u = 0$ in Ω vanishing on $\Delta_{2r}(Q)$ for some $Q \in S$. Then there exist $\alpha = \alpha(n, \lambda) > 0$ and $C = C(n, \lambda) > 0$ such that*

$$|u(X)| \lesssim \left(\frac{\delta(X)}{r}\right)^\alpha \sup_{\Psi_r(Q)} u \tag{2.2}$$

for every $X \in \Psi_r(Q)$

From now on, we will use the notation $\lesssim$ or $\gtrsim$ to write an inequality with constants, where the constants involved depend at most on known features such as dimension, ellipticity and the Lipschitz character of Ω.

Some of the results stated above have a counterpart that holds for adjoint solutions, that is, solutions to $\mathcal{L}^* v = 0$, where $\mathcal{L}^* v = \operatorname{div} A\nabla v + \partial_t v$, see e.g. [24, p. 202]. If needed, we will explicitly mention each of the results that we may use, and at this point we include an instance of the *reflecting-in-time-variable technique* that we adopt in a couple of results later in the paper (see the first few paragraphs of §4 below, as well as the proof of Lemma 5.2).

Theorem 2.9. *Let $Q \in S_T$ and v_1, v_2 be two positive solutions of $\mathcal{L}^* v = 0$ in $\Psi_{2r}(Q)$ vanishing continuously on $\Delta_{2r}(Q)$. Then, for each $X \in \Psi_{r/8}(Q)$ and $0 < r < \frac{1}{2}\min\left\{r_0, \sqrt{s}, \sqrt{T-s}\right\}$, we have*

$$\frac{v_1(X)}{v_2(X)} \le c\frac{v_1(\underline{\mathcal{A}}_r(Q))}{v_2(\overline{\mathcal{A}}_r(Q))}$$

where the constant $c = c(n, \lambda, m, r_0) > 0$.

Proof. Define $u_i(x,t) = v_i(x, -t)$ for $i = 1, 2$ and $-T < t < 0$. Then u_i are positive solutions to $\mathcal{L}u = 0$ in $D \times (-T, 0)$, for the same operator $\mathcal{L}$. Inserting u_i in Theorem 2.6, we deduce the desired estimate. □

Theorem 2.10. *Let $Q \in S$ and $0 < r < \min\left\{r_0, \sqrt{s}\right\}$. Then for any nonnegative solution of $\mathcal{L}^* v = 0$ in Ω vanishing continuously on $\Delta(Q, 2r)$, we have*

$$\sup_{\Psi_r(Q)} v \leq c\, v(\underline{\mathcal{A}}_r(Q))$$

where the constant $c = c(n, \lambda, m, r_0) > 0$.

Finally, we include what might as well be called a *local backward Harnack inequality*. We adapt the statement from [2, Theorem 1].

Lemma 2.11. *Pick $Q = (q, s) \in S_T$ with $0 < r < \frac{1}{2}\min\left\{r_0, \sqrt{s}, \sqrt{T-s}\right\}$. Assume u is a nonnegative solution to $\mathcal{L}u = 0$ in $\Psi_{2r}(Q)$ which continuously vanishes on $\Delta_{2r}(Q)$ and such that $m_r = u(\underline{\mathcal{A}}_r) > 0$. Then, for $0 < \rho \leq \frac{1}{2} r$ we have*

$$u(\overline{\mathcal{A}}_\rho(Q)) \lesssim \left(1 + \frac{M_r}{m_r}\right) u(\underline{\mathcal{A}}_\rho(Q))$$

where $M_r = \sup_{\Psi_{2r}(Q)} u$.

We have an application of Lemma 2.11 for very specific solutions in which the factor $\left(1 + \frac{M_r}{m_r}\right)$ may be dispensed of the explicit dependance on u or r. This will prove to be useful in the next section.

3 Poincaré type inequalities

The first Poincaré type inequality that we state takes place in the boundary of a Lipschitz cylinder and has nothing to do with the properties of solutions of any parabolic operator. Rather, it is a first instance where the definition of the norm in (1.7) becomes convenient. The inequality contained in the following theorem is an auxiliary result to prove Lemma 5.4 which in turn helps to prove Theorem 1.4, and is inspired on a result in [4, p. 17-18]. It may be of independent interest.

Theorem 3.1. *Let $f \in C_0^\infty(\mathbb{R}^n \times (0, \infty))$. Then there exists $\beta = \beta_r \in \mathbb{R}$ such that for $p \geq 1$ and $r < r_0$, we have*

$$\int_{\Delta_r} |f(q,s) - \beta|^p \, d\sigma(q,s) \lesssim r^p \int_{\Delta_r} |\nabla_{tan} f(q,s)|^p \, d\sigma(q,s) + r^p \int_{\Delta_r} |\partial_t^{\frac{1}{2}} f(q,s)|^p \, d\sigma(q,s)$$

where $\Delta_r = \Delta_r(Q_0)$ with $Q_0 \in S_T$.

Proof. Let $Q_0 = (q_0, s_0) \in S_T$. Notice that $\Delta_r = \widetilde{\Delta}_r \times I_r$, where $\widetilde{\Delta}_r = \widetilde{\Delta}_r(q_0)$ is a surface ball with radius r on ∂D (the boundary of the Lipschitz domain D), and $I_r = (s_0 - r^2 + s_0 + r^2)$ is a time-interval. Define

$$\beta_r = \fint_{\Delta_r} f(q,s)\, d\sigma(q,s) \qquad \text{and} \qquad \beta_r(s) \equiv \beta(s) = \fint_{\widetilde{\Delta}_r} f(q,s)\, d\widetilde{\sigma}(q).$$

Observe that

$$\int_{\Delta_r} |f(q,s) - \beta_r|^q \, d\sigma(q,s) \lesssim \int_{\Delta_r} |f(q,s) - \beta_r(s)|^q \, d\sigma(q,s) + \int_{\Delta_r} |\beta_r(s) - \beta_r|^q \, d\sigma(q,s) \equiv \mathcal{I} + \mathcal{II}.$$

By Fubini's theorem and a Poincaré type inequality for the boundary of a Lipschitz domain in $\mathbb{R}^n$ (see the proof of Theorem 5.19 in [16]) we see that

$$\mathcal{I} \lesssim r^p \int_{\Delta_r} |\nabla_{tan} f(q,s)|^p \, d\sigma(q,s).$$

To handle $\mathcal{II}$, we notice that

$$\begin{aligned}
|\beta_r - \beta_r(s)| &\lesssim \fint_{\widetilde{\Delta}_r} \fint_{I_r} |f(q,\tau) - f(q,s)| \, d\tau \, d\widetilde{\sigma}(q) \\
&\lesssim \fint_{\widetilde{\Delta}_r} \left\{ \left(\fint_{I_r} \frac{|f(q,\tau) - f(q,s)|}{|\tau - s|^2} \, d\tau \right)^{\frac{1}{2}} \left(\fint_{I_r} |\tau - s|^2 \, d\tau \right)^{\frac{1}{2}} \right\} d\widetilde{\sigma}(q) \\
&\lesssim r \fint_{\widetilde{\Delta}_r} \partial_t^{\frac{1}{2}} f(q,s) \, d\widetilde{\sigma}(q).
\end{aligned}$$

Finally, integrating over Δ_r and applying Hölder's inequality and Fubini's theorem we obtain,

$$\begin{aligned}
\int_{\Delta_r} |\beta_r - \beta_r(s)|^q \, d\sigma(q,s) &\lesssim r^q \int_{\Delta_r} \left| \fint_{\widetilde{\Delta}_r} \partial_t^{\frac{1}{2}} f(q,s) \, d\widetilde{\sigma}(q) \right|^q d\sigma \\
&\lesssim r^q \int_{\Delta_r} \fint_{\widetilde{\Delta}_r} \left| \partial_t^{\frac{1}{2}} f(q,s) \right|^q d\widetilde{\sigma}(q) \, d\sigma(q,s) \\
&\lesssim r^q \int_{\Delta_r} \left| \partial_t^{\frac{1}{2}} f(q,s) \right|^q d\sigma(q,s).
\end{aligned}$$

□

The next result is a sort of weighted Poincaré-type inequality. It originates from the elliptic result in [16].

Lemma 3.2. *Let Ω be a Lipschitz cylinder, $Q = (q', q_n, s) \in S$ and $0 < r < \min\{r_0, \sqrt{s}\}$. If $u \in C^\infty(\Psi_r(Q)) \cap C(\overline{\Psi_r(Q)})$ and $u \equiv 0$ on $\Delta(Q,r)$, then*

$$\int_{\Psi_r(Q)} \delta(X)^\alpha u^2(X) \, dX \lesssim \frac{r^2}{1-\alpha} \int_{\Psi_r(Q)} \delta(X)^\alpha |\nabla u(X)|^2 \, dX$$

for each $0 \le \alpha < 1$.

Proof. Fix $X' = (x', x_n', t) \in \Psi_r(Q)$. We first note that

$$u(X') = u(x', x_n', t) - u(x', \psi(x',t), t) = \int_{\psi(x',t)}^{x_n'} \frac{\partial}{\partial x_n} u(x', x_n, t) \, dx_n.$$

Setting $X = (x', x_n, t)$, by the Cauchy–Schwarz inequality,

$$|u(X')| \le \int_{\psi(x',t)}^{x_n'} |\nabla u(X)| \, dx_n \le \left(\int_{\psi(x',t)}^{x_n'} \delta^\alpha(X) |\nabla u(X)|^2 \, dx_n \right)^{\frac{1}{2}} \left(\int_{\psi(x',t)}^{x_n'} \delta^{-\alpha}(X) \, dx_n \right)^{\frac{1}{2}}. \quad (3.1)$$

If $\alpha > 0$, then

$$\int_{\psi(x',t)}^{x_n'} \delta^{-\alpha}(X)\,dx_n \lesssim \int_{\psi(x',t)}^{x_n'} \frac{dx_n}{(x_n - \psi(x',t))^\alpha}$$
$$= \int_0^{x_n'-\psi(x',t)} y^{-\alpha}\,dy \le \int_0^r y^{-\alpha}\,dy = \frac{r^{1-\alpha}}{1-\alpha}. \tag{3.2}$$

By (3.1) and (3.2)

$$\begin{aligned}
\int_{\Psi_r(Q)} \delta^\alpha(X')u^2(X')\,dX' &\le \int_{\Psi_r(Q)} r^\alpha u^2(X')\,dX' \\
&\lesssim \frac{r^{1-\alpha}r^\alpha}{1-\alpha}\int_{\Psi_r(Q)}\int_{\psi(x',t)}^{x_n'} \delta^\alpha(X)|\nabla u(X)|^2\,dx_n\,dX' \\
&\le \frac{r}{1-\alpha}\int_{E_r(Q)}\int_{\psi(x',t)}^{\psi(x',t)+r}\int_{\psi(x',t)}^{\psi(x',t)+r} \delta^\alpha(X)|\nabla u(X)|^2\,dx_n\,dx_n'\,d\sigma(x',t) \\
&\le \frac{r^2}{1-\alpha}\int_{E_r(Q)}\int_{\psi(x',t)}^{\psi(x',t)+r} \delta^\alpha(X)|\nabla u(X)|^2\,dx_n\,d\sigma(x',t) \\
&= \frac{r^2}{1-\alpha}\int_{\Psi_r(Q)} \delta^\alpha(X)|\nabla u(X)|^2\,dX.
\end{aligned}$$

Here, $E_r(Q) = \{(x',t) \in S : (x',x_n,t) \in \Psi_r(Q)\}$. If $\alpha = 0$, we argue the same way using (3.1). □

In order to prove Theorem 1.3 we will go through a series of lemmas and observations concerning a very precise type of solution of (1.3).

If $Q = (q,s) \in S_T$ and $0 < r < \min\{r_0, \sqrt{s}, \sqrt{T-s}\}$, we define $\overrightarrow{Q}(r) = (q, s + r^2)$ and $\overleftarrow{Q}(r) = (q, s - r^2)$. Now, take $f \in C_0^\infty(\mathbb{R}^n \times (0,\infty))$ such that $f \equiv 1$ in $\Delta_r(\overleftarrow{Q_0}(5r))$ and $f \equiv 0$ in $\Delta_{2r}(\overleftarrow{Q_0}(5r))^c$ for some $Q_0 = (q_0, s_0) \in S_T$ with $0 < 6r < \min\{r_0, \sqrt{s_0}, \sqrt{T-s_0}\}$.

For the remaining of this section and the next one, u will denote the solution to the Dirichlet problem over the domain $\Omega \cap \{-1 < t < 2T\}$, with boundary datum given by a function f as described above.

Remark 3.3. Observe that with the notation just introduced, the conclusion of the Corollary 2.11 yields

$$u(\overline{\mathcal{A}}_\rho(Q)) \lesssim u(\underline{\mathcal{A}}_\rho(Q)), \qquad \rho \le 2r$$

for $Q = (q,s) \in S_T$ with $s_0 + 43r^2 \le s \le s_0 + 200r^2$ and $|q - q_0| \lesssim r$.

Proof. Note that $s_0 + 43r^2 \le s$ implies $\Delta_{8r}(Q) \subset \Delta_{2r}(\overleftarrow{Q_0}(5r))^c$. Clearly $M_r \le 1$. By Theorem 2.3, for $X \in \Psi_{r/2}(\overleftarrow{Q_0}(5r))$ we have

$$\omega^X(\Delta_r(\overleftarrow{Q_0}(5r))) \gtrsim 1.$$

As a consequence

$$u(X) = \int_{S_T} f\,d\omega^X \ge \omega^X(\Delta_r(\overleftarrow{Q_0}(5r))) \gtrsim 1.$$

With this at hand, Carleson-type estimate and Harnack inequality (Theorems 2.2 and 2.1) yield $m_r = u(\underline{\mathcal{A}}_{4r}(Q)) \gtrsim 1$, and the proof of the Remark is finished. □

Lemma 3.4. *Let $Q=(q,s)\in S_T$ with $s_0+43r^2\le s\le s_0+200r^2$. Then, there exists $0<\alpha<1$ depending on n and λ only, such that*

$$\fint_{\Psi_r(Q)} u^2(X)\,dX \le c\,\frac{1}{r^\alpha}\fint_{\Psi_r(Q)} \delta(X)^\alpha u^2(X)\,dX.$$

Proof. Consider $K=\Psi_r(Q)\setminus S_{\frac{1}{2}r}(Q)$ where

$$S_{ar}(Q)=\{X\in\Psi_r(Q):\delta(X)<ar\},\quad 0<a<1. \tag{3.3}$$

Note that $|K|\approx r^{n+2}\approx|\Psi_r(P)|$. By the Theorem 2.2, Remark 3.3 and Harnack's inequality we obtain

$$\sup_{\Psi_r(Q)} u^2 \lesssim u^2(\overline{\mathcal{A}}_r(Q)) \lesssim u^2(\underline{\mathcal{A}}_r(Q)) \lesssim \inf_K u^2 \le \fint_K u^2(X)\,dX. \tag{3.4}$$

This and (2.2) imply that for $X\in\Psi_r(Q)$ one has

$$u(X)\lesssim\left(\frac{\delta(X)}{r}\right)^\alpha\left(\fint_K u^2(Y)\,dY\right)^{1/2}. \tag{3.5}$$

Since $\delta(X)\approx\delta(Y)$, for $X,Y\in K$, we can use (3.4) and (3.5), to obtain

$$\begin{aligned}\fint_{\Psi_r(Q)} u^2(X)\,dX &\lesssim \fint_K\fint_K\left(\frac{\delta(X)}{r}\right)^{2\alpha} u^2(Y)\,dY\,dX\\ &\lesssim \frac{1}{r^\alpha}\fint_K \delta(Y)^\alpha u^2(Y)\,dY \lesssim \frac{1}{r^\alpha}\fint_{\Psi_r(Q)} \delta(Y)^\alpha u^2(Y)\,dY.\end{aligned}$$

The lemma follows. □

Lemma 3.5. *Let $Q=(q,s)\in S_T$ with $s_0+43r^2\le s\le s_0+200r^2$. Then we have*

$$\fint_{\Psi_{2r}(Q)} u^2(X)\,dX \lesssim \fint_{\Psi_r(Q)} u^2(X)\,dX,\qquad \fint_{\Psi_r(Q)} |\nabla u(X)|^2\,dX \lesssim \fint_{\Psi_{\frac{3}{4}r}(Q)} |\nabla u(X)|^2\,dX. \tag{3.6}$$

Proof. The first assertion follows using (3.4). For the second assertion we use Caccioppoli's at the boundary inequality to obtain

$$\fint_{\Psi_r(Q)} |\nabla u(X)|^2\,dX \lesssim r^{-2}\fint_{\Psi_{\frac{3}{2}r}(Q)} u^2(X)\,dX.$$

With this, the second assertion of this lemma may be derived from the first assertion and Lemma 3.2 for $\alpha=0$. □

Lemma 3.6. *Let $Q=(q,s)\in S_T$ with $s_0+43r^2\le s\le s_0+200r^2$. Then, there exists $\epsilon=\epsilon(n,\lambda,m)>0$ such that*

$$\fint_{\Psi_r(Q)} |\nabla u(X)|^2\,dX \lesssim \fint_{\Psi_r(Q)\setminus S_{\epsilon r}(Q)} |\nabla u(X)|^2\,dX.$$

Proof. According to Lemma 3.5 and a Caccioppoli's inequality at the boundary inequality, we have

$$\fint_{\Psi_r(Q)} |\nabla u(X)|^2\, dX \lesssim \fint_{\Psi_{r/2}(Q)} |\nabla u(X)|^2\, dX \lesssim \frac{1}{r^2}\fint_{\Psi_r(Q)} u^2(X)\, dX.$$

From Lemmas 3.4 and 3.2

$$\frac{1}{r^2}\fint_{\Psi_r(Q)} u^2(X)\, dX \lesssim \frac{1}{r^{2+\alpha}}\fint_{\Psi_r(Q)} \delta^\alpha(X) u^2(X)\, dX \lesssim \frac{1}{r^\alpha}\fint_{\Psi_r(Q)} \delta^\alpha(X)|\nabla u(X)|^2\, dX.$$

Thus, we have

$$\begin{aligned}
\fint_{\Psi_r(Q)} |\nabla u(X)|^2\, dX &\lesssim \frac{1}{r^\alpha}\fint_{\Psi_r(Q)\setminus S_{\epsilon r}(Q)} \delta^\alpha(X)|\nabla u(X)|^2\, dX + \frac{1}{r^\alpha}\fint_{S_{\epsilon r}(Q)} \delta^\alpha(X)|\nabla u(X)|^2\, dX \\
&\lesssim \fint_{\Psi_r(Q)\setminus S_{\epsilon r}(Q)} |\nabla u(X)|^2\, dX + \epsilon^\alpha \fint_{S_{\epsilon r}(Q)} |\nabla u(X)|^2\, dX \\
&\leq \fint_{\Psi_r(Q)\setminus S_{\epsilon r}(Q)} |\nabla u(X)|^2\, dX + \epsilon^\alpha \fint_{\Psi_r(Q)} |\nabla u(X)|^2\, dX.
\end{aligned}$$

Finally, choosing $\epsilon > 0$ very small we can hide the second term in the right hand side into the left hand side, and hence we conclude the desired estimate. □

4 Proof of Theorem 1.3

We retain notations from the previous sections. In particular, recall that we have stated right before the Remark 3.3 that u denotes a solution over $\Omega \cap \{-1 < t < 2T\}$ with a very particular prescribed data function f. It is convenient now to impose some extra conditions on this data, namely $|\nabla f| \lesssim \frac{1}{r}$ and $|f_t| \lesssim \frac{1}{r^2}$.

We are interested in the norm of f as an element of $W_p^{1,\frac{1}{2}}(S_T)$, because once this norm is computed, the fact that $(R)_p$ is solvable will provide us a precise estimate of the L^p norm of $\mathcal{N}(\nabla u)$. Indeed, observe that with these new conditions on f, we have

$$\int_{S_T} |\nabla f(Q)|^p\, d\sigma(Q) \lesssim r^{n+1-p}, \qquad \left|\partial_t^{\frac{1}{2}} f(x,s)\right| \lesssim \left(\int_{s_0-29r^2}^{s_0-21r^2} \left(\frac{1}{r^2}\right)^2 dt\right)^{\frac{1}{2}} \lesssim \frac{1}{r}.$$

This implies that

$$\|f\|^p_{W_p^{1,1/2}(S_{T+1})} \lesssim r^{n+1-p}.$$

When attempting to prove that $(R)_p$ implies $(D)_{p'}$, using the techniques from [16], one finds some difficulties when trying to use the Theorem 2.4. Indeed, in the elliptic case several times it is used the fact that the divergence form second order linear operators similar to $\mathcal{L}$ are selfadjoint. For the parabolic operators this is not the case, and actually the Green's function is not symmeteric in its arguments; that is, the order of the argument variables is essential.

One way to tackle this obstacle is to use an auxiliary solution v to the adjoint equation $\mathcal{L}^* v = 0$, defined in terms of the particular solution u, by a *reflection in time* change of variables, as mentioned earlier. And in the estimates (4.1) below is crucial our use of Theorem 2.9, thus the independence of the coefficients matrix A from the t variable.

To be more precise, we think of Ω as the extended domain $D\times(-\infty,\infty)$. For $X=(x,t)\in\overline{\Omega}\cap\{-2T<t<2T\}$, define $\widetilde{X}=(x,\widetilde{t})$ to be the reflection of X with respect to the hyperplane for which $t=s_0+25r^2$, that is, $\widetilde{X}=(x,2(s_0+25r^2)-t)$. Here it is important to recall that $Q_0=(q_0,s_0)\in S_T$ has been fixed when defining the support of f.

Now define $v(X)=u(\widetilde{X})$. Observe that $\mathcal{L}^*v=0$ in $\overline{\Omega}\cap\{-2T<t<2T\}$ and that by the support definition of f the boundary values of $v(x,t)$ vanish for $2(s_0+25r^2)-t\geq s_0-21r^2$ or equivalently $t\leq s_0+71r^2$.

Take $Q\in\Delta_{r/16}(Q_0)$ and $\rho<\frac{r}{16}$. From Theorems 2.4, 2.5 and 2.9 , we have

$$\begin{aligned}\frac{\omega^{\Xi}(\Delta_\rho(Q))}{\rho^{n+1}}&\lesssim\frac{G(\Xi,\underline{\mathcal{A}}_\rho(Q))}{\rho}=\frac{v(\underline{\mathcal{A}}_\rho(Q))}{\rho}\frac{G(\Xi;\underline{\mathcal{A}}_\rho(Q))}{v(\underline{\mathcal{A}}_\rho(Q))}\\&\lesssim\frac{v(\underline{\mathcal{A}}_\rho(Q))}{\rho}\frac{G(\Xi;\underline{\mathcal{A}}_r(Q_0))}{v(\overline{\mathcal{A}}_r(Q_0))}\lesssim\frac{v(\underline{\mathcal{A}}_\rho(Q))}{\rho}G(\Xi;\overline{\mathcal{A}}_r(Q_0))\\&\lesssim\frac{v(\underline{\mathcal{A}}_\rho(Q))}{\rho}\frac{\omega^{\Xi}(\Delta_r(Q_0))}{r^n},\end{aligned}\tag{4.1}$$

where we have used the fact that $v(\overline{\mathcal{A}}_r(Q_0))=u(\underline{\mathcal{A}_r}(\widetilde{Q}_0))\gtrsim 1$. On the other hand, similarly to (3.4) we may conclude

$$v(\underline{\mathcal{A}}_\rho(Q))=u(\overline{\mathcal{A}}_\rho(\widetilde{Q}))\lesssim\left(\fint_{\Psi_\rho(\widetilde{Q})}u^2(Y)\,dY\right)^{\frac{1}{2}}.$$

From this and lemmas 3.2 and 3.6 we obtain

$$v(\underline{\mathcal{A}}_\rho(Q))\lesssim\rho\left(\fint_{\Psi_\rho(\widetilde{Q})\setminus S_{\epsilon\rho}(\widetilde{Q})}|\nabla u(Y)|^2\,dY\right)^{\frac{1}{2}}.\tag{4.2}$$

Plugging (4.1) and (4.2) together, we get

$$\frac{\omega^{\Xi}(\Delta_\rho(Q))}{\rho^{n+1}}\lesssim\frac{\omega^{\Xi}(\Delta_r(Q_0))}{r^n}\left(\fint_{\Psi_\rho(\widetilde{Q})\setminus S_{\epsilon\rho}(\widetilde{Q})}|\nabla u(Y)|^2\,dY\right)^{\frac{1}{2}}.\tag{4.3}$$

This suggests that we introduce the next two maximal functions:

$$\mathcal{M}_\sigma\omega(Q)\equiv\mathcal{M}_{\sigma,\frac{r}{16}}\omega(Q)=\sup_{0<\rho<\frac{r}{16}}\frac{\omega^{\Xi}(\Delta_\rho(Q))}{\rho^{n+1}}\qquad\text{where }\omega\text{ is the parabolic measure, and}$$

$$\mathcal{N}^\epsilon\varphi(Q)\equiv\mathcal{N}^\epsilon_\alpha\varphi(Q)=\sup_{X\in\Gamma_\alpha(Q)}\left(\fint_{\Psi_{\delta(X)}(P_X)\setminus S_{\epsilon\delta(X)}(P_X)}\varphi^2(Y)\,dY\right)^{\frac{1}{2}}$$

for φ any function defined on Ω, and where $P_X=(x',\psi(x',t),t)$ if $X=(x',x_n,t)$. Notice that we have included the aperture α in the notation. The reason will be clear shortly. With this definitions, (4.3) yields

$$\mathcal{M}_\sigma\omega(Q)\lesssim\frac{\omega^{\Xi}(\Delta_r(Q_0))}{r^n}\mathcal{N}^\epsilon(\nabla u)(\widetilde{Q}).\tag{4.4}$$

In the last step to prove Theorem 1.3, we make use of the following

Lemma 4.1. *For any function φ defined on Ω*

$$\mathcal{N}^{\epsilon}_{\alpha}\varphi(Q) \lesssim \mathcal{N}_{\beta}\varphi(Q)$$

for some $\beta = \beta(n,\lambda,m,r_0) > \alpha$, where ϵ is as in Theorem 3.6.

Proof. First note that the region

$$\widetilde{\Psi}_r(Q) = \Big\{(x,t) \in \mathbb{R}^{n+1} : |x_i - q_i| < r, i = 1,\cdots,n-1, \\ \psi(x',t) - r < x_n < \psi(x',t) + 4nmr, |s-t| < r^2\Big\}$$

can be covered with $N = N(\epsilon,n,m)$ parabolic cubes of radius $r' \geq \epsilon c^{-2} r$, independently of r. Hence, for $X \in \Gamma_\alpha(Q)$, as we just observed, we can cover $\Psi_{\delta(X)}(P_X) \setminus S_{\epsilon\delta(X)}(P_X)$ with N parabolic cubes Q_i, each one with radius $c^{-2}\epsilon\delta(X_i)$, centered at points of $\Psi_{\delta(X)}(P_X) \setminus S_{\epsilon\delta(X)}(P_X)$. Now, from the fact that $\epsilon\delta(X) \leq \delta(X_i) \leq \delta(X)$, it follows that

$$\left|\Psi_{\delta(X)}(P_X) \setminus S_{\epsilon\delta(X)}(P_X)\right| \approx \left|\Psi_{\delta(X)}(P_X)\right| \approx |Q_i|$$

where the comparability constants depend again only on ϵ and n. As a consequence, for $\beta > 0$ big enough

$$\fint_{\Psi_{\delta(X)}(P_X)\setminus S_{\epsilon\delta(X)}(P_X)} |\varphi|^2\, dY \lesssim c\sum_1^N \fint_{Q_i} |\varphi|^2 dY \lesssim \mathcal{N}_\beta\varphi(Q).$$

Taking the supremum over $X \in \Gamma_\alpha(Q)$, the inequality is proved. □

To finish the proof of Theorem 1.3, we can apply the previous lemma and (4.4), along with the Remark 1.2 to obtain by the very definition of $\mathcal{M}_\sigma\omega$ that $\omega^\Xi << \sigma$, because $\mathcal{N}_\beta(\nabla u) \in L^p(S_T)$. In fact, $\omega^\Xi \in A_\infty(d\sigma)$ (see [24, p. 224]). Now

$$\left(\fint_{\Delta_{\frac{r}{16}}(Q_0)} \left(\frac{d\omega^\Xi}{d\sigma}\right)^p d\sigma\right)^{\frac{1}{p}} \lesssim \left(\frac{1}{r^{n+1}}\right)^{\frac{1}{p}} \frac{\omega(\Delta)}{r^n} \|\mathcal{N}(\nabla u)\|_{L^p(S_{T+1})} \lesssim \left(\frac{1}{r^{n+1}}\right)^{\frac{1}{p}} \frac{\omega(\Delta)}{r^n} \|f\|_{W_p^{1,\frac{1}{2}}(S_{T+1})}$$
$$\lesssim \left(\frac{1}{r^{n+1}}\right)^{\frac{1}{p}} \frac{\omega(\Delta)}{r^n} \left(r^{n+1-p}\right)^{\frac{1}{p}} \approx \frac{\omega(\Delta)}{r^{n+1}} \approx \fint_{\Delta_{\frac{r}{16}}(Q_0)} \left(\frac{d\omega^\Xi}{d\sigma}\right) d\sigma,$$

where $\Delta = \Delta_r(Q_0)$, thus finishing the proof.

5 $(D)_{p'}$ and $(R)_q$ implies $(R)_p$

Let us first make a couple of observations about the behavior of solutions near the boundary. While the second observation depends on properties of solutions, the first one does not, and it depends purely on the geometric features of Ω_T. Here is our first observation:

Lemma 5.1. *For any function u such that ∇u exists almost everywhere and $r < r_0$, we have,*

$$\fint_{\Psi_{28r}(Q_0)} |\nabla u(Y)|\, dY \lesssim \fint_{\Delta_{32r}(Q_0)} \mathcal{N}(\nabla u)(P)\, d\sigma(P) \tag{5.1}$$

where $Q_0 \in S_T$.

Proof. Inequality (5.1) can be broken into two assertions:

$$\fint_{\Psi_{28r}(Q_0)} |\nabla u(Y)|\, dY \lesssim \fint_{\Psi_{28r}(Q_0)} \fint_{Q(X)} |\nabla u(Y)|\, dY\, dX, \tag{5.2}$$

$$\fint_{\Psi_{28r}(Q_0)} \fint_{Q(X)} |\nabla u(Y)|\, dY\, dX \lesssim \fint_{\Delta_{32r}(Q_0)} N(\nabla u)(P)\, d\sigma(P). \tag{5.3}$$

Taking into account that $\delta(X) \approx \delta(Y)$ if $Y \in Q(X)$, (5.2) is proved as follows:

$$\begin{aligned}
\fint_{\Psi_{28r}(Q_0)} \fint_{Q(X)} |\nabla u(Y)|\, dY\, dX &\approx \fint_{\Psi_{28r}(Q_0)} \frac{1}{\delta(X)^{n+2}} \int_{\Psi_{30r}(Q_0)} |\nabla u(Y)| \chi_{Q(X)}(Y)\, dY\, dX \\
&\approx \fint_{\Psi_{30r}(Q_0)} \int_{\Psi_{28r}(Q_0)} \frac{1}{\delta(X)^{n+2}} |\nabla u(Y)| \chi_{Q(Y)}(X)\, dX\, dY \\
&\gtrsim \fint_{\Psi_{28r}(Q_0)} |\nabla u(Y)|\, dY.
\end{aligned}$$

To prove (5.3) we observe that for $P = (p', p_n, s) \in \Delta_{32r}(Q_0)$

$$\mathcal{N}(\nabla u)(P) = \fint_{p_n}^{p_n+112nmr} \mathcal{N}(\nabla u)(p', \rho, s)\, d\rho \geq \fint_{p_n}^{p_n+112nmr} \fint_{Q(p',\rho,s)} |\nabla u(Y)| dY\, d\rho.$$

Integrating this, we have

$$\begin{aligned}
\fint_{\Delta_{32r}(Q_0)} \mathcal{N}(\nabla u)(P)\, d\sigma(P) &\geq \fint_{\Delta_{32r}(Q_0)} \fint_{p_n}^{p_n+112nmr} \fint_{Q(p',\rho,s)} |\nabla u(Y)| dY\, d\rho\, d\sigma(P) \\
&\gtrsim \fint_{\Psi_{28r}(Q_0)} \fint_{Q(X)} |\nabla u(Y)|\, dY\, dX.
\end{aligned}$$

□

Here is the second observation of the behavior of solutions near the boundary:

Lemma 5.2. *Assume that u is a solution of $\mathcal{L}u = 0$ in Ω and that $u = 0$ continuously on $\Delta_{32r}(Q_0)$ for some $Q_0 = (q_0, s_0) \in S_T$. Then we have,*

$$|u(X)| \lesssim \frac{G(\Xi; \widetilde{X})}{G(\Xi; \overline{\mathcal{A}}_{12r}(\widetilde{Q_0}))} \left(\fint_{\Psi_{18r}(Q_0)} |u(Y)|^2\, dY \right)^{\frac{1}{2}} \tag{5.4}$$

for $X \in \Psi_{\frac{3}{2}r}(Q_0)$, and where $\widetilde{X}$ and $\widetilde{Q}_0$ were defined in page 49.

Proof. Let u_1 and u_2 be the solutions of $\mathcal{L}u = 0$ in $\Psi_{32r} = \Psi_{32r}(Q_0)$ with data $f_1 = \max\{u, 0\}$ and $f_2 = \max\{-u, 0\}$ on $\partial\Psi_{16r}(Q_0)$ respectively. Note that $u = u_1 - u_2$ in $\partial\Psi_{32r}$, by uniqueness $u = u_1 - u_2$ in Ψ_{32r}.

Now we perform again a reflection to X with respect to the time variable of Q_0. In this instance we define $v_i(X) = u_i(\widetilde{X})$ for $X \in \Psi_{32r}(\widetilde{Q_0})$ where $\widetilde{X} = (x, 2(s_0 + (32)^2 r^2) - t)$ if $X = (x, t)$. By the comparison principle

$$v_i(\widetilde{X}) \lesssim \frac{G(\Xi; \widetilde{X})}{G(\Xi; \overline{\mathcal{A}}_{12r}(\widetilde{Q_0}))} v_i(\underline{\mathcal{A}}_{12r}(\widetilde{Q_0})) \qquad i = 1, 2$$

for $X \in \Psi_{\frac{3}{2}r}(Q_0)$. This is the same as

$$u_i(X) \lesssim \frac{G(\Xi;\widetilde{X})}{G(\Xi;\overline{\mathcal{A}}_{12r}(\widetilde{Q_0}))} u_i(\overline{\mathcal{A}}_{12r}(Q_0)) \qquad i = 1,2. \tag{5.5}$$

Now, note that by Harnack's inequality we have

$$u_i(\overline{\mathcal{A}}_{12r}(Q_0)) \lesssim \inf_K u_i \le \left(\fint_K u_i^2(Y)\,dY \right)^{\frac{1}{2}} \lesssim \left(\fint_{\Psi_{18r}(Q_0)} u_i^2(Y)\,dY \right)^{\frac{1}{2}} \qquad i = 1,2 \tag{5.6}$$

where $K \subset \Psi_{18r}(Q_0)$ is an appropiate compact set to the right of $\overline{\mathcal{A}}_{16r}(Q_0)$. Putting (5.5) and (5.6) together we obtain the lemma. □

Back to our main goal, which is to prove Theorem 1.4, we now state two lemmas, whose proof is provided in the last section.

Lemma 5.3. *Let $1 < p < \infty$. Assume that $(D)_{p'}$ is solvable in Ω_T for $\mathcal{L}$. Let u be a solution of $\mathcal{L}u = 0$ in Ω_T that vanishes continuously in $\Delta_{32r} = \Delta_{32r}(Q_0)$ with $0 < 16r < \frac{1}{2}\min\{r_0, \sqrt{s_0}, \sqrt{T - s_0}\}$. Then*

$$\left(\fint_{\Delta_r} |\mathcal{N}(\nabla u)|^p \, d\sigma \right)^{\frac{1}{p}} \lesssim \fint_{\Delta_{32r}} \mathcal{N}(\nabla u)\, d\sigma. \tag{5.7}$$

Lemma 5.4. *Retaining the notation from the previous lemma, define*

$$E(\lambda) = \{Q \in \Delta_{2r} : \mathcal{M}_{\Delta_{2r}}(|\mathcal{N}(\nabla u)|^q)(Q) > \lambda\}. \tag{5.8}$$

Let $1 < q < p < \infty$ and suppose that $(D)_{p'}$ and $(R)_q$ are solvable in Ω_T. Then there exists constants $\epsilon, \gamma, \alpha > 0$ such that

$$\begin{aligned} |E(A\lambda)| \le \epsilon |E(\lambda)| &+ |\{Q \in \Delta_r : \mathcal{M}_{\Delta_{2r}}(|\nabla_{tan} f|^q)(Q) > \gamma\lambda\}| \\ &+ |\{Q \in \Delta_r : \mathcal{M}_{\Delta_{2r}}(|\partial_t^{\frac{1}{2}} f|^q)(Q) > \gamma\lambda\}| \end{aligned} \tag{5.9}$$

for $\lambda \ge \lambda_0$, where $A = (2\epsilon)^{-\frac{q}{p}}$ and

$$\lambda_0 = \alpha \fint_{\Delta_{2r}} |\mathcal{N}(\nabla u)|^q \, d\sigma. \tag{5.10}$$

Assuming temporarily these results, we now provide the

Proof of Theorem 1.4. Multiplying both sides of (5.9) by $\lambda^{\frac{p}{q}-1}$, integrating, and using the L^p boundedness of Hardy-Littlewood operator we obtain

$$\begin{aligned} \int_{\lambda_0}^{\Lambda} |E(A\lambda)| \lambda^{\frac{p}{q}-1} \, d\lambda \le \epsilon & \int_{\lambda_0}^{\Lambda} |E(\lambda)| \lambda^{\frac{p}{q}-1} \, d\lambda \\ &+ c \int_{\Delta_{2r}} |\nabla_{tan} f|^p \, d\sigma + c \int_{\Delta_{2r}} |\partial_t^{\frac{1}{2}} f|^p \, d\sigma, \end{aligned} \tag{5.11}$$

with Λ sufficiently large. Applying a change of variables, and noting that $A^{\frac{p}{q}} = \frac{1}{2\epsilon} > 1$, we find that

$$2\epsilon \int_{A\lambda_0}^{A\Lambda} |E(\lambda)|\lambda^{\frac{p}{q}-1}\, d\lambda \leq \epsilon \int_{\lambda_0}^{A\Lambda} |E(\lambda)|\lambda^{\frac{p}{q}-1}\, d\lambda \\ + c \int_{\Delta_{2r}} |\nabla_{tan} f|^p \, d\sigma + c \int_{\Delta_{2r}} |\partial_t^{\frac{1}{2}} f|^p \, d\sigma. \tag{5.12}$$

Splitting the first integral of the right hand side (5.12), hiding a small term we get

$$\epsilon \int_{A\lambda_0}^{A\Lambda} |E(\lambda)|\lambda^{\frac{p}{q}-1}\, d\lambda \leq \int_{\lambda_0}^{A\lambda_0} |E(\lambda)|\lambda^{\frac{p}{q}-1}\, d\lambda \\ + \frac{1}{\epsilon} \int_{\Delta_{2r}} |\nabla_{tan} f|^p \, d\sigma + \frac{1}{\epsilon} \int_{\Delta_{2r}} |\partial_t^{\frac{1}{2}} f|^p \, d\sigma. \tag{5.13}$$

From (5.13) and (5.8) we see that

$$\int_{A\lambda_0}^{A\Lambda} |E(\lambda)|\lambda^{\frac{p}{q}-1}\, d\lambda \lesssim |\Delta_{2r}|\lambda_0^{\frac{p}{q}} + \int_{\Delta_{2r}} |\nabla_{tan} f|^p \, d\sigma + \int_{\Delta_{2r}} |\partial_t^{\frac{1}{2}} f|^p \, d\sigma,$$

where the dependance on ϵ has been incorporated to the constants of the inequality. From here we obtain

$$\int_{0}^{A\Lambda} |E(\lambda)|\lambda^{\frac{p}{q}-1}\, d\lambda \lesssim |\Delta_{2r}|\lambda_0^{\frac{p}{q}} + \int_{\Delta_{2r}} |\nabla_{tan} f|^p \, d\sigma + \int_{\Delta_{2r}} |\partial_t^{\frac{1}{2}} f|^p \, d\sigma.$$

Letting $\Lambda \to \infty$, from (5.9) we get

$$\int_{\Delta_{2r}} |\mathcal{N}(\nabla u)|^p \, d\sigma \lesssim |\Delta_{2r}|\lambda_0^{\frac{p}{q}} + \int_{\Delta_{2r}} |\nabla_{tan} f|^p \, d\sigma + \int_{\Delta_{2r}} |\partial_t^{\frac{1}{2}} f|^p \, d\sigma. \tag{5.14}$$

Substituting the value of λ_0 in the first term of (5.10), using the hypothesis $(R)_q$ and then Hölder's inequality,

$$\begin{aligned} |\Delta_{2r}|\lambda_0^{\frac{p}{q}} &= |\Delta_{2r}| \left(\alpha \fint_{\Delta_{2r}} |\mathcal{N}(\nabla u)|^q \, d\sigma \right)^{\frac{p}{q}} \\ &\lesssim |\Delta_{2r}| \left(\fint_{\Delta_{2r}} |\nabla_{tan} f|^q \, d\sigma \right)^{\frac{p}{q}} + |\Delta_{2r}| \left(\fint_{\Delta_{2r}} |\partial_t^{\frac{1}{2}} f|^q \, d\sigma \right)^{\frac{p}{q}} \\ &\lesssim |\Delta_{2r}| \fint_{\Delta_{2r}} |\nabla_{tan} f|^p \, d\sigma + |\Delta_{2r}| \fint_{\Delta_{2r}} |\partial_t^{\frac{1}{2}} f|^p \, d\sigma \end{aligned}$$

we conclude that

$$\int_{\Delta_{2r}} |\mathcal{N}(\nabla u)|^p \, d\sigma \lesssim \int_{\Delta_{2r}} |\nabla_{tan} f|^p \, d\sigma + \int_{\Delta_{2r}} |\partial_t^{\frac{1}{2}} f|^p \, d\sigma. \tag{5.15}$$

Finally, by covering S_T with a finite number of surface balls, we obtain $(R)_p$ and the proof is finished. □

6 Proofs of Technical Lemmas 5.3 and 5.4

Proof of Lemma 5.3. It is enough to prove the following two estimates:

$$\mathcal{N}(\nabla u)(Q) \lesssim \left(\frac{u}{\delta}\right)^*(Q) + \fint_{\Delta_{32r}} \mathcal{N}(\nabla u)\, d\sigma(Q), \qquad Q \in \Delta_r, \tag{6.1}$$

$$\left(\fint_{\Delta_r} \left|\left(\frac{u}{\delta}\right)^*(Q)\right|^p d\sigma(Q)\right)^{\frac{1}{p}} \lesssim \fint_{\Delta_{32r}} \mathcal{N}(\nabla u)(Q)\, d\sigma(Q) \tag{6.2}$$

where

$$\left(\frac{u}{\delta}\right)^*(Q) = \sup_{Y \in \Gamma_{20\alpha}(Q)} \left\{\frac{|u(Y)|}{\delta(Y)} : \delta(Y) \le r\right\}.$$

Let us begin by establishing (6.1). For $Q \in \Delta_r$ pick $X \in \Gamma(Q)$ with $\delta(X) \ge r$. This way, if $A = \{P \in \Delta_{32r} : X \in \Gamma(P)\}$ then we will have $|A| \gtrsim r^{n+1}$. Hence we have

$$\begin{aligned}\fint_{\Delta_{32r}} |\mathcal{N}(\nabla u)(P)|\, d\sigma(P) &\ge \frac{1}{|\Delta_{32r}|} \int_A \left(\fint_{Q(X)} |\nabla u(Y)|^2 dY\right)^{\frac{1}{2}} d\sigma(P) \\ &\gtrsim \left(\fint_{Q(X)} |\nabla u(Y)|^2 dY\right)^{\frac{1}{2}}.\end{aligned} \tag{6.3}$$

On the other hand, if $\delta(X) \le r$, by Caccioppoli's inequality we see that

$$\begin{aligned}\left(\fint_{Q(X)} |\nabla u(Y)|^2 dY\right)^{\frac{1}{2}} &\lesssim \left(\frac{1}{|Q_{\frac{3}{4}\delta(X)}(X)|} \int_{Q_{\frac{3}{4}\delta(X)}(X)} \frac{|u(Y)|^2}{\delta(X)^2}\, dY\right)^{\frac{1}{2}} \\ &\lesssim \left(\frac{1}{|Q_{\frac{3}{4}\delta(X)}(X)|} \int_{Q_{\frac{3}{4}\delta(X)}(X)} \frac{|u(Y)|^2}{\delta(Y)^2}\, dY\right)^{\frac{1}{2}} \lesssim \left(\frac{u}{\delta}\right)^*(Q).\end{aligned} \tag{6.4}$$

Now (6.1) follows from (6.3) and (6.4).

We now focus on proving (6.2). By (5.4) one deduces that

$$\left(\frac{u}{\delta}\right)^*(Q) \lesssim \frac{1}{G(\Xi; \overline{\mathcal{A}}_{12r}(\widetilde{Q}_0))} \left(\fint_{\Psi_{18r}(Q_0)} u^2(Y)\, dY\right)^{\frac{1}{2}} \left(\frac{G(\Xi, \cdot)}{\delta}\right)^*(\widetilde{Q}) \tag{6.5}$$

for $Q \in \Delta_r(Q_0)$. By Theorem 2.4 we know that

$$\frac{G(\Xi, \overline{\mathcal{A}}_{\delta(X)}(\widetilde{Q}))}{\delta(X)} \lesssim \frac{\omega(\Delta_{\delta(X)}(\widetilde{Q}))}{\delta(X)^{n+1}}.$$

Using this and the adjoint version of Carleson estimate (Theorem 2.10 above) for $X \in \Gamma(Q)$ with $\delta(X) \le r$ we have

$$\frac{G(\Xi; \widetilde{X})}{\delta(\widetilde{X})} \lesssim \frac{G(\Xi; \underline{\mathcal{A}}_{\delta(X)}(\widetilde{Q}))}{\delta(\widetilde{X})} \lesssim \frac{G(\Xi; \overline{\mathcal{A}}_{\delta(X)}(\widetilde{Q}))}{\delta(\widetilde{X})} \lesssim \frac{\omega(\Delta_{\delta(X)}(\widetilde{Q}))}{\delta(\widetilde{X})^{n+1}}.$$

Therefore we obtain,

$$\left(\frac{G(\Xi,\cdot)}{\delta(\cdot)}\right)^*(\widetilde{Q}) \lesssim \sup_{0<\rho<r}\left\{\frac{\omega(\Delta_\rho(\widetilde{Q}))}{\rho^{n+1}}\right\} \equiv \mathcal{M}_{\sigma,r}(\omega)(\widetilde{Q}) \tag{6.6}$$

for $Q \in \Delta_r(Q_0)$. By hypotesis $(D)_{p'}$ is solvable in Ω_T, so we know that

$$\left(\fint_{\Delta_r(\widetilde{Q}_0)}\left|\frac{d\omega}{d\sigma}\right|^p d\sigma\right)^{\frac{1}{p}} \lesssim \frac{\omega(\Delta_r(\widetilde{Q}_0))}{|\Delta_r(\widetilde{Q}_0)|}. \tag{6.7}$$

By (6.5), (6.6), (6.7) and the L^p boundedness of $M_{\sigma,r}(\omega)$ (where this last maximal operator is the same we defined at the end of section 4) we get

$$\begin{aligned}
&\left(\fint_{\Delta_r}\left|\left(\frac{u}{\delta}\right)^*(Q)\right|^p d\sigma(Q)\right)^{\frac{1}{p}} \lesssim \\
&\lesssim \left(\fint_{\Delta_r}\left(\frac{1}{G(\Xi;\overline{\mathcal{A}}_{12r}(\widetilde{Q}_0))}\left(\fint_{\Psi_{18r}(Q_0)} u^2(Y)\,dY\right)^{\frac{1}{2}}\left(\frac{G(\Xi,\cdot)}{\delta(\cdot)}\right)^*(\widetilde{Q})\right)^p d\sigma(Q)\right)^{\frac{1}{p}} \\
&\lesssim \frac{1}{G(\Xi;\overline{\mathcal{A}}_{12r}(\widetilde{Q}_0))}\left(\fint_{\Psi_{18r}(Q_0)} u^2(Y)\,dY\right)^{\frac{1}{2}}\left(\fint_{\Delta_r}(M_{\sigma,r}(\omega)(\widetilde{Q}))^p\,d\sigma(Q)\right)^{\frac{1}{p}} \\
&\lesssim \frac{1}{G(\Xi;\overline{\mathcal{A}}_{12r}(\widetilde{Q}_0))}\left(\fint_{\Psi_{18r}(Q_0)} u^2(Y)\,dY\right)^{\frac{1}{2}}\left(\fint_{\Delta_r(\widetilde{Q}_0)}\left(\frac{d\omega}{d\sigma}\right)^p(Q)\,d\sigma(Q)\right)^{\frac{1}{p}} \\
&\lesssim \frac{1}{G(\Xi;\underline{\mathcal{A}}_{12r}(\widetilde{Q}_0))}\left(\fint_{\Psi_{18r}(Q_0)} u^2(Y)\,dY\right)^{\frac{1}{2}}\frac{\omega(\Delta_r(\widetilde{Q}_0))}{|\Delta_r(\widetilde{Q}_0)|} \\
&\lesssim \frac{1}{r}\left(\fint_{\Psi_{18r}(Q_0)} u^2(Y)\,dY\right)^{\frac{1}{2}}
\end{aligned}$$

where the last two inequalities are consequence of Theorems 2.4 and 2.5 respectively. We can continue this sequence of inequalities making use of Poincaré's inequality in Lemma 3.2 with $\alpha = 0$, and obtain

$$\left(\fint_{\Delta_r}\left|\left(\frac{u}{\delta}\right)^*(Q)\right|^p d\sigma(Q)\right)^{\frac{1}{p}} \lesssim \left(\fint_{\Psi_{18r}(Q_0)}|\nabla u(Y)|^2\,dY\right)^{\frac{1}{2}}. \tag{6.8}$$

By Caccioppoli's inequality at the boundary, and arguing as in (5.6) we get

$$\begin{aligned}
\left(\fint_{\Psi_{18r}(Q_0)}|\nabla u(Y)|^2\,dY\right)^{\frac{1}{2}} &\lesssim r^{-1}\left(\fint_{\Psi_{19r}(Q_0)} u^2(Y)\,dY\right)^{\frac{1}{2}} \lesssim r^{-1}\sup_{\Psi_{19r}(Q_0)}|u| \\
&\lesssim r^{-1}\fint_{\Psi_{28r}(Q_0)}|u(Y)|\,dY \lesssim \fint_{\Psi_{28r}(Q_0)}|\nabla u(Y)|\,dY.
\end{aligned} \tag{6.9}$$

The proof is complete once we put together (6.8),(6.9) and apply (5.1). □

Proof of Lemma 5.4. Let $\epsilon > 0$ be a small constant to be chosen later. By the weak $(1,1)$ estimate for the maximal operator $\mathcal{M}_{\Delta_{2r}}$ we have $|E(\lambda)| \leq \epsilon|\Delta_r|$ for $\lambda \geq \lambda_0$ if we choose α big enough. Now we apply the Calderón-Zygmund type decomposition (as described for instance in [27, p. 210]) and obtain a collection of disjoint cubes $\{Q_k\}_k$ contained in Δ_r such that $E(\lambda) = \bigcup_k Q_k$ and each Q_k is maximal. We may also choose ϵ small enough so that $64Q_k \subset \Delta_{2r}$.

The key statement of this proof is that there exist constants $\epsilon, \gamma, \alpha > 0$ such that if Q_k is a cube that satisfies

$$F_k = \left\{ Q \in Q_k : \mathcal{M}_{\Delta_{2r}}(|\nabla_{tan} f|^q)(Q) \leq \gamma\lambda, \quad \mathcal{M}_{\Delta_{2r}}(|\partial_t^{\frac{1}{2}} f|^q)(Q) \leq \gamma\lambda \right\} \neq \emptyset \tag{6.10}$$

then

$$|E(A\lambda) \cap Q_k| \lesssim \epsilon |Q_k|. \tag{6.11}$$

From this, setting $C_\lambda = \bigcup_k F_k$, we have

$$|E(A\lambda) \cap C_\lambda| \leq \sum_k |E(A\lambda) \cap Q_k| \leq \epsilon \sum_k |Q_k| = \epsilon |E(\lambda)|$$

and (5.9) follows. To prove (6.11) we notice that for $Q \in Q_k$

$$\mathcal{M}_{\Delta_{2r}}(|\mathcal{N}(\nabla u)|^q)(Q) \leq \max\left\{ \mathcal{M}_{2Q_k}(|\mathcal{N}(\nabla u)|^q)(Q), \beta\lambda \right\}. \tag{6.12}$$

For ϵ small enough $A = (2\epsilon)^{-q/p} \geq \beta$, and so in view of (6.12), we get

$$|E(A\lambda) \cap Q_k| \leq |\{ Q \in Q_k : \mathcal{M}_{2Q_k}(|\mathcal{N}(\nabla u)|^q)(Q) > A\lambda \}|. \tag{6.13}$$

Now, for each k consider the smooth function $\phi_k : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $\phi_k = 1$ in $64Q_k$, $\phi_k = 0$ in $(66Q_k)^c$, $|\nabla\phi_k| \lesssim |Q_k|^{-\frac{1}{n+1}}$ and $|(\phi_k)_t| \lesssim |Q_k|^{-\frac{2}{n+1}}$. Let v_k be the solution to $\mathcal{L}v = 0$ in Ω_T with boundary data $\phi_k(f - \alpha_k)$ where $\alpha_k = \fint_{64Q_k} f\, d\sigma$. Let $\overline{p} > p$. By (6.13), we obtain

$$\begin{aligned} |E(A\lambda) \cap Q_k| &\leq \left| \left\{ Q \in Q_k : \mathcal{M}_{2Q_k}(|\mathcal{N}(\nabla u - \nabla v_k)|^q)(Q) > \frac{A\lambda}{2^{q+1}} \right\} \right| \\ &\quad + \left| \left\{ Q \in Q_k : \mathcal{M}_{2Q_k}(|\mathcal{N}(\nabla v_k)|^q)(Q) > \frac{A\lambda}{2^{q+1}} \right\} \right| \\ &\lesssim \frac{1}{(A\lambda)^{\frac{\overline{p}}{q}}} \int_{2Q_k} |\mathcal{N}(\nabla u - \nabla v_k)|^{\overline{p}} d\sigma + \frac{1}{A\lambda} \int_{2Q_k} |\mathcal{N}(\nabla v_k)|^q d\sigma \equiv \mathcal{I} + \mathcal{II}. \end{aligned} \tag{6.14}$$

First, let's handle $\mathcal{II}$. By hypotesis $(R)_q$ is solvable, which yields

$$\mathcal{II} \lesssim \frac{1}{A\lambda} \int_{66Q_k} |\nabla_{tan} \phi(f - \alpha_k)|^q\, d\sigma + \frac{1}{A\lambda} \int_{66Q_k} \left| \partial_t^{\frac{1}{2}} \phi(f - \alpha_k) \right|^q d\sigma \equiv \mathcal{III} + \mathcal{IV}.$$

From the Poincaré inequality in Theorem 3.1, we can see that

$$\begin{aligned} \mathcal{III} &\lesssim \frac{1}{A\lambda} \int_{66Q_k} |\phi \nabla_{tan} f|^q\, d\sigma + \frac{1}{A\lambda} \int_{66Q_k} |(f - \alpha_k) \nabla_{tan} \phi|^q\, d\sigma \\ &\lesssim \frac{1}{A\lambda} \int_{66Q_k} |\nabla_{tan} f|^q d\sigma + \frac{1}{A\lambda} |Q_k|^{\frac{-q}{n+1}} \int_{66Q_k} |f - \alpha_k|^q d\sigma \\ &\lesssim \frac{1}{A\lambda} \int_{66Q_k} |\nabla_{tan} f|^q d\sigma + \frac{1}{A\lambda} \int_{66Q_k} \left| \partial_t^{\frac{1}{2}} f \right|^q d\sigma. \end{aligned}$$

In order to bound $\mathcal{IV}$, we first notice that

$$\begin{aligned}
\partial_t^{\frac{1}{2}}(\phi_k(f-\alpha_k))(q,s) &= \left(\int_{I_k} \frac{|\phi_k(f-\alpha_k)(q,\tau)-\phi_k(f-\alpha_k)(q,s)|^2}{|\tau-s|^2}\,d\tau\right)^{\frac{1}{2}} \\
&\lesssim \left\{\int_{I_k}\left(|\phi_k(q,\tau)|^2\frac{|f(q,\tau)-f(q,s)|^2}{|\tau-s|^2}+|f(q,s)-\alpha_k|^2\frac{|\phi_k(q,\tau)-\phi_k(q,s)|^2}{|\tau-s|^2}\right)d\tau\right\}^{\frac{1}{2}} \\
&\lesssim \left\{\int_{I_k}\frac{|f(q,\tau)-f(q,s)|^2}{|\tau-s|^2}\,d\tau\right\}^{\frac{1}{2}}+|f(q,s)-\alpha_k|\left\{\int_{I_k}\frac{|\phi_k(q,\tau)-\phi_k(q,s)|^2}{|\tau-s|^2}\,d\tau\right\}^{\frac{1}{2}} \\
&\lesssim \partial_t^{\frac{1}{2}}f(q,s)+|Q_k|^{-\frac{1}{n+1}}|f(q,s)-\alpha_k|
\end{aligned}$$

where I_k is the projection over the t axis of $66Q_k$. Consequently,

$$\mathcal{IV}\lesssim\frac{1}{A\lambda}\int_{66Q_k}|\nabla_{tan}f|^q d\sigma+\frac{1}{A\lambda}\int_{66Q_k}\left|\partial_t^{\frac{1}{2}}f\right|^q d\sigma.$$

The estimates for $\mathcal{III}$ and $\mathcal{IV}$ together with (6.10) give,

$$\mathcal{II}\lesssim\frac{\gamma|Q_k|}{A}.$$

Now, let's handle $\mathcal{I}$. Note that the hypotesis $(D)_{p'}$ and well-known properties of the L^p-Dirichlet problem implies $(D)_{\overline{p}'}$ for some $\overline{p}>p$. Also observe that $u-v_k-\alpha_k$ is a solution with boundary data $(f-\alpha_k)(1-\phi)$ and it vanishes on $64Q_k$. By (5.7) we find that

$$\begin{aligned}
\mathcal{I} &\lesssim \frac{|2Q_k|}{(A\lambda)^{\frac{\overline{p}}{q}}}\left(\fint_{64Q_k}|\mathcal{N}(\nabla u-\nabla v_k)|d\sigma\right)^{\overline{p}} \\
&\lesssim \frac{|2Q_k|}{(A\lambda)^{\frac{\overline{p}}{q}}}\left\{\left(\fint_{64Q_k}|\mathcal{N}(\nabla u)|^q\,d\sigma\right)^{\frac{\overline{p}}{q}}+\left(\fint_{64Q_k}|\mathcal{N}(\nabla v_k)|^q\,d\sigma\right)^{\frac{\overline{p}}{q}}\right\} \\
&\lesssim \frac{|2Q_k|}{(A\lambda)^{\frac{\overline{p}}{q}}}\left(\fint_{66Q_k}|\nabla_{tan}f|^q d\sigma+\fint_{66Q_k}\left|\partial_t^{\frac{1}{2}}f\right|^q d\sigma\right)^{\frac{\overline{p}}{q}}
\end{aligned}$$

where the last inequality is due to $(R)_q$. Using (6.10) again,

$$\mathcal{I}\lesssim\frac{|Q_k|}{A^{\frac{\overline{p}}{q}}}.$$

Finally, since $A=(2\epsilon)^{-\frac{q}{p}}$,

$$|E(A\lambda)\cap Q_k|\lesssim\left[\gamma\epsilon^{\frac{q}{p}-1}+\epsilon^{\frac{\overline{p}}{p}-1}\right]\epsilon|Q_k|.$$

We fix $\epsilon>0$ so small such that $\epsilon^{\frac{\overline{p}}{p}-1}<1$, and then we choose $\gamma>0$ such that $\gamma\epsilon^{\frac{q}{p}-1}<1$ and (6.11) follows. □

Acknowledgments

This article contains part of the results in the Doctoral Dissertation of the first author, who gratefully acknowledges partial financial support from *Instituto de Matemáticas - Unidad Cuernavaca, UNAM*, to finish this work. The second author received support from *Asociación Mexicana de Cultura* during the latter part of this work.

Both authors thank Prof. Steve Hofmann for useful conversations and feedback related to the topics included herein. The authors also thank the referees for their suggestions to improve the presentation of this work.

References

[1] D. G. Aronson, *Non-negative solutions of linear parabolic equations*, Ann. Scuola Norm. Sup. Pisa **22** (1968), 607–694.

[2] I. Athanasopoulos, L. A. Cafarelli and S. Salsa, *Caloric functions in Lipschitz domains and the regularity of solutions to phase transition problems*, Ann. of Math. **143** (1996), 413–434.

[3] R. M. Brown, *The Method of Layer Potentials for the Heat Equation in Lipschitz Cylinders*, Amer. J. Math. **111** (1989), 339–379.

[4] R. M. Brown, *The initial Neumann problem for the heat equation in Lipschitz cylinders*, Trans. Am. Math. Soc. **320** (1990), 1–52.

[5] L. A. Cafarelli and S. Salsa, *A Geometric Approach to Free Boundary Problems*, American Mathematical Society (2005).

[6] A. Castro, S. Rodriguez-López, and W. Staubach, *L^2 solvability of the Dirichlet, Neumann and regularity problems for parabolic equations with time independent Hölder continuous coefficients*, Trans. Amer. Math. Soc. **370** (2018), 265–319.

[7] B.E.J. Dahlberg, *Estimates of harmonic measure*, Arch. Rat. Mech. Anal. **65** (1977), 275–288.

[8] B.E.J. Dahlberg, *On the Poisson integral for Lipschitz and C^1 doamins*, Studia Math. **66** (1979), 13–24.

[9] E. B. Fabes, N. Garofalo, and E. Lanconelli, *Wiener's criterion for divergence form parabolic operators with C^1-Dini continuous coefficients*, Duke Math. J. **59** (1989),191-232.

[10] E. B. Fabes, N. Garofalo, and S. Salsa, *A backward Harnack inequality and Fatou theorem for non-negative solutions of parabolic equations*, Illinois J. Math. **30** (1986), 536–565.

[11] E. B. Fabes and M. Jodeit, *L^p boundary value problems for parabolic equations*, Bull. Amer. Math. Soc. **74** (1968), 1098–1102.

[12] E. B. Fabes, M. Jodeit and N.M. Riviére *Potential techniques for boundary value problems on C^1 domains*, Acta Math. **141** (1978), 165–186.

[13] E.B. Fabes and M.V. Safonov, *Behavior near boundary of positive solutions of second order parabolic equations*, J. Fourier Anal. and Appl., Special Issue: Proceedings of the Conference El Escorial 96, **3** (1997), 871–882.

[14] E.B. Fabes and S. Salsa, *Estimates of caloric measure and the initial-Dirichlet problem for the heat equation in Lipschitz cylinders*, Trans. Amer. Math Soc. **279** (1983), 635–650.

[15] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137-193.

[16] C. E. Kenig and J. Pipher, *The Neumann problem for eliptic equations with non-smoth coefficients*, Invent. Math. **113** (1993), 447–509.

[17] S. Hofmann and J. L. Lewis, *L^2 solvability and representation by caloric layer potentials in time-varying domains*, Ann. Math **144** (1996), 349–420.

[18] S. Hofmann and J. L. Lewis, *The L^p regularity problem for the heat equation in non-cylindrical domains*, Illinois J. Math **43** (1999), 752–769.

[19] R. A. Hunt and R. L. Wheeden, *On the Boundary values of harmonic functions*, Trans. Amer. Math. Soc. **132** (1967).

[20] D.S. Jerison and C.E. Kenig, *The Neumann problem on Lipschitz domains*, Bull. Amer. Math. Soc. **4** (1981), 203–207.

[21] J. T. Kemper, *Temperatures in several variables: kernel functions, representations, and parabolic boundary values*, Trans. Amer. Math. Soc. **167** (1972), 243–262.

[22] J. L. Lewis, and K. Nyström, *On a parabolic symmetry problem*, Rev. Mat. Iberoamericana **23** (2007), 513-536.

[23] J. Moser, *A Harnack inequality for parabolic differential equations*, Comm. Pure. Appl. Math. **17** (1964), 101-134.

[24] K. Nyström, *The Dirichlet problem for second order parabolic operators*, Indiana Univ. Math. J. **46** (1997), 183–245.

[25] J. Rivera-Noriega, *Absolute continuity of parabolic measure and area integral estimates in non-cylindrical domains*, Indiana Univ. Math. J. **52** (2003), 477–525.

[26] J. Rivera-Noriega, *Perturbation and solvability of initial L^p Dirichlet problems for parabolic equations over non-cylindrical domains*, Canadian J. Math. **66** (2014), 429–452.

[27] Z. Shen, *A relationship between the Dirichlet and regularity problems for elliptic equations*, Math. Res. Lett **14** (2007), 205–213.

[28] R. S. Strichartz. *Multipliers on fractional Sobolev spaces*, J. Math. Mech. **16** (1967),1031–1060.

[29] Torchinsky. *Real Variable Methods in Harmonic Analysis*, Academic Press, 1986.

[30] G. Verchota, *Layer potentials and Regularity for the Dirichlet problem for Laplace's equation in Lipschitz doamins*, J. Funct. Anal. **59** (1984), 572–611.