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Existence of Standing Waves in DNLS with Saturable Nonlinearity on 2D-Lattice

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Abstract

In this paper we obtain results on existence of standing waves in Discrete Nonlinear Shrödinger equation (DNLS) with saturable nonlinearity on a two-dimensional lattice. We consider two types of solutions: with periodic amplitude and vanishing at infinity (localized solution). Sufficient conditions for the existence of such solutions are obtained with the aid of Nehari manifold and periodic approximations.

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1 Introduction

Recently, considerable attention has been paid to models that are discrete in the spatial variables. Among the equations that describe such models, the most famous are the equations of chains of oscillators, the Discrete Sine–Gordon equation, the Fermi–Pasta–Ulam system and the Discrete Nonlinear Shrödinger equation.

Among the solutions of such systems, traveling waves deserve special attention. In papers [3], [15], [18], [19] traveling waves for infinite systems of linearly coupled oscillators on 2D–lattice are studied, while [9] and [26] deal with periodic in time solutions for such

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systems. Papers [11], [13], [16] are devoted to the well-posedness of initial value problem for infinite systems of linearly coupled nonlinear oscillators on 2D–lattice. In [7] it is obtained a result on the existence of subsonic periodic traveling waves for the system of nonlinearly coupled nonlinear oscillators on 2D–lattice, while in [8] supersonic periodic traveling waves for such systems are studied. Paper [5] is devoted to the existence of solitary traveling waves for such systems.

In paper [1] it is obtained a result on the existence of heteroclinic traveling waves for the discrete sine–Gordon equation with linear interaction on 2D–lattice. Paper [10] is devoted to the existence of periodic traveling waves for the discrete sine–Gordon equation with nonlinear interaction on 2D–lattice, while in [12] it is obtained a result on existence of heteroclinic traveling waves for such equation.

The existence of periodic and solitary traveling waves in Fermi–Pasta–Ulam system on 2D–lattice is studied in [2], [14]. A survey of existing results on the classical $1D$ Fermi-Pasta-Ulam lattice can be found in [21].

Another important class of solutions is standing waves. The existence of standing waves in DNLS is studied in [4], [22], [23], [24], [28], while [6] and [20] deal with standing waves for DNLS on 2D–lattice. In particular, in the paper [6] it is obtained a result on the existence of standing waves for such equations with cubic nonlinearity, and in [20] two-dimensional solitons in such systems are studied.

In the present paper we study the Discrete Nonlinear Shrödinger equation on a two-dimensional lattice

$$i\dot{\psi}_{n,m}(t) - a_1\triangle_{(1)}\psi_{n,m}(t) - a_2\triangle_{(2)}\psi_{n,m}(t) + \theta f(\psi_{n,m}(t)) = 0,\ (n,m)\in\mathbb{Z}^2, \tag{1.1}$$

where $\psi_{n,m}(t)$ is the wave function, $a_1 > 0, a_2 > 0$, $f:\mathbb{C}\to\mathbb{C}$ is a gauge invariant function, that is,

$$f(e^{i\omega t}z) = e^{i\omega t}f(z)$$

for all real ω, and

$$(\triangle_{(1)}\psi)_{n,m} = \psi_{n+1,m} + \psi_{n-1,m} - 2\psi_{n,m},$$

$$(\triangle_{(2)}\psi)_{n,m} = \psi_{n,m+1} + \psi_{n,m-1} - 2\psi_{n,m}$$

are the discrete Laplacians with respect to the variables n and m. If $a_1 = a_2 = 1$, then the sum of these operators will be a two-dimensional discrete Laplacian

$$(\triangle\psi)_{n,m} = \psi_{n+1,m} + \psi_{n-1,m} + \psi_{n,m+1} + \psi_{n,m-1} - 4\psi_{n,m}.$$

In addition, we assume that $f(\mathbb{R})\subset\mathbb{R}$. We introduce the parameter $\theta = \pm 1$ to distinguish between self-focusing ($\theta = 1$) and defocusing ($\theta = -1$) cases.

In this paper we deal with saturable nonlinearities which means that at infinity $f(z)$ growth as $const\cdot|z|$. Important examples of such nonlinearities are the following

$$f(u) = \frac{\nu|u|^p}{1+\mu|u|^p}u,\ \mu > 0, \nu > 0, p > 1, \tag{1.2}$$

and

$$f(u) = \chi(1 - \exp(-a|u|^p)u,\ \chi > 0, a > 0, p > 0. \tag{1.3}$$

Note that nonlinearity (1.2) with $1 < p \leq 2$, and nonlinearities (1.2), with $p > 2$, and (1.3) have different nature with respect to our approach.

A standing wave solution of Eq. (1.1) is a solution of the form

$$\psi_{n,m} = u_{n,m} \exp(-i\omega t), \tag{1.4}$$

where $u_{n,m} \in \mathbb{R}$ is called the amplitude of the standing wave and ω is the frequency of wave. Sometimes such solutions are called breathers.

We consider two types of solutions to (1.1): with k-periodic amplitude (periodic solution)

$$u_{n+k,m} = u_{n,m+k} = u_{n,m} \tag{1.5}$$

and vanishing at infinity (localized solution)

$$\lim_{n,m \to \pm\infty} u_{n,m} = 0. \tag{1.6}$$

Making use of the standing wave Ansatz (1.4) for Eq. (1.1), we obtain the equation for the amplitude

$$Lu_{n,m} - \omega u_{n,m} = \theta f(u_{n,m}), \tag{1.7}$$

where

$$(Lu)_{n,m} = -a_1 \triangle_{(1)} u_{n,m} - a_2 \triangle_{(2)} u_{n,m}.$$

Therefore, in this paper we are looking for nontrivial solutions of Eq. (1.7).

Now we introduce our basic assumptions. Let $F(t)$ be the primitive function for $f(t)$, that is,

$$F(t) = \int_0^t f(s)ds.$$

Then throughout the paper we will assume that the function $f(t)$ satisfies the following conditions:

(i) $f(t) = o(t)$, $t \to 0$;

(ii) $\lim\limits_{t \to \pm\infty} \frac{f(t)}{t} = l < \infty$;

(iii) *$f \in C^1(\mathbb{R})$ and $f(t)t < f'(t)t^2$, $t \neq 0$;*

Also we shall accept one of the following assumptions:

(iv) $\lim\limits_{t \to \pm\infty} (\frac{1}{2} f(t)t - F(t)) = \infty$.

(v) *the function $g(t) = f(t) - lt$ is bounded.*

The main result of the paper is the following theorem.

Theorem 1.1. *Assume (i) – (iii) and either (iv) or (v). Furthermore, assume that $\omega < 0$ and $\omega + l > 0$ if $\theta = 1$, or $\omega > 4(a_1 + a_2)$ and $\omega - l < 4(a_1 + a_2)$ if $\theta = -1$. Then Eq. (1.7) has a nontrivial solution $u \in l^2$. Moreover, if f is odd, then Eq. (1.7) has two nontrivial solutions $\pm u \in l^2$, and one of them is nonnegative provided $\theta = 1$.*

The theorem is a consequence of Theorems 5.2, 6.4 and 6.7. Some additional results can be found in the main body of the paper.

It is easily verified that nonlinearities (1.2) and (1.3) satisfy $(i)-(iii)$. Furthemore, (1.2) satisfies (iv) if $1 < p \leq 2$, and (v) if $p > 2$. Nonlinearity (1.3) satisfies (v). Therefore, we have the folollowing.

Corollary 1.2. *Assume that* $\omega < 0$ *and* $\omega + l > 0$ *if* $\theta = 1$, *or* $\omega > 4(a_1 + a_2)$ *and* $\omega - l < 4(a_1 + a_2)$ *if* $\theta = -1$. *Then Eq. (1.7) with either nonlinearity (1.2) or (1.3) has two nontrivial solutions* $\pm u \in l^2$, *and one of them is nonnegative provided* $\theta = 1$.

Notice that $u \in l^2$ implies immediately boundary condition (1.6) at infinity.

Our approach to the existence of localized solutions is variational. More precisely, we make use of Nehari manifold approach (references). First, we prove the existence of spatially periodic solutions with arbitrarily large periods (wave lengths). Then we pass to the limit as the period tends to infinity to obtain localized solutions. In the main body of the paper we deal with the self-focusing case under assumption (iv). On the end of the paper we indicate the changes needed to cover defocusing case and assumption (v).

The paper is organized as follows. In Section 2, we present some preliminaries, while in Section 3 we introduce the Nehari manifolds. In Sections 4 and 5 we prove the existence of periodic and localized solutions respectively in the self-focusing case ($\theta = 1$ under assumption (iv). Section 6 is devoted to the case of assumption (v) and defocusing case ($\theta = -1$).

2 Preliminaries

Let $k \geq 2$ be an integer. Then we denote by E_k the finite dimensional space of all $k-$periodic sequences $\{u_{n,m}\}$ (satisfying (1.5)) with the scalar product

$$(u, v)_k = \sum_{(n,m) \in Q_k} u_{n,m} v_{n,m}$$

and corresponding norm $\|u\|_k = (u, u)_k^{\frac{1}{2}}$, where

$$Q_k := \{(n,m) \in \mathbb{Z}^2 | -[\frac{k}{2}] \leq n, m \leq k - [\frac{k}{2}] - 1\},$$

$[\cdot]$ is the integer part.

We denote by E the space $l^2 = l^2(\mathbb{Z}^2)$ of all sequences $\{u_{n,m}\}$ (satisfying (1.6)) with the scalar product

$$(u, v) = \sum_{(n,m) \in \mathbb{Z}^2} u_{n,m} v_{n,m}$$

and corresponding norm $\|u\| = (u, u)^{\frac{1}{2}}$.

Sometimes, we will consider the spaces l_k^p and l^p with norms

$$\|u\|_{l_k^p} = \left(\sum_{(n,m) \in Q_k} |u_{n,m}|^p \right)^{\frac{1}{p}}, 1 \leq p \leq \infty,$$

and

$$\|u\|_{l^p} = \left(\sum_{(n,m)\in\mathbb{Z}^2} |u_{n,m}|^p \right)^{\frac{1}{p}}, 1 \le p \le \infty,$$

with well-known changes when $p = \infty$. Notice that $\|\cdot\|_{l_k^p}$ is an equivalent norm on E_k but not uniformly with respect to k. We note that

$$\|u\|_{l_k^q} \le \|u\|_{l_k^p},\ \|u\|_{l^q} \le \|u\|_{l^p},\ 1 \le p \le q \le \infty. \tag{2.1}$$

Remark 2.1. The operator L is bounded and self-adjoint in both spaces E_k and E. Elementary Fourier analysis shows that the spectrum of L in E coincides with the interval $[0, 4(a_1 + a_2)]$ and is absolutely continuous. In particular, L does not have eigenvectors in E. Under assumptions (i) – (iii) the function $\frac{f(t)}{|t|}$ is strictly increasing, while the function $\frac{1}{2}f(t)t - F(t)$ strictly increases for $t \ge 0$, strictly decreases for $t \le 0$ and, hence, is nonnegative.

On E_k and E, respectively, we consider the functionals

$$J_k(u) = \frac{1}{2}(L_k u - \omega u, u)_k - \sum_{(n,m)\in Q_k} \theta F(u_{n,m})$$

and

$$J(u) = \frac{1}{2}(Lu - \omega u, u) - \sum_{(n,m)\in\mathbb{Z}^2} \theta F(u_{n,m}),$$

where L_k is the operator L acting in the space E_k.

The following lemma can be obtained by a straightforward calculation.

Lemma 2.2. *Under assumptions J_k and J are the C^1-functionals on E_k and E, respectively, and their derivatives are given by*

$$\langle J_k'(u), h\rangle = (L_k u, h)_k - \sum_{(n,m)\in Q_k} \theta f(u_{n,m}) h_{n,m},\ u, h \in E_k, \tag{2.2}$$

$$\langle J'(u), h\rangle = (Lu, h) - \sum_{(n,m)\in\mathbb{Z}^2} \theta f(u_{n,m}) h_{n,m},\ u, h \in E. \tag{2.3}$$

Moreover, critical points of J_k and J are solutions of Eq. (1.7) satisfying (1.5) and (1.6), respectively.

3 Nehari manifolds

In this section we assume that $\theta = 1$ and assumptions (i)–(iii) hold.

For J_k and J we define the Nehari manifolds

$$N_k := \{u \in E_k | \langle J_k'(u), u\rangle = 0, u \ne 0\} \subset E_k$$

and

$$N := \{u \in E | \langle J'(u), u\rangle = 0, u \ne 0\} \subset E,$$

respectively. An interesting, simply verified property of Nehari manifolds is that all nontrivial solutions of Eq. 1.7 in E_k (respectively, in E) belong to N_k (respectively, N_k).

We introduce the notations $I_k(u) := \langle J_k'(u), u\rangle$ and $I(u) := \langle J'(u), u\rangle$. These are the C^1-functionals and their derivatives are given by

$$\langle I_k'(u), h\rangle = 2(L_k u, h)_k - \sum_{(n,m)\in Q_k} [f(u_{n,m}) + f'(u_{n,m})u_{n,m}]h_{n,m}, \tag{3.1}$$

$$\langle I'(u), h\rangle = 2(Lu, h) - \sum_{(n,m)\in \mathbb{Z}^2} [f(u_{n,m}) + f'(u_{n,m})u_{n,m}]h_{n,m}. \tag{3.2}$$

Lemma 3.1. *Assume (i) – (iii). Furthermore, assume that $\theta = 1$, $\omega < 0$ and $\omega + l > 0$. Then the sets N_k and N are nonempty closed C^1-submanifolds in E_k and in E, respectively. The derivatives I_k' and I' are nonzero on N_k and N, respectively. Moreover, there exists $\beta_0 > 0$ independent of k such that $\|u\|_k \geq \beta_0, u \in N_k$, and $\|u\| \geq \beta_0, u \in N$.*

Proof. We consider the case of N_k, the case of N is similar. First we show that the manifold N_k is nonempty. Let $\delta \in (-\omega, l)$ and E_δ be the spectral subspace of $L_k - \omega$ in the space E_k that corresponds to $[0, \delta]$. Since $-\omega \in \sigma(L_k - \omega)$, we have that $E_\delta \neq \{0\}$. Let $v \in E_\delta \setminus \{0\}$. Due to (i),

$$\langle J_k'(tv), tv\rangle = t^2(L_k v - \omega v, v)_k - \sum_{(n,m)\in Q_k} f(tv_{n,m})tv_{n,m} =$$

$$= t^2(L_k v - \omega v, v)_k - o(t^2) > 0,$$

as $t > 0$ small enough. But

$$\langle J_k'(tv), tv\rangle = t^2(L_k v - \omega v, v)_k - \sum_{(n,m)\in Q_k} f(tv_{n,m})tv_{n,m} =$$

$$\leq t^2 \left(\delta\|v\|_k^2 - \sum_{(n,m)\in Q_k} \frac{f(tv_{n,m})v_{n,m}^2}{tv_{n,m}} \right).$$

By assumption (ii), the sum above tends to $l\|v\|_k^2$. This implies that $\langle J_k'(tv), tv\rangle < 0$, as $t > 0$ large enough. Thus, there is $t^* > 0$ such that $\langle J_k'(t^*v), t^*\rangle = 0$ and $t^*v \in N_k$, i.e., $N_k \neq \varnothing$.

Let $u \in N_k$, then, by (2.2), (3.1) and the definition of N_k, we obtain

$$\langle I_k'(u), u\rangle = \langle I_k'(u), u\rangle - 2I_k(u) = \sum_{(n,m)\in Q_k} \left(f(u_{n,m})u_{n,m} - f'(u_{n,m})u_{n,m}^2 \right).$$

Due to (iii), this quantity is negative. Therefore $I_k'(u) \neq 0$, and, by the (infinite dimensional) implicit function theorem (see, e.g., [17]), N_k is a C^1-submanifold of E_k.

Now we prove the last statement of the lemma. We set $\phi(r) = \sup\limits_{|t|\leq r} \frac{f(t)}{t}$. This is an increasing function of $r \geq 0$, and, by (i), $\phi(r) \to 0$, as $r \to 0$. Let $u \in N_k$. We note that $L_k - \omega$ is the positive definite operator, and, by the definition of N_k and Eq. (2.1), we have

$$|\omega|\|u\|_k^2 \leq (L_k u - \omega u, u)_k = \sum_{(n,m)\in Q_k} f(u_{n,m})u_{n,m} \leq$$

$$\leq \phi(\|u\|_{l_k^\infty}) \cdot \|u\|_k^2 \leq \phi(\|u\|_k) \cdot \|u\|_k^2.$$

This implies that $\phi(\|u\|_k) \geq |\omega|$. Since function ϕ is an incresing, there exists $\beta_0 > 0$ such that $\|u\|_k \geq \beta_0$, $u \in N_k$.

It is obvious that the set N_k is closed. □

From the proof of Lemma 3.1 we obtain the following statement.

Corollary 3.2. *If $I_k(v) < 0$ (resp., $I(v) < 0$), then there exists a unique $t^* \in (0,1)$ such that $t^* v \in N_k$ (resp., $t^* v \in N$). Furthermore, there exists $v \in E_k \setminus \{0\}$ (resp., $v \in E \setminus \{0\}$) such that $J_k(v) < 0$ (resp., $J(v) < 0$).*

From the definitions of J_k and I_k it follows that on N_k

$$J_k(u) = J_k(u) - \frac{1}{2} I_k(u) = \sum_{(n,m)\in Q_k} \left(\frac{1}{2} f(u_{n,m}) u_{n,m} - F(u_{n,m}) \right). \tag{3.3}$$

By (iii), $J_k(u) \geq 0, u \in N_k$. Similarly, from the definitions of J and I it follows that on N

$$J(u) = J(u) - \frac{1}{2} I(u) = \sum_{(n,m)\in \mathbb{Z}^2} \left(\frac{1}{2} f(u_{n,m}) u_{n,m} - F(u_{n,m}) \right), \tag{3.4}$$

and $J(u) \geq 0, u \in N$.

Lemma 3.3. *Assume (i) – (iii). Furthermore, assume that $\theta = 1$, $\omega < 0$ and $\omega + l > 0$. Then there exists $\alpha_0 = \alpha_0(k) > 0$ such that $J_k(u) \geq \alpha_0$ for all $u \in N_k$.*

Proof. Let $u \in N_k$, then

$$J_k(u) = J_k(u) - \frac{1}{2} I_k(u) = \sum_{(n,m)\in Q_k} \left(\frac{1}{2} f(u_{n,m}) u_{n,m} - F(u_{n,m}) \right).$$

By Lemma 3.1, $\|u\|_k \geq \beta_0 > 0$. Therefore there exist $(n_0, m_0) \in Q_k$ (depending on u) and $\delta_0 = \delta_0(k, \beta_0) >$ (independent of u), such that $|u_{n_0,m_0}| \geq \delta_0$. Then we set $\alpha_0 = \frac{1}{2} f(\delta_0)\delta_0 - F(\delta_0)$, and, by Remark 2.1, we obtain that $J_k(u) \geq \alpha_0$ for $u \in N_k$.

□

Now we introduce the following minimization problems.

$$m_k = \inf\{J_k(u) : u \in N_k\}, \tag{3.5}$$

$$m = \inf\{J(u) : u \in N\}. \tag{3.6}$$

It turns that solutions of these problems are solutions of Eq. (1.7) as well.

Lemma 3.4. *Assume that (i)–(iii) hold, $\theta = 1$, $\omega < 0$ and $\omega + l > 0$. Then solutions of problems (3.5) and (3.6) are solutions of Eq. (1.7) in the spaces E_k and E, respectively.*

Proof. We consider the case of problem (3.6), the other one being similar. Let $u \in N$ be a solution of minimization problem (3.6). By the Lagrange multiplier method (see, e.g., [17]), there exists $\lambda \in \mathbb{R}$ such that

$$J'(u) + \lambda I'(u) = 0.$$

Since $\langle J'(u), u\rangle = I(u) = 0$, this and Eq. (3.2) imply that

$$0 = \lambda \langle I'(u), u\rangle = \lambda \sum_{(n,m)\in\mathbb{Z}^2} \left[f(u_{n,m})u_{n,m} - f'(u_{n,m})u_{n,m}^2\right].$$

By assumption (iii), the sum in the right-hand side is negative and, hence, $\lambda = 0$.

□

4 Periodic solutions

Lemma 4.1. *Assume (i) – (iv). Furthermore, assume that $\theta = 1$, $\omega < 0$ and $\omega + l > 0$. Then problem (3.5) has a solution.*

Proof. Let $\{u^j\} \subset N_k$ be a minimizing sequence for the functional J_k, i.e., $J_k(u^j) \to m_k$. From Eq. (3.3) we obtain

$$J_k(u^j) = \sum_{(n,m)\in Q_k} \left(\frac{1}{2} f(u^j_{n,m})u^j_{n,m} - F(u^j_{n,m})\right). \tag{4.1}$$

Let us prove that the sequence $\{u^j\}$ is bounded in E_k. Assume the contrary. Since all l^p-norms on E_k are equivalent, passing to a subsequence we have that $\|u^j\|_{l^\infty} \to \infty$. Then, for a further subsequence still denoted by $\{u^j\}$, there exists $(n_0, m_0) \in Q_k$ such that $u^j(n_0, m_0) \to \infty$. By Eq. (4.1), this and assumption (iv) imply that $J_k(u^j) \to \infty$. This is a contradiction because $J_k(u^j) \to m_k$ and, hence, is bounded.

Since E_k is a finite dimensional space and $\{u^j\}$ is bounded, we can assume, passing to a subsequence, that $u^j \to u \in E_k$. Since the Nehari manifold N_k is closed and J_k is continuous, we have that $u \in N_k$ and $J_k(u) = m_k$. □

The main result of this section is the following.

Theorem 4.2. *Assume (i) – (iv). Furthermore, assume that $\theta = 1$, $\omega < 0$ and $\omega + l > 0$. Then for every $k \geq 2$ Eq. (1.7) has a nontrivial k-periodic solution $u \in E_k$. Moreover, if f is odd, then Eq. (1.7) has two nontrivial solutions $\pm u \in E_k$, and one of them is nonnegative.*

Proof. The existence of nontrivial k-periodic solution $u \in E_k$ follows from Lemma 4.1.

Let f is odd. Then F is even and it is obvious that Eq. (1.7) has two nontrivial solutions $\pm u \in E_k$. It is easy to see that

$$(L|u|, |u|)_k \leq (Lu, u)_k.$$

Besides, $f(|t|)|t| = f(t)t$ and $F(|t|) = F(t)$. This means that

$$I_k(|u|) \leq I_k(u) = 0.$$

On the other hand,

$$\sum_{(n,m)\in Q_k}\left(\frac{1}{2}f(|u_{n,m}|)|u_{n,m}|-F(|u_{n,m}|)\right)=\sum_{(n,m)\in Q_k}\left(\frac{1}{2}f(u_{n,m})u_{n,m}-F(u_{n,m})\right)=m_k.$$

By Corollary 3.2, there is $t^*\in(0,1]$ such that $u^*=t^*|u|\in N_k$. Then, by Remark 2.1 and equation (3.3), we have that

$$J_k(u^*)\le\sum_{(n,m)\in Q_k}\left(\frac{1}{2}f(|u_{n,m}|)|u_{n,m}|-F(|u_{n,m}|)\right)=m_k.$$

Thus, $J_k(u^*)=m_k$ and u^* is a nonnegative solution, and we can assume that $u=u^*$. The proof is complete. □

Corollary 4.3. *Assume that* $\theta=1$, $\omega<0$ *and* $\omega+l>0$. *Then Eq. (1.7) with either nonlinearity (1.2), where* $1<p\le 2$, *or (1.3), where any* $p>0$, *has two nontrivial k-periodic solutions* $\pm u\in E_k$, *and one of them is nonnegative.*

5 Long wave length limit and localized solutions

We note that it is difficult to prove a result similar to Lemma 4.1 for problem (3.6). Therefore, in this case, critical points of J will be constructed in a different way, namely, by passing to the limit as $k\to\infty$ in the critical points of J_k. For this we need the following lemma.

Lemma 5.1. *Assume* (i) – (iv). *Furthermore, assume that* $\theta=1$, $\omega<0$ *and* $\omega+l>0$. *Let* u^k *be a k-periodic solution of problem (3.5), then the sequences* $\{m_k\}=\{J_k(u^k)\}$ *and* $\{\|u^k\|_k\}$ *are bounded.*

Proof. First we recall that the spectrum of L is absolutely continuous and coincides with the interval $[0,4(a_1+a_2)]$. Hence, for any $\delta\in(-\omega,l)$ the spectral subspace of $L-\omega$ that corresponds to $[0,\delta]$ is nonzero. Let $w\neq 0$ be any vector in that subspace. We have

$$\begin{aligned}I(tw)=\langle J'(tw),tw\rangle&=t^2(Lw-\omega w,w)-\sum_{(n,m)\in\mathbb{Z}^2}f(tw_{n,m})tw_{n,m}\\&\le t^2\left(\delta\|w\|^2-\sum_{(n,m)\in\mathbb{Z}^2}\frac{f(tw_{n,m})}{tw_{n,m}}w_{n,m}^2\right).\end{aligned}\tag{5.1}$$

By assumptions (i) and (ii), there is a constant $C>0$ independent of m, n and t, and such that

$$\left|\frac{f(tw_{n,m})}{tw_{n,m}}\right|\le C$$

for all $t\in\mathbb{R}$ and $(n,m)\in\mathbb{Z}^2$. Hence, the series in the right-hand side of Eq. (5.1) converges uniformly with respect to $t\in\mathbb{R}$. Therefore, by assumption (ii), the sum of this series converges to $l\|w\|^2$, and Eq. (5.1) implies that $I(tw)<0$ for all $t>0$ large enough. Fix any t with that property. By the density of finitely supported sequences in the space E, there exists a finitely supported vector $\tilde{w}$ sufficiently close to tw with the property that $I(\tilde{w})$. Corollary 3.2 implies that there exists $t^* \in (0,1)$ such that $I(v) = 0$, with $v = t^*\tilde{w}$. Since v is finitely supported, its support is contained in Q_k for all k large enough. For any such k, we define $v^k \in E_k$ as a unique element such that $v^k_{n,m} = v_{n,m}$ for $(n,m) \in Q_k$. Then $I_k(v^k) = I(v) = 0$ and $m_k \le J_k(v^k) = J(v)$. Thus, $\{m_k\}$ is bounded.

Now, we will prove that $\{\|u^k\|_k\}$ is bounded. Assume the contrary. Then, along a subsequence, $\|u^k\|_k \to \infty$. Letting $v^k = \frac{u^k}{\|u^k\|_k}$, we have that $\|v^k\|_k = 1$ and one of the following properties holds:

(vi) Vanishing: *the sequence* $\{v^k\}$ *satisfies the condition* $\|v^k\|_{l^\infty_k} = \|v^k\|_{l^\infty} \to 0$ *as* $k \to \infty$, or

(vii) Non-vanishing: *there exist* $\delta > 0$ *and* $(x_k, y_k) \in \mathbb{Z}^2$ *such that* $|v^k_{x_k,y_k}| \ge \delta$ *for all* k.

Consider the first case. Since the operator L is nonnegative and

$$0 = \frac{1}{\|u^k\|_k^2} I_k(u^k) = (L_k v^k - \omega v^k, v^k)_k - \sum_{(n,m)\in Q_k} \frac{f(u^k_{n,m})}{u^k_{n,m}} (v^k_{n,m})^2,$$

we have

$$|\omega| = |\omega| \|v^k\|_k^2 \le (L_k v^k - \omega v^k, v^k)_k = \sum_{(n,m)\in Q_k} \frac{f(u^k_{n,m})}{u^k_{n,m}} (v^k_{n,m})^2. \tag{5.2}$$

By assumption (i), there exists $t_0 > 0$ such that $\frac{f(t)}{t} \le \frac{|\omega|}{2}$ for $|t| < t_0$. Let

$$A_k = \{(n,m) \in Q_k : |u^k_{n,m}| < t_0\},$$

$$B_k = \{(n,m) \in Q_k : |u^k_{n,m}| \ge t_0\}.$$

Then

$$\sum_{(n,m)\in A_k} \frac{f(u^k_{n,m})}{u^k_{n,m}} (v^k_{n,m})^2 \le \frac{|\omega|}{2} \sum_{(n,m)\in A_k} (v^k_{n,m})^2 \le \frac{|\omega|}{2} \|v^k\|_k^2 = \frac{|\omega|}{2}.$$

By (5.2), this implies that

$$\liminf_{k\to\infty} \sum_{(n,m)\in B_k} \frac{f(u^k_{n,m})}{u^k_{n,m}} (v^k_{n,m})^2 \ge \frac{|\omega|}{2}. \tag{5.3}$$

On the other hand, $|f(t)| \le C_0|u|$ with a constant $C_0 > 0$. By the Hölder's inequality, we have

$$\sum_{(n,m)\in B_k} \frac{f(u^k_{n,m})}{u^k_{n,m}} (v^k_{n,m})^2 \le C_0 |B_k|^{\frac{p-2}{p}} \|v^k\|_{l^p_k}^{\frac{2}{p}} \tag{5.4}$$

for all $p > 2$, where $|B_k|$ is a number of the elements in B_k. Since $\|v^k\|_{l^\infty_k} \to 0$, inequalities (5.3) and (5.4), and the following simple inequality

$$\|w\|_{l^p_k} \le \|w\|_{l^\infty_k}^{\frac{p-2}{p}} \|w\|_k^{\frac{2}{p}} \tag{5.5}$$

imply that $|B_k| \to \infty$.

By Eq. (3.4) and Remark 2.1, we obtain

$$m_k = \sum_{(n,m)\in Q_k} \left(\frac{1}{2}f(u_{n,m})u_{n,m} - F(u_{n,m})\right) \geq \sum_{(n,m)\in B_k} \left(\frac{1}{2}f(u_{n,m})u_{n,m} - F(u_{n,m})\right) \geq$$

$$\geq \alpha_0 |B_k| \to \infty,$$

where

$$\alpha_0 = \min\left\{\frac{1}{2}f(\pm t_0)(\pm t_0) - F(\pm t_0)\right\}.$$

This is a contradiction.

Now we consider the second case (condition (vii)). By discrete translation invariance, we can assume that $(x_k, y_k) = (0,0)$. Since $\|v^k\|_k = 1$, we can assume that there exists $v = (v_{n,m})$ such that $v^k_{n,m} \to v_{n,m}$ for all $(n,m) \in \mathbb{Z}^2$ (passing to a subsequence if needed). Besides, it is obvious that $v \in E$, $\|v\| \leq 1$ and $|v_{0,0}| \geq \delta$. Thus, $v \neq 0$.

Since $u^k \in E_k$ is a solution of Eq. (1.7), we have

$$Lv^k_{n,m} - (\omega + l)v^k_{n,m} = \frac{g(u^k_{n,m})}{\|u^k\|_k}, \tag{5.6}$$

where $g(t) = f(t) - lt$. By assumption (i), $\lim\limits_{t\to\pm\infty} \frac{g(t)}{t} = 0$. If $v_{n,m} \neq 0$ for some $(n,m) \in \mathbb{Z}^2$, then $|u^k_{n,m}| \to \infty$. Passing to the limit in Eq. (5.6), we have that

$$Lv_{n,m} - (\omega + l)v_{n,m} = 0.$$

This implies that $v \in E$ is a nonzero eigenvector of the operator L, with the eigenvalue $\omega + l$. However, by Remark 2.1, the spectrum of L in E is absolutely continuous in E and, hence, all eigenvectors are tirvial. Again we have got a contradiction. Hence, $\{\|u^k\|_k\}$ is bounded. This completes the proof. □

The main result of the section is the following.

Theorem 5.2. *Assume (i) – (iv). Furthermore, assume that $\theta = 1$, $\omega < 0$ and $\omega + l > 0$. Then Eq. (1.7) has a nontrivial solution $u \in E$. Moreover, if f is odd, then Eq. (1.7) has two nontrivial solutions $\pm u \in E$, and one of them is nonnegative.*

Proof. Let $u^k \in E_k$ be a solution of Eq. (1.7). Then, by Lemma 5.1, the sequence $\{\|u^k\|_k\}$ is bounded and $\{u^k\}$ satisfies (vi) or (vii). In case (vi) inequality (5.5) implies that $\|u^k\|_{l^p_k} \to 0$ as $k \to \infty$ for all $p > 2$. By assumption (i), for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|f(t)| \leq \varepsilon|t| + C_\varepsilon |t|^{p-1}.$$

Since $u^k \in E_k$ is a solution of Eq. (1.7), we obtain that

$$|\omega| \|u^k\|_k^2 \leq (L_k u^k - \omega u^k, u^k)_k = \sum_{(n,m)\in Q_k} f(u^k_{n,m})u^k_{n,m} \leq \varepsilon \|u^k\|_k^2 + C_\varepsilon \|u^k\|^p_{l^p_k}.$$

Taking $\varepsilon = \frac{|\omega|}{2}$, we have that

$$\frac{|\omega|}{2}\|u^k\|_k^2 \le C_\varepsilon \|u^k\|_{l_k^p}^p \to 0$$

as $k \to \infty$. But this contradicts to Lemma 3.1. Thus, condition (vi) is not possible. Hence, the sequence $\{u^k\}$ satisfies (vii). Passing to a subsequence and using the discrete translation invariance, we can assume that $|u_{0,0}^k| \ge \delta$ for some $\delta > 0$. Making use of passage to a further subsequence, we also can assume that there is an $u = \{u_{n,m}\} \in E$ such that $u_{n,m}^k \to u_{n,m}$ for every $(n,m) \in \mathbb{Z}^2$. Obviously, $u \in E \setminus \{0\}$. Since point-wise limits preserve Eq. (1.7), u is a nontrivial solution of problem (3.6).

To prove the last part of the theorem, it is enough to take as u^k nonnegative periodic solutions that exist due to Theorem 4.2. Then the limit solution u is nonnegative.

The proof is complete. □

We complement Theorem 5.2 with the following result.

Theorem 5.3. *Under the assumptions of Theorem 5.2, $m_k \to m$. Furthermore, the solution $u \in E$ obtained in that theorem is a minimizer for problem (3.6), i.e., $J(u) = m$.*

Proof. Let $k_j \to \infty$ be a sequence of positive integers, and let $u^{k_j} \in E_{k_j}$ be a solution of problem (3.5) with $k = k_j$. From the proof of Theorem 5.2 it follows that, passing to a subsequence and making appropriate translations, we can assume that $u^{k_j} \to u \neq 0$ point-wise, where $u \in E$ is a solution of Eq. (1.7).

Let $(N,M) \in \mathbb{Z}_+^2$. Then, due to Remark 2.1, we obtain

$$\liminf_{j\to\infty} J_{k_j}(u^{k_j}) = \liminf_{j\to\infty} \sum_{(n,m)\in Q_{k_j}} \left(\frac{1}{2} f(u_{n,m}^{k_j}) u_{n,m}^{k_j} - F(u_{n,m}^{k_j})\right) \ge$$

$$\ge \liminf_{j\to\infty} \sum_{n=-N}^{N} \sum_{m=-M}^{M} \left(\frac{1}{2} f(u_{n,m}^{k_j}) u_{n,m}^{k_j} - F(u_{n,m}^{k_j})\right) =$$

$$= \sum_{n=-N}^{N} \sum_{m=-M}^{M} \left(\frac{1}{2} f(u_{n,m}) u_{n,m} - F(u_{n,m})\right).$$

In the limit $N \to \infty$ and $M \to \infty$, this implies that

$$\liminf_{j\to\infty} J_{k_j}(u^{k_j}) \ge J(u) \ge m.$$

Therefore,

$$\liminf_{k\to\infty} m_k \ge J(u) \ge m. \tag{5.7}$$

On the other hand, given $\varepsilon > 0$, let $w \in N$ be such that

$$J(w) = \sum_{(n,m)\in\mathbb{Z}^2} \left(\frac{1}{2} f(w_{n,m}) w_{n,m} - F(w_{n,m})\right) < m + \varepsilon.$$

Choose $t_1 > 1$ sufficiently close to 1 and such that

$$\sum_{(n,m)\in\mathbb{Z}^2} \left(\frac{1}{2} f(t_1 w_{n,m}) t_1 w_{n,m} - F(t_1 w_{n,m})\right) < m + \varepsilon.$$

Notice that $I(t_1 w) < 0$. Since finitely supported sequences are dense in E, there exists a finitely supported sequence $v = \{v_{n,m}\}$ sufficiently close to $t_1 w$ in E, and such that $I(v) < 0$ and

$$\sum_{(n,m)\in\mathbb{Z}^2} \left(\frac{1}{2} f(v_{n,m}) v_{n,m} - F(v_{n,m})\right) < m + \varepsilon.$$

Then there is $t_2 \in (0,1)$ such that $I(t_2 v) = 0$, and, by Remark 2.1, we have

$$J(t_2 v) = \sum_{(n,m)\in\mathbb{Z}^2} \left(\frac{1}{2} f(t_2 v_{n,m}) t_2 v_{n,m} - F(t_2 v_{n,m})\right) \le$$

$$\le \sum_{(n,m)\in\mathbb{Z}^2} \left(\frac{1}{2} f(v_{n,m}) v_{n,m} - F(v_{n,m})\right) < m + \varepsilon.$$

Define $v^k \in E_k$ $v^k = v$ as Q_k. Then for all sufficiently large k,

$$I_k(t_2 v^k) = I(t_2 v) = 0$$

and

$$J_k(t_2 v^k) = J(t_2 v) < m + \varepsilon.$$

Thus, we have

$$\limsup_{k\to\infty} m_k \le J(u) \le m. \tag{5.8}$$

Now from (5.7) and (5.8) we obtain the required. The proof is complete. □

6 Assumption (v) and defocusing case

In this section we replace assumption (iv) by (v).

Let σ_k be the spectrum of the operator L_k in the space E_k that consists of eigenvalues

$$4(a_1 \sin^2 \frac{\pi j}{k} + a_2 \sin^2 \frac{\pi l}{k}),$$

where $j, l = 0, 1, \dots, k-1$. It is easily seen that $\cup_k \sigma_k$ is a countable, dense subset of the spectrum $\sigma(L) = [0, 4(a_1 + a_2)]$ of L in E. Then $\sigma(L) \setminus \cup_k \sigma_k$ is dense in $\sigma(L)$ as well.

We need the following lemma.

Lemma 6.1. *Assume (i) – (iii) and (v). Furthermore, assume that $\theta = 1$, $\omega < 0$, $\omega + l > 0$ and $\omega + l \notin \sigma_k$. Then the functional J_k satisfies the Palais–Smale condition, i.e., every sequence $\{u^j\} \subset E_k$ such that $\{J_k(u^j)\}$ is bounded and $J_k'(u^j) \to 0$ (a Palais–Smale sequence) contains a convergent subsequence.*

Proof. To prove of the lemma it is enough to show that every Palais–Smale sequence $\{u^j\} \subset E_k$ is bounded, because the space E_k is finite dimensional.

Let E_k^+ and E_k^- be spectral subspaces of the operator $L_k - \omega$ that correspond to eigenvalues $\lambda > l$ and $\lambda < l$, respectively. Since $\omega + l$ is not an eigenvalue of L_k, l is not eigenvalue of $L_k - \omega$ and we have the orthogonal sum $E_k = E_k^+ \oplus E_k^-$. Any $u \in E_k$ has a unique decomposition $u = u^+ + u^-$, where $u^\pm \in E_k^\pm$. Accordingly, the Palais–Smale sequence $\{u^j\}$ splits as $u^j = u^{j+} + u^{j-}$. We have

$$\langle J_k'(u), v\rangle = ((L_k - \omega - l)u, v)_k - \sum_{(n,m)\in Q_k} g(u_{n,m})v_{n,m}.$$

Taking $u = u^j$ and $v = u^{j+}$, we obtain, due to the orthogonality of E_k^+ and E_k^-,

$$\langle J_k'(u^j), u^{j+}\rangle = \big((L_k - \omega - l)u^j, u^{j+}\big)_k - \sum_{(n,m)\in Q_k} g(u^j_{n,m})u^{j+}_{n,m} =$$

$$= \big((L_k - \omega - l)u^{j+}, u^{j+}\big)_k - \sum_{(n,m)\in Q_k} g(u^j_{n,m})u^{j+}_{n,m}.$$

On E_k^+

$$((L_k - \omega - l)v, v)_k \geq \alpha \|v\|_k^2,$$

where $\alpha > 0$. Since $\|J_k'(u^j)\| \leq 1$ for all sufficiently large j, and since all norms on a finite dimensional space are equivalent, assumption (v) implies

$$\alpha\|u^{j+}\|_k^2 \leq |\langle J_k'(u^j), u^{j+}\rangle| + |\sum_{(n,m)\in Q_k} g(u^j_{n,m})u^{j+}_{n,m}| \leq \|u^{j+}\|_k + c\|u^{j+}\|_k.$$

As consequence, $\{u^{j+}\}$ is bounded.

Similarly, making use of the inequality

$$((L_k - \omega - l)v, v)_k \leq -\alpha\|v\|_k^2, \quad v \in E_k^-,$$

where $\alpha > 0$, we obtain

$$\alpha\|u^{j-}\|_k^2 \leq \|u^{j-}\|_k + c\|u^{j+}\|_k.$$

Since u^{j+} is bounded, $\{u^{j-}\}$ is bounded as well. Therefore, $\{u^j\}$ is bounded, the lemma is proved. □

Making use of the mountain pass theorem, we shall prove the following analogue of Theorem 4.2.

Theorem 6.2. *Assume (i) – (iii) and (v). Furthermore, assume that $\theta = 1$, $\omega < 0$, $\omega + l > 0$ and $\omega + l \notin \sigma_k$. Then for every $k \geq 2$ Eq. (1.7) has a nontrivial k-periodic solution $u \in E_k$. Moreover, if f is odd, then Eq. (1.7) has two nontrivial solutions $\pm u \in E_k$, and one of them is nonnegative.*

Proof. Let

$$\Gamma = \{\gamma \in C([0,1]; E_k) : \gamma(0) = 0, J_k(\gamma(1)) < 0\}$$

and

$$b_k = \inf_{\gamma \in \Gamma_k} \max_{t \in [0,1]} J_k(\gamma(t)).$$

As in the proof of Lemma 3.3, we have that $J_k(u) \geq \alpha_0 > 0$ on the sphere $\|u\|_k = \beta_0 > 0$ (here we only need assumption (i)). On the other hand, by Corollary 3.2, there exists $v \in E_k \setminus \{0\}$ such that $J_k(v) < 0$. This implies that the functional J_k possess the mountain pass geometry. By Lemma 6.1, J_k satisfies Palais–Smale condition. Thus, the functional J_k satisfies all the assumptions of mountain pass theorem (see, e.g., [25] and [27]). Hence, $b_k > 0$ is a critical value of J_k and there exists a critical point $u^k \in E_k \setminus \{0\}$ of J_k with $J_k(u^k) = b_k$. By Lemma 2.2, u^k is a solution of Eq. (1.7).

The same arguments as in the proof of Theorem 4.2 prove the remaining part of the theorem. □

Corollary 6.3. *Assume that* $\theta = 1$, $\omega < 0$, $\omega + l > 0$ *and* $\omega + l \notin \sigma_k$. *Then Eq. (1.7) with either nonlinearity (1.2), where* $p > 2$, *or (1.3), where* $p > 0$, *has two nontrivial solutions* $\pm u \in E_k$.

Theorem 6.4. *Assume* (i) – (iii) *and* (v). *Furthermore, assume that* $\theta = 1$, $\omega < 0$ *and* $\omega + l > 0$. *Then Eq. (1.7) has a nontrivial solution* $u \in E$. *Moreover, if* f *is odd, then Eq. (1.7) has two nontrivial solutions* $\pm u \in E$.

Proof. For every k there exists ω_k such that $\omega_k \to \omega$ and $\omega_k + l \notin \sigma_k$. Then, by Theorem 6.2, there exists a periodic solution $u^k \in E_k$ of Eq. (1.7), with ω replaced by ω_k. Now we can use the same arguments as in the proofs of Lemma 5.1 Theorem 5.2 to obtain the boundedness of $\|u^k\|_k$ and then pass to the limit as $k \to \infty$. □

In the proofs given above we consider the case $\theta = 1$ only. The case $\theta = -1$ is similar with the functionals J and J_k replaced by $-J$ and $-J_k$. In this case conditions $\omega < 0$ and $\omega + l > 0$ are replaced by $\omega > 4(a_1 + a_2)$ and $\omega - l < 4(a_1 + a_2)$, respectively. Making use of essentially the same arguments as in the self-focusing case, we obtain the following results in the defocusing case.

Theorem 6.5. *Assume* (i) – (iii) *and either* (iv) *or* (v) *holds, and* $\theta = -1$, $\omega > 4(a_1 + a_2)$ *and* $\omega - l < 4(a_1 + a_2)$. *Furthermore, in the case of assumption* (v) *assume that* $\omega - l \notin \sigma_k$. *Then Eq. (1.7) has a nontrivial k-periodic solution* $u \in E_k$. *Moreover, if* f *is odd, then Eq. (1.7) has two nontrivial solutions* $\pm u \in E_k$.

Corollary 6.6. *Under assumptions* (i) – (iii) *and either* (iv) *or* (v), *assume that* $\theta = -1$, $\omega > 4(a_1 + a_2)$ *and* $\omega - l < 4(a_1 + a_2)$. *In addition, assume that* $\omega - l \notin \sigma_k$ *in case of assumption* (v). *Then Eq. (1.7) with nonlinearities (1.2) and (1.3) has two nontrivial k-periodic solutions* $\pm u \in E_k$.

Theorem 6.7. *Assume* (i) – (iii) *and either* (iv) *or* (v). *Furthermore, assume that* $\theta = -1$, $\omega > 4(a_1 + a_2)$ *and* $\omega - l < 4(a_1 + a_2)$. *Then Eq. (1.7) has a nontrivial solution* $u \in E$. *Moreover, if* f *is odd, then Eq. (1.7) has two nontrivial solutions* $\pm u \in E$.

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