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Generalizations of Majorization Inequality via Lidstone's Polynomial and Their Applications

M. Adil Khan 1*
Department of Mathematics
University of Peshawar
Peshawar, Pakistan

N. Latif 2†
Department of Mathematics
Govt. College University
Faisalabad 38000, Pakistan

J. Pečarić 3‡
Faculty of Textile Technology Zagreb
University of Zagreb
Prilaz Baruna Filipovića 28A, 10000 Zagreb, Croatia

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Abstract

In this paper, we obtain the generalizations of majorization inequalities by using Lidstone's interpolating polynomials and conditions on Green's functions. We give bounds for identities related to the generalizations of majorization inequalities by using *Č*eby*š*ev functionals. We also give Gr*ü*ss type inequalities and Ostrowski-type inequalities for these functionals. We present mean value theorems and n-exponential convexity which leads to exponential convexity and then log-convexity for these functionals. We give some families of functions which enable us to construct a large families of functions that are exponentially convex and also give Stolarsky type means with their monotonicity.

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*E-mail address: adilbandai@yahoo.com
†E-mail address: naveed707@gmail.com
‡E-mail address: pecaric@element.hr

1 Introduction and Preliminaries

Majorization makes precise the vague notion that the components of a vector **x** are ”less spread out” or ”more nearly equal” than the components of a vector **y**. For fixed $m \geq 2$ let

$$\mathbf{x} = (x_1, ..., x_m), \ \mathbf{y} = (y_1, ..., y_m)$$

denote two m-tuples. Let

$$x_{[1]} \geq x_{[2]} \geq ... \geq x_{[m]}, \ \ y_{[1]} \geq y_{[2]} \geq ... \geq y_{[m]},$$

$$x_{(1)} \leq x_{(2)} \leq ... \leq x_{(m)}, \ \ y_{(1)} \leq y_{(2)} \leq ... \leq y_{(m)}$$

be their ordered components.
Majorization: [20, p.319] **x** is said to majorize **y** (or **y** is said to be majorized by **x**), in symbol, $\mathbf{x} \succ \mathbf{y}$, if

$$\sum_{i=1}^{l} y_{[i]} \leq \sum_{i=1}^{l} x_{[i]} \tag{1.1}$$

holds for $l = 1, 2, ..., m-1$ and

$$\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} y_i.$$

Note that (1.1) is equivalent to

$$\sum_{i=m-l+1}^{m} y_{(i)} \leq \sum_{i=m-l+1}^{m} x_{(i)}$$

holds for $l = 1, 2, ..., m-1$.

There are several equivalent characterizations of the majoriztion relation $\mathbf{x} \succ \mathbf{y}$ in addition to the conditions given in definition of majorization. One is actually the answer of a question posed and answered in 1929 by Hardy, Littlewood and Polya [11, 12]: **x** majorizes **y** if

$$\sum_{i=1}^{m} \phi(y_i) \leq \sum_{i=1}^{m} \phi(x_i) \tag{1.2}$$

for all continuous convex functions ϕ. Another interesting characterization of $\mathbf{x} \succ \mathbf{y}$, also by Hardy, Littlewood, and Polya [11, 12], is that $\mathbf{y} = \mathbf{Px}$ for some double stochastic matrix **P**. In fact, the previous characterization implies that the set of vectors **x** that satisfy $\mathbf{x} \succ \mathbf{y}$ is the convex hull spanned by the $n!$ points formed from the permutations of the elements of **x**.
The following theorem is well-known as the majorization theorem and a convenient reference for its proof is given by Marshall and Olkin [17, p.11] (see also [20, p.320]):

Theorem 1.1. *Let* $\mathbf{x} = (x_1, ..., x_m), \mathbf{y} = (y_1, ..., y_m)$ *be two m-tuples such that* $x_i, \ y_i \in [a, b]$ $(i = 1, ..., m)$. *Then*

$$\sum_{i=1}^{m} \phi(y_i) \leq \sum_{i=1}^{m} \phi(x_i) \tag{1.3}$$

holds for every continuous convex function $\phi : [a, b] \to \mathbb{R}$ *iff* $\mathbf{x} \succ \mathbf{y}$ *holds.*

The following theorem can be regarded as a generalization of Theorem 1.1 known as Weighted Majorization Theorem and is proved by Fuchs in [10] (see also [20, p.323]):

Theorem 1.2. *Let* $\mathbf{x} = (x_1,...,x_m)$, $\mathbf{y} = (y_1,...,y_m)$ *be two decreasing real m-tuples with* x_i, $y_i \in [a,b]$ $(i = 1,...,m)$, *let* $\mathbf{w} = (w_1,...,w_m)$ *be a real m-tuple such that*

$$\sum_{i=1}^{l} w_i y_i \leq \sum_{i=1}^{l} w_i x_i \text{ for } l = 1,...,m-1; \tag{1.4}$$

and

$$\sum_{i=1}^{m} w_i y_i = \sum_{i=1}^{m} w_i x_i. \tag{1.5}$$

Then for every continuous convex function $\phi : [a,b] \to \mathbb{R}$, *we have*

$$\sum_{i=1}^{n} w_i \phi(y_i) \leq \sum_{i=1}^{n} w_i \phi(x_i). \tag{1.6}$$

Consider the Green function G defined on $[\alpha,\beta]\times[\alpha,\beta]$ by

$$G(t,s) = \begin{cases} \frac{(t-\beta)(s-\alpha)}{\beta-\alpha}, & \alpha \leq s \leq t; \\ \frac{(s-\beta)(t-\alpha)}{\beta-\alpha}, & t \leq s \leq \beta. \end{cases} \tag{1.7}$$

The function G is convex in s, it is symmetric, so it is also convex in t. The function G is continuous in s and continuous in t.

Theorem 1.3 ([2]). *Let* $\phi : [a,b] \to \mathbb{R}$ *be a continuous convex function on the interval* $[a,b]$ *and* $\mathbf{x} = (x_1,...,x_m)$, $\mathbf{y} = (y_1,...,y_m)$ *and* $\mathbf{w} = (w_1,...,w_m)$ *be m-tuples such that* x_i, $y_i \in [a,b]$ *and* $w_i \in \mathbb{R}$ $(i = 1,...,m)$ *which satisfies (1.5) and also G is defined in (1.7).*

Then the following two statements are equivalent.

(i) For every continuous convex function $\phi : [a,b] \to \mathbb{R}$, *it holds*

$$\sum_{i=1}^{m} w_i\phi(y_i) \leq \sum_{i=1}^{m} w_i\phi(x_i) \tag{1.8}$$

(ii) For all $\tau \in [a,b]$, *it holds*

$$\sum_{i=1}^{m} w_i G(y_i,\tau) \leq \sum_{i=1}^{m} w_i G(x_i,\tau). \tag{1.9}$$

Moreover, the statements (i) and (ii) are also equivalent if we change the sign of inequality in both inequalities, in (1.8) and in (1.9).

For integral version and generalization of majorization theorem see [17, p.583] ,[1, 3, 4, 7, 16, 18, 19]).

Bernstein had proved that if all the even derivatives are at least 0 in (a,b), then f has an analytic continuation into the complex plane. Boas suggested to Widder that this might be proved by use of the Lidstone series. This seemed plausible because the Lidstone series, a generalization of the Taylor series, approximates a given function in the neighborhood of two points instead of one by using the even derivatives. Such series have been studied by G. J. Lidstone (1929), H. Poritsky (1932), J. M. Wittaker (1934) and others (see [6]).

Definition 1.4. Let $\phi \in C^{\infty}([0,1])$, then the Lidstone series has the form

$$\sum_{k=o}^{\infty}\left(\phi^{(2k)}(0)\Lambda_k(1-x)+\phi^{(2k)}(1)\Lambda_k(x)\right), \tag{1.10}$$

where, Λ_n is a polynomial of degree $2n+1$ defined by the relations

$$\Lambda_0(t)=t,\quad \Lambda_n''(t)=\Lambda_{n-1}(t),\quad \Lambda_n(0)=\Lambda_n(1)=0,\ n\geq 1.$$

In [23], Widder proved the fundamental lemma:

Lemma 1.5. *if $\phi \in C^{2n}([0,1])$, then*

$$\phi(t)=\sum_{k=0}^{n-1}\left[\phi^{(2k)}(0)\Lambda_k(1-t)+\phi^{(2k)}\Lambda_k(t)\right]+\int_0^1 G_n(t,s)\phi^{(2n)}(t)dt,$$

where,

$$G_1(t,s)=G(t,s)=\begin{cases}(t-1)s\,, & s\leq t,\\ (s-1)t, & t\leq s,\end{cases} \tag{1.11}$$

is homogeneous Green's function of the differential operator $\frac{d^2}{ds^2}$ on $[0,1]$, and with the successive iterates of $G(t,s)$

$$G_n(t,s)=\int_0^1 G_1(t,p)G_{n-1}(p,s)dp,\quad n\geq 2. \tag{1.12}$$

The Lidstone polynomial can be expressed in terms of $G_n(t,s)$ as

$$\Lambda_n(t)=\int_0^1 G_n(t,s)s\,ds. \tag{1.13}$$

When dealing with functions with different degree of smoothness divided differences are found to be very useful.

Definition 1.6. Let ϕ be a real-valued function defined on the segment $[a,b]$. The divided difference of order n of the function ϕ at distinct points $x_0,...,x_n \in [a,b]$ is defined recursively (see [5], [20]) by

$$\phi[x_i]=\phi(x_i),\quad (i=0,...,n)$$

and

$$\phi[x_0,...,x_n]=\frac{\phi[x_1,...,x_n]-\phi[x_0,...,x_{n-1}]}{x_n-x_0}.$$

The value $\phi[x_0,...,x_n]$ is independent of the order of the points $x_0,...,x_n$.

The definition may be extended to include the case that some (or all) the points coincide. Assuming that $\phi^{(j-1)}(x)$ exists, we define

$$\phi\underbrace{[x,...,x]}_{j-times} = \frac{\phi^{(j-1)}(x)}{(j-1)!}. \tag{1.14}$$

The notion of n-convexity goes back to Popoviciu [22]. We follow the definition given by Karlin [15]:

Definition 1.7. A function $\phi : [a,b] \to \mathbb{R}$ is said to be n-convex on $[a,b]$, $n \geq 0$ if for all choices of $(n+1)$ distinct points in $[a,b]$, the n-th order divided difference of ϕ satisfies

$$\phi[x_0,...,x_n] \geq 0.$$

In fact, Popoviciu proved that each continuous n-convex function on $[a,b]$ is the uniform limit of the sequence of n-convex polynomials. Many related results, as well as some important inequalities due to Favard, Berwald and Steffensen can be found in [14].
The aim of this paper is to obtain some new identities by using Lidstone's interpolating polynomial. We arrange the paper in this manner: in Section 2, we give generalized results of majorization theorem by using Lidstone's polynomial and conditions on Green's function. We also give results for $(2n)$-convex functions and get classical and weighted majorization theorems as its special cases. In Section 3, we give bounds for identities related to the generalizations of majorization inequalities by using *Čebyšev* functionals. We also give Gr*ü*ss type inequalities and Ostrowski-type inequalities for these functionals. In Section 4, we present mean value theorems and n-exponential convexity for these functionals which leads to exponential convexity and then log-convexity. Finally, in Section 5, we present several families of functions which construct to a large families of functions that are exponentially convex. We give classes of Cauchy type means and prove their monotonicity.

2 Main Results

Theorem 2.1. *Let $n \in \mathbb{N}$, $\boldsymbol{x} = (x_1,...,x_m)$, $\boldsymbol{y} = (y_1,...,y_m)$ and $\mathbf{w} = (w_1,...,w_m)$ be m-tuples such that x_i, $y_i \in [a,b]$ and $w_i \in \mathbb{R}$ $(i = 1,...,m)$ and $\phi \in C^{2n}[a,b]$. Then*

$$\begin{aligned}
&\sum_{i=1}^{m} w_i\phi(x_i) - \sum_{i=1}^{m} w_i\phi(y_i) \\
&\quad = \sum_{k=0}^{n-1}(b-a)^{2k}\phi^{(2k)}(a)\left[\sum_{i=1}^{m} w_i\Lambda_k\left(\frac{b-x_i}{b-a}\right) - \sum_{i=1}^{m} w_i\Lambda_k\left(\frac{b-y_i}{b-a}\right)\right] \\
&\quad + \sum_{k=0}^{n-1}(b-a)^{2k}\phi^{(2k)}(b)\left[\sum_{i=1}^{m} w_i\Lambda_k\left(\frac{x_i-a}{b-a}\right) - \sum_{i=1}^{m} w_i\Lambda_k\left(\frac{y_i-a}{b-a}\right)\right] \\
&\quad + (b-a)^{2n-1}\int_a^b \left[\sum_{i=1}^{m} w_iG_n\left(\frac{x_i-a}{b-a}, \frac{t-a}{b-a}\right) - \sum_{i=1}^{m} w_iG_n\left(\frac{y_i-a}{b-a}, \frac{t-a}{b-a}\right)\right]\phi^{(2n)}(t)dt.
\end{aligned} \tag{2.1}$$

Proof. Consider

$$\sum_{i=1}^{m} w_i \phi(x_i) - \sum_{i=1}^{m} w_i \phi(y_i). \tag{2.2}$$

We use Widder's Lemma for representation of function in the form:

$$\phi(x) = \sum_{k=0}^{n-1} (b-a)^{2k} \left[\phi^{(2k)}(a) \Lambda_k \left(\frac{b-x}{b-a} \right) + \phi^{(2k)}(b) \Lambda_k \left(\frac{x-a}{b-a} \right) \right]$$
$$+ (b-a)^{2n-1} \int_a^b G_n \left(\frac{x-a}{b-a}, \frac{t-a}{b-a} \right) \phi^{(2n)}(t) dt, \tag{2.3}$$

where, Λ_k is a Lidstone polynomial.
Using value of $\phi(x)$ from (2.3) in (2.2), we have

$$\sum_{i=1}^{m} w_i \phi(x_i) - \sum_{i=1}^{m} w_i \phi(y_i)$$
$$= \sum_{i=1}^{m} w_i \left\{ \sum_{k=0}^{n-1} (b-a)^{2k} \left[\phi^{(2k)}(a) \Lambda_k \left(\frac{b-x_i}{b-a} \right) + \phi^{(2k)}(b) \Lambda_k \left(\frac{x_i-a}{b-a} \right) \right] \right\}$$
$$+ \sum_{i=1}^{m} w_i \left[(b-a)^{2n-1} \int_a^b G_n \left(\frac{x_i-a}{b-a}, \frac{t-a}{b-a} \right) \phi^{(2n)}(t) dt \right]$$
$$- \sum_{i=1}^{m} w_i \left\{ \sum_{k=0}^{n-1} (b-a)^{2k} \left[\phi^{(2k)}(a) \Lambda_k \left(\frac{b-y_i}{b-a} \right) + \phi^{(2k)}(a) \Lambda_k \left(\frac{y_i-a}{b-a} \right) \right] \right\}$$

$$- \sum_{i=1}^{m} w_i \left[(b-a)^{2n-1} \int_a^b G_n \left(\frac{y_i-a}{b-a}, \frac{t-a}{b-a} \right) \phi^{(2n)}(t) dt \right],$$

after some arrangement we get (2.1). □

Integral version of the above theorem can be stated as:

Theorem 2.2. *Let $n \in \mathbb{N}$, $x, y : [\alpha, \beta] \to [a, b]$, $w : [\alpha, \beta] \to \mathbb{R}$ be continuous functions and $\phi \in C^{2n}[a, b]$. Then*

$$\int_\alpha^\beta w(t) \phi(x(t)) dt - \int_\alpha^\beta w(t) \phi(y(t)) dt$$
$$= \sum_{k=0}^{n-1} (b-a)^{2k} \phi^{(2k)}(a) \left[\int_\alpha^\beta w(t) \Lambda_k \left(\frac{b-x(t)}{b-a} \right) dt - \int_\alpha^\beta w(t) \Lambda_k \left(\frac{b-y(t)}{b-a} \right) dt \right]$$
$$+ \sum_{k=0}^{n-1} (b-a)^{2k} \phi^{(2k)}(b) \left[\int_\alpha^\beta w(t) \Lambda_k \left(\frac{x(t)-a}{b-a} \right) dt - \int_\alpha^\beta w(t) \Lambda_k \left(\frac{y(t)-a}{b-a} \right) dt \right]$$
$$+ (b-a)^{2n-1} \int_a^b \phi^{(2n)}(s) \left[\int_\alpha^\beta w(t) G_n \left(\frac{x(t)-a}{b-a}, \frac{s-a}{b-a} \right) dt - \int_\alpha^\beta w(t) G_n \left(\frac{y(t)-a}{b-a}, \frac{s-a}{b-a} \right) dt \right] ds.$$

We give generalization of majorization theorem for $2n-$convex function.

Theorem 2.3. *Let $n \in \mathbb{N}$, , $\boldsymbol{x} = (x_1,...,x_m)$,$\boldsymbol{y} = (y_1,...,y_m)$ and $\mathbf{w} = (w_1,...,w_m)$ be m-tuples such that x_i, $y_i \in [a,b]$ and $w_i \in \mathbb{R}$ $(i = 1,...,m)$.*
If for all $t \in [a,b]$

$$\sum_{i=1}^{m} w_i G_n\left(\frac{y_i - a}{b-a}, \frac{t-a}{b-a}\right) \leq \sum_{i=1}^{m} w_i G_n\left(\frac{x_i - a}{b-a}, \frac{t-a}{b-a}\right) \tag{2.4}$$

then for every $(2n)$-convex function $\phi : [a,b] \to \mathbb{R}$, we have

$$\begin{aligned} &\sum_{i=1}^{m} w_i \phi(x_i) - \sum_{i=1}^{m} w_i \phi(y_i) \\ &\quad \geq \sum_{k=0}^{n-1} (b-a)^{2k} \phi^{(2k)}(a) \left[\sum_{i=1}^{m} w_i \Lambda_k \left(\frac{b-x_i}{b-a}\right) - \sum_{i=1}^{m} w_i \Lambda_k \left(\frac{b-y_i}{b-a}\right) \right] \\ &\quad + \sum_{k=0}^{n-1} (b-a)^{2k} \phi^{(2k)}(b) \left[\sum_{i=1}^{m} w_i \Lambda_k \left(\frac{x_i - a}{b-a}\right) - \sum_{i=1}^{m} w_i \Lambda_k \left(\frac{y_i - a}{b-a}\right) \right]. \end{aligned} \tag{2.5}$$

If the reverse inequality in (2.4) holds, then also the reverse inequality in (2.5) holds.

Proof. If the function ϕ is $2n$-convex, without loss of generality we can assume that ϕ is $2n$-times differentiable, therefore we have $\phi^{(2n)}(x) \geq 0$, for all $x \in [a,b]$, and by using (2.4), we get (2.5). □

Integral version can be stated as:

Theorem 2.4. *Let $n \in \mathbb{N}$, $x,y : [\alpha,\beta] \to [a,b]$ and $w : [\alpha,\beta] \to \mathbb{R}$ be any continuous functions. If for all $s \in [a,b]$*

$$\int_{\alpha}^{\beta} w(t) G_n\left(\frac{y(t)-a}{b-a}, \frac{s-a}{b-a}\right) dt \leq \int_{\alpha}^{\beta} w(t) G_n\left(\frac{x(t)-a}{b-a}, \frac{s-a}{b-a}\right) dt \tag{2.6}$$

then for every $(2n)$-convex function $\phi : [a,b] \to \mathbb{R}$,

$$\begin{aligned} &\int_{\alpha}^{\beta} w(t)\phi(x(t))dt \geq \int_{\alpha}^{\beta} w(t)\phi(y(t))dt \\ &\quad + \sum_{k=0}^{n-1} (b-a)^{2k} \phi^{(2k)}(a) \left[\int_{\alpha}^{\beta} w(t) \Lambda_k \left(\frac{b-x(t)}{b-a}\right) dt - \int_{\alpha}^{\beta} w(t) \Lambda_k \left(\frac{b-y(t)}{b-a}\right) dt \right] \\ &\quad + \sum_{k=0}^{n-1} (b-a)^{2k} \phi^{(2k)}(b) \left[\int_{\alpha}^{\beta} w(t) \Lambda_k \left(\frac{x(t)-a}{b-a}\right) dt - \int_{\alpha}^{\beta} w(t) \Lambda_k \left(\frac{y(t)-a}{b-a}\right) dt \right]. \end{aligned} \tag{2.7}$$

If the reverse inequality in (2.6) holds, then also the reverse inequality in (2.7) holds.

The following theorem is majorization theorem for $2n$-convex function:

Theorem 2.5. *Let $n \in \mathbb{N}$, Let $\boldsymbol{x} = (x_1, ..., x_m)$, $\boldsymbol{y} = (y_1, ..., y_m)$ be two decreasing real m-tuples with x_i, $y_i \in [a,b]$ $(i = 1, ..., m)$, let $\mathbf{w} = (w_1, ..., w_m)$ be a real m-tuple such that which satisfies (1.4), (1.5) and G_n be defined in (1.12).*

(i) If n is odd, then for every $2n$-convex function $\phi : [a,b] \to \mathbb{R}$, it holds

$$\begin{aligned}\sum_{i=1}^{m} w_i \phi(x_i) - \sum_{i=1}^{m} w_i \phi(y_i) &\geq \sum_{k=1}^{n-1} (b-a)^{2k} \phi^{(2k)}(a) \left[\sum_{i=1}^{m} w_i \Lambda_k \left(\frac{b-x_i}{b-a} \right) - \sum_{i=1}^{m} w_i \Lambda_k \left(\frac{b-y_i}{b-a} \right) \right] \\ &+ \sum_{k=1}^{n-1} (b-a)^{2k} \phi^{(2k)}(b) \left[\sum_{i=1}^{m} w_i \Lambda_k \left(\frac{x_i-a}{b-a} \right) - \sum_{i=1}^{m} w_i \Lambda_k \left(\frac{y_i-a}{b-a} \right) \right].\end{aligned} \tag{2.8}$$

(ii) Consider the inequality (2.8) be satisfied and let $\vartheta : [a,b] \to \mathbb{R}$ be a function defined by

$$\vartheta(.) := \sum_{k=1}^{n-1} (b-a)^{2k} \left(\phi^{(2k)}(a) \Lambda_k \left(\frac{b-.}{b-a} \right) + \phi^{(2k)}(b) \Lambda_k \left(\frac{.-a}{b-a} \right) \right). \tag{2.9}$$

If ϑ is a convex function, then the right hand side of (2.8) is non-negative that is the following weighted majorization inequality holds

$$\sum_{i=1}^{m} w_i \phi(y_i) \leq \sum_{i=1}^{m} w_i \phi(x_i). \tag{2.10}$$

(iii) If n is even, then for every $2n$-convex function $\phi : [a,b] \to \mathbb{R}$, it holds

$$\begin{aligned}\sum_{i=1}^{m} w_i \phi(x_i) - \sum_{i=1}^{m} w_i \phi(y_i) &\leq \sum_{k=1}^{n-1} (b-a)^{2k} \phi^{(2k)}(a) \left[\sum_{i=1}^{m} w_i \Lambda_k \left(\frac{b-x_i}{b-a} \right) - \sum_{i=1}^{m} w_i \Lambda_k \left(\frac{b-y_i}{b-a} \right) \right] \\ &+ \sum_{k=1}^{n-1} (b-a)^{2k} \phi^{(2k)}(b) \left[\sum_{i=1}^{m} w_i \Lambda_k \left(\frac{x_i-a}{b-a} \right) - \sum_{i=1}^{m} w_i \Lambda_k \left(\frac{y_i-a}{b-a} \right) \right].\end{aligned} \tag{2.11}$$

(iv) Consider the inequality (2.11) be satisfied and let $\vartheta : [a,b] \to \mathbb{R}$ be a function defined in (2.9). If ϑ is a concave function, then the right hand side of (2.11) is non-positive that is reverse inequality in (2.10) is valid.

Proof. (i) By (1.11), $G_1(t,s) \leq 0$, for $0 \leq t, s \leq 1$.
By using (1.12), we have $G_n(t,s) \leq 0$ for odd n and $G_n(t,s) \geq 0$ for even n.

Now as G_1 is convex and G_{n-1} is positive for odd n, therefore by using (1.12), G_n is convex in first variable if n is odd. Similarly G_n is concave in first variable if n is even.
Hence if n is odd then by majorization theorem we have

$$\sum_{i=1}^{m} w_i G_n\left(\frac{y_i - a}{b-a}, \frac{t-a}{b-a}\right) \leq \sum_{i=1}^{m} w_i G_n\left(\frac{x_i - a}{b-a}, \frac{t-a}{b-a}\right). \tag{2.12}$$

Therefore if n is odd, then by Theorem 2.3, (2.8) holds.
(ii) We can easily get the equivalent form of the inequality (2.8) as

$$\sum_{i=1}^{m} w_i\, \phi(x_i) - \sum_{i=1}^{m} w_i\, \phi(y_i) \geq \sum_{i=1}^{m} w_i\, \vartheta(x_i) - \sum_{i=1}^{m} w_i\, \vartheta(y_i).$$

By using (1.4), (1.5) and the fact that ϑ is a convex function, so by applying weighted majorization inequality, we get immediately the non-negativity of the right hand side of (2.8) and we have the inequality (2.10).
Similarly we can prove (iii) and (iv). □

The following theorem is majorization theorem for $2n$-convex function in integral case:

Theorem 2.6. *Let $n \in \mathbb{N}$, $x, y : [\alpha,\beta] \to [a,b]$ be decreasing and $w : [\alpha,\beta] \to \mathbb{R}$ be any continuous functions and G_n be defined in (1.12). Let*

$$\int_{\alpha}^{\upsilon} w(t)y(t)dt \leq \int_{\alpha}^{\upsilon} w(t)x(t)dt, \quad for\ \upsilon \in [\alpha,\beta] \tag{2.13}$$

and

$$\int_{\alpha}^{\beta} w(t)y(t)dt = \int_{\alpha}^{\beta} w(t)x(t)dt. \tag{2.14}$$

(i) If n is odd, then for every $2n$-convex function $\phi : [a,b] \to \mathbb{R}$, it holds

$$\begin{aligned} &\int_{\alpha}^{\beta} w(t)\phi(x(t))dt - \int_{\alpha}^{\beta} w(t)\phi(y(t))dt \\ &\quad \geq \sum_{k=1}^{n-1} (b-a)^{2k}\phi^{(2k)}(a)\left[\int_{\alpha}^{\beta} w(t)\Lambda_k\left(\frac{b-x(t)}{b-a}\right)dt - \int_{\alpha}^{\beta} w(t)\Lambda_k\left(\frac{b-y(t)}{b-a}\right)dt\right] \\ &\quad + \sum_{k=1}^{n-1} (b-a)^{2k}\phi^{(2k)}(b)\left[\int_{\alpha}^{\beta} w(t)\Lambda_k\left(\frac{x(t)-a}{b-a}\right)dt - \int_{\alpha}^{\beta} w(t)\Lambda_k\left(\frac{y(t)-a}{b-a}\right)dt\right]. \end{aligned} \tag{2.15}$$

(ii) Consider the inequality (2.15) be satisfied and let $\vartheta : [a,b] \to \mathbb{R}$ be a function defined in (2.9) is a convex function, then the right hand side of (2.15) is non-negative that is the following weighted majorization inequality in integral case holds

$$\int_{\alpha}^{\beta} w(t)\phi(y(t))dt \leq \int_{\alpha}^{\beta} w(t)\phi(x(t))dt. \tag{2.16}$$

(iii) If n is even, then for every $2n$-convex function $\phi : [a,b] \to \mathbb{R}$, it holds

$$\int_\alpha^\beta w(t)\phi(x(t))dt - \int_\alpha^\beta w(t)\phi(y(t))dt$$
$$\leq \sum_{k=1}^{n-1}(b-a)^{2k}\phi^{(2k)}(a)\left[\int_\alpha^\beta w(t)\Lambda_k\left(\frac{b-x(t)}{b-a}\right)dt - \int_\alpha^\beta w(t)\Lambda_k\left(\frac{b-y(t)}{b-a}\right)dt\right]$$
$$+ \sum_{k=1}^{n-1}(b-a)^{2k}\phi^{(2k)}(b)\left[\int_\alpha^\beta w(t)\Lambda_k\left(\frac{x(t)-a}{b-a}\right)dt - \int_\alpha^\beta w(t)\Lambda_k\left(\frac{y(t)-a}{b-a}\right)dt\right]. \tag{2.17}$$

(iv) Consider the inequality (2.17) be satisfied and let $\vartheta : [a,b] \to \mathbb{R}$ be a function defined in (2.9). If ϑ is a concave function, then the right hand side of (2.17) is non-positive that is reverse inequality in (2.16) is valid.

3 BOUNDS FOR IDENTITIES RELATED TO GENERALIZATION OF MAJORIZATION INEQUALITY

For two Lebesgue integrable functions $f, h : [a,b] \to \mathbb{R}$ we consider the Čebyšev functional

$$\Omega(f,h) = \frac{1}{b-a}\int_a^b f(t)h(t)dt - \frac{1}{b-a}\int_a^b f(t)dt . \frac{1}{b-a}\int_a^b h(t)dt. \tag{3.1}$$

In [8], the authors proved the following theorems:

Theorem 3.1. *Let $f : [a,b] \to \mathbb{R}$ be a Lebesgue integrable function and $h : [a,b] \to \mathbb{R}$ be an absolutely continuous function with $(.-a)(b-.)[h']^2 \in L[a,b]$. Then we have the inequality*

$$|\,\Omega(f,h)\,| \leq \frac{1}{\sqrt{2}}\,[\Omega(f,f)]^{\frac{1}{2}}\,\frac{1}{\sqrt{b-a}}\left(\int_a^b (x-a)(b-x)\left[h'(x)\right]^2 dx\right)^{\frac{1}{2}}. \tag{3.2}$$

The constant $\frac{1}{\sqrt{2}}$ in (3.2) is the best possible.

Theorem 3.2. *Assume that $h : [a,b] \to \mathbb{R}$ is monotonic nondecreasing on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ is absolutely continuous with $f' \in L_\infty[a,b]$. Then we have the inequality*

$$|\,\Omega(f,h)\,| \leq \frac{1}{2(b-a)}\,\|\,f'\,\|_\infty \int_a^b (x-a)(b-x)dh(x). \tag{3.3}$$

The constant $\frac{1}{2}$ in (3.3) is the best possible.

In the sequel we use the above theorems to obtain generalizations of the results proved in the previous section.
For m-tuples $\mathbf{w} = (w_1, ..., w_m)$, $\mathbf{x} = (x_1, ..., x_m)$ and $\mathbf{y} = (y_1, ..., y_m)$ with $x_i,\ y_i \in [a,b], w_i \in \mathbb{R}$ $(i = 1, ..., m)$ and the function G_n as defined above, denote

$$\Upsilon(t) = \sum_{i=1}^m w_i G_n\left(\frac{x_i-a}{b-a}, \frac{t-a}{b-a}\right) - \sum_{i=1}^m w_i G_n\left(\frac{y_i-a}{b-a}, \frac{t-a}{b-a}\right), \tag{3.4}$$

similarly for $x,y : [\alpha,\beta] \to [a,b]$ and $w : [\alpha,\beta] \to \mathbb{R}$ be continuous functions and for all $s \in [a,b]$, denote

$$\widetilde{\Upsilon}(s) = \int_{\alpha}^{\beta} w(t) G_n\left(\frac{x(t)-a}{b-a}, \frac{s-a}{b-a}\right) dt - \int_{\alpha}^{\beta} w(t) G_n\left(\frac{y(t)-a}{b-a}, \frac{s-a}{b-a}\right) dt. \tag{3.5}$$

Theorem 3.3. *Let $\phi : [a,b] \to \mathbb{R}$ be such that $\phi \in C^{2n}[a,b]$ for $n \in \mathbb{N}$ with $(.-a)(b-.)\left[\phi^{(2n+1)}\right]^2 \in L[a,b]$, and $x_i, y_i \in [a,b]$ and $w_i \in \mathbb{R}$ $(i = 1,2,...,m)$ and let the functions G_n and Υ be defined in (1.12) and (3.4) respectively. Then the remainder $H_n^1(\phi; a,b)$ defined by*

$$\begin{aligned} H_n^1(\phi;a,b) &= \sum_{i=1}^{m} w_i \phi(x_i) - \sum_{i=1}^{m} w_i \phi(y_i) \\ &- \sum_{k=0}^{n-1} (b-a)^{2k} \phi^{(2k)}(a) \left[\sum_{i=1}^{m} w_i \Lambda_k\left(\frac{b-x_i}{b-a}\right) - \sum_{i=1}^{m} w_i \Lambda_k\left(\frac{b-y_i}{b-a}\right) \right] \\ &- \sum_{k=0}^{n-1} (b-a)^{2k} \phi^{(2k)}(b) \left[\sum_{i=1}^{m} w_i \Lambda_k\left(\frac{x_i-a}{b-a}\right) - \sum_{i=1}^{m} w_i \Lambda_k\left(\frac{y_i-a}{b-a}\right) \right] \\ &- (b-a)^{2n-2} \left(\phi^{(2n-1)}(b) - \phi^{(2n-1)}(a)\right) \int_a^b \Upsilon(t) dt \end{aligned} \tag{3.6}$$

satisfies the estimation

$$\left|H_n^1(\phi;a,b)\right| \le \frac{(b-a)^{2n-\frac{1}{2}}}{\sqrt{2}} \left[\Omega(\Upsilon,\Upsilon)\right]^{\frac{1}{2}} \left| \int_a^b (t-a)(b-t) \left[\phi^{(2n+1)}(t)\right]^2 dt \right|^{\frac{1}{2}}. \tag{3.7}$$

Proof. Comparing (2.1) and (3.6) we get

$$\begin{aligned} H_n^1(\phi;a,b) &= (b-a)^{2n-1} \int_a^b \Upsilon(t) \phi^{(2n)}(t) dt - (b-a)^{2n-2} \left(\phi^{(2n-1)}(b) - \phi^{(2n-1)}(a)\right) \int_a^b \Upsilon(t) dt \\ &= (b-a)^{2n-1} \int_a^b \Upsilon(t) \phi^{(2n)}(t) dt - (b-a)^{2n-2} \int_a^b \phi^{(2n)} dt \int_a^b \Upsilon(t) dt \\ &= (b-a)^{2n} \Omega(\Upsilon, \phi^{(2n)}). \end{aligned}$$

Applying Theorem 3.1 on the functions Υ and $\phi^{(2n)}$ we get (3.7). □

Integral case of the above theorem can be given:

Theorem 3.4. *Let $\phi : [a,b] \to \mathbb{R}$ be such that $\phi \in C^{2n}[a,b]$ for $n \in \mathbb{N}$ with $(.-a)(b-.)\left[\phi^{(2n+1)}\right]^2 \in L[a,b]$, and $x,y : [\alpha,\beta] \to [a,b]$, $w : [\alpha,\beta] \to \mathbb{R}$ be continuous functions and let the functions G_n and $\widetilde{\Upsilon}$ be defined in (1.12) and (3.5) respectively. Then the remainder*

$\widetilde{H}_n^1(\phi;a,b)$ *defined by*

$$\begin{aligned}\widetilde{H}_n^1(\phi;a,b) &= \int_\alpha^\beta w(t)\phi(x(t))\,dt - \int_\alpha^\beta w(t)\phi(y(t))\,dt \\ &- \sum_{k=0}^{n-1}(b-a)^{2k}\phi^{(2k)}(a)\left[\int_\alpha^\beta w(t)\Lambda_k\left(\frac{b-x(t)}{b-a}\right)dt - \int_\alpha^\beta w(t)\Lambda_k\left(\frac{b-y(t)}{b-a}\right)dt\right] \\ &- \sum_{k=0}^{n-1}(b-a)^{2k}\phi^{(2k)}(b)\left[\int_\alpha^\beta w(t)\Lambda_k\left(\frac{x(t)-a}{b-a}\right)dt - \int_\alpha^\beta w(t)\Lambda_k\left(\frac{y(t)-a}{b-a}\right)dt\right] \\ &- (b-a)^{2n-2}\left(\phi^{(2n-1)}(b)-\phi^{(2n-1)}(a)\right)\int_a^b \widetilde{\Upsilon}(s)ds \end{aligned} \tag{3.8}$$

satisfies the estimation

$$\left|\widetilde{H}_n^1(\phi;a,b)\right| \le \frac{(b-a)^{2n-\frac{1}{2}}}{\sqrt{2}}\left[\Omega(\widetilde{\Upsilon},\widetilde{\Upsilon})\right]^{\frac{1}{2}}\left|\int_a^b (t-a)(b-t)\left[\phi^{(2n+1)}(t)\right]^2 dt\right|^{\frac{1}{2}}.$$

Use Theorem 3.2 we obtain the following Gr*ü*ss type inequality.

Theorem 3.5. *Let $\phi:[a,b]\to\mathbb{R}$ be such that $\phi\in C^{2n}[a,b]$ $(n\in\mathbb{N})$ and $\phi^{(2n+1)}\ge 0$ on $[a,b]$ and let the function Υ be defined by (3.4). Then the remainder $H_n^1(\phi;a,b)$ defined by (3.6) satisfies the estimation*

$$\left|H_n^1(\phi;a,b)\right| \le (b-a)^{2n-1}\left\|\Upsilon'\right\|_\infty\left\{\frac{\phi^{(2n-1)}(b)+\phi^{(2n-1)}(a)}{2} - \frac{\phi^{(2n-2)}(b)-\phi^{(2n-2)}(a)}{b-a}\right\}. \tag{3.9}$$

Proof. Since $H_n^1(\phi;a,b) = (b-a)^{2n}\,\Omega(\Upsilon,\phi^{(2n)})$, applying Theorem 3.2 on the functions Υ and $\phi^{(2n)}$ we get (3.9). □

Integral version of the above theorem can be given as:

Theorem 3.6. *Let $\phi:[a,b]\to\mathbb{R}$ be such that $\phi\in C^{2n}[a,b]$ $(n\in\mathbb{N})$ and $\phi^{(2n+1)}\ge 0$ on $[a,b]$ and let the function $\widetilde{\Upsilon}$ be defined by (3.5). Then the remainder $\widetilde{H}_n^1(\phi;a,b)$ defined by (3.8) satisfies the estimation*

$$\left|\widetilde{H}_n^1(\phi;a,b)\right| \le (b-a)^{2n-1}\left\|\widetilde{\Upsilon}'\right\|_\infty\left\{\frac{\phi^{(2n-1)}(b)+\phi^{(2n-1)}(a)}{2} - \frac{\phi^{(2n-2)}(b)-\phi^{(2n-2)}(a)}{b-a}\right\}.$$

We give the Ostrowski-type inequality related to the generalization of majorization inequality.

Theorem 3.7. *Suppose that all the assumptions of Theorem 2.1 hold. Assume (p,q) is a pair of conjugate exponents, that is $1\le p,q\le\infty$, $\frac{1}{p}+\frac{1}{q}=1$. Let $\left|\phi^{(2n)}\right|^p:[a,b]\to\mathbb{R}$ be an*

R-integrable function for some $n \in \mathbb{N}$. Then we have

$$\begin{aligned}
&\left| \sum_{i=1}^{m} w_i \phi(x_i) - \sum_{i=1}^{m} w_i \phi(y_i) \right. \\
&- \sum_{k=0}^{n-1} (b-a)^{2k} \phi^{(2k)}(a) \left[\sum_{i=1}^{m} w_i \Lambda_k \left(\frac{b-x_i}{b-a} \right) - \sum_{i=1}^{m} w_i \Lambda_k \left(\frac{b-y_i}{b-a} \right) \right] \\
&\left. - \sum_{k=0}^{n-1} (b-a)^{2k} \phi^{(2k)}(b) \left[\sum_{i=1}^{m} w_i \Lambda_k \left(\frac{x_i-a}{b-a} \right) - \sum_{i=1}^{m} w_i \Lambda_k \left(\frac{y_i-a}{b-a} \right) \right] \right| \\
&\leq (b-a)^{2n-1} \left\| \phi^{(2n)} \right\|_p \left(\int_a^b \left| \sum_{i=1}^{m} w_i G_n \left(\frac{x_i-a}{b-a}, \frac{t-a}{b-a} \right) - \sum_{i=1}^{m} w_i G_n \left(\frac{y_i-a}{b-a}, \frac{t-a}{b-a} \right) \right|^q dt \right)^{\frac{1}{q}}.
\end{aligned} \tag{3.10}$$

The constant on the right-hand side of (3.10) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. Let us denote

$$\Psi(t) = (b-a)^{2n-1} \left[\sum_{i=1}^{m} w_i G_n \left(\frac{x_i-a}{b-a}, \frac{t-a}{b-a} \right) - \sum_{i=1}^{m} w_i G_n \left(\frac{y_i-a}{b-a}, \frac{t-a}{b-a} \right) \right].$$

Using the identity (2.1) and applying Hölder's inequality we obtain

$$\begin{aligned}
&\left| \sum_{i=1}^{m} w_i \phi(x_i) - \sum_{i=1}^{m} w_i \phi(y_i) \right. \\
&- \sum_{k=0}^{n-1} (b-a)^{2k} \phi^{(2k)}(a) \left[\sum_{i=1}^{m} w_i \Lambda_k \left(\frac{b-x_i}{b-a} \right) - \sum_{i=1}^{m} w_i \Lambda_k \left(\frac{b-y_i}{b-a} \right) \right] \\
&\left. - \sum_{k=0}^{n-1} (b-a)^{2k} \phi^{(2k)}(b) \left[\sum_{i=1}^{m} w_i \Lambda_k \left(\frac{x_i-a}{b-\alpha} \right) - \sum_{i=1}^{m} w_i \Lambda_k \left(\frac{y_i-a}{b-a} \right) \right] \right| \\
&= \left| \int_a^b \Psi(t) \phi^{(2n)}(t) dt \right| \leq \left\| \phi^{(2n)} \right\|_p \left(\int_a^b |\Psi(t)|^q \, dt \right)^{\frac{1}{q}}.
\end{aligned}$$

For the proof of the sharpness of the constant $\left(\int_a^b |\Psi(t)|^q \, dt \right)^{\frac{1}{q}}$ let us find a function ϕ for which the equality in (3.10) is obtained.
For $1 < p < \infty$ take ϕ to be such that

$$\phi^{(2n)}(t) = sgn\Psi(t) \, |\Psi(t)|^{\frac{1}{p-1}}.$$

For $p = \infty$ take $\phi^{(2n)}(t) = sgn\Psi(t)$.
For $p = 1$ we prove that

$$\left| \int_a^b \Psi(t) \phi^{(2n)}(t) \right| \leq \max_{t \in [a,b]} |\Psi(t)| \left(\int_a^b \left| \phi^{(2n)}(t) \right| dt \right) \tag{3.11}$$

is the best possible inequality. Suppose that $|\Psi(t)|$ attains its maximum at $t_0 \in [a,b]$. First we assume that $\Psi(t_0) > 0$. For ϵ small enough we define $\phi_\epsilon(t)$ by

$$\phi_\epsilon(t) := \begin{cases} 0, & a \le t \le t_0, \\ \frac{1}{\epsilon n!}(t-t_0)^n, & t_0 \le t \le t_0+\epsilon, \\ \frac{1}{n!}(t-t_0)^{n-1}, & t_0+\epsilon \le t \le b. \end{cases}$$

Then for ϵ small enough

$$\left|\int_a^b \Psi(t)\phi^{(2n)}(t)\right| = \left|\int_{t_0}^{t_0+\epsilon} \Psi(t)\frac{1}{\epsilon}dt\right| = \frac{1}{\epsilon}\int_{t_0}^{t_0+\epsilon} \Psi(t)dt.$$

Now from the inequality (3.11) we have

$$\frac{1}{\epsilon}\int_{t_0}^{t_0+\epsilon} \Psi(t)dt \le \Psi(t_0)\int_{t_0}^{t_0+\epsilon} \frac{1}{\epsilon}dt = \Psi(t_0).$$

Since

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{t_0}^{t_0+\epsilon} \Psi(t)dt = \Psi(t_0)$$

the statement follows. In the case $\Psi(t_0) < 0$, we define $\phi_\epsilon(t)$ by

$$\phi_\epsilon(t) := \begin{cases} \frac{1}{n!}(t-t_0-\epsilon)^{n-1}, & a \le t \le t_0, \\ -\frac{1}{\epsilon n!}(t-t_0-\epsilon)^n, & t_0 \le t \le t_0+\epsilon, \\ 0, & t_0+\epsilon \le t \le b, \end{cases}$$

and the rest of the proof is the same as above. □

Integral version of the above theorem can be stated as:

Theorem 3.8. *Suppose that all the assumptions of Theorem 2.2 hold. Assume (p,q) is a pair of conjugate exponents, that is $1 \le p,q \le \infty$, $\frac{1}{p}+\frac{1}{q}=1$. Let $\left|\phi^{(2n)}\right|^p : [a,b] \to \mathbb{R}$ be an R-integrable function for some $n \in \mathbb{N}$. Then we have*

$$\begin{aligned} &\left|\int_\alpha^\beta w(t)\phi(x(t))\,dt - \int_\alpha^\beta w(t)\phi(y(t))\,dt \right. \\ &- \sum_{k=0}^{n-1}(b-a)^{2k}\phi^{(2k)}(a)\left[\int_\alpha^\beta w(t)\Lambda_k\left(\frac{b-x(t)}{b-a}\right)dt - \int_\alpha^\beta w(t)\Lambda_k\left(\frac{b-y(t)}{b-a}\right)dt\right] \\ &\left. - \sum_{k=0}^{n-1}(b-a)^{2k}\phi^{(2k)}(b)\left[\int_\alpha^\beta w(t)\Lambda_k\left(\frac{x(t)-a}{b-a}\right)dt - \int_\alpha^\beta w(t)\Lambda_k\left(\frac{y(t)-a}{b-a}\right)dt\right]\right| \\ &\le (b-a)^{2n-1}\left\|\phi^{(2n)}\right\|_p\left(\int_a^b\left|\int_\alpha^\beta w(t)G_n\left(\frac{x(t)-a}{b-a},\frac{s-a}{b-a}\right)dt - \int_\alpha^\beta w(t)G_n\left(\frac{y(t)-a}{b-a},\frac{s-a}{b-a}\right)dt\right|^q ds\right)^{\frac{1}{q}}. \end{aligned} \tag{3.12}$$

The constant on the right-hand side of (3.12) is sharp for $1 < p \le \infty$ and the best possible for $p = 1$.

4 n-Exponential Convexity and Exponential Convexity

Motivated by the inequality (2.5) and (2.7), we define functional $\Theta_1(\phi)$ and $\Theta_2(\phi)$ by

$$\begin{aligned}\Theta_1(\phi) &= \sum_{i=1}^{m} w_i\phi(x_i) - \sum_{i=1}^{m} w_i\phi(y_i)\\ &- \sum_{k=0}^{n-1}(b-a)^{2k}\phi^{(2k)}(a)\left[\sum_{i=1}^{m} w_i\Lambda_k\left(\frac{b-x_i}{b-a}\right) - \sum_{i=1}^{m} w_i\Lambda_k\left(\frac{b-y_i}{b-a}\right)\right]\\ &- \sum_{k=0}^{n-1}(b-a)^{2k}\phi^{(2k)}(b)\left[\sum_{i=1}^{m} w_i\Lambda_k\left(\frac{x_i-a}{b-a}\right) - \sum_{i=1}^{m} w_i\Lambda_k\left(\frac{y_i-a}{b-a}\right)\right]\end{aligned} \tag{4.1}$$

$$\begin{aligned}\Theta_2(\phi) &= \int_\alpha^\beta w(t)\phi(x(t))dt - \int_\alpha^\beta w(t)\phi(y(t))dt\\ &- \sum_{k=0}^{n-1}(b-a)^{2k}\phi^{(2k)}(a)\left[\int_\alpha^\beta w(t)\Lambda_k\left(\frac{b-x(t)}{b-a}\right)dt - \int_\alpha^\beta w(t)\Lambda_k\left(\frac{b-y(t)}{b-a}\right)dt\right]\\ &- \sum_{k=0}^{n-1}(b-a)^{2k}\phi^{(2k)}(b)\left[\int_\alpha^\beta w(t)\Lambda_k\left(\frac{x(t)-a}{b-a}\right)dt - \int_\alpha^\beta w(t)\Lambda_k\left(\frac{y(t)-a}{b-a}\right)dt\right].\end{aligned} \tag{4.2}$$

Theorem 4.1. *Let $\phi : [a,b] \to \mathbb{R}$ be such that $\phi \in C^{2n}[a,b]$. If the inequalities in (2.4) $(i = 1)$, (2.6) $(i = 2)$ hold, then there exist $\xi_i \in [a,b]$ such that*

$$\Theta_i(\phi) = \phi^{(2n)}(\xi)\Theta_i(\eta), \;\; i = 1,2. \tag{4.3}$$

where $\eta(x) = \frac{x^{2n}}{(2n)!}$.

Proof. Similar to the proof of Theorem 7 in [6]. □

Theorem 4.2. *Let $\phi,\psi : [a,b] \to \mathbb{R}$ be such that $\phi,\psi \in C^{2n}[a,b]$. If the inequalities in (2.4) $(i = 1)$, (2.6) $(i = 2)$ hold, then there exist $\xi_i \in [a,b]$ such that*

$$\frac{\Theta_i(\phi)}{\Theta_i(\varphi)} = \frac{\phi^{(2n)}(\xi)}{\psi^{(2n)}(\xi)}, \;\; i = 1,2. \tag{4.4}$$

provided that the denominators are not zero.

Proof. Similar to the proof of Corollary 12 in [6]. □

Definition 4.3 ([20, p. 2])**.** A function $\phi : I \to \mathbb{R}$ is convex on an interval I if

$$\phi(x_1)(x_3-x_2) + \phi(x_2)(x_1-x_3) + \phi(x_3)(x_2-x_1) \ge 0, \tag{4.5}$$

holds for all $x_1,x_2,x_3 \in I$ such that $x_1 < x_2 < x_3$.

Now, let us recall some definitions and facts about exponentially convex functions (see [13]):

Definition 4.4. A function $\phi : I \to \mathbb{R}$ is *n-exponentially convex* in the Jensen sense on I if

$$\sum_{k,l=1}^{n} \alpha_k \alpha_l \phi\left(\frac{x_k + x_l}{2}\right) \geq 0$$

holds for $\alpha_k \in \mathbb{R}$ and $x_k \in I$, $k = 1,2,...,n$.

Definition 4.5. A function $\phi : I \to \mathbb{R}$ is *n-exponentially convex* on I if it is *n-exponentially convex* in the Jensen sense and continuous on I.

Remark 4.6. From the definition it is clear that 1-exponentially convex functions in the Jensen sense are in fact non-negative functions. Also, n-exponentially convex functions in the Jensen sense are m-exponentially convex in the Jensen sense for every $m \in \mathbb{N}, m \leq n$.

Proposition 4.7. *If $\phi : I \to \mathbb{R}$ is an n-exponentially convex in the Jensen sense, then the matrix $\left[\phi\left(\frac{x_k+x_l}{2}\right)\right]_{k,l=1}^{m}$ is a positive semi-definite matrix for all $m \in \mathbb{N}, m \leq n$. Particularly,*

$$\det\left[\phi\left(\frac{x_k + x_l}{2}\right)\right]_{k,l=1}^{m} \geq 0$$

for all $m \in \mathbb{N}$, $m = 1,2,...,n$.

Definition 4.8. A function $\phi : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

Definition 4.9. A function $\phi : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Remark 4.10. It is easy to show that $\phi : I \to \mathbb{R}$ is log-convex in the Jensen sense if and only if

$$\alpha^2\phi(x) + 2\alpha\beta\phi\left(\frac{x+y}{2}\right) + \beta^2\phi(y) \geq 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a function is log-convex in the Jensen-sense if and only if it is 2-exponentially convex in the Jensen sense.

Also, using basic convexity theory it follows that a function is log-convex if and only if it is 2-exponentially convex.

Corollary 4.11. *If $\phi : I \to (0,\infty)$ is an exponentially convex function, then ϕ is a* log-*convex function that is*

$$\phi(\lambda x + (1-\lambda)y) \leq \phi^{\lambda}(x)\phi^{1-\lambda}(y), \ for\ all\ x,y \in I,\ \lambda \in [0,1].$$

We use an idea from [13] to give an elegant method of producing an n-exponentially convex functions and exponentially convex functions applying the above functionals on a given family with the same property (see [19]):

Theorem 4.12. *Let $\Phi = \{\phi_s : s \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval $[a,b]$ in $\mathbb{R}$, such that the function $s \mapsto \phi_s[x_0,...,x_{2l}]$ is an n-exponentially convex in the Jensen sense on J for every $(2l+1)$ mutually different points $x_0,...,x_{2l} \in [a,b]$. Let $\Theta_i(\phi)$, $i = 1,2$ be linear functionals defined as in (4.1) and (4.2). Then $s \mapsto \Theta_i(\phi_s)$ is an n-exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Theta_i(\phi_s)$ is continuous on J, then it is n-exponentially convex function on J.*

Proof. For $\vartheta_i \in \mathbb{R}$, $i = 1,...,n$ and $s_i \in J$, $i = 1,...,n$ we define the function

$$\delta(x) = \sum_{i,j=1}^{n} \vartheta_i \vartheta_j \phi_{\frac{s_i+s_j}{2}}(x).$$

Using the assumption that the function $s \mapsto f_s[x_0,...,x_{2l}]$ is n-exponentially convex in the Jensen sense, we have

$$\delta[x_0,...,x_{2l}] = \sum_{i,j=1}^{n} \vartheta_i \vartheta_j \phi_{\frac{s_i+s_j}{2}}[x_0,...,x_{2l}] \geq 0,$$

which in turn implies that δ is a $2l$-convex function on J, so it is $\Theta_k(\delta) \geq 0$, hence

$$\sum_{i,j=1}^{n} \vartheta_i \vartheta_j \Theta_k(\phi_{\frac{s_i+s_j}{2}}) \geq 0.$$

We conclude that the function $s \mapsto \Theta_k(\phi_s)$ is n-exponentially convex function in the Jensen sense on J.
If the function $s \mapsto \Theta_k(\phi_s)$ is also continuous on J, then $s \mapsto \Theta_k(\phi_s)$ is n-exponentially convex by definition. □

The following corollaries are an immediate consequences of the above theorem:

Corollary 4.13. *Let $\Phi = \{\phi_s : s \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval $[a,b]$ in $\mathbb{R}$, such that the function $s \mapsto \phi_s[x_0,...,x_{2l}]$ is an exponentially convex in the Jensen sense on J for every $(2l+1)$ mutually different points $x_0,...,x_{2l} \in [a,b]$. Let $\Theta_i(\phi)$, $i = 1,2$ be linear functionals defined as in (4.1) and (4.2). Then $s \mapsto \Theta_i(\phi_s)$ is an exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Theta_i(\phi_s)$ is continuous on J, then it is exponentially convex function on J.*

Corollary 4.14. *Let $\Phi = \{\phi_s : s \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval $[a,b]$ in $\mathbb{R}$, such that the function $s \mapsto \phi_s[x_0,...,x_{2l}]$ is an 2-exponentially convex in the Jensen sense on J for every $(2l+1)$ mutually different points $x_0,...,x_{2l} \in [a,b]$. Let $\Theta_i(\phi)$, $i = 1,2$ be linear functionals defined as in (4.1) and (4.2). Then the following statements hold:*

(i) If the function $s \mapsto \Theta_i(\phi_s)$ is continuous on J, then it is 2-exponentially convex function on J. If $s \mapsto \Theta_i(\phi_s)$ is additionally strictly positive, then it is log-*convex on J. Furthermore, the Lypunov's inequality holds true:*

$$[\Theta_i(\phi_s)]^{t-r} \leq [\Theta_i(\phi_r)]^{t-s} [\Theta_i(\phi_t)]^{s-r} \tag{4.6}$$

for every choice $r,s,t \in J$, such that $r < s < t$.

(ii) If the function $s \mapsto \Theta_i(\phi_s)$ is strictly positive and differentiable on J, then for every $s,q,u,v \in J$, such that $s \leq u$ and $q \leq v$, we have

$$\mu_{s,q}(\Theta_i,\Phi) \leq \mu_{u,v}(\Theta_i,\Phi), \tag{4.7}$$

where

$$\mu_{s,q}(\Theta_i,\Phi)=\begin{cases}\left(\frac{\Theta_i(\phi_s)}{\Theta_i(\phi_q)}\right)^{\frac{1}{s-q}}, & s\neq q,\\ \exp\left(\frac{\frac{d}{ds}\Theta_i(\phi_s)}{\Theta_i(\phi_q)}\right), & s=q,\end{cases} \tag{4.8}$$

for $\phi_s,\phi_q\in\Phi$.

Proof. (i) This is an immediate consequence of Theorem 4.12 and Remark 4.10.
(ii) Since by (i) the function $s\mapsto\Theta_i(\phi_s)$ is log-convex on J. So, we get

$$\frac{\log\Theta_i(\phi_s)-\log\Theta_i(\phi_q)}{s-q}\leq\frac{\log\Theta_i(\phi_u)-\log\Theta_i(\phi_v)}{u-v} \tag{4.9}$$

for $s\leq u$ and $q\leq v$, $s\neq q$ $u\neq v$, and there form conclude that

$$\mu_{s,q}(\Theta_i,\Phi)\leq\mu_{u,v}(\Theta_i,\Phi).$$

Cases $s=q$ and $u=v$ follows from (4.9) as limiting cases. □

Remark 4.15. Note that the results from theorem and corollaries still hold when two of the points $x_0,...,x_{2l}\in[a,b]$ coincide, say $x_1=x_0$, for a family of differentiable functions ϕ_s such that the function $s\mapsto\phi_s[x_0,...,x_{2l}]$ is an n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all $(2l+1)$ points coincide for a family of $2l$ differentiable functions with the same property. The proofs are obtained by (1.14) and suitable characterization of convexity.

5 Applications to Stolarsky type means

In this section, we present several families of functions which fulfill the conditions of Theorem 4.12, Corollary 4.13, Corollary 4.14 and Remark 4.15. This enable us to construct a large families of functions which are exponentially convex. For a discussion related to this problem see [9].

Example 5.1. Let

$$\Lambda_1=\{\psi_t:\mathbb{R}\to[0,\infty):t\in\mathbb{R}\}$$

be a family of functions defined by

$$\psi_t(x)=\begin{cases}\frac{e^{tx}}{t^{2n}}, & \text{t}\neq 0;\\ \frac{x^{2n}}{(2n)!}, & \text{t}=0.\end{cases}$$

We have $\frac{d^{2n}\psi_t}{dx^{2n}}(x)=e^{tx}>0$ which shows that ψ_t is $2n$-convex on $\mathbb{R}$ for every $t\in\mathbb{R}$ and $t\mapsto\frac{d^{2n}\psi_t}{dx^{2n}}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 4.12 we also have that $t\mapsto\psi_t[x_0,...,x_{2n}]$ is exponentially convex (and so exponentially convex in the Jensen sense). Using Corollary 4.13 we conclude that $t\mapsto\Theta_i(\psi_t)$, $i=1,2$ are exponentially convex in the Jensen sense. It is easy to verify that this mapping

is continuous (although mapping $t \mapsto \psi_t$ is not continuous for $t = 0$), so it is exponentially convex.
For this family of functions, $\mu_{t,q}(\Theta_i, \Lambda_1)$, $i = 1,2$ from (4.8), becomes

$$\mu_{t,q}(\Theta_i,\Lambda_1) = \begin{cases} \left(\frac{\Theta_i(\psi_t)}{\Theta_i(\psi_q)}\right)^{\frac{1}{t-q}}, & t \neq q; \\ exp\left(\frac{\Theta_i(id.\psi_t)}{\Theta_i(\psi_t)} - \frac{2n}{t}\right), & t = q \neq 0; \\ exp\left(\frac{1}{2n+1}\frac{\Theta_i(id.\psi_0)}{\Theta_i(\psi_0)}\right), & t = q = 0. \end{cases}$$

Now, using (4.7) it is monotone function in parameters t and q.
We observe here that $\left(\frac{\frac{d^{2n}\psi_t}{dx^{2n}}}{\frac{d^{2n}\psi_q}{dx^{2n}}}\right)^{\frac{1}{t-q}}(lnx) = x$ so using Theorem 4.2 it follows that

$$M_{t,q}(\Theta_i,\Lambda_1) = ln\mu_{t,q}(\Theta_i,\Lambda_1), \quad i = 1,2$$

satisfy

$$a \le M_{t,q}(\Theta_i,\Lambda_1) \le b, \quad i = 1,2.$$

This shows that $M_{t,q}(\Theta_i,\Lambda_1)$ is mean for $i = 1,2$. Because of the above inequality (4.7), this mean is also monotonic.

Example 5.2. Let

$$\Lambda_2 = \{\lambda_t : (0,\infty) \to \mathbb{R} : t \in \mathbb{R}\}$$

be a family of functions defined by

$$\lambda_t(x) = \begin{cases} \frac{x^t}{t(t-1)...(t-2n+1)}, & \text{t} \notin \{0,1,..., 2n-1\}; \\ \frac{x^j lnx}{(-1)^{2n-1-j} j!(2n-1-j)!}, & \text{t = j} \in \{0,1,..., 2n-1\}. \end{cases}$$

Here, $\frac{d^{2n}\lambda_t}{dx^{2n}}(x) = x^{t-2n} = e^{(t-2n)lnx} > 0$ which shows that λ_t is $2n$-convex on $(0,\infty)$ for every $t \in \mathbb{R}$ and $t \mapsto \frac{d^{2n}\psi_t}{dx^{2n}}(x)$ is exponentially convex by definition. Arguing as in Example 5.1 we get the mappings $t \mapsto \Theta_i(\lambda_t)$, $i = 1,2$ are exponentially convex. In this case we assume that $[a,b] \in \mathbb{R}^+$.
For this family of functions, $\mu_{t,q}(\Theta_i, \Lambda_1)$, $i = 1,2$ from (4.8), becomes

$$\mu_{t,q}(\Theta_i,\Lambda_2) = \begin{cases} \left(\frac{\Theta_i(\lambda_t)}{\Theta_i(\lambda_q)}\right)^{\frac{1}{t-q}}, & t \neq q; \\ exp\left(-(2n-1)!\frac{\Theta_i(\lambda_0\lambda_t)}{\Theta_i(\lambda_t)} + \sum_{k=0}^{2n-1}\frac{1}{k-t}\right), & \text{t=q} \notin \{0,1,..., 2n-1\}; \\ exp\left(-(2n-1)!\frac{\Theta_i(\lambda_0\lambda_t)}{2\Theta_i(\lambda_t)} + \sum_{k=0,k\neq t}^{2n-1}\frac{1}{k-t}\right), & \text{t=q} \in \{0,1,..., 2n-1\}. \end{cases}$$

We observe that $\left(\frac{\frac{d^{2n}\lambda_t}{dx^{2n}}}{\frac{d^{2n}\lambda_q}{dx^{2n}}}\right)^{\frac{1}{t-q}}(x) = x$, so if Θ_i $(i = 1,2)$ are positive, then Theorem 4.2 yield that there exists some $\xi_i \in [a,b]$, $i = 1,2$ such that

$$\xi_i^{t-q} = \frac{\Theta_i(\lambda_t)}{\Theta_i(\lambda_q)}, \quad i = 1,2.$$

Since the function $\xi \to \xi^{t-q}$ is invertible for $t \neq q$, we then have

$$a \leq \left(\frac{\Theta_i(\lambda_t)}{\Theta_i(\lambda_q)}\right)^{\frac{1}{t-q}} \leq b, \quad i = 1,2.$$

This shows that $\mu_{t,q}(\Theta_i, \Lambda_2)$ is mean for $i = 1,2$. Because of the above inequality (4.7), this mean is also monotonic.

Example 5.3. Let

$$\Lambda_3 = \{\zeta_t : (0,\infty) \to (0,\infty) : t \in (0,\infty)\}$$

be a family of functions defined by

$$\zeta_t(x) = \begin{cases} \frac{t^{-x}}{(lnt)^{2n}}, & \text{t} \neq 1; \\ \\ \frac{x^{2n}}{(2n)!}, & \text{t} = 1. \end{cases}$$

Since $\frac{d^{2n}\zeta_t}{dx^{2n}}(x) = t^{-x}$ is the Laplace transform of a non-negative function (see [24]) it is exponentially convex. Obviously ζ_t are $2n$-convex functions for every $t > 0$.
For this family of functions, $\mu_{t,q}(\Theta_i, \Lambda_3)$, $i = 1,2$, in this case for $[a,b] \in \mathbb{R}^+$, from (4.8) becomes

$$\mu_{t,q}(\Theta_i, \Lambda_3) = \begin{cases} \left(\frac{\Theta_i(\zeta_t)}{\Theta_i(\zeta_q)}\right)^{\frac{1}{t-q}}, & t \neq q; \\ exp\left(-\frac{\Theta_i(id.\zeta_t)}{t\Theta_i(\zeta_t)} - \frac{2n}{t\,lnt}\right), & t = q \neq 1; \\ exp\left(-\frac{1}{2n+1}\frac{\Theta_i(id.\zeta_1)}{\Theta_i(\zeta_1)}\right), & t = q = 1. \end{cases}$$

This is monotonous function in parameters t and q by (4.7).
Using Theorem 4.2 it follows that

$$M_{t,q}(\Theta_i, \Lambda_3) = -L(t,q)ln\mu_{t,q}(\Theta_i, \Lambda_3), \quad i = 1,2.$$

satisfy

$$a \leq M_{t,q}(\Theta_i, \Lambda_3) \leq b, \quad i = 1,2.$$

This shows that $M_{t,q}(\Theta_i, \Lambda_3)$ is mean for $i = 1,2$. Because of the above inequality (4.7), this mean is also monotonic. $L(t,q)$ is logarithmic mean defined by

$$L(t,q) = \begin{cases} \frac{t-q}{\log t - \log q}, & \text{t} \neq \text{q}; \\ \\ t, & \text{t} = \text{q}. \end{cases}$$

Example 5.4. Let

$$\Lambda_4 = \{\gamma_t : (0,\infty) \to (0,\infty) : t \in (0,\infty)\}$$

be a family of functions defined by

$$\gamma_t(x) = \frac{e^{-x\sqrt{t}}}{t^n}.$$

Since $\frac{d^{2n}\gamma_t}{dx^{2n}}(x) = e^{-x\sqrt{t}}$ is the Laplace transform of a non-negative function (see [24]) it is exponentially convex. Obviously γ_t are $2n$-convex function for every $t > 0$.
For this family of functions, $\mu_{t,q}(\Theta_i, \Lambda_4)$, $i = 1,2$, in this case for $[a,b] \in \mathbb{R}^+$, from (4.8) becomes

$$\mu_{t,q}(\Theta_i, \Lambda_4) = \begin{cases} \left(\frac{\Theta_i(\gamma_t)}{\Theta_i(\gamma_q)}\right)^{\frac{1}{t-q}}, & t \neq q; \\ exp\left(-\frac{\Theta_i(id.\gamma_t)}{2\sqrt{t}\Theta_i(\gamma_t)} - \frac{n}{t}\right), & t = q. \end{cases}$$

This is monotonous function in parameters t and q by (4.7).
Using Theorem 4.2 it follows that

$$M_{t,q}(\Theta_i, \Lambda_4) = -\left(\sqrt{t} + \sqrt{q}\right) ln\mu_{t,q}(\Theta_i, \Lambda_4), \quad i = 1,2.$$

satisfy

$$a \leq M_{t,q}(\Theta_i, \Lambda_4) \leq b, \quad i = 1,2.$$

This shows that $M_{t,q}(\Theta_i, \Lambda_4)$ is mean for $i = 1,2$. Because of the above inequality (4.7), this mean is also monotonic.

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