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Commutators Generated by Singular Integral Operators with Variable Kernels and Local Campanato Functions on Generalized Local Morrey Spaces

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Abstract

In this paper, we obtain the boundedness for the singular integral operator with rough variable kernel T_Ω on the generalized local Morrey spaces, as well as the boundedness for the multilinear commutators generated by T_Ω and local Campanato functions.

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1 Introduction

Suppose that S^{n-1} is the unit sphere in $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma$. We say that a function $\Omega(x,z)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ belongs to $L^\infty(\mathbb{R}^n) \times L^s(S^{n-1})$, if $\Omega(x,z)$ satisfies the following conditions:
(i) for any $x, z \in \mathbb{R}^n$, $\Omega(x,\lambda z) = \Omega(x,z)$ for all $\lambda > 0$;
(ii) $\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^s(S^{n-1})} := \sup_{x\in\mathbb{R}^n}\Big(\int_{S^{n-1}} |\Omega(x,z')|^s d\sigma(z')\Big)^{1/s} < \infty.$

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Then, the singular integral operator with variable kernel is defined by

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x, x-y)}{|x-y|^n} f(y)dy. \tag{1.1}$$

Moreover, let $\overrightarrow{b} = (b_1, b_2, ..., b_m)$, where $b_i \in L_{loc}(\mathbb{R}^n)$ for$1 \le i \le m$. Then the multilinear commutator generated by $\vec{b}$ and T_Ω can be defined as

$$T_\Omega^{\vec{b}} f(x) = \int_{\mathbb{R}^n} \prod_{i=1}^{m} (b_i(x) - b_i(y)) \frac{\Omega(x, x-y)}{|x-y|^n} f(y)dy. \tag{1.2}$$

In [1, 2, 3], Calderón and Zygmund investigated the L^p boundedness of the singular integral operator T_Ω with the variable kernel. They found that these operators connected closely with the problem about the second order linear elliptic equations with variable coefficients. In 1971, Muckenhoupt and Wheeden [4] studied the weighted norm inequalities for T_Ω with power weights. Ding, Chen and Fan [5] established the (H^p, L^p) boundedness of T_Ω.

Recently, the commutators generated by singular integral with variable kernel also attract much attention. In 1993, Fazio and Raguse [6] obtained the weighted L^p boundedness of commutators generated by T_Ω and BMO functions. And, Zhang and Zhao [7] studied the commutators of integral operators with variable kernels on Hardy spaces.

Moreover, the classical Morrey space $M_{p,\lambda}$ were first introduced by Morrey in [8] to study the local behavior of solutions to second order elliptic partial differential equations. And, in [9] the authors considered the boundedess of the commutators gereated by singular integral operators with variable kernel and BMO functions on the classical Morrey space $M_{p,\lambda}$. Moreover, in [10], the authors introduced the local generalized Morrey space $LM_{p,\varphi}^{\{x_0\}}$, and they also studied the boundedness of the homogeneous singular integrals with rough kernel on these spaces.

Motivated by the works of [8, 9, 10], we are going to consider the boundedness of the singular integral operator T_Ω with variable kernel on the local generalized Morrey space $LM_{p,\varphi}^{\{x_0\}}$. Furtherly, we also obtain the boundedness of the commutators generated by T_Ω and local Campanato functions on the local generalized Morrey space $LM_{p,\varphi}^{\{x_0\}}$.

2 Some notations and lemmas

Definition 2.1. [10] Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and $1 \le p \le \infty$. For any fixed $x_0 \in \mathbb{R}^n$, a function $f \in L_{loc}^q$ is said to belong to the local Morrey space, if

$$\|f\|_{LM_{p,\varphi}^{\{x_0\}}} = \sup_{r>0} \varphi(x_0, r)^{-1} |B(x_0, r)|^{-\frac{1}{p}} \|f\|_{L^p(B(x_0,r))} < \infty.$$

And, we denote

$$LM_{p,\varphi}^{\{x_0\}} \equiv LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n) = \{f \in L_{loc}^q(\mathbb{R}^n) : \|f\|_{LM_{p,\varphi}^{\{x_0\}}} < \infty\}.$$

According to this definition, we recover the local Morrey space $LM_{p,\lambda}^{\{x_0\}}$ under the choice $\varphi(x_0, r) = r^{\frac{\lambda-n}{p}}$.

Definition 2.2. [10] Let $1 \le q < \infty$ and $0 \le \lambda < 1/n$. A function $f \in L^q_{loc}(\mathbb{R}^n)$ is said to belong to the space $LC^{\{x_0\}}_{q,\lambda}$ (local Campanato space), if

$$\|f\|_{LC^{\{x_0\}}_{q,\lambda}} = \sup_{r>0}\Big(\frac{1}{|B(x_0,r)|^{1+\lambda q}}\int_{B(x_0,r)}|f(y)-f_{B(x_0,r)}|^q dy\Big)^{1/q} < \infty,$$

where

$$f_{B(x_0,r)} = \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} f(y)dy.$$

Define

$$LC^{\{x_0\}}_{q,\lambda}(\mathbb{R}^n) = \{f \in L^q_{loc}(\mathbb{R}^n) : \|f\|_{LC^{\{x_0\}}_{q,\lambda}} < \infty\}.$$

Remark[10] Note that, the central BMO space $CBMO_q(\mathbb{R}^n) = LC^{\{0\}}_{q,0}(\mathbb{R}^n)$ and $CBMO^{\{x_0\}}_q(\mathbb{R}^n) = LC^{\{x_0\}}_{q,0}(\mathbb{R}^n)$. Moreover, one can imagine that the behavior of $CBMO^{\{x_0\}}_q(\mathbb{R}^n)$ may be quite different from that of $BMO(\mathbb{R}^n)$, since there is no analogy of the John-Nirenberg inequality of BMO for the space $CBMO^{\{x_0\}}_q(\mathbb{R}^n)$.

Lemma 2.3. *[10] Let $1 < q < \infty$, $0 < r_2 < r_1$ and $b \in LC^{\{x_0\}}_{q,\lambda}$, then*

$$\Big(\frac{1}{|B(x_0,r_1)|^{1+\lambda q}}\int_{B(x_0,r_1)}|b(x)-b_{B(x_0,r_2)}|^q dx\Big)^{1/q} \le C\Big(1+\ln\frac{r_1}{r_2}\Big)\|b\|_{LC^{\{x_0\}}_{q,\lambda}}.$$

And, from this inequality, we have

$$|b_{B(x_0,r_1)} - b_{B(x_0,r_2)}| \le C\Big(1+\ln\frac{r_1}{r_2}\Big)|B(x_0,r_1)|^{\lambda}\|b\|_{LC^{\{x_0\}}_{q,\lambda}}.$$

In this section, we are going to use the following statement on the boundedness of the weighted Hardy operator:

$$H_w g(t) := \int_t^{\infty} g(s)w(s)ds,\ 0 < t < \infty,$$

where w is a fixed function non-negative and measurable on $(0,\infty)$.

Lemma 2.4. *[11, 12] Let v_1, v_2 and w be positive almost everywhere and measurable functions on $(0,\infty)$. The inequality*

$$ess\sup_{t>0} v_2(t)H_w g(t) \le C ess\sup_{t>0} v_1(t)g(t)$$

holds for some $C > 0$ and all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := ess\sup_{t>0} v_2(t)\int_t^{\infty}\frac{w(s)}{ess\sup_{s<\tau<\infty} v_1(\tau)}ds < \infty.$$

Lemma 2.5. *[3] Let $\Omega \in L^\infty(\mathbb{R}^n)\times L^s(S^{n-1})$, $s > 1$, and for any $x \in \mathbb{R}^n$, $\int_{S^{n-1}}\Omega(x,z')d\sigma(z') = 0$, where $z' = z/|z|$ for any $z \in \mathbb{R}^n$. Then T_Ω is bounded on $L^p(\mathbb{R}^n)$ for all $p \ge s'$, where $s' = s/(s-1)$ is the conjugate exponent of s.*

Let us formulate our main results in section 3 and section 4.

3 Singular integral operators with variable kernels on generalized local Morrey spaces

Theorem 3.1. *Let* $x_0 \in \mathbb{R}^n$, $\Omega \in L^\infty(\mathbb{R}^n)\times L^s(S^{n-1})$, $s>1$ *and for any* $x \in \mathbb{R}^n$, $\int_{S^{n-1}} \Omega(x,z')d\sigma(z') = 0$, *where* $z' = z/|z|$ *for any* $z \in \mathbb{R}^n$. *If* $p>1$ *and* $s' \le p$, *then the inequality*

$$\|T_\Omega f\|_{L^p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))} t^{-\frac{n}{p}-1} dt$$

holds for any ball $B(x_0,r)$ *and for all* $f \in L^p_{loc}(\mathbb{R}^n)$.

Proof. Let $B = B(x_0,r)$. We write $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{(2B)^c}$.
Thus, we have

$$\|T_\Omega f\|_{L^p(B)} \le \|T_\Omega f_1\|_{L^p(B)} + \|T_\Omega f_2\|_{L^p(B)}.$$

And, from the boundedness of T_Ω on $L^p(\mathbb{R}^n)$(see Lemma 2.5) it follows that

$$\|T_\Omega f_1\|_{L^p(B)} \lesssim \|f\|_{L^p(2B)} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}. \tag{3.1}$$

Moreover, it is obvious that

$$\begin{aligned} \|\Omega(x,x-\cdot)\|_{L^s(B(x_0,t))} &= \left(\int_{B(0,t+|x-x_0|)} |\Omega(x,u)|^s du \right)^{\frac{1}{s}} \\ &\approx \left(\int_0^{t+|x-x_0|} r^{n-1} dr \int_{S^{n-1}} |\Omega(x,u')|^s d\sigma(u') \right)^{\frac{1}{s}} \\ &\approx \|\Omega\|_{L^\infty\times L^s(S^{n-1})} |B(0,t+|x-x_0|)|^{\frac{1}{s}}. \end{aligned} \tag{3.2}$$

Note that $\frac{1}{2}|x_0-y| \le |x-y| \le \frac{3}{2}|x_0-y|$ for $x \in B, y \in (2B)^c$. Then, by (1.1), (3.2), the Fubini theorem and Hölder's inequality, we have

$$\begin{aligned} |T_\Omega f_2(x)| &\approx \int_{(2B)^c} \frac{|f(y)||\Omega(x,x-y)|}{|x_0-y|^n} dy \\ &\approx \int_{(2B)^c} |f(y)||\Omega(x,x-y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} dy \\ &\approx \int_{2r}^{\infty} \int_{2r\le|x_0-y|\le t} |f(y)||\Omega(x,x-y)| dy \frac{dt}{t^{n+1}} \\ &\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)||\Omega(x,x-y)| dy \frac{dt}{t^{n+1}}. \\ &\lesssim \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))} \|\Omega(x,x-\cdot)\|_{L^s(B(x_0,t))} |B(x_0,t)|^{1-\frac{1}{p}-\frac{1}{s}} \frac{dt}{t^{n+1}} \\ &\lesssim \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))} |B(0,t+|x-x_0|)|^{\frac{1}{s}} |B(x_0,t)|^{1-\frac{1}{p}-\frac{1}{s}} \frac{dt}{t^{n+1}} \\ &\approx \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}. \end{aligned} \tag{3.3}$$

Therefore,

$$\|T_\Omega f_2\|_{L^p(B)} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}. \tag{3.4}$$

So, combining (3.1) and (3.4) we have

$$\|T_\Omega f\|_{L^p(B)} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}.$$

□

Theorem 3.2. *Let* $x_0 \in \mathbb{R}^n$, $\Omega \in L^\infty(\mathbb{R}^n) \times L^s(S^{n-1})$, $s > 1$ *and for any* $x \in \mathbb{R}^n$, $\int_{S^{n-1}} \Omega(x, z')d\sigma(z') = 0$, *where* $z' = z/|z|$ *for any* $z \in \mathbb{R}^n$. *If functions* φ, $\psi : \mathbb{R}^n \times (0,\infty) \to (0,+\infty)$ *satisfy the inequality*

$$\int_r^\infty \frac{ess\inf_{t<\tau<\infty} \varphi(x_0,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C\psi(x_0, r), \tag{3.5}$$

where C *does not depend on* r, *then the operator* T_Ω *is bounded from* $LM^{\{x_0\}}_{p,\varphi}$ *to* $LM^{\{x_0\}}_{p,\psi}$ *for* $p \ge s'$.

Proof. Taking $v_1(t) = \varphi(x_0,t)^{-1}t^{-\frac{n}{p}}$, $v_2(t) = \psi(x_0,t)^{-1}$, $g(t) = \|f\|_{L^p(B(x_0,t))}$ and $w(t) = t^{-\frac{n}{p}-1}$, then from (3.5) we have

$$\operatorname*{ess\,sup}_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Thus, from Lemma 2.4, it follows that

$$\operatorname*{ess\,sup}_{t>0} v_2(t)H_w g(t) \le C \operatorname*{ess\,sup}_{t>0} v_1(t)g(t).$$

Therefore,

$$\begin{aligned}
\|T_\Omega f\|_{LM^{\{x_0\}}_{p,\psi}} &= \sup_{r>0} \psi(x_0,r)^{-1}|B(x_0,r)|^{-\frac{1}{p}}\|T_\Omega f\|_{L^p(B(x_0,r))} \\
&\lesssim \sup_{r>0} \psi(x_0,r)^{-1} \int_r^\infty \|f\|_{L^p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}} \\
&\lesssim \sup_{r>0} \varphi(x_0,r)^{-1} r^{-\frac{n}{p}} \|f\|_{L^p(B(x_0,r)} = \|f\|_{LM^{\{x_0\}}_{p,\varphi}}.
\end{aligned}$$

□

4 Multilinear commutators of singular integral operators with variable kernels on generalized local Morrey spaces

In this section, we will consider the boundedness of the multinear commutators generated by singular integral operators with variable kernels and Campanato functions on generalized local Morrey spaces.

Theorem 4.1. *Let* $x_0 \in \mathbb{R}^n$, $1 < p, q, p_1, p_2, \dots, p_m < \infty$, *such that* $1/q = 1/p_1 + 1/p_2 + \cdots + 1/p_m + 1/p$, *and* $b_i \in LC^{\{x_0\}}_{p_i,\lambda_i}$ *for* $0 < \lambda_i < 1/n$, $i = 1, 2, \cdots, m$. *Assume that* $\Omega \in L^\infty(\mathbb{R}^n) \times L^s(S^{n-1})$, $s > 1$ *and for any* $x \in \mathbb{R}^n$, $\int_{S^{n-1}} \Omega(x, z')d\sigma(z') = 0$, *where* $z' = z/|z|$ *for any* $z \in \mathbb{R}^n$. *Then, for* $1 \le s' \le q$ *the inequality*

$$\|T^{\vec{b}}_\Omega f\|_{L^q(B(x_0,r))} \lesssim \prod_{i=1}^m \|b_i\|_{LC^{\{x_0\}}_{p_i,\lambda_i}} r^{\frac{n}{q}} \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right)^m \|f\|_{L^p(B(x_0,r))} t^{-\frac{n}{p}+\lambda n - 1} dt$$

holds for any ball $B(x_0, r)$, *where* $\lambda = \lambda_1 + \lambda_2 + \dots + \lambda_m$.

Proof. Without loss of generality, it is sufficient for us to show that the conclusion holds for $m = 2$.

Let $B = B(x_0, r)$. And, we write $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$, $f_2 = f\chi_{(2B)^c}$. Thus, we have

$$\|T^{(b_1,b_2)}_\Omega f\|_{L^q(B)} \le \|T^{(b_1,b_2)}_\Omega f_1\|_{L^q(B)} + \|T^{(b_1,b_2)}_\Omega f_2\|_{L^q(B)} =: I + II.$$

Let us estimate I and II, respectively.

It is obvious that

$$\begin{aligned}
&\|T^{(b_1,b_2)}_\Omega f_1\|_{L^q(B)} \\
=&\|(b_1 - (b_1)_B)(b_2 - (b_2)_B)T_\Omega f_1\|_{L^q(B)} + \|(b_1 - (b_1)_B)T_\Omega((b_2 - (b_2)_B)f_1)\|_{L^q(B)} \\
&+ \|(b_2 - (b_2)_B)T_\Omega((b_1 - (b_1)_B)f_1)\|_{L^q(B)} + \|T_\Omega((b_1 - (b_1))(b_2 - (b_2)_B)f_1)\|_{L^q(B)} \\
=:&I_1 + I_2 + I_3 + I_4. \qquad (4.1)
\end{aligned}$$

Since $1/q = 1/p_1 + 1/p_2 + 1/p$ and $q \ge s'$, then $p \ge s'$. And, from Definition 2.1, it is easy to see that

$$\|b_i - (b_i)_B\|_{L^{p_i}(B)} \le Cr^{n/p_i + n\lambda_i} \|b_i\|_{LC^{\{x_0\}}_{p_i,\lambda_i}}, \text{ for } i = 1, 2. \qquad (4.2)$$

Thus, using Hölder's inequality, Lemma 2.5 and (4.2), we have

$$\begin{aligned}
I_1 &\lesssim \|b_1 - (b_1)_B\|_{L^{p_1}(B)} \|b_2 - (b_2)_B\|_{L^{p_2}(B)} \|T_\Omega f_1\|_{L^p(B)} \\
&\lesssim \|b_1 - (b_1)_B\|_{L^{p_1}(B)} \|b_2 - (b_2)_B\|_{L^{p_2}(B)} \|f\|_{L^p(2B)} \\
&\lesssim \|b_1 - (b_1)_B\|_{L^{p_1}(B)} \|b_2 - (b_2)_B\|_{L^{p_2}(B)} r^{\frac{n}{p}} \int_{2r}^\infty \|f\|_{L^p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}} \\
&\lesssim \|b_1\|_{LC^{\{x_0\}}_{p_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{p_2,\lambda_2}} r^{\frac{n}{q}} \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n - \frac{n}{p} - 1} \|f\|_{L^p(B(x_0,t))} dt. \qquad (4.3)
\end{aligned}$$

Moreover, from Lemma 2.3, it is easy to see that

$$\|b_i - (b_i)_B\|_{L^{p_i}(2B)} \le Cr^{n/p_i + n\lambda_i} \|b_i\|_{LC^{\{x_0\}}_{p_i,\lambda_i}}, \text{for } i = 1, 2. \qquad (4.4)$$

And, let $1 < \bar{q} < \infty$, such that $1/q = 1/p_1 + 1/\bar{q}$. It is easy to see that $1/\bar{q} = 1/p_2 + 1/p$ and $\bar{q} \ge s'$. Then similarly to the estimate of (4.3), we have

$$\begin{aligned}
I_2 &\lesssim \|b_1 - (b_1)_B\|_{L^{p_1}(B)} \|T_\Omega((b_2 - (b_2)_B)f_1)\|_{L^{\bar{q}}(B)} \\
&\lesssim \|b_1 - (b_1)_B\|_{L^{p_1}(B)} \|b_2 - (b_2)_B\|_{L^{p_2}(2B)} \|f\|_{L^p(2B)} \\
&\lesssim \|b_1\|_{LC^{\{x_0\}}_{p_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{p_2,\lambda_2}} r^{\frac{n}{q}} \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n - \frac{n}{p} - 1} \|f\|_{L^p(B(x_0,t))} dt.
\end{aligned}$$

Similarly,

$$I_3 \lesssim \|b_1\|_{LC^{\{x_0\}}_{p_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{p_2,\lambda_2}} r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-\frac{n}{p}-1} \|f\|_{L^p(B(x_0,t))} dt.$$

Moreover, by Lemma 2.5, Hölder's inequality and (4.4), we obtain

$$\begin{aligned} I_4 &\lesssim \|(b_1-(b_1)_B)(b_2-(b_2)_B)f_1\|_{L^q(\mathbb{R}^n)} \\ &\lesssim \|b_1-(b_1)_B\|_{L^{p_1}(2B)}\|b_2-(b_2)_B\|_{L^{p_2}(2B)}\|f\|_{L^p(2B)} \\ &\lesssim \|b_1\|_{LC^{\{x_0\}}_{p_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{p_2,\lambda_2}} r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-\frac{n}{p}-1} \|f\|_{L^p(B(x_0,t))} dt. \end{aligned}$$

Therefore, combining the estimates of I_1, I_2, I_3 and I_4, we have

$$I \lesssim \|b_1\|_{LC^{\{x_0\}}_{p_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{p_2,\lambda_2}} r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-\frac{n}{p}-1} \|f\|_{L^p(B(x_0,t))} dt.$$

Let us estimate II.
It's analogues to (4.1), we have

$$\begin{aligned} & \|T_\Omega^{(b_1,b_2)} f_2\|_{L^q(B)} \\ \lesssim\ & \|(b_1-(b_1)_B)(b_2-(b_2)_B)T_\Omega f_2\|_{L^q(B)} + \|(b_1-(b_1)_B)T_\Omega((b_2-(b_2)_B)f_2)\|_{L^q(B)} \\ & +\|(b_2-(b_2)_B)T_\Omega((b_1-(b_1)_B)f_2)\|_{L^q(B)} + \|T_\Omega((b_1-(b_1)_B)(b_2-(b_2)_B)f_2)\|_{L^q(B)} \\ =:\ & II_1+II_2+II_3+II_4. \end{aligned}$$

Then, using the Hölder's inequality and (3.4), we have

$$\begin{aligned} II_1 &\lesssim \|b_1-(b_1)_B\|_{L^{p_1}(B)}\|b_2-(b_2)_B\|_{L^{p_2}(B)}\|T_\Omega f_2\|_{L^p(B)} \\ &\lesssim \|b_1-(b_1)_B\|_{L^{p_1}(B)}\|b_2-(b_2)_B\|_{L^{p_2}(B)} r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}} \\ &\lesssim \|b_1\|_{LC^{\{x_0\}}_{p_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{p_2,\lambda_2}} r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-\frac{n}{p}-1} \|f\|_{L^p(B(x_0,t))} dt. \end{aligned}$$

It is obvious that for any $x \in B$,

$$\begin{aligned} & |T_\Omega((b_2-(b_2)_B)f_2)(x)| \\ \lesssim & \int_{(2B)^c} |b_2(y)-(b_2)_B||\Omega(x,x-y)|\frac{|f(y)|}{|x_0-y|^n} dy \\ \approx & \int_{2r}^{\infty}\left[\int_{2r<|x_0-y|<t} |b_2(y)-(b_2)_B||\Omega(x,x-y)||f(y)|dy\right]\frac{dt}{t^{n+1}} \\ \lesssim & \int_{2r}^{\infty}\left[\int_{B(x_0,t)} |b_2(y)-(b_2)_{B(x_0,t)}||\Omega(x,x-y)||f(y)|dy\right]\frac{dt}{t^{n+1}} \\ & + \int_{2r}^{\infty}\left[\int_{B(x_0,t)} |(b_2)_{B(x_0,t)}-(b_2)_B||\Omega(x,x-y)||f(y)|dy\right]\frac{dt}{t^{n+1}} \\ =: & E_1+E_2. \end{aligned} \tag{4.5}$$

Then, it is analogues to (3.3), we have

$$\begin{aligned} E_1 &\lesssim \int_{2r}^{\infty} \|b_2-(b_2)_{B(x_0,t)}\|_{L^{p_2}(B(x_0,t))}\|\Omega(x,x-\cdot)\|_{L^s(B(x_0,t))} \\ &\quad \times \|f\|_{L^p(B(x_0,t))}|B(x_0,t)|^{1-\frac{1}{p_2}-\frac{1}{s}-\frac{1}{p}}\frac{dt}{t^{n+1}} \\ &\lesssim \int_{2r}^{\infty} \|b_2-(b_2)_{B(x_0,t)}\|_{L^{p_2}(B(x_0,t))}\|f\|_{L^p(B(x_0,t))}\frac{dt}{t^{1+\frac{n}{p}}} \\ &\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda_2}} \int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)t^{n\lambda_2-\frac{n}{p}-1}\|f\|_{L^p(B(x_0,t))}dt. \end{aligned} \tag{4.6}$$

And, from Lemma 2.3 and (3.3), it follows that

$$E_2 \lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda_2}} \int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)t^{n\lambda_2-\frac{n}{p}-1}\|f\|_{L^p(B(x_0,t))}dt.$$

Therefore, we get

$$|T_\Omega((b_2-(b_2)_B)f_2)(x)| \lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda_2}} \int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)t^{n\lambda_2-\frac{n}{p}-1}\|f\|_{L^p(B(x_0,t))}dt. \tag{4.7}$$

Let $1<\bar{q}<\infty$, such that $1/q = 1/p_1 + 1/\bar{q}$. Then, using Hölder's inequality and (4.7), we have

$$\begin{aligned} II_2 &\lesssim \|b_1-(b_1)_B\|_{L^{p_1}(B)}\|T_\Omega((b_2-(b_2)_B)f_2)\|_{L^{\bar{q}}(B)} \\ &\lesssim \|b_1\|_{LC^{\{x_0\}}_{p_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{p_2,\lambda_2}} r^{\frac{n}{q}} \int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)^2 t^{(\lambda_1+\lambda_2)n-\frac{n}{p}-1}\|f\|_{L^p(B(x_0,t))}dt. \end{aligned}$$

Similarly, we have

$$II_3 \lesssim \|b_1\|_{LC^{\{x_0\}}_{p_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{p_2,\lambda_2}} r^{\frac{n}{q}} \int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)^2 t^{(\lambda_1+\lambda_2)n-\frac{n}{p}-1}\|f\|_{L^p(B(x_0,t))}dt.$$

Let us estimate II_4.

It is analogue to the estimate of (4,5), for any $x\in B$,

$$\begin{aligned} &|T_\Omega((b_1-(b_1)_B)(b_2-(b_2)_B)f_2)(x)| \\ &\lesssim \int_{2r}^{\infty}\Big[\int_{B(x_0,t)}|b_1(y)-(b_1)_{B(x_0,t)}||b_2(y)-(b_2)_{B(x_0,t)}||\Omega(x,x-y)||f(y)|dy\Big]\frac{dt}{t^{n+1}} \\ &\quad + \int_{2r}^{\infty}\Big[\int_{B(x_0,t)}|b_1(y)-(b_1)_{B(x_0,t)}||(b_2)_{B(x_0,t)}-(b_2)_B||\Omega(x,x-y)||f(y)|dy\Big]\frac{dt}{t^{n+1}} \\ &\quad + \int_{2r}^{\infty}\Big[\int_{B(x_0,t)}|(b_1)_{B(x_0,t)}-(b_1)_B||b_2(y)-(b_2)_{B(x_0,t)}||\Omega(x,x-y)||f(y)|dy\Big]\frac{dt}{t^{n+1}} \\ &\quad + \int_{2r}^{\infty}\Big[\int_{B(x_0,t)}|(b_1)_{B(x_0,t)}-(b_1)_B||(b_2)_{B(x_0,t)}-(b_2)_B||\Omega(x,x-y)||f(y)|dy\Big]\frac{dt}{t^{n+1}} \\ &=: U_1+U_2+U_3+U_4. \end{aligned}$$

Similar to the estimate of (4.6), we have

$$U_1 \lesssim \|b_1\|_{LC^{\{x_0\}}_{p_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{p_2,\lambda_2}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-\frac{n}{p}-1} \|f\|_{L^p(B(x_0,t))} dt.$$

From (4.6) and Lemma 2.3, it follows that

$$U_2 \lesssim \|b_1\|_{LC^{\{x_0\}}_{p_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{p_2,\lambda_2}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-\frac{n}{p}-1} \|f\|_{L^p(B(x_0,t))} dt,$$

and

$$U_3 \lesssim \|b_1\|_{LC^{\{x_0\}}_{p_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{p_2,\lambda_2}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-\frac{n}{p}-1} \|f\|_{L^p(B(x_0,t))} dt.$$

Moreover, from (3.3) and Lemma 2.3, we obtain

$$U_4 \lesssim \|b_1\|_{LC^{\{x_0\}}_{p_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{p_2,\lambda_2}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-\frac{n}{p}-1} \|f\|_{L^p(B(x_0,t))} dt.$$

Therefore, combining the estimates of U_1, U_2, U_3 and U_4, we have

$$\begin{aligned} &|T_\Omega((b_1-(b_1)_B)(b_2-(b_2)_B)f_2)(x)| \\ &\lesssim \|b_1\|_{LC^{\{x_0\}}_{p_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{p_2,\lambda_2}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-\frac{n}{p}-1} \|f\|_{L^p(B(x_0,t))} dt. \end{aligned}$$

So,

$$\begin{aligned} II_4 = \ & \|T_\Omega((b_1-(b_1)_B)(b_2-(b_2)_B)f_2)\|_{L^q(B)} \\ & \lesssim \|b_1\|_{LC^{\{x_0\}}_{p_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{p_2,\lambda_2}} r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-\frac{n}{p}-1} \|f\|_{L^p(B(x_0,t))} dt. \end{aligned}$$

Therefore, combining the estimates of II_1, II_2, II_3 and II_4, we have

$$II \lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-\frac{n}{p}-1} \|f\|_{L^p(B(x_0,t))} dt.$$

Thus, from the estimates of I and II, we obtain

$$\|T_\Omega^{(b_1,b_2)} f\|_{L^q(B(x_0,r))} \lesssim \|b_1\|_{LC^{\{x_0\}}_{p_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{p_2,\lambda_2}} r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 t^{(\lambda_1+\lambda_2)n-\frac{n}{p}-1} \|f\|_{L^p(B(x_0,t))} dt.$$

□

Theorem 4.2. *Let $x_0 \in \mathbb{R}^n$, $1 < p, q, p_1, p_2, \ldots, p_m < \infty$, such that $1/q = 1/p_1 + 1/p_2 + \cdots + 1/p_m + 1/p$ and $b_i \in LC^{\{x_0\}}_{p_i,\lambda_i}$ for $0 < \lambda_i < 1/n$, $i = 1, 2, \cdots, m$. Assume that $\Omega \in L^\infty(\mathbb{R}^n) \times L^s(S^{n-1})$, $s > 1$ and for any $x \in \mathbb{R}^n$, $\int_{S^{n-1}} \Omega(x, z')d\sigma(z') = 0$, where $z' = z/|z|$ for any $z \in \mathbb{R}^n$. If functions $\varphi, \psi : \mathbb{R}^n \times (0,\infty) \to (0,+\infty)$, satisfy the conditions*

$$\int_r^{\infty} \left(1+\ln\frac{t}{r}\right)^m \frac{ess\inf_{t<s<\infty} \varphi(x_0, s)s^{n/p}}{t^{\frac{n}{p}-n\lambda+1}} dt \le C\psi(x_0, r),$$

where $\lambda = \sum_{i=1}^{m} \lambda_i$ and the constant $C > 0$ doesn't depend on r. Then the commutator $T_\Omega^{\vec{b}}$ is bounded from $LM^{\{x_0\}}_{p,\varphi}$ to $LM^{\{x_0\}}_{q,\psi}$ for $q \ge s'$.

Proof. Taking $v_1(t) = \varphi(x_0,t)^{-1}t^{-\frac{n}{p}}$, $v_2(t) = \psi(x_0,t)^{-1}$, $g(t) = \|f\|_{L^q(B(x_0,t))}$ and $w(t) = (1 + \ln\frac{t}{r})^m t^{n\lambda-\frac{n}{p}-1}$, then we have

$$\operatorname*{ess\,sup}_{t>0} v_2(t)\int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Thus, from Lemma 2.4, it follows that

$$\operatorname*{ess\,sup}_{t>0} v_2(t)H_w g(t) \le C\operatorname*{ess\,sup}_{t>0} v_1(t)g(t).$$

So,

$$\begin{aligned}
&\|T_\Omega^{\vec{b}}(\vec{f})\|_{LM_{q,\psi}^{\{x_0\}}}\\
= &\sup_{r>0}\psi(x_0,r)^{-1}|B(x_0,r)|^{-1/q}\|T_\Omega(\vec{f})\|_{L^q(B(x_0,r))}\\
\lesssim &\prod_{i=1}^m \|b_i\|_{LC_{p_i,\lambda_i}^{\{x_0\}}}\sup_{r>0}\psi(x_0,r)^{-1}\int_{2r}^\infty\left(1+\ln\frac{t}{r}\right)^m t^{n\lambda-\frac{n}{p}-1}\|f\|_{L^p(B(x_0,t))}dt\\
\lesssim &\prod_{i=1}^m \|b_i\|_{LC_{p_i,\lambda_i}^{\{x_0\}}}\sup_{r>0}\varphi(x_0,r)^{-1}r^{-\frac{n}{p}}\|f\|_{L^p(B(x_0,r))}\\
= &\prod_{i=1}^m \|b_i\|_{LC_{p_i,\lambda_i}^{\{x_0\}}}\|f\|_{LM_{p,\varphi}^{\{x_0\}}}.
\end{aligned}$$

□

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