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LITTLE HANKEL OPERATORS AND ASSOCIATED INTEGRAL INEQUALITIES

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Abstract

In this paper we consider a class of integral operators on $L^2(0,\infty)$ that are unitarily equivalent to little Hankel operators between weighted Bergman spaces. We calculate the norms of such integral operators and as a by-product obtain a generalization of the Hardy-Hilbert's integral inequality. We also consider the discrete version of the inequality which give the norms of the companion matrices of certain generalized Bergman-Hilbert matrices. These results are then generalized to vector valued case and operator valued case.

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1 Introduction

Let $\mathbb{C}_+ = \{z \in \mathbb{C} : \text{Re} z > 0\}$ be the right half plane. Let $d\widetilde{A}(s) = dxdy$ be the area measure. Let $L^2(\mathbb{C}_+, d\widetilde{A})$ be the space of complex valued, square-integrable, measurable functions on $\mathbb{C}_+$ with respect to the area measure. Let $L^2_a(\mathbb{C}_+)$ be the closed subspace of $L^2(\mathbb{C}_+, d\widetilde{A})$ consisting of those functions in $L^2(\mathbb{C}_+, d\widetilde{A})$ that are analytic. The space $L^2_a(\mathbb{C}_+)$ is referred to as the Bergman space of the right half plane. Let P_+ denote the orthogonal projection

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of $L^2(\mathbb{C}_+, d\widetilde{A})$ onto $L_a^2(\mathbb{C}_+)$. The functions $\mathcal{K}_w(z) = \frac{1}{(\overline{w}+z)^2}, z \in \mathbb{C}_+$ are the reproducing kernel [6] for $L_a^2(\mathbb{C}_+)$. Let $L^\infty(\mathbb{C}_+)$ be the space of complex-valued, essentially bounded, Lebesgue measurable functions on $\mathbb{C}_+$. For $\phi \in L^\infty(\mathbb{C}_+)$, the little Hankel operator $\widetilde{h}_\phi$ is a mapping from $L_a^2(\mathbb{C}_+)$ into $\overline{L_a^2(\mathbb{C}_+)}$ defined by $\widetilde{h}_\phi f = \overline{P}_+(\phi f)$, where $\overline{P}_+$ is the projection operator from $L^2(\mathbb{C}_+, d\widetilde{A})$ onto $\overline{L_a^2(\mathbb{C}_+)} = \{\overline{f} : f \in L_a^2(\mathbb{C}_+)\}$. Let $\widetilde{S}_\phi$ be the mapping from $L_a^2(\mathbb{C}_+)$ into $L_a^2(\mathbb{C}_+)$ defined by $\widetilde{S}_\phi f = P_+(\widetilde{J}(\phi f))$ where $\widetilde{J}$ is the mapping from $L^2(\mathbb{C}_+, d\widetilde{A})$ into $L^2(\mathbb{C}_+, d\widetilde{A})$ such that $\widetilde{J}f(s) = f(\overline{s})$. Notice that $\widetilde{J}$ is unitary and $\widetilde{J}\widetilde{S}_\phi f = \widetilde{J}(P_+(\widetilde{J}(\phi f))) = \widetilde{J}P_+\widetilde{J}(\phi f) = \overline{P}_+(\phi f) = \widetilde{h}_\phi f$ for all $f \in L_a^2(\mathbb{C}_+)$. Let $\widetilde{\Gamma}_\phi$ be the mapping from $L_a^2(\mathbb{C}_+)$ into $L_a^2(\mathbb{C}_+)$ defined by $\widetilde{\Gamma}_\phi f = P_+\widetilde{M}_\phi\widetilde{J}f$, where $\widetilde{M}_\phi$ is the mapping from $L^2(\mathbb{C}_+, d\widetilde{A})$ into $L^2(\mathbb{C}_+, d\widetilde{A})$ defined by $\widetilde{M}_\phi f = \phi f$. Thus $\widetilde{\Gamma}_\phi f = P_+\widetilde{M}_\phi\widetilde{J}f = P_+(\phi(s)f(\overline{s})) = P_+(\widetilde{J}(\phi(\overline{s})f(s))) = \widetilde{S}_{\widetilde{J}\phi}f$ for all $f \in L_a^2(\mathbb{C}_+)$. Hence $\widetilde{\Gamma}_\phi = \widetilde{S}_{\widetilde{J}\phi}$.

For $\alpha > -1$, let $L_a^2(\mathbb{C}_+, x^\alpha d\widetilde{A}(s))$ be the space of complex analytic functions F on $\mathbb{C}_+$ such that $\int |F(s)|^2 x^\alpha d\widetilde{A}(s) < \infty$, where $s = x + iy$. One can also define little Hankel operators $\widetilde{S}_\phi$ on this space as we did in $L_a^2(\mathbb{C}_+, d\widetilde{A}(s))$. We shall use the same notation $\widetilde{S}_\phi, \widetilde{\Gamma}_\phi, \widetilde{h}_\phi$ to denote little Hankel operators on $L_a^2(\mathbb{C}_+, x^\alpha d\widetilde{A}(s))$ and it will be clear from the context on which space we are considering these operators. Finally, let $L^2\left((0,\infty), \frac{dt}{t^{\alpha+1}}\right)$ be the space of complex-valued, absolutely square-integrable, measurable functions on $(0,\infty)$ with respect to the measure $\frac{dt}{t^{\alpha+1}}$.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Let $L^2(\mathbb{D}, dA)$ be the space of complex-valued, square-integrable, measurable functions on $\mathbb{D}$ with respect to the normalized area measure $dA(z) = \frac{1}{\pi}dxdy$. Let $L_a^2(\mathbb{D})$ be the closed subspace consisting of those functions in $L^2(\mathbb{D}, dA)$ that are analytic. The space $L_a^2(\mathbb{D})$ is called the Bergman space of the open unit disk $\mathbb{D}$. The functions $\{e_n(z)\}_{n=0}^\infty = \{\sqrt{n+1}z^n\}_{n=0}^\infty$ form an orthonormal basis for $L_a^2(\mathbb{D})$. The function $K(z,w) = \frac{1}{(1-z\overline{w})^2}$, $z, w \in \mathbb{D}$ is the reproducing kernel [21] of $L_a^2(\mathbb{D})$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic in $\mathbb{D}$, a simple calculation show that $\int_{\mathbb{D}} |f(z)|^2 dA(z) = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}$. Consequently, $f \in L_a^2(\mathbb{D})$ if and only if the last expression is finite. The scalar product of f and $g(z) = \sum_{n=0}^{\infty} b_n z^n, f, g \in L_a^2(\mathbb{D})$, is given by $\langle f, g \rangle_{L_a^2(\mathbb{D})} = \sum_{n=0}^{\infty} \frac{a_n \overline{b_n}}{n+1}$. The polynomials are dense in $L_a^2(\mathbb{D})$. If $f(z) = \sum_{n=0}^{\infty} a_n e_n(z) \in L_a^2(\mathbb{D})$ then a_n is called the n^{th} Fourier coefficient of f. Let $L^\infty(\mathbb{D})$ be the space of complex-valued, essentially bounded, Lebesgue measurable functions on $\mathbb{D}$. For $\phi \in L^\infty(\mathbb{D})$, the little Hankel operator h_ϕ is a mapping from $L_a^2(\mathbb{D})$ into $\overline{L_a^2(\mathbb{D})}$ defined by $h_\phi f = \overline{P}(\phi f)$, where $\overline{P}$ is the projection operator from $L^2(\mathbb{D}, dA)$ onto $\overline{L_a^2(\mathbb{D})} = \{\overline{f} : f \in L_a^2(\mathbb{D})\}$. Let S_ϕ be the mapping from $L_a^2(\mathbb{D})$ into $L_a^2(\mathbb{D})$ defined by $S_\phi f = P(J(\phi f))$ where J is the mapping from $L^2(\mathbb{D}, dA)$ into itself such that $Jf(z) = f(\overline{z})$. Notice that J is unitary and $JS_\phi f = J(P(J(\phi f))) = JPJ(\phi f) = \overline{P}(\phi f) = h_\phi f$ for all $f \in L_a^2(\mathbb{D})$. Let Γ_ϕ be the mapping from $L_a^2(\mathbb{D})$ into $L_a^2(\mathbb{D})$ defined by $\Gamma_\phi f = PM_\phi Jf$, where M_ϕ is the mapping from $L^2(\mathbb{D}, dA)$ into $L^2(\mathbb{D}, dA)$ defined by $M_\phi f = \phi f$. Thus $\Gamma_\phi f = PM_\phi Jf = P(\phi(z)f(\overline{z})) = P(J(\phi(\overline{z})f(z))) = S_{J\phi}f$ for all $f \in L_a^2(\mathbb{D})$.

Hence $\Gamma_\phi = S_{J\phi}$.

For $-1<\alpha<\infty$, let dA_α be the probability measure on $\mathbb{D}$ defined by

$$dA_\alpha(z)=(\alpha+1)(1-|z|^2)^\alpha dA(z).$$

Let $L^2(\mathbb{D},dA_\alpha)$ be the space of all measurable functions on the unit disk $\mathbb{D}$ for which the norm

$$\|f\|_\alpha^2=(\alpha+1)\int_{\mathbb{D}}|f(z)|^2(1-|z|^2)^\alpha dA(z)<\infty.$$

The weighted Bergman space $L_a^2(\mathbb{D},dA_\alpha)$ is the subspace of functions in $L^2(dA_\alpha)$ that are analytic and $L_a^2(dA_\alpha)$ is a closed subspace of $L^2(dA_\alpha)$. For convenience, we shall write $L^2(\mathbb{D},dA_\alpha)=L^{2,\alpha}(\mathbb{D})$ and $L_a^2(\mathbb{D},dA_\alpha)=L_a^{2,\alpha}(\mathbb{D})$. Let P_α be the orthogonal projection from the Hilbert space $L^2(dA_\alpha)$ onto the closed subspace $L_a^2(dA_\alpha)$, given by

$$P_\alpha f(z)=\int_{\mathbb{D}}K^\alpha(z,w)f(w)dA_\alpha(w),$$

where $K^\alpha(z,w)=K(z,w)^{1+\frac{\alpha}{2}}=\frac{1}{(1-z\overline{w})^{\alpha+2}}, z,w\in\mathbb{D}$ is the reproducing kernel of $L_a^2(dA_\alpha)$. Let ϕ be a measurable function on $\mathbb{D}$. The little Hankel operator with symbol ϕ denoted by h_ϕ is defined by $h_\phi f=\overline{P_\alpha}(\phi f), f\in L_a^2(dA_\alpha)$ where $\overline{P_\alpha}$ is the orthogonal projection from the Hilbert space $L^2(dA_\alpha)$ onto $\overline{L_a^2(dA_\alpha)}$, conjugates of functions in $L_a^2(dA_\alpha)$. Let $L^\infty(dA_\alpha)$ be the space of complex-valued, essentially bounded, measurable functions on $\mathbb{D}$ with respect to the measure dA_α and $H^\infty(dA_\alpha)$ be the subspace consisting of those functions that are analytic in $L^\infty(dA_\alpha)$. In this paper we shall consider only those symbols ϕ that are bounded and lie in $H^\infty+\overline{H^\infty}$, where $\overline{H^\infty(dA_\alpha)}$ constitutes the conjugates of functions in $H^\infty(dA_\alpha)$. If $\phi\in H^\infty$, then $h_\phi=0$. Let Γ_ϕ be the map from $L_a^2(dA_\alpha)$ into $L_a^2(dA_\alpha)$ such that $\Gamma_\phi f=P_\alpha(\phi Jf)$ for all $f\in L_a^2(dA_\alpha)$ where J is the mapping from $L^2(dA_\alpha)$ onto $L^2(dA_\alpha)$ such that $Jf(z)=f(\overline{z})$. Note that J is unitary. It can be checked that the operators Γ_ϕ is unitarily equivalent to an operator h_ψ for some $\psi\in L^\infty(dA_\alpha)$.

Let $z=\frac{1-s}{1+s}$. Hence $2\mathrm{Re}s=\frac{2(1-|z|^2)}{|1+z|^2}$. Recall that an analytic function $F\in L_a^{2,\alpha}(\mathbb{C}_+)$ if and only if $\int_{\mathbb{C}_+}|F(s)|^2x^\alpha dxdy<\infty$. Let $f(z)=F\left(\frac{1-z}{1+z}\right), s=\frac{1-z}{1+z}$. Thus $F\in L_a^{2,\alpha}(\mathbb{C}_+)$ if and only if

$$\int_{\mathbb{D}}|f(z)|^2\frac{(1-|z|^2)^\alpha}{|1+z|^{2\alpha}}\frac{4}{|1+z|^4}dA(z)<\infty.$$

This is possible if and only if $\int_{\mathbb{D}}\left|\frac{2f(z)}{|1+z|^{\alpha+2}}\right|^2(1-|z|^2)^\alpha dA(z)<\infty$. Hence $F\in L_a^{2,\alpha}(\mathbb{C}_+)$ if and only if $\frac{2f(z)}{(1+z)^{\alpha+2}}\in L_a^{2,\alpha}(\mathbb{D})$. Therefore $f\in L_a^{2,\alpha}(\mathbb{D})$ if and only if $\frac{2^{\alpha+1}}{(1+s)^{\alpha+2}}F(s)\in L_a^{2,\alpha}(\mathbb{C}_+)$. For $G\in H^\infty(\mathbb{C}_+)$, the little Hankel operator

$$\widetilde{\Gamma}_G : L_a^2(\mathbb{C}_+,x^\alpha d\widetilde{A}(s))\to L_a^2(\mathbb{C}_+,x^\beta d\widetilde{A}(s))$$

is defined by

$$(\widetilde{\Gamma}_G U)(s)=P_{\alpha\beta}(G(s)U(\overline{s}))$$

where $U\in L_a^2(\mathbb{C}_+,x^\alpha d\widetilde{A}(s))$ where $P_{\alpha\beta}$ is the orthogonal projection of $L_a^2(\mathbb{C}_+,x^\alpha d\widetilde{A}(s))$ onto $L_a^2(\mathbb{C}_+,x^\beta d\widetilde{A}(s))$. The operator $\widetilde{\Gamma}_G$ is bounded. For proof see [11].

For $h(t) \in L^2((0,\infty), dt)$, we define the Laplace transform $H(s) = (\mathcal{L}h)(s) = \int_0^\infty e^{-st} h(t)dt$. Then $(\mathcal{L}^{-1}H)(t) = \frac{1}{2\pi i}\int_\Omega H(s)e^{st}ds$, where Ω is the contour $\{\text{Re}s = \gamma\}$ for any $\gamma > 0$.

The layout of this paper is as follows: In §2, we consider a class of integral operators

$$(K_g u)(t) = \int_0^\infty \frac{t^{\frac{\beta+1}{2}} \tau^{\frac{\alpha+1}{2}}}{(t+\tau)^{\frac{\alpha+\beta+2}{2}}} g(t+\tau)u(\tau)d\tau,\ \alpha,\beta > -1$$

defined on $L^2(0,\infty)$ and show that these integral operators K_g are unitarily equivalent to the little Hankel operators $\widetilde{\Gamma}_G$ defined from $L_a^{2,\alpha}(\mathbb{C}_+)$ into $L_a^{2,\beta}(\mathbb{C}_+)$, where $G = \mathcal{L}\left(t^{\frac{\beta-\alpha}{2}} g(t)\right)$ and the little Hankel operator $\widetilde{\Gamma}_G$ is unitarily equivalent to the little Hankel operator Γ_ϕ defined from $L_a^{2,\alpha}(\mathbb{D})$ into $L_a^{2,\beta}(\mathbb{D})$ where $\phi(z) = \left(\frac{1+\bar{z}}{1+z}\right)^{\alpha+2} G(Mz)$. In §3, we calculate the norm of the integral operator K_g and obtain a generalization of Hardy-Hilbert's integral inequality. Applications of the inequality are also established . In §4, we concentrate on weighted Bergman-Hilbert matrices. We obtain the corresponding discrete version Hardy-Hilbert inequality which gives the norm of the companion matrices of the weighted Bergman-Hilbert matrices. We show that the Bergman-Hilbert matrix A has no maximizing vector and $\|A\| < \frac{\pi^2}{6}$ as an operator from l^2 into itself and the corresponding companion matrix B has norm 1. In section §5 and §6 we obtain generalizations of Hardy-Hilbert inequality for vector-valued functions and operator-valued functions.

2 Little Hankel operators between weighted Bergman spaces

In this section we consider a class of bounded integral operators defined on $L^2(0,\infty)$ (called weighted Hankel integral operators) and show that these operators are unitarily equivalent to little Hankel operators between weighted Bergman spaces of the open unit disk $\mathbb{D}$. The weighted Hankel integral operator K_g from $L^2((0,\infty), dt)$ into itself is defined by

$$(K_g u)(t) = \int_0^\infty \frac{t^{\frac{\beta+1}{2}} \tau^{\frac{\alpha+1}{2}}}{(t+\tau)^{\frac{\alpha+\beta+2}{2}}} g(t+\tau)u(\tau)d\tau.$$

We have shown that these operators are unitarily equivalent to little Hankel operators between weighted Bergman spaces of the disk. In Theorem 2.1, we show that for $g \in L^1 \cap L^2$, the operator K_g is bounded and $\|K_g\| \le \|g\|_1$.

Theorem 2.1. *If $g(t) \in L^1((0,\infty), dt) \cap L^2((0,\infty), dt)$ then the weighted Hankel integral operator K_g is well-defined and bounded with $\|K_g\| \le \|g\|_1$.*

Proof. Let $f, h \in L^2((0,\infty), dt)$ be such that $\|f\|_{L^2} \le 1$ and $\|h\|_{L^2} \le 1$. Then,

$$\left|\int_0^\infty \overline{(K_g f)(t)} h(t)dt\right| = \left|\int_0^\infty \int_0^\infty \frac{t^{\frac{\beta+1}{2}} \tau^{\frac{\alpha+1}{2}}}{(t+\tau)^{\frac{\alpha+\beta+2}{2}}} \overline{g(t+\tau)f(\tau)} h(t)dtd\tau\right|.$$

This result follows from [8] since the modulus of $\frac{t^{\frac{\beta+1}{2}} \tau^{\frac{\alpha+1}{2}}}{(t+\tau)^{\frac{\alpha+\beta+2}{2}}}$ does not exceed 1. □

In Theorem 2.2, we show that for $G \in H^\infty(\mathbb{C}_+)$, the little Hankel operator $\widetilde{\Gamma}_G$ from $L^2_a(\mathbb{C}_+, x^\alpha d\widetilde{A}(s))$ into $L^2_a(\mathbb{C}_+, x^\beta d\widetilde{A}(s))$, with $\beta > \alpha > -1$ is unitarily equivalent to the integral operator K_g where $G = \mathcal{L}\left(t^{\frac{\beta-\alpha}{2}} g(t)\right)$.

Theorem 2.2. *For $\beta > \alpha > -1$, the little Hankel operator $\widetilde{\Gamma}_G$ from $L^2_a(\mathbb{C}_+, x^\alpha d\widetilde{A}(s))$ into $L^2_a(\mathbb{C}_+, x^\beta d\widetilde{A}(s))$ with symbol $G \in H^\infty(\mathbb{C}_+)$ is unitarily equivalent to the integral operator K_g defined above where $G = \mathcal{L}\left(t^{\frac{\beta-\alpha}{2}} g(t)\right)$.*

Proof. For $\alpha > -1$, notice that $\frac{1}{t^{\alpha+1}} = \mathcal{L}(x^\alpha)(2t)$. Let $S : L^2((0,\infty), dt) \to L^2\left((0,\infty), \frac{dt}{t^{\alpha+1}}\right)$ be such that

$$(Sf)(t) = t^{\frac{\alpha+1}{2}} f(t).$$

Let $T : L^2\left((0,\infty), \frac{dt}{t^{\beta+1}}\right) \to L^2((0,\infty), dt)$ be such that

$$(Tf)(t) = t^{-\frac{\beta+1}{2}} f(t).$$

It can easily be checked that S and T are unitary maps. Let $\widetilde{K_h}$ be the operator unitarily equivalent to K_h by the relation

$$\widetilde{K_h} = T^{-1} K_h S^{-1}.$$

Then the operator

$$\widetilde{K_h} : L^2\left((0,\infty), \frac{dt}{t^{\alpha+1}}\right) \to L^2\left((0,\infty), \frac{dt}{t^{\beta+1}}\right)$$

satisfies

$$\begin{aligned}(\widetilde{K_h}u)(s) &= (T^{-1} K_h S^{-1} u)(s) \\ &= \int_0^\infty \frac{s^{\beta+1}}{(s+t)^{\frac{\alpha+\beta+2}{2}}} h(s+t) u(t) dt.\end{aligned}$$

Let $G(s) = \mathcal{L}\left(t^{\frac{\beta-\alpha}{2}} g(t)\right)$, $U(s) = \mathcal{L}\left(t^{\frac{\alpha+1}{2}} u(t)\right)$ and $(\widetilde{\Gamma}_G U)(s) = P_{\alpha\beta}(G(s)U(\overline{s})) = R(s)$. Then

$$\langle G(s)U(\overline{s}), F(s)\rangle = \langle R(s), F(s)\rangle$$

for all $F \in L^2_a(\mathbb{C}_+, x^\beta d\widetilde{A}(s))$. Thus

$$\langle G(s), \overline{U(\overline{s})} F(s)\rangle = \langle R(s), F(s)\rangle$$

for all $F \in L^2_a(\mathbb{C}_+, x^\beta d\widetilde{A}(s))$. Also $\overline{U(\overline{s})} = \mathcal{L}\left(t^{\frac{\alpha+1}{2}} \overline{u}\right)(s)$. Thus

$$\int_0^\infty t^{\frac{\beta-\alpha}{2}} g(t) \overline{\left(t^{\frac{\alpha+1}{2}} \overline{u}(t)\right) * \left(t^{\frac{\beta+1}{2}} f(t)\right)} \frac{dt}{t^{\beta+1}}$$

$$= \int_0^\infty t^{\frac{\beta+1}{2}} r(t) t^{\frac{\beta+1}{2}} \overline{f(t)} \frac{dt}{t^{\beta+1}}$$

where $*$ denotes convolution, $t^{\frac{\beta+1}{2}} f(t) = \mathcal{L}^{-1}\{F(s)\}$, $t^{\frac{\beta+1}{2}} r(t) = \mathcal{L}^{-1}\{R(s)\}$ and

$$\begin{aligned}\overline{\left(t^{\frac{\alpha+1}{2}}\overline{u}(t)\right)*\left(t^{\frac{\beta+1}{2}}f(t)\right)} &= \int_0^t \overline{\tau^{\frac{\alpha+1}{2}}\overline{u}(\tau)(t-\tau)^{\frac{\beta+1}{2}}f(t-\tau)d\tau}\\ &= \int_0^t \tau^{\frac{\alpha+1}{2}}u(\tau)(t-\tau)^{\frac{\beta+1}{2}}\overline{f(t-\tau)}d\tau.\end{aligned}$$

Hence

$$\begin{aligned}&\int_0^\infty t^{\frac{\beta-\alpha}{2}}g(t)\overline{\left(t^{\frac{\alpha+1}{2}}\overline{u}(t)\right)*\left(t^{\frac{\beta+1}{2}}f(t)\right)}\frac{dt}{t^{\beta+1}}\\ &= \int_0^\infty t^{\frac{\beta-\alpha}{2}}g(t)\left(\int_0^t \tau^{\frac{\alpha+1}{2}}u(\tau)(t-\tau)^{\frac{\beta+1}{2}}\overline{f(t-\tau)}d\tau\right)\frac{dt}{t^{\beta+1}}\\ &= \int_{x=0}^\infty\int_{\tau=0}^\infty (x+\tau)^{\frac{\beta-\alpha}{2}}g(x+\tau)\tau^{\frac{\alpha+1}{2}}u(\tau)x^{\frac{\beta+1}{2}}\overline{f(x)}\frac{d\tau}{(x+\tau)^{\beta+1}}dx\\ &= \int_{x=0}^\infty\left[\int_{\tau=0}^\infty \frac{(x+\tau)^{\frac{\beta-\alpha}{2}}}{(x+\tau)^{\beta+1}}g(x+\tau)\tau^{\frac{\alpha+1}{2}}u(\tau)\right]x^{\frac{\beta+1}{2}}\overline{f(x)}dx\\ &= \int_{x=0}^\infty \frac{1}{x^{\beta+1}}\left(\widetilde{K}_g\left(x^{\frac{\alpha+1}{2}}u\right)\right)(x)x^{\frac{\beta+1}{2}}\overline{f(x)}dx\\ &= \int_{x=0}^\infty \left(\widetilde{K}_g\left(x^{\frac{\alpha+1}{2}}u\right)\right)(x)x^{\frac{\beta+1}{2}}\overline{f(x)}\frac{dx}{x^{\beta+1}}\\ &= \left\langle\left(\widetilde{K}_g\left(x^{\frac{\alpha+1}{2}}u\right)\right)(x), x^{\frac{\beta+1}{2}}f(x)\right\rangle_{L^2\left((0,\infty),\frac{dt}{t^{\beta+1}}\right)}.\end{aligned}$$

Thus $\left\langle\left(\widetilde{K}_g\left(x^{\frac{\alpha+1}{2}}u\right)\right)(x), x^{\frac{\beta+1}{2}}f(x)\right\rangle_{L^2\left((0,\infty),\frac{dt}{t^{\beta+1}}\right)} = \left\langle x^{\frac{\beta+1}{2}}r(x), x^{\frac{\beta+1}{2}}f(x)\right\rangle_{L^2\left((0,\infty),\frac{dt}{t^{\beta+1}}\right)}$.
Hence $\left(\widetilde{K}_g\left(x^{\frac{\alpha+1}{2}}u\right)\right)(x) = x^{\frac{\beta+1}{2}}r(x) = \mathcal{L}^{-1}\{R(s)\}$, and $\mathcal{L}\left(\widetilde{K}_g\left(x^{\frac{\alpha+1}{2}}u\right)\right)(s) = R(s) = (\widetilde{\Gamma}_G U)(s)$. □

In Theorem 2.3, we have shown that for $G \in L^\infty(\mathbb{C}_+)$, the little Hankel operator $\widetilde{\Gamma}_G$ from $L_a^{2,\alpha}(\mathbb{C}_+)$ into $L_a^{2,\beta}(\mathbb{C}_+)$ is unitarily equivalent to the little Hankel operator Γ_ϕ from $L_a^{2,\alpha}(\mathbb{D})$ into $L_a^{2,\beta}(\mathbb{D})$, where $\phi(z) = \left(\frac{1+\bar{z}}{1+z}\right)^{\alpha+2} G(Mz)$.

Theorem 2.3. *Let $G(s) \in L^\infty(\mathbb{C}_+)$. Then the little Hankel operator $\widetilde{\Gamma}_G$ defined from $L_a^{2,\alpha}(\mathbb{C}_+)$ into $L_a^{2,\beta}(\mathbb{C}_+)$ by G is equivalent to the little Hankel operator Γ_ϕ from $L_a^{2,\alpha}(\mathbb{D})$ into $L_a^{2,\beta}(\mathbb{D})$ determined by the function $\phi(z) = \left(\frac{1+\bar{z}}{1+z}\right)^{\alpha+2} G(Mz)$.*

Proof. Let $W : L_a^{2,\alpha}(\mathbb{D}) \to L_a^{2,\alpha}(\mathbb{C}_+)$ be defined by

$$(Wg)(s) = \frac{2^{\frac{\alpha}{2}+1}}{\sqrt{\pi}} g(Ms)\frac{1}{(1+s)^{\alpha+2}},$$

where $Ms = \frac{1-s}{1+s}$. The inverse map $W^{-1} : L_a^{2,\alpha}(\mathbb{C}_+) \to L_a^{2,\alpha}(\mathbb{D})$ satisfies

$$(W^{-1}G)(z) = 2^{\frac{\alpha}{2}+1}\sqrt{\pi}G(Mz)\frac{1}{(1+z)^{\alpha+2}},$$

where $Mz = \frac{1-z}{1+z}$. Further, we shall define $V : L^{2,\beta}_a(\mathbb{C}_+) \to L^{2,\beta}_a(\mathbb{D})$ by $(VG)(z) = 2^{\frac{\beta}{2}+1}\sqrt{\pi}G(Mz)\frac{1}{(1+z)^{\beta+2}}$ where $Mz = \frac{1-z}{1+z}$. The inverse map $V^{-1} : L^{2,\beta}_a(\mathbb{D}) \to L^{2,\beta}_a(\mathbb{C}_+)$ satisfies $(V^{-1}g)(s) = \frac{2^{\frac{\beta}{2}+1}}{\sqrt{\pi}}g(Ms)\frac{1}{(1+s)^{\beta+2}}$. It can easily be checked that V and W are unitary maps. Notice that the operator W can also be defined from $L^{2,\alpha}(\mathbb{D})$ into $L^{2,\alpha}(\mathbb{C}_+)$ and similarly V can be defined from $L^{2,\beta}(\mathbb{C}_+)$ into $L^{2,\beta}(\mathbb{D})$ and are also unitary on these spaces. Then $\nu^2_{n,\alpha} = \|z^n\|^2_\alpha = (\alpha+1)\int_{\mathbb{D}}|z|^{2n}(1-|z|^2)^\alpha dA(z) = (\alpha+1)\int_0^1 x^n(1-x)^\alpha dx = (\alpha+1)\frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)} \sim (n+1)^{-\alpha-1}$. Hence $\nu_{n,\alpha} \sim n^{-\frac{\alpha+1}{2}}, n \geq 1$ and $\left\{\frac{z^n}{\nu_{n,\alpha}}\right\}$ is an orthonormal basis for $L^{2,\alpha}_a(\mathbb{D})$.

Let $\widetilde{P}_{\alpha\beta}$ be the orthogonal projection of $L^{2,\alpha}_a(\mathbb{C}_+)$ onto $L^{2,\beta}_a(\mathbb{C}_+)$ and $P_{\alpha\beta}$ be the orthogonal projection of $L^{2,\alpha}_a(\mathbb{D})$ onto $L^{2,\beta}_a(\mathbb{D})$. Define the map $\widetilde{J} : L^{2,\alpha}(\mathbb{C}_+) \to L^{2,\alpha}(\mathbb{C}_+)$ such that $\widetilde{J}f(s) = f(\overline{s})$. We shall show that $V\widetilde{\Gamma}_G W\left(\frac{z^n}{\nu_{n,\alpha}}\right) = \Gamma_\phi\left(\frac{z^n}{\nu_{n,\alpha}}\right)$. That is, $\widetilde{\Gamma}_G W\left(\frac{z^n}{\nu_{n,\alpha}}\right) = V^{-1}\Gamma_\phi\left(\frac{z^n}{\nu_{n,\alpha}}\right)$. Notice that

$$\begin{aligned}
\widetilde{\Gamma}_G W\left(\tfrac{z^n}{\nu_{n,\alpha}}\right) &= \widetilde{P}_{\alpha\beta}G\widetilde{J}\left(W\left(\tfrac{z^n}{\nu_{n,\alpha}}\right)\right) \\
&= \widetilde{P}_{\alpha\beta}G\widetilde{J}\left(\tfrac{2^{\frac{\alpha}{2}+1}}{\sqrt{\pi}}\tfrac{1}{\nu_{n,\alpha}}(Ms)^n\tfrac{1}{(1+s)^{\alpha+2}}\right) \\
&= \widetilde{P}_{\alpha\beta}G\widetilde{J}\left(\tfrac{2^{\frac{\alpha}{2}+1}}{\sqrt{\pi}}\tfrac{1}{\nu_{n,\alpha}}\left(\tfrac{1-s}{1+s}\right)^n\tfrac{1}{(1+s)^{\alpha+2}}\right) \\
&= \widetilde{P}_{\alpha\beta}G\left(\tfrac{2^{\frac{\alpha}{2}+1}}{\sqrt{\pi}}\tfrac{1}{\nu_{n,\alpha}}\left(\tfrac{1-\overline{s}}{1+\overline{s}}\right)^n\tfrac{1}{(1+\overline{s})^{\alpha+2}}\right) \\
&= V^{-1}P_{\alpha\beta}W^{-1}\left(G(s)\tfrac{2^{\frac{\alpha}{2}+1}}{\sqrt{\pi}}\tfrac{1}{\nu_{n,\alpha}}\left(\tfrac{1-\overline{s}}{1+\overline{s}}\right)^n\tfrac{1}{(1+\overline{s})^{\alpha+2}}\right) \\
&= V^{-1}P_{\alpha\beta}\left(\tfrac{2^{\frac{\alpha}{2}+1}}{\sqrt{\pi}}\tfrac{1}{\nu_{n,\alpha}}2^{\frac{\alpha}{2}+1}\sqrt{\pi}\left(\tfrac{1-\frac{1-\overline{z}}{1+\overline{z}}}{1+\frac{1-\overline{z}}{1+\overline{z}}}\right)^n\tfrac{1}{\left(1+\frac{1-\overline{z}}{1+\overline{z}}\right)^{\alpha+2}}G(Mz)\tfrac{1}{(1+z)^{\alpha+2}}\right) \\
&= V^{-1}P_{\alpha\beta}\left(2^{\alpha+2}\tfrac{1}{\nu_{n,\alpha}}\overline{z}^n\left(\tfrac{1+\overline{z}}{2}\right)^{\alpha+2}G(Mz)\tfrac{1}{(1+z)^{\alpha+2}}\right) \\
&= V^{-1}P_{\alpha\beta}\left(G(Mz)\left(\tfrac{1+\overline{z}}{1+z}\right)^{\alpha+2}J\left(\tfrac{z^n}{\nu_{n,\alpha}}\right)\right).
\end{aligned}$$

Let $\phi(z) = G(Mz)\left(\frac{1+\overline{z}}{1+z}\right)^{\alpha+2}$. Then

$$\begin{aligned}
\widetilde{\Gamma}_G W\left(\frac{z^n}{\nu_{n,\alpha}}\right) &= V^{-1}P_{\alpha\beta}\left(\phi J\left(\frac{z^n}{\nu_{n,\alpha}}\right)\right) \\
&= V^{-1}\Gamma_\phi\left(\frac{z^n}{\nu_{n,\alpha}}\right).
\end{aligned}$$

Thus $V\widetilde{\Gamma}_G W\left(\frac{z^n}{\nu_{n,\alpha}}\right) = \Gamma_\phi\left(\frac{z^n}{\nu_{n,\alpha}}\right)$ and $\widetilde{\Gamma}_G$ is unitarily equivalent to Γ_ϕ. □

3 Hardy-Hilbert's integral inequality

In this section we calculate the norm of the integral operator K_g and obtain a generalization of Hardy-Hilbert's integral inequality. Applications of the inequality are also established. If $p > 1, \frac{1}{p} + \frac{1}{q} = 1$ and $f(t), g(t) \geq 0, 0 < \int_0^\infty f^p(t)dt < \infty$ and $0 < \int_0^\infty g^q(t)dt < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\frac{\pi}{p})} \left(\int_0^\infty f^p(t)dt \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(t)dt \right)^{\frac{1}{q}}, \tag{3.1}$$

where the constant factor $\frac{\pi}{\sin(\frac{\pi}{p})}$ is still best possible (see [10]). The integral inequality (3.1) is known as Hardy-Hilbert's integral inequality. The inequality plays an important role in analysis and its application (see [14]). In the last decade many generalizations and refinements of the inequality were also obtained. We formulate the $\beta-$function as (see [13]):

$$B(p,q) = \int_0^\infty \frac{1}{(1+t)^{p+q}} t^{p-1} dt = B(q,p), \quad p,q > 0. \tag{3.2}$$

Further, the Hölder's inequality with weight (see [13]) is as follows:

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \omega(t) > 0, f, g \geq 0, f \in L^q_\omega(E)$ and $g \in L^q_\omega(E)$, then

$$\int_E \omega(t) f(t) g(t) d(t) \leq \left\{ \int_E \omega(t) f^p(t) d(t) \right\}^{\frac{1}{p}} \left\{ \int_E \omega(t) g^q(t) d(t) \right\}^{\frac{1}{q}}; \tag{3.3}$$

if $p < 1 (p \neq 0)$; with the above assumption, the reverse of (3.3) holds, where the equality in the above two cases holds if and only if there exists non-negative real numbers c_1 and c_2 such that they are not all zero and

$$c_1 f^p(t) = c_2 g^q(t), a.e. \text{ in } E.$$

In Theorem 3.1, we obtain a generalization of Hardy-Hilbert's integral inequality.

Theorem 3.1. *Suppose* $\frac{1}{p} + \frac{1}{q} = 1, 1 < p < \infty, f \in L^p(0,\infty), g \in L^q(0,\infty), \alpha > -\frac{1}{q}, \beta > -\frac{1}{p}, f, g \geq 0$. *Then*

$$\int_0^\infty \int_0^\infty \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}} e^{-(x+y)} f(x) g(y) dxdy$$

$$\leq B\left(\alpha + \frac{1}{q}, \beta + \frac{1}{p}\right) \left(\int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy \right)^{\frac{1}{q}} \tag{3.4}$$

and the constant factor $B\left(\alpha + \frac{1}{q}, \beta + \frac{1}{p}\right)$ *is the best possible.*

Proof. We shall first establish that if $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha > -\frac{1}{q}, \beta > -\frac{1}{p}, f, g \geq 0$, satisfy $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$ then $\int_0^\infty \int_0^\infty \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}} f(x)g(y)dxdy$

$$< B\left(\alpha + \frac{1}{q}, \beta + \frac{1}{p}\right) \left(\int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(x)dx \right)^{\frac{1}{q}}; \tag{3.5}$$

where the constant factor $B\left(\alpha+\frac{1}{q},\beta+\frac{1}{p}\right)$ is the best possible.

By Hölder's inequality (3.3), we obtain

$$\int_0^\infty\int_0^\infty \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}} f(x)g(y)dxdy$$
$$= \int_0^\infty\int_0^\infty \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}} \left(\frac{x}{y}\right)^{\frac{1}{pq}} f(x)\left(\frac{y}{x}\right)^{\frac{1}{pq}} g(y)dxdy$$

$$\leq \left(\int_0^\infty\int_0^\infty \frac{x^{\alpha+\frac{1}{q}}y^{\beta-\frac{1}{q}}}{(x+y)^{\alpha+\beta+1}} f^p(x)dxdy\right)^{\frac{1}{p}} \tag{3.6}$$
$$\left(\int_0^\infty\int_0^\infty \frac{x^{\alpha-\frac{1}{p}}y^{\beta+\frac{1}{p}}}{(x+y)^{\alpha+\beta+1}} g^q(y)dxdy\right)^{\frac{1}{q}} \tag{3.7}$$

$$= \left(\int_0^\infty\left[\int_0^\infty \frac{x^{\alpha+\frac{1}{q}}y^{\beta-\frac{1}{q}}}{(x+y)^{\alpha+\beta+1}}dy\right] f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty\left[\int_0^\infty \frac{x^{\alpha-\frac{1}{p}}y^{\beta+\frac{1}{p}}}{(x+y)^{\alpha+\beta+1}}dx\right] g^q(y)dy\right)^{\frac{1}{q}}.$$

If equality holds in (3.6), then there exists non-negative constants c_1 and c_2, such that they are not all zero and

$$c_1\frac{x^{\alpha+\frac{1}{q}}y^{\beta-\frac{1}{q}}}{(x+y)^{\alpha+\beta+1}} f^p(x) = c_2\frac{x^{\alpha-\frac{1}{q}}y^{\beta+\frac{1}{q}}}{(x+y)^{\alpha+\beta+1}} g^q(y),\ a.e. \text{ in } (0,\infty)\times(0,\infty).$$

It follows therefore that

$$c_1xf^p(x) = c_2yg^q(y) = c_3, a.e.\text{in } (0,\infty)\times(0,\infty),$$

where c_3 is a constant. Without loss of generality, suppose that $c_1 \neq 0$. Then we have

$$\int_0^\infty f^p(x)dx = \frac{c_3}{c_1}\int_0^\infty \frac{1}{x}dx = \infty,$$

which contradicts our assumption that $0 < \int_0^\infty f^p(x)dx < \infty$. Hence strict inequality holds in (3.6). Putting $t = \frac{y}{x}$, we get from (3.2) that

$$\int_0^\infty \frac{x^{\alpha+\frac{1}{q}}y^{\beta-\frac{1}{q}}}{(x+y)^{\alpha+\beta+1}}dy = \int_0^\infty \frac{1}{(1+t)^{\alpha+\beta+1}} t^{(\beta+\frac{1}{p})-1}dt = B\left(\alpha+\frac{1}{q},\beta+\frac{1}{p}\right). \tag{3.8}$$

Similarly, we have

$$\int_0^\infty \frac{x^{\alpha-\frac{1}{p}}y^{\beta+\frac{1}{p}}}{(x+y)^{\alpha+\beta+1}}dx = \int_0^\infty \frac{1}{(1+t)^{\alpha+\beta+1}} t^{(\beta+\frac{1}{p})-1}dt = B\left(\alpha+\frac{1}{q},\beta+\frac{1}{p}\right). \tag{3.9}$$

Then from (3.6), we get (3.5). For the best constant factor, let for $0 < \epsilon < q(\beta+\frac{1}{p})$,

$$f_\epsilon(x) = \begin{cases} 0 & \text{if } x\in(0,1), \\ x^{-\frac{1+\epsilon}{p}} & \text{if } x\in[1,\infty). \end{cases}$$

$$g_\epsilon(x) = \begin{cases} 0 & \text{if } x \in (0,1), \\ x^{-\frac{1+\epsilon}{q}} & \text{if } x \in [1,\infty). \end{cases}$$

Then

$$\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}} = \frac{1}{\epsilon}. \tag{3.10}$$

Also

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}} f_\epsilon(x) g_\epsilon(y) dxdy \\
&= \int_1^\infty \int_1^\infty \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}} x^{-\frac{1+\epsilon}{p}} y^{-\frac{1+\epsilon}{q}} dxdy \\
&= \int_1^\infty x^{\alpha-\frac{1+\epsilon}{p}} \left(\int_1^\infty \frac{y^{\beta-\frac{1+\epsilon}{q}}}{(x+y)^{\alpha+\beta+1}} dy\right) dx \\
&= \int_1^\infty x^{-(1+\epsilon)} \left(\int_{\frac{1}{x}}^\infty \frac{t^{\beta-\frac{1+\epsilon}{q}}}{(1+t)^{\alpha+\beta+1}} dt\right) dx \quad (\text{Setting } t = \frac{y}{x}, x > 1) \\
&= \int_1^\infty x^{-(1+\epsilon)} dx \left(\int_0^\infty \frac{t^{\beta-\frac{1+\epsilon}{q}}}{(1+t)^{\alpha+\beta+1}} dt\right) - \int_1^\infty x^{-(1+\epsilon)} \left(\int_0^{\frac{1}{x}} \frac{t^{\beta-\frac{1+\epsilon}{q}}}{(1+t)^{\alpha+\beta+1}} dt\right) dx \\
&= I_1 - I_2 \quad (\text{say}).
\end{aligned}$$

By (3.2), we have

$$I_1 = \frac{1}{\epsilon} B\left(\alpha + \frac{1}{q} + \frac{\epsilon}{q}, \beta + \frac{1}{p} - \frac{\epsilon}{q}\right)$$

and

$$\begin{aligned}
I_2 &\le \int_1^\infty x^{-(1+\epsilon)} \left(\int_0^{\frac{1}{x}} t^{\beta-\frac{1+\epsilon}{q}} dt\right) dx \\
&= \frac{1}{\beta + \frac{1}{p} - \frac{\epsilon}{q}} \int_1^\infty \int_0^\infty x^{-(1+\beta+\frac{1+\epsilon}{p})} dx \\
&= \frac{1}{(\beta + \frac{1}{p} - \frac{\epsilon}{q})(\beta + \frac{1}{p} + \frac{\epsilon}{q})} \\
&= \bigcirc(1).
\end{aligned}$$

Hence

$$\int_0^\infty \int_0^\infty \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}} f_\epsilon(x) g_\epsilon(y) dxdy \ge \frac{1}{\epsilon} B\left(\alpha + \frac{1}{q} + \frac{\epsilon}{q}, \beta + \frac{1}{p} - \frac{\epsilon}{q}\right) - \bigcirc(1). \tag{3.11}$$

If the constant factor $B\left(\alpha + \frac{1}{q}, \beta + \frac{1}{p}\right)$ in (3.5) is not the best possible, then there exists a positive constant $C < B\left(\alpha + \frac{1}{q}, \beta + \frac{1}{p}\right)$, such that (3.5) is still valid if we replace $B\left(\alpha + \frac{1}{q}, \beta + \frac{1}{p}\right)$

by C. In particular, by (3.10) and (3.11), we have

$$B\left(\alpha+\frac{1}{q}+\frac{\epsilon}{q},\beta+\frac{1}{p}-\frac{\epsilon}{q}\right)-\epsilon\bigcirc(1)$$
$$\leq \epsilon\int_0^\infty\int_0^\infty \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}}f_\epsilon(x)g_\epsilon(y)dxdy$$
$$< \epsilon C\left(\int_0^\infty f_\epsilon^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g_\epsilon^q(x)dx\right)^{\frac{1}{q}}.$$

Hence $B\left(\alpha+\frac{1}{q},\beta+\frac{1}{p}\right)\leq C$ as $\epsilon\to 0^+$. This contradiction leads to the conclusion that the constant factor in (3.5) is the best possible. It now follows from (3.5) that

$$\int_0^\infty\int_0^\infty \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}}e^{-(x+y)}f(x)g(y)dxdy$$
$$=\int_0^\infty\int_0^\infty \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}}e^{-x}f(x)e^{-y}g(y)dxdy$$

$$\leq B\left(\alpha+\frac{1}{q},\beta+\frac{1}{p}\right)\left(\int_0^\infty e^{-px}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty e^{-qy}g^q(y)dy\right)^{\frac{1}{q}} \tag{3.12}$$

$$\leq B\left(\alpha+\frac{1}{q},\beta+\frac{1}{p}\right)\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}.$$

It thus remains to show that the constant factor 1 in the inequality

$$\int_0^\infty e^{-px}f^p(x)dx\leq\int_0^\infty f^p(x)dx \tag{3.13}$$

is the best possible.

Suppose there exists a constant $k, 0<k<1$ such that

$$\int_0^\infty e^{-px}f^p(x)dx< k\int_0^\infty f^p(x)dx \tag{3.14}$$

for all $f\in L^p(0,\infty)$.

Setting

$$f^\dagger(x)=\begin{cases}1, & 0\leq x\leq \frac{1}{p}\log\frac{1}{k}\\ 0, & x>\frac{1}{p}\log\frac{1}{k},\end{cases}$$

we have $\int_0^\infty (f^\dagger)^p(x)dx=\int_0^{\frac{1}{p}\log\frac{1}{k}}dx=\frac{1}{p}\log\frac{1}{k}$; hence $f^\dagger\in L^p(0,\infty)$. Now

$$\int_0^\infty (e^{-px}-k)(f^\dagger)^p(x)dx=\frac{1}{p}+\frac{k}{p}\log\left(\frac{k}{e}\right). \tag{3.15}$$

Consider the function $g(t) = -e^{-pt} + 1 - kpt, t \in [0,\infty)$. Then $g'(t) = pe^{-pt} - kp = 0$ for $t = \frac{1}{p}\log\frac{1}{k}$ and $g''(t) = -p^2e^{-pt} < 0$ for $t = \frac{1}{p}\log\frac{1}{k}$. Hence $g(t) > g(0)$ for $t = \frac{1}{p}\log\frac{1}{k}$. Therefore $1 + k\log(\frac{k}{e}) > 0$. Now from (3.15) we get

$$\int_0^\infty (e^{-px} - k)(f^\dagger)^p(x)dx > 0.$$

This is a contradiction to the assumption (3.14) and we thus show that the constant factor 1 in the inequality (3.13) is the best possible. Again the constant factor $\frac{\pi}{\sin(\frac{\pi}{p})}$ is the best possible in the Hardy-Hilbert's integral inequality (3.1). Hence the result follows.

□

Corollary 3.2. *If $f, g \in L^2(-\infty,\infty)$, then*

$$\left|\int_\infty^\infty [\cosh(t-s)]^{-2} f(s)g(t)dsdt\right| \leq 2\|f\|_{L^2(-\infty,\infty)}\|g\|_{L^2(-\infty,\infty)}.$$

Proof. Consider the map $\mathbb{W} : L^2(0,\infty) \to L^2(-\infty,\infty)$ defined by

$$\mathbb{W}f(t) = \sqrt{2}e^t f(e^{2t}).$$

The operator $\mathbb{W}$ is an unitary operator . Let f be a continuous function with compact support in $(0,\infty)$ and $h(x+y) = \frac{1}{(x+y)^2}, x = e^{2t}, y = e^{2s}$. Define $K_h : L^2(0,\infty) \longrightarrow L^2(0,\infty)$ by

$$(K_hf)(x) = \int_0^\infty \frac{\sqrt{x}\sqrt{y}}{(x+y)^2} f(y)dy. \tag{3.16}$$

We proceed to show that $K_h = \mathbb{W}^*C\mathbb{W}$, where $C : L^2(-\infty,\infty) \longrightarrow L^2(-\infty,\infty)$ is defined as $(Cf)(t) = \frac{1}{2}\int_{-\infty}^\infty [\cosh(t-s)]^{-2} f(s)ds$.

Notice that

$$\begin{aligned}
(K_hf)(x) &= \int_0^\infty \frac{\sqrt{x}\sqrt{y}f(y)}{(x+y)^2}dy \\
&= \int_{-\infty}^\infty \frac{e^te^sf(e^{2s})2e^{2s}}{(e^{2t}+e^{2s})^2}ds \\
&= \frac{1}{\sqrt{2}e^t}\int_{-\infty}^\infty \frac{\sqrt{2}e^te^te^sf(e^{2s})2e^{2s}}{(e^{2t}+e^{2s})^2}ds \\
&= \frac{1}{\sqrt{2}e^t}\int_{-\infty}^\infty \frac{\sqrt{2}e^sf(e^{2s})2e^{2s}e^{2t}}{(e^{2t}+e^{2s})^2}ds \\
&= \frac{1}{2\sqrt{2}e^t}\int_{-\infty}^\infty \frac{\mathbb{W}f(s)ds}{\left(\frac{e^{2t}+e^{2s}}{2e^te^s}\right)^2} \\
&= \frac{1}{2\sqrt{2}e^t}\int_{-\infty}^\infty \frac{\mathbb{W}f(s)ds}{\left(\frac{e^{t-s}+e^{s-t}}{2}\right)^2} \\
&= \frac{1}{2\sqrt{2}e^t}\int_{-\infty}^\infty [\cosh(t-s)]^{-2}\mathbb{W}f(s)ds \\
&= (\mathbb{W}^*C\mathbb{W}f)(x),
\end{aligned}$$

since if $g \in L^2(-\infty,\infty)$ then $\frac{g(t)}{\sqrt{2e^t}} = \frac{1}{\sqrt{2x}} g\left(\frac{1}{2}\log x\right) = \mathbb{W}^* g(x)$. Thus $K_h = \mathbb{W}^* C \mathbb{W}$, where C is the convolution with $\frac{(\cosh t)^{-2}}{2}$. That is,

$$(Cf)(t) = \frac{1}{2}\int_{-\infty}^{\infty} [\cosh(t-s)]^{-2} f(s)ds.$$

Since K_h and C are unitarily equivalent hence $\|C\| = 1$ and

$$|\langle Cf, g\rangle| \le \|f\|_{L^2(-\infty,\infty)} \|g\|_{L^2(-\infty,\infty)}.$$

Thus

$$\left| \int_{\infty}^{\infty} [\cosh(t-s)]^{-2} f(s) g(t) ds dt \right| \le 2\|f\|_{L^2(-\infty,\infty)} \|g\|_{L^2(-\infty,\infty)}.$$

□

Theorem 3.3 shows also that the integral operator $(\underline{K}_u f)(x) = \int_0^\infty u(x,y) f(x)dx$, where $u(x,y) = \frac{e^{-(\sqrt{x}+\sqrt{y})}}{x+y}$ is also bounded from $L^p(0,\infty)$ into $L^q(0,\infty)$ and $\|\underline{K}_u\| = \frac{\pi}{\sin\frac{\pi}{p}}$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 3.3. *Let* $\frac{1}{p} + \frac{1}{q} = 1, 1 < p < \infty, f \in L^p(0,\infty), g \in L^q(0,\infty), f, g \ge 0$, *then* $\int_0^\infty \int_0^\infty \frac{e^{-(\sqrt{x}+\sqrt{y})}}{x+y} f(x)g(y)dxdy \le \frac{\pi}{\sin(\frac{\pi}{p})} \left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}$ *and the constant factor* $\frac{\pi}{\sin(\frac{\pi}{p})}$ *is the best possible.*

Proof. Using Hardy-Hilbert's inequality (3.1), we obtain

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{e^{-(\sqrt{x}+\sqrt{y})}}{x+y} f(x)g(y)dxdy \\
&< \frac{\pi}{\sin(\frac{\pi}{p})} \left(\int_0^\infty e^{-p\sqrt{x}} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty e^{-q\sqrt{y}} g^q(y)dy\right)^{\frac{1}{q}} \\
&\le \frac{\pi}{\sin(\frac{\pi}{p})} \left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}
\end{aligned}$$

as $e^{-p\sqrt{t}} \le 1$ for $t \in (0,\infty)$. It remains to show that the constant factor 1 in the inequality

$$\int_0^\infty e^{-p\sqrt{x}} f^p dx \le \int_0^\infty f^p(x)dx \tag{3.17}$$

is the best possible. Suppose there exists a constant $k, 0 < k < 1$, such that

$$\int_0^\infty e^{-p\sqrt{x}} f^p dx < k\int_0^\infty f^p(x)dx \tag{3.18}$$

for all $f \in L^p(0,\infty)$. Setting

$$f^\dagger(x) = \begin{cases} 1, & 0 \le x \le \left(\frac{1}{p}\log\frac{1}{k}\right)^2 \\ 0, & x > \left(\frac{1}{p}\log\frac{1}{k}\right)^2, \end{cases}$$

we have $\int_0^\infty (f^\dagger)^p(x)dx = \int_0^{\left(\frac{1}{p}\log\frac{1}{k}\right)^2} dx = \left(\frac{1}{p}\log\frac{1}{k}\right)^2$. Hence $f^\dagger \in L^p(0,\infty)$. Now

$$\int_0^\infty (e^{-p\sqrt{x}} - k)(f^\dagger)^p(x)dx = \frac{2}{p} + \frac{k}{p}\left(\log\frac{k}{e}\right).$$

Consider the function

$$g(t) = -2\left(\sqrt{t}e^{-p\sqrt{t}} + \frac{1}{p}e^{-p\sqrt{t}}\right) + \frac{2}{p} - kpt; \text{ hence } g(0) = 0.$$

Further,

$$\begin{aligned} g'(t) &= -2\left[\frac{e^{-p\sqrt{t}}}{2\sqrt{t}} + \frac{\sqrt{t}\,e^{-p\sqrt{t}}(-p)}{2\sqrt{t}} + \frac{(-p)e^{-p\sqrt{t}}}{2p\sqrt{t}}\right] - kp \\ &= -2\left[\frac{1}{2\sqrt{t}} - \frac{p}{2} - \frac{1}{2\sqrt{t}}\right]e^{-p\sqrt{t}} - kp \\ &= pe^{-p\sqrt{t}} - kp. \end{aligned}$$

Therefore $g''(t) = pe^{-p\sqrt{t}}(-p)\cdot\frac{1}{2\sqrt{t}}$. Now putting $t = \left(\frac{1}{p}\log\frac{1}{k}\right)^2$, we have

$$\begin{aligned} g''\left(\frac{1}{p}\log\frac{1}{k}\right)^2 &= pe^{-p(\frac{1}{p}\log\frac{1}{k})}(-p)\cdot\frac{1}{2(\frac{1}{p}\log\frac{1}{k})} \\ &= \frac{-p^2}{2}k\left(\frac{1}{-\frac{1}{p}\log\frac{1}{k}}\right) \\ &= \frac{-p^2}{2}k\frac{1}{\frac{1}{p}(-\log k)} \\ &= \frac{p^3k}{2\log k} < 0. \end{aligned}$$

Hence $g'(t) = 0$ for $t = \left(\frac{1}{p}\log\frac{1}{k}\right)^2$ and $g''(t) < 0$ for $t = \left(\frac{1}{p}\log\frac{1}{k}\right)^2$. Thus $g(t) > g(0)$ for $t = \left(\frac{1}{p}\log\frac{1}{k}\right)^2$. Therefore $\int_0^\infty (e^{-p\sqrt{x}} - k)(f^{\dagger p})dx > 0$. This is a contradiction to the assumption (3.1) which shows that the constant factor 1 in the inequality (3.17) is the best possible. Again the constant factor $\frac{\pi}{\sin(\frac{\pi}{p})}$ is the best possible in the Hardy-Hilbert's integral inequality. The result follows: □

In Corollary 3.4, we further generalize the inequality obtained in Corollary 3.2.

Corollary 3.4. *If $f, g \in L^2(-\infty,\infty)$ and $\alpha, \beta > -1$, then*

$$\left|\int_0^\infty [\cosh(t-s)]^{-(\alpha+\beta+1)} f(s)g(t)dsdt\right| \le 2^{\alpha+\beta}\|f\|_{L^2(-\infty,\infty)}\|g\|_{L^2(-\infty,\infty)}.$$

Proof. Consider the map $\mathbb{W} : L^2(0,\infty) \to L^2(-\infty,\infty)$ defined by $\mathbb{W}f(t) = \sqrt{2}e^t f(e^{2t})$. The operator $\mathbb{W}$ is an unitary operator. Let f be a continuous function with compact support in $(0,\infty)$ and $x = e^{2t}, y = e^{2s}$. Then

$$\begin{aligned}
(K_h f)(x) &= \int_0^\infty \frac{x^\alpha y^\beta f(y)dy}{(x+y)^{\alpha+\beta+1}} \\
&= \int_{-\infty}^\infty \frac{e^{2\alpha t}\cdot e^{2\beta s}\cdot f(e^{2s})\cdot 2e^{2s}ds}{(e^{2t}+e^{2s})^{\alpha+\beta+1}} \\
&= \frac{1}{\sqrt{2}}\int_{-\infty}^\infty \frac{\sqrt{2}e^s f(e^{2s})\cdot 2e^s\cdot e^{2\alpha t}\cdot e^{2\beta s}ds}{(e^{2t}+e^{2s})^{\alpha+\beta+1}} \\
&= \frac{1}{\sqrt{2}}\int_{-\infty}^\infty \frac{\mathbb{W}f(s)ds\cdot e^{(2\beta+1)s}\cdot 2e^{2\alpha t}}{(e^{2t}+e^{2s})^{\alpha+\beta+1}} \\
&= \frac{1}{\sqrt{2}}\cdot\frac{1}{2^{\alpha+\beta}}\cdot\frac{1}{e^t}\int_{-\infty}^\infty \frac{\mathbb{W}f(s)ds\cdot e^{(2\beta+1)s}\cdot 2e^{(2\alpha+1)t}\cdot 2^{\alpha+\beta+1}}{(e^{2t}+e^{2s})^{\alpha+\beta+1}} \\
&= \frac{1}{\sqrt{2}}\cdot\frac{1}{2^{\alpha+\beta}}\cdot\frac{1}{e^t}\int_{-\infty}^\infty \frac{\mathbb{W}f(s)\cdot e^{(2\beta+1)s}\cdot e^{\alpha-\beta}\cdot 2e^{(2\alpha+1)t}\cdot e^{\beta-\alpha}}{(e^{2t}+e^{2s})^{\alpha+\beta+1}} \\
&= \frac{1}{\sqrt{2}e^t}\cdot\frac{1}{2^{\alpha+\beta}}\int_{-\infty}^\infty \frac{\mathbb{W}f(s)ds}{\left(\frac{e^{2t}+e^{2s}}{2e^te^s}\right)^{\alpha+\beta+1}} \\
&= \frac{1}{\sqrt{2}e^t}\cdot\frac{1}{2^{\alpha+\beta}}\int_{-\infty}^\infty [\cosh(t-s)]^{-(\alpha+\beta+1)}\cdot\mathbb{W}f(s)ds \\
&= (\mathbb{W}^*C\mathbb{W}f)(t),
\end{aligned}$$

since if $g \in L^2(-\infty,\infty)$, then $\frac{g(t)}{\sqrt{2}e^t} = \frac{1}{\sqrt{2x}}g(\frac{1}{2}\log x) = \mathbb{W}^*g(x)$. Thus $K_h = \mathbb{W}^*C\mathbb{W}$, where C is the convolution with $(\cosh t)^{-(\alpha+\beta+1)}$. That is,

$$(Cf)(t) = \frac{1}{2^{\alpha+\beta}}\int_{-\infty}^\infty [\cosh(t-s)]^{-(\alpha+\beta+1)}f(s)ds.$$

Since K_h and C are unitarily equivalent, hence $\|C\| = 1$ and

$$|\langle Cf, g\rangle| \le \|f\|_{L^2(-\infty,\infty)}\|g\|_{L^2(-\infty,\infty)}.$$

Thus $\left|\int_0^\infty [\cosh(t-s)]^{-(\alpha+\beta+1)}f(s)g(t)dsdt\right| \le 2^{\alpha+\beta}\|f\|_{L^2(-\infty,\infty)}\|g\|_{L^2(-\infty,\infty)}$. □

For $\alpha,\beta > 0$, Aleksandrov and Peller [1] studied the integral operator

$$(\mathfrak{I}_h^{\alpha,\beta}f)(x) = \int_0^\infty h(x^\alpha + y^\beta)f(y)dy. \tag{3.19}$$

Clearly, if h is a locally integrable function on $(0,\infty)$, the right hand side of (3.19) is well defined for smooth functions f with compact support in $(0,\infty)$. The integral on the right hand side of (3.19) also makes sense if h is an infinitely differentiable function with compact support in $(0,\infty)$. Integral operator $\mathfrak{I}_h^{\alpha,\beta}$ are called distorted Hankel integral operators. These operators are studied in detail in Aleksandrov and Peller [1].

For $\alpha = \beta = 1$, the operator $\mathfrak{I}_h^{\alpha,\beta}$ coincides with the Hankel integral operator $\tilde{\tilde{\mathcal{K}}}_h$, where $\tilde{\tilde{\mathcal{K}}}_h : L^2(0,\infty) \longrightarrow L^2(0,\infty)$ is defined as $(\tilde{\tilde{\mathcal{K}}}_h f)(x) = \int_0^\infty h(x+y)f(y)dy$. For a locally integrable function h on $(0,\infty)$, the weighted Hankel integral operator $K_h^{\alpha,\beta}$ is defined by

$$(K_h^{\alpha,\beta} f)(x) = \int_0^\infty x^\alpha y^\beta h(x+y)f(y)dy,$$

where $h(x+y) = \frac{e^{-(x+y)}}{(x+y)^2}$ for smooth functions f with compact support in $(0,\infty)$. The operator $K_h^{\alpha,\beta}$ are analogous of weighted Hankel matrices form $\{(j+1)^\alpha(k+1)^\beta\widehat{\psi}(j+k)\}_{j,k\geq 0}$, where Ψ is a function analytic in the unit disk. For $\alpha = \beta = 0$, the operator $K_h^{\alpha,\beta} = \tilde{\tilde{\mathcal{K}}}_h$. Let $\alpha,\beta > 0$. We introduce the unitary operator $\mathbb{A}_\alpha$ on $L^2(0,\infty)$ defined by

$$(\mathbb{A}_\alpha f)(x) = \frac{1}{\sqrt{\alpha}} x^{\frac{1}{2\alpha}-\frac{1}{2}} f(x^{\frac{1}{\alpha}}), f \in L^2(0,\infty).$$

Suppose h is a locally integrable function on $(0,\infty)$. Then

$$K_h^{\frac{1}{2\alpha}-\frac{1}{2},\frac{1}{2\beta}-\frac{1}{2}} = \sqrt{\alpha\beta}\, \mathbb{A}_\alpha \mathfrak{I}_h^{\alpha,\beta} \mathbb{A}_\beta^*.$$

This can be verified as follows: Observe that $(\mathbb{A}_\beta^* f)(x) = \sqrt{\beta}\, x^{\frac{\beta}{2}-\frac{1}{2}} f(x^\beta)$. Hence

$$\begin{aligned}
(\mathbb{A}_\alpha \mathfrak{I}_h^{\alpha,\beta} \mathbb{A}_\beta^* f)(x) &= \mathbb{A}_\alpha \mathfrak{I}_h^{\alpha,\beta} \sqrt{\beta}\, x^{\frac{\beta}{2}-\frac{1}{2}} f(x^\beta) \\
&= \sqrt{\beta}\, \mathbb{A}_\alpha \left(\int_0^\infty h(x^\alpha + y^\beta) y^{\frac{\beta}{2}-\frac{1}{2}} f(y^\beta) dy \right) \\
&= \frac{\sqrt{\beta}}{\sqrt{\alpha}} x^{\frac{1}{2\alpha}-\frac{1}{2}} \int_0^\infty h(x + y^\beta) y^{\frac{\beta}{2}-\frac{1}{2}} f(y^\beta) dy \\
&= \frac{\sqrt{\beta}}{\sqrt{\alpha}} \frac{1}{\beta} \int_0^\infty x^{\frac{1}{2\alpha}-\frac{1}{2}} z^{\frac{1}{\beta}(\frac{\beta}{2}-\frac{1}{2})} h(x+z) f(z) z^{\frac{1}{\beta}-1} dz \\
&= \frac{1}{\sqrt{\alpha\beta}} \int_0^\infty x^{\frac{1}{2\alpha}-\frac{1}{2}} z^{\frac{1}{2\beta}-\frac{1}{2}} h(x+z) f(z) dz \\
&= \frac{1}{\sqrt{\alpha\beta}} \left(K_h^{\frac{1}{2\alpha}-\frac{1}{2},\frac{1}{2\beta}-\frac{1}{2}} \right)(x).
\end{aligned}$$

As a result of this it is not difficult to find the norm of a weighted Hankel integral operator if we can calculate the norm of the corresponding distorted Hankel operator and vice versa.

4 Norm of the Bergman Hilbert matrix

Let $\mathcal{H}^2(U)$ be the Hardy space of functions which are holomorphic in the upper half palne U and for which

$$\|f\|^2_{\mathcal{H}^2(U)} = \sup_{y>0} \int_{-\infty}^\infty |f(x+iy)|^2 dx < \infty.$$

For $0 < p < \infty$ and $\alpha > -1$, let $A^{p\alpha}$ be the Bergman space of functions f which are holomorphic in U and which satisfy

$$\|f\|_{p\alpha}^p = \int_U |f(x+iy)|^p y^\alpha dxdy < \infty.$$

We define integration of arbitrary order using the Fourier transform. For any complex number w with $Re(w) > 0$ and function f in any of the $A^{p\alpha}$ we define the integral of f of order $w, \mathcal{I}^w f$, by

$$(\mathcal{I}^w f)\widehat{}(t) = t^{-w}\widehat{f}(t).$$

Here $\widehat{f}$ is the Fourier transform of the distributional boundary values $\lim_{y\to 0} f(x+iy)$. These operators have the expected action on basic building blocks. That is,

$$\mathcal{I}^w((z-\overline{\zeta})^{-a}) = c(z-\overline{\zeta})^{-a+w},$$

where c is a constant. We define the general differentiation operators D^w by $D^w = \mathcal{I}^{-w}$. Rochberg [17] studied the Schatten class properties of weighted Hankel integral operators for complex α,β. He showed that the operator $K_b^{\alpha,\beta}$ acting on functions defined on $(0,\infty)$ by

$$(K_b^{\alpha,\beta} f)(x) = \int_0^\infty \frac{s^\alpha t^\beta}{(s+t)^{\alpha+\beta}} \overline{\overline{b}}(s+t) f(t) dt$$

is equal to $D^\alpha \mathcal{H}_c D^\beta$ with $D^{\alpha+\beta} c = b$ and $\mathcal{H}_c$ is the Hankel operator defined on $\mathcal{H}^2(U)$ by $\mathcal{H}_c f = Q(\overline{c} f)$ and Q is the orthogonal projection from $L^2(\mathbb{R}, dx)$ onto $\overline{\mathcal{H}^2(U)} = \left\{\overline{f} : f \in \mathcal{H}^2(U)\right\}$. Alternatively, these operators $K_b^{\alpha,\beta}$ can be regarded as Hankel type operators on the Bergman space $A^{p\alpha}$. Fractional integration gives a unitary equivalence of $A^{p\alpha}$ and $\mathcal{H}^2(U)$ and hence can be used to pull these operators over to $\mathcal{H}^2(U)$. When this is done (by straight forward Fourier transform calculation) the resulting operators are of the form $K_b^{\alpha,\beta}$. For $g \in L^1 \cap L^2$, Partington [15] has shown that the integral operator

$$(\widetilde{\widetilde{\mathcal{K}}}_g f)(x) = \int_0^\infty g(x+y) f(y) dy$$

on $L^2(0,\infty)$ is unitarily equivalent to the Hankel operator $\widetilde{\Gamma}_G$ defined on $\mathcal{H}^2(\mathbb{C}_+)$ where $G = \mathcal{L}g$ and $\widetilde{\Gamma}_G$ is unitarily equivalent to the Hankel operator Γ_ϕ defined on $\mathcal{H}^2(\mathbb{D})$, where $\phi(z) = \frac{G(Mz)}{z}$.

In this paper we establish that for $\alpha,\beta > -1$ the integral operator

$$(K_g f)(x) = \int_0^\infty \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta}} g(x+y) f(y) dy$$

defined on $L^2(0,\infty)$ is unitarily equivalent to the little Hankel operator $\widetilde{\Gamma}_G$ defined from $L_a^{2,\alpha}(\mathbb{C}_+)$ into $L_a^{2,\beta}(\mathbb{C}_+)$ where $G = \mathcal{L}\left(t^{\frac{\beta-\alpha}{2}} g\right)$ and $\widetilde{\Gamma}_G$ is unitarily equivalent to the little Hankel operator Γ_ϕ defined from $L_a^{2,\alpha}(\mathbb{D})$ into $L_a^{2,\beta}(\mathbb{D})$ where $\phi(z) = \left(\frac{1+\overline{z}}{1+z}\right)^{\alpha+2} G(Mz)$. From Theorem 2.2 and Theorem 2.3, it follows that for $h \in L^1 \cap L^2$, the integral operators $K_h^{\alpha,\beta}, \alpha,\beta >$

-1 on $L^2(0,\infty)$ are unitarily equivalent to little Hankel operators $\Gamma_\phi = S_{z\phi}$ defined from the weighted Bergman space $L_a^{2,\alpha}(\mathbb{D})$ into $L_a^{2,\beta}(\mathbb{D})$. For $\phi \in \overline{\mathcal{H}^\infty(\mathbb{D})}, \phi(z) = \sum_{n=0}^{\infty} \widehat{\phi}(-n)\bar{z}^n$, the matrix of S_ϕ with respect to the orthonormal basis $\{e_n(z)\}_{n=0}^{\infty} = \left\{\sqrt{n+1}z^n\right\}_{n=0}^{\infty}$ of $L_a^2(\mathbb{D})$ is given by

$$\langle S_\phi e_j, e_i\rangle = \frac{\sqrt{i+1}\sqrt{j+1}}{i+j+1}\widehat{\phi}(-(i+j)),\ i,j \geq 0.$$

Thus $S_\phi = D_2 B_{\widetilde{\psi}} D_2$, where $\widetilde{\psi}(e^{i\theta}) = \sum_{k=0}^{\infty} \frac{1}{k+1}\widehat{\phi}(-k)e^{-ik\theta} = \widetilde{\phi} * \widetilde{\phi}_1$. The function $\widetilde{\psi}$ is the convolution on the circle of $\widetilde{\phi} = \sum_{k=0}^{\infty} \widehat{\phi}(-k)e^{-ik\theta}$ (the boundary value function of ϕ) with the function $\widetilde{\phi}_1(e^{i\theta}) = \sum_{k=0}^{\infty} \frac{1}{k+1}e^{-ik\theta}$, $B_{\widetilde{\psi}}$ is the operator on $L_a^2(\mathbb{D})$ having a classical Hankel matrix with respect to the standard orthonormal basis of $L_a^2(\mathbb{D})$ with symbol $\widetilde{\psi}$ and $D_2 e_j = \sqrt{j+1}e_j$ for all $j \geq 0$. Hence

$$\langle S_{z\phi} e_j, e_i\rangle = \frac{\sqrt{i+1}\sqrt{j+1}}{i+j+2}\widehat{\phi}(-(i+j+1)), i,j \geq 0.$$

For example, if we take $\widetilde{\phi}(e^{i\theta}) = -i(\pi-\theta), 0 \leq \theta < 2\pi$. Then $\widetilde{\phi} \in L^\infty(\mathbb{T})$, where $\mathbb{T}$ be the unit circle and if

$$\widetilde{\phi}(e^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta},$$

then

$$a_n = \begin{cases} 0 & \text{if } n = 0; \\ -\frac{1}{n} & \text{if } n \neq 0, \end{cases}$$

and the matrix of $\mathcal{S}_{e^{i\theta}\widetilde{\phi}}$ with respect to the orthonormal basis of $\mathcal{H}^2(\mathbb{T})$ is the Hilbert matrix $\Gamma = \left[\frac{1}{i+j+1}\right]_{i,j=0}^{\infty}$. Let $\phi_2 = z\phi$ be the harmonic extension of $e^{i\theta}\,\widetilde{\phi}$ into $\mathbb{D}$. That is, $\widetilde{\phi}_2 = e^{i\theta}\widetilde{\phi}$(the boundary value function of ϕ_2). Notice that the matrix of the little Hankel operator $S_{z\phi}$ with respect to the standard orthonormal basis of $L_a^2(\mathbb{D})$ is equal to

$$A = [a_{ij}] = \langle D_2 B_{e^{i\theta}\widetilde{\phi} * \widetilde{\phi}_1} D_2 e_j, e_i\rangle = \frac{\sqrt{i+1}\sqrt{j+1}}{(i+j+1)^2},\ i,j \geq 0$$

which is called the Bergman Hilbert matrix. Thus A is the Schur multiplication of the matrices $[m_{ij}]$ and the Hilbert matrix $\Gamma = \left[\frac{1}{i+j+1}\right]$. Let $B = [b_{ij}]$, where $b_{ij} = \frac{\sqrt{i+1}\sqrt{j+1}}{(i+j+2)^2}$. The matrix B is called the homogeneous companion of A. Notice that $a_{ij} = m_{ij}\frac{1}{i+j+1}$ and $0 < m_{ij} \leq 1$ for all i and j. Since $\|\Gamma\| = \pi$ (see [4]), hence $\|A\| \leq \|\Gamma\|$. It is not difficult to see that the Hilbert matrix Γ as an operator on $l^2(\mathbb{Z}_+)$ is unitarily equivalent to the integral operator

$$(\widetilde{\widetilde{\mathcal{K}}}_{\widetilde{h}} f)(x) = \int_0^\infty \widetilde{h}(x+y)f(y)dy,\ f \in L^2(0,\infty),$$

where $\widetilde{h}(x) = \frac{e^{-x}}{x}$. On the other hand, the Carleman's operator on $L^2(0,\infty)$ given by

$$(\mathfrak{G}_h f)(x) = \int_0^\infty h(x+y)f(y)dy,$$

where $h(x) = \frac{1}{x}$ and the operator $\mathfrak{G}_h$ is unitarily equivalent to the Hankel operator H defined on $\mathcal{H}^2(\mathbb{T})$ whose matrix representation with respect to the standard orthonormal basis is

$$\mathcal{S} = 2\begin{pmatrix} 1 & 0 & \frac{1}{3} & 0 & \frac{1}{5} & \cdots \\ 0 & \frac{1}{3} & 0 & \frac{1}{5} & \cdots & \cdots \\ \frac{1}{3} & 0 & \frac{1}{5} & \cdots & \cdots & \cdots \\ 0 & \frac{1}{5} & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{5} & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Let $\mathbb{M}$ denotes the Mellin transform on $L^2(0,\infty)$ defined by

$$\mathbb{M}_f(s) = \int_0^\infty x^{s-1} f(x)dx.$$

and

$$(Ef)(x) = \int_0^\infty \frac{\sqrt{x}\sqrt{y}}{(x+y)^2} f(y)dy,$$

for $f \in L^2(0,\infty)$. It is easy to see that $\mathbb{M}_{Ef}(s) = m(s)\mathbb{M}_f(s)$. This can be verified as follows: Notice that

$$\begin{aligned}\mathbb{M}_{Ef}(s) &= \int_0^\infty x^{s-1}(Ef)(x)dx \\ &= \int_0^\infty \int_0^\infty x^{s-1} \frac{\sqrt{x}\sqrt{y}}{(x+y)^2} f(y)dydx \\ &= \int_0^\infty \frac{x^{s-\frac{1}{2}}}{(1+x)^2}dx \int_0^\infty y^{s-1} f(y)dy.\end{aligned}$$

Thus $\mathbb{M}_{Ef}(s) = m(s)\mathbb{M}_f(s)$, where

$$\begin{aligned}m(s) &= \int_0^\infty \frac{x^{s-\frac{1}{2}}}{(x+1)^2}dx \\ &= \left(\frac{1}{2} - s\right)\pi\, cosec\, \pi\left(s - \frac{1}{2}\right).\end{aligned}$$

Hence

$$\begin{aligned}\sigma(E) &= \text{ closure of range}\left\{m\left(\frac{1}{2} + it\right) : t \in \mathbb{R}\right\} \\ &= \overline{Range}\{t cosech\, t : t \in (0,\infty)\} = [0,1].\end{aligned}$$

The operator B is not unitarily equivalent to the integral operator E and the kernel $\frac{\sqrt{x}\sqrt{y}}{(x+y)^2}$ is not a decreasing function in either variable.

Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the set of all bounded linear operators from the Hilbert space $\mathcal{H}$ into itself and $\mathcal{LC}(\mathcal{H})$ denote the set of all compact operators in $\mathcal{L}(\mathcal{H})$. Let $T \in \mathcal{L}(\mathcal{H})$. A maximizing vector for T is a non-zero vector $x \in \mathcal{H}$ such that $\|Tx\| = \|T\|\,\|x\|$. Thus a maximizing vector for T is one at which T attains its norm. On a Banach space, even rank 1 operators need not have maximizing vectors. The operator $Mx(t) = tx(t), 0 < t < 1$, is bounded on $L^2(0,1)$ but has no maximizing vector. However, compact operators on Hilbert spaces do have maximizing vectors.

Suppose $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ and $\sigma(T)$ denote the spectrum of T. To determine $\|T\|$, one may investigate the spectrum of the operator T^*T. Since T^*T is self-adjoint, its spectral radius equals $\|T^*T\| = \|T\|^2$. We define the essential norm of $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ denoted by $\|T\|_e$ as

$$\|T\|_e = \inf\{\|T - K\| : K \in \mathcal{LC}(L^2_a(\mathbb{D}))\}.$$

The essential spectrum of T (denoted by $\sigma_e(T)$) is defined to be the spectrum of the element $T + \mathcal{LC}(L^2_a(\mathbb{D}))$ in $\mathcal{L}(L^2_a(\mathbb{D}))/\mathcal{LC}(L^2_a(\mathbb{D}))$. The essential spectral radius of T, which we write $r_\sigma(T) = \sup\{|\lambda| : \lambda \in \sigma_e(T)\}$. If T is self-adjoint, $\sigma_e(T)$ consists of limit points of $\sigma(T)$ or eigenvalues of infinite multiplicity and $\sigma(T)\backslash\sigma_e(T)$ consists of isolated eigenvalues of finite multiplicity. Further, $\|T\| = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ and $\|T\|_e = \sup\{|\lambda| : \lambda \in \sigma_e(T)\}$. It is not difficult to see that $\sigma_e(T) \subseteq \sigma(T)$. Whenever T is a normal operator, any point in the spectrum of T that does not belong to $\sigma_e(T)$ must be an eigenvalue of finite multiplicity. It is not difficult to show that $\|T^*T\|_e = \|T\|_e^2$ for any bounded operator T. Hence $r_\sigma(T) = \|T\|_e$ whenever T is self-adjoint. Similarly, the spectral radius of $T = r(T) = \|T\|$, if T is a self-adjoint operator.

Lemma 4.1. *Let $T \in \mathcal{L}(L^2_a(\mathbb{D}))$. The function $f \in L^2_a(\mathbb{D})$ is a maximizing vector for T if and only if $T^*Tf = \|T\|^2 f$.*

Proof. Suppose $T^*Tf = \|T\|^2 f$. Then

$$\begin{aligned}\|Tf\|^2 &= \langle Tf, Tf\rangle \\ &= \langle T^*Tf, f\rangle \\ &= \langle \|T\|^2 f, f\rangle \\ &= \|T\|^2\|f\|^2.\end{aligned}$$

Hence $\|Tf\| = \|T\|\,\|f\|$ and f is maximizing vector for T.
Conversely, suppose that $\|Tf\| = \|T\|\,\|f\|$. Then

$$\begin{aligned}\|T\|^2\|f\|^2 &= \|Tf\|^2 \\ &= \langle Tf, Tf\rangle \\ &= \langle T^*Tf, f\rangle \\ &\leq \|T^*Tf\|\,\|f\| \\ &\leq \|T\|^2\|f\|^2.\end{aligned}$$

Thus T^*Tf is a scalar multiple of f and in fact $\|T^*Tf\| = \|T\|^2\|f\|$ and since T^*T is a positive operator, we obtain $T^*Tf = \|T\|^2 f$. □

Proposition 4.2. *If $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ and $\|T\|_e < \|T\|$ then T has a maximizing vector.*

Proof. Consider the positive operator T^*T. Notice that

$$r_\sigma(T^*T) = \|T^*T\|_e = \|T\|_e^2 < \|T\|^2 = \|T^*T\| = r(T^*T).$$

Therefore $\|T\|^2$, the largest element of the spectrum of T^*T, does not belong to the essential spectrum. Since any self adjoint operator is normal, $\|T\|^2$ must be an eigenvalue of finite multiplicity. Consequently, T^*T has an eigenvector corresponding to $\|T\|^2$ on which the operator T attains its norm. □

Lemma 4.3. *Let $R = \left(r_{ij}\right)_{i,j=0}^{\infty}$, is self-adjoint, $r_{ij} > 0$ and $\sum_{j=0}^{\infty} r_{ij} p_j \le M p_i$ for all $i = 0, 1, 2, \cdots$. Then $Rf = Mf, f \in L^2_a(\mathbb{D})$, implies $\langle f, e_j\rangle = kp_j, j = 0, 1, 2\cdots$. for some constant k.*

Proof. Let $f_j = \langle f, e_j\rangle, j = 0, 1, 2, \cdots$. Then

$$\begin{aligned}
\sum_{i=0}^{\infty}\left|\sum_{j=0}^{\infty} r_{ij} f_j\right|^2 &= \sum_{i=0}^{\infty}\left|\sum_{j=0}^{\infty} \sqrt{r_{ij}}\sqrt{p_j}\sqrt{r_{ij}}\frac{f_j}{\sqrt{p_j}}\right|^2 \\
&\le \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} r_{ij}p_j\right)\left(\sum_{j=0}^{\infty}\frac{r_{ij}|f_j|^2}{p_j}\right) \\
&\le \sum_{i=0}^{\infty} M p_i \sum_{j=0}^{\infty} r_{ij}\frac{|f_j|^2}{p_j} \\
&\le M\left(\sum_{j=0}^{\infty}\frac{|f_j|^2}{p_j}\right)\left(\sum_{i=0}^{\infty} r_{ij}p_i\right) \\
&\le M^2\left(\sum_{j=0}^{\infty}|f_j|^2\right).
\end{aligned}$$

Now $\|Rf\| = M\|f\|$ implies $\sum_{j=0}^{\infty}\left(\sqrt{r_{ij}}\sqrt{p_j}\right)\left(\sqrt{r_{ij}}\frac{f_j}{\sqrt{p_j}}\right) = \left(\sum_{j=0}^{\infty} r_{ij}p_j\right)^{\frac{1}{2}}\left(\sum_{j=0}^{\infty}\frac{r_{ij}|f_j|^2}{p_j}\right)^{\frac{1}{2}}$. That is, equality holds in the Cauchy-Schwarz inequality. Hence $f_j = kp_j$ for all $j = 0, 1, 2, \cdots$ and for some constant k. □

Lemma 4.4. *The following hold: (i) $\|A\| < \frac{\pi^2}{6}$ (ii) $\|B\| = 1$. (iii) The norm $\|A\|$ is an isolated eigenvalue of A of finite multiplicity. (iv) The operator A as an operator from l^2 into l^2 has a maximizing vector.*

Proof. To prove (i), let $p_i = q_i = \frac{1}{\sqrt{i+1}}$. Applying Schur test (see [3], p. 30), we obtain

$$\sum_{i=0}^{\infty} a_{ij} p_i = \sqrt{j+1}\sum_{i=0}^{\infty}\frac{1}{(i+j+1)^2}$$

and

$$\sum_{i=0}^{\infty} b_{ij} p_i = \sqrt{i+1}\sum_{i=0}^{\infty}\frac{1}{(i+j+2)^2}.$$

Since $\frac{1}{r^{1-p}}\sum_{k\geq r}\frac{1}{k^p}$ is a strictly decreasing function of r, we obtain

$$(j+1)\sum_{i=0}^{\infty}\frac{1}{(i+j+1)^2}\leq\sum_{i=0}^{\infty}\frac{1}{(i+1)^2}=\frac{\pi^2}{6}.$$

Thus it follows that $\sum_{i=0}^{\infty}a_{ij}p_i\leq\left(\frac{\pi^2}{6}\right)p_j$. By symmetry,

$$\sum_{j=0}^{\infty}a_{ij}p_j\leq\left(\frac{\pi^2}{6}\right)p_i \textit{ and } \|A\|\leq\frac{\pi^2}{6}.$$

Further, since $\sum_{i=0}^{\infty}\frac{1}{(i+j+2)^2}\leq\frac{1}{j+1}$, we obtain

$$\sqrt{j+1}\sum_{i=0}^{\infty}\frac{1}{(i+j+2)^2}\leq\frac{1}{\sqrt{j+1}}.$$

Hence $\|B\|\leq 1$. Now let $K(x,y)=\frac{\sqrt{x}\sqrt{y}}{(x+y)^2}$. The kernel K satisfies the hypothesis of Theorem 318 of [10] with $p=2$ and

$$K=\int_0^{\infty}K(x,1)x^{-\frac{1}{2}}dx=\int_0^{\infty}\frac{1}{(1+x)^2}dx=1.$$

Using [10] one can show that $\|B\|\geq 1$. Therefore $\|B\|=1$. This proves (ii).
Since $a_{00}=1$, we have $\|A\|>1$. Let $C=[c_{ij}]$, where $c_{ij}=a_{ij}-b_{ij}$. Thus $c_{ij}=\frac{\sqrt{i+1}\sqrt{j+1}}{i+j+1}\frac{2(i+j)+3}{(i+j+1)(i+j+2)}$. Since $\sum_{i,j=0}^{\infty}c_{ij}^2<\infty$, the matrix C is Hilbert-Schmidt.That is, B is a compact perturbation of A. It is also not difficult to see that $\|A\|_e=\|B\|_e=1$. To verify this, suppose $\|B\|_e<\|B\|=1$. Then it follows that 1 is an eigenvalue of B. Now, since $\sum_{j=0}^{\infty}p_j^2=\sum_{j=0}^{\infty}\frac{1}{j+1}$ is divergent, it follows from Lemma 4.3 that this is impossible. Thus $\|A\|_e=\|B\|_e=1$ and $\|A-C\|=\|A\|_e$, giving the best compact approximant of A. We also have $1=\|A\|_e<\|A\|$ and hence there are points in $\sigma(A)$ which do not belong to $\sigma_e(A)$. In particular, $\|A\|$ is such a point. Since A is self-adjoint, all these points are eigenvalues of A. It follows from Proposition 4.2 that the operator A has a maximizing vector and $\|A\|$ is an isolated eigenvalue of finite multiplicity. This proves (iii) and (iv). It follows by Lemma 4.3 that $\frac{\pi^2}{6}$ cannot be an eigenvalue and hence $\|A\|<\frac{\pi^2}{6}$. This proves (i). □

Remark 4.5. The matrix B as an operator on l^2 is self-adjoint, positive, $\sigma(B)=\sigma_e(B)=\sigma_e(A)=[0,1]$ and B dose not have isolated eigenvalues of finite multiplicity in $[0,1]$.

In general, one can consider the generalized companion matrices $\left(\frac{m^{\alpha}n^{\beta}}{(m+n)^{\alpha+\beta+1}}\right)_{m,n=1}^{\infty}$ of the weighted Bergman Hilbert matrices $\frac{m^{\alpha}n^{\beta}}{(m+n-1)^{\alpha+\beta+1}}$. In the following theorem, we establish that the norm of the matrix $\left(\frac{m^\alpha n^\beta}{(m+n)^{\alpha+\beta+1}}\right)_{m,n=1}^{\infty}$ as an operator from l^2 into itself is $B\left(\alpha+\frac{1}{2},\beta+\frac{1}{2}\right)$ where $-\frac{1}{2}<\alpha,\beta\le\frac{1}{2}$. In fact, we prove a more general result.

Theorem 4.6. *Let* $p>1, \frac{1}{p}+\frac{1}{q}=1, -\frac{1}{q}<\alpha\le\frac{1}{p}, -\frac{1}{p}<\beta\le\frac{1}{q}$. *If* $a_m, b_n\ge 0, m,n=1,2,3,\cdots$ *satisfy* $0<\sum_{m=1}^{\infty}a_m^p<\infty$ *and* $0<\sum_{n=1}^{\infty}b_n^q<\infty$, *then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^\alpha n^\beta}{(m+n)^{\alpha+\beta+1}}a_m b_n < B\left(\alpha+\frac{1}{q},\beta+\frac{1}{p}\right)\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}, \tag{4.1}$$

where the constant factor $B\left(\alpha+\frac{1}{q},\beta+\frac{1}{p}\right)$ *is the best possible. In particular*

i) for $\alpha=\frac{1}{p}$ *and* $\beta=\frac{1}{q}$, *we have*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^{\frac{1}{p}}n^{\frac{1}{q}}}{(m+n)^2}a_m b_n < \left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}; \tag{4.2}$$

ii) for $\alpha=\beta=\frac{1}{2}$ *and* $p=q=2$, *we have*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sqrt{m}\sqrt{n}}{(m+n)^2}a_m b_n < \left(\sum_{m=1}^{\infty}a_m^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}b_n^2\right)^{\frac{1}{2}}. \tag{4.3}$$

Proof. Rearranging the terms and using Hölder's inequality, we obtain

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^\alpha n^\beta}{(m+n)^{\alpha+\beta+1}}a_m b_n$$

$$\le\left(\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\frac{m^\alpha n^\beta}{(m+n)^{\alpha+\beta+1}}\left(\frac{m}{n}\right)^{\frac{1}{q}}\right]a_m^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\frac{m^\alpha n^\beta}{(m+n)^{\alpha+\beta+1}}\left(\frac{n}{m}\right)^{\frac{1}{p}}\right]b_n^q\right)^{\frac{1}{q}}. \tag{4.4}$$

For $\beta\le\frac{1}{q}$, using (3.8) we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{m^\alpha n^\beta}{(m+n)^{\alpha+\beta+1}}\left(\frac{m}{n}\right)^{\frac{1}{q}} &= m^{\alpha+\frac{1}{q}}\sum_{n=1}^{\infty}\frac{1}{(m+n)^{\alpha+\beta+1}}\cdot\frac{1}{n^{\frac{1}{q}-\beta}}\\ &< m^{\alpha+\frac{1}{q}}\sum_{n=1}^{\infty}\int_{n-1}^{n}\frac{1}{(m+t)^{\alpha+\beta+1}}\cdot\frac{1}{t^{\frac{1}{q}-\beta}}dt\\ &= \int_0^{\infty}\frac{m^{\alpha+\frac{1}{q}}t^{\beta-\frac{1}{q}}}{(m+t)^{\alpha+\beta+1}}dt\\ &= B\left(\alpha+\frac{1}{q},\beta+\frac{1}{p}\right).\end{aligned}$$

Similarly for $\alpha \le \frac{1}{p}$, using (3.9) we obtain

$$\sum_{m=1}^{\infty} \frac{m^{\alpha} n^{\beta}}{(m+n)^{\alpha+\beta+1}} \left(\frac{n}{m}\right)^{\frac{1}{p}} < B\left(\alpha+\frac{1}{q},\beta+\frac{1}{p}\right).$$

Hence (4.1) follows from (4.4). For the best constant factor, we take for $0 < \epsilon < q(\beta + 1/p)$,

$$\widetilde{a}_m = m^{-\frac{1+\epsilon}{p}} \quad (m \ge 1)$$

and

$$\widetilde{b}_n = n^{-\frac{1+\epsilon}{q}} \quad (n \ge 1).$$

Then

$$\sum_{m=1}^{\infty} \widetilde{a}_m^p = 1 + \sum_{m=1}^{\infty} \frac{1}{m^{1+\epsilon}} < 1 + \int_1^{\infty} x^{-1-\epsilon} dx = 1 + \frac{1}{\epsilon}.$$

Similarly

$$\sum_{n=1}^{\infty} \widetilde{b}_n^q < 1 + \frac{1}{\epsilon}.$$

Hence

$$\left(\sum_{m=1}^{\infty} \widetilde{a}_m^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \widetilde{b}_n^q\right)^{\frac{1}{q}} < 1 + \frac{1}{\epsilon}. \tag{4.5}$$

Again by (3.11), we have

$$\begin{aligned} &\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m^{\alpha} n^{\beta}}{(m+n)^{\alpha+\beta+1}} \widetilde{a}_m \widetilde{b}_n \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(m+n)^{\alpha+\beta+1}} \cdot \frac{1}{m^{\frac{\epsilon}{p}+\frac{1}{p}-\alpha}} \cdot \frac{1}{n^{\frac{\epsilon}{q}+\frac{1}{q}-\beta}} \end{aligned}$$

$$> \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \int_m^{m+1} \int_n^{n+1} \frac{1}{(x+y)^{\alpha+\beta+1}} \cdot \frac{1}{x^{\frac{\epsilon}{p}+\frac{1}{p}-\alpha}} \cdot \frac{1}{y^{\frac{\epsilon}{q}+\frac{1}{q}-\beta}} dxdy \tag{4.6}$$

$$\begin{aligned} &= \int_1^{\infty}\int_1^{\infty} \frac{x^{\alpha} y^{\beta}}{(x+y)^{\alpha+\beta+1}} \cdot x^{-\frac{1+\epsilon}{p}} \cdot y^{-\frac{1+\epsilon}{q}} dxdy \\ &\ge \frac{1}{\epsilon} B\left(\alpha+\frac{1}{q}+\frac{\epsilon}{q},\beta+\frac{1}{p}-\frac{\epsilon}{q}\right) - \bigcirc(1). \end{aligned}$$

If the constant factor $B\left(\alpha+\frac{1}{q},\beta+\frac{1}{p}\right)$ in (4.1) is not the best possible, then there exists a positive constant $C < B\left(\alpha+\frac{1}{q},\beta+\frac{1}{p}\right)$, such that (4.1) is still valid if we replace $B\left(\alpha+\frac{1}{q},\beta+\frac{1}{p}\right)$

by C. In particular, by (4.5) and (4.6), we have

$$\begin{aligned} & B\left(\alpha+\frac{1}{q}+\frac{\epsilon}{q},\beta+\frac{1}{p}-\frac{\epsilon}{q}\right)-\epsilon\bigcirc(1) \\ & < \epsilon\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^{\alpha}n^{\beta}}{(m+n)^{\alpha+\beta+1}}\widetilde{a}_m\widetilde{b}_n \\ & < \epsilon C\left(\sum_{m=1}^{\infty}\widetilde{a}_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\widetilde{b}_n^q\right)^{\frac{1}{q}} \\ & < (\epsilon+1)C. \end{aligned}$$

Hence $B\left(\alpha+\frac{1}{q},\beta+\frac{1}{p}\right)\leq C$ as $\epsilon\to 0^{+}$. This contradiction leads to the conclusion that the constant factor in (4.1) is the best possible.

□

We shall refer the inequality (4.2) as Bergman-Hilbert inequality as it involves the companion matrix of the Bergman-Hilbert matrix.

5 Generalized Hilbert inequality for vector valued functions

In this section, we generalize the Bergman-Hilbert inequality (4.2) for vector-valued functions. Here we consider sequences (x_n) whose terms are elements of a separable Hilbert spaces $\mathcal{H}$ and such that $0<\sum_{n=o}^{\infty}\|x_n\|^2<\infty$. We observe that in the discrete case the inequality involves inner products and in the continuous case the inequality involves integral operator with matrix-valued kernels.

Theorem 5.1. *Let (x_n) and (y_n) be two sequences in the separable Hilbert space $\mathcal{H}$ such that $0<\sum_{0}^{\infty}\|x_n\|^2<\infty$ and $0<\sum_{0}^{\infty}\|y_n\|^2<\infty$. Then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sqrt{m+1}\,\sqrt{n+1}|\langle x_m,y_n\rangle|}{(m+n+2)^2}\leq\left\{\sum_{m=1}^{\infty}\|x_m\|^2\right\}^{\frac{1}{2}}\left\{\sum_{n=1}^{\infty}\|y_n\|^2\right\}^{\frac{1}{2}} \tag{5.1}$$

where the constant factor 1 *is the best possible.*

Proof. Let $\mathcal{H}\neq\{0\}$ be a Hilbert space and $\mathcal{E}$ be an orthonormal basis for $\mathcal{H}$. The set $\{e\in\mathcal{E}|\langle z,e\rangle\neq 0$ for some $z=x_m$ or $y_n\}$ is countable, let us enumerate this set as the sequence $\{\epsilon_1,\epsilon_2,\epsilon_3,\cdots\}$. Then every x_m and y_n can be expressed as

$$x_m=\sum_{k=1}^{\infty}a_{mk}\epsilon_k;\; y_n=\sum_{k=1}^{\infty}b_{nk}\epsilon_k,$$

where $a_{mk}=\langle x_m,\epsilon_k\rangle, b_{nk}=\langle y_n,\epsilon_k\rangle$. Then $\langle x_m,y_n\rangle=\sum_{k=1}^{\infty}a_{mk}\overline{b}_{nk}$. By Parseval's identity $\|x_m\|^2=\sum_{k=1}^{\infty}|a_{mk}|^2$, for every m and $\|y_n\|^2=\sum_{k=1}^{\infty}|b_{nk}|^2$, for every n. So we have $|a_{mk}|\leq\|x_m\|$ for all m

and $|b_{nk}| \leq \|y_n\|$ for all n. Hence for each k, $\sum_{m=1}^{\infty} |a_{mk}|^2 < \infty$ and $\sum_{n=1}^{\infty} |b_{nk}|^2 < \infty$. Now using Hilbert's inequality (3.1), we have for each k,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\sqrt{m+1}\,\sqrt{n+1}}{(m+n+2)^2}|a_{mk}||b_{nk}| < \left\{\sum_{m=1}^{\infty} |a_{mk}|^2\right\}^{\frac{1}{2}} \left\{\sum_{n=1}^{\infty} |b_{nk}|^2\right\}^{\frac{1}{2}}.$$

Taking summation over k from 1 to p and using Cauchy-Schwarz inequality, we get

$$\begin{aligned}\sum_{k=1}^{p}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\sqrt{m+1}\,\sqrt{n+1}}{(m+n+2)^2}|a_{mk}||b_{nk}| &< \left\{\sum_{k=1}^{p}\sum_{m=1}^{\infty} |a_{mk}|^2\right\}^{\frac{1}{2}} \left\{\sum_{k=1}^{p}\sum_{n=1}^{\infty} |b_{nk}|^2\right\}^{\frac{1}{2}} \\ &= \left\{\sum_{m=1}^{\infty}\sum_{k=1}^{p} |a_{mk}|^2\right\}^{\frac{1}{2}} \left\{\sum_{n=1}^{\infty}\sum_{k=1}^{p} |b_{nk}|^2\right\}^{\frac{1}{2}}.\end{aligned}$$

Thus for every $p \geq 1$,

$$\sum_{k=1}^{p}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\sqrt{m+1}\,\sqrt{n+1}}{(m+n+2)^2}|a_{mk}||b_{nk}| < \left\{\sum_{m=1}^{\infty} \|x_m\|^2\right\}^{\frac{1}{2}} \left\{\sum_{n=1}^{\infty} \|y_n\|^2\right\}^{\frac{1}{2}}. \tag{5.2}$$

Notice that

$$|\langle x_m, y_n\rangle| = \left|\sum_{k=1}^{\infty} a_{mk}\overline{b}_{nk}\right| \leq \sum_{k=1}^{\infty} |a_{mk}||b_{nk}|.$$

It follows from the relation $|a_{mk}||b_{nk}| \leq \frac{1}{2}(|a_{mk}|^2 + |b_{nk}|^2)$ and the convergence of the series $\sum_{k=1}^{\infty} |a_{mk}|^2$ and $\sum_{k=1}^{\infty} |b_{nk}|^2$. Hence letting $p \to \infty$ in (5.2), we obtain (5.1). In particular for the Hilbert space $\mathcal{H} = \mathbb{R}$, (5.1) reduces to the Hilbert's inequality (3.1). Since the constant factor 1 in (3.1) is the best possible, so we conclude that the constant factor 1 in (5.1) is the best possible. □

We shall now present the integral version of the inequality (5.1) and derive some related inequalities using tensor products.

Let $L^{2,\mathbb{C}^n}(\mathbb{D}, dA)$ denote the Hilbert space of $\mathbb{C}^n-$valued, norm-square integrable, measurable functions on $\mathbb{D}$ and $L_a^{2,\mathbb{C}^n}(\mathbb{D})$ the corresponding Bergman space. We notice that $L^{2,\mathbb{C}^n}(\mathbb{D}, dA) = L^2(\mathbb{D}, dA) \otimes \mathbb{C}^n$ and $L_a^{2,\mathbb{C}^n}(\mathbb{D}, dA) = L_a^2(\mathbb{D}, dA) \otimes \mathbb{C}^n$ where the Hilbert space tensor product is used. When endowed with the inner product defined by

$$\langle f, g\rangle_{L^{2,\mathbb{C}^n}(\mathbb{D},dA)} = \int_{\mathbb{D}} \langle f(z), g(z)\rangle_{\mathbb{C}^n} dA(z), \text{ for } f, g \in L^{2,\mathbb{C}^n}(\mathbb{D}, dA),$$

the spaces $L^{2,\mathbb{C}^n}(\mathbb{D}, dA)$ and $L_a^{2,\mathbb{C}^n}(\mathbb{D}, dA)$ become separable Hilbert spaces. Here the measures $dA(z)$ denotes the normalized area measure on $\mathbb{D}$. If Φ is a bounded, measurable $M_n = M_n(\mathbb{C})-$valued function (the algebra of $n \times n$ matrices with complex entries) in $L^{\infty}_{M_n}(\mathbb{D}) = L^{\infty}(\mathbb{D}) \otimes M_n$, then $\mathbb{S}_{\Phi}$ denotes the Hankel operator defined on $L_a^{2,\mathbb{C}^n}(\mathbb{D}, dA)$ by

$$\mathbb{S}_{\Phi} f = \widetilde{P}\widetilde{J}(\Phi f) \text{ for } f \in L_a^{2,\mathbb{C}^n}(\mathbb{D}, dA),$$

where $\widetilde{P}$ is the orthonormal projection of $L^{2,\mathbb{C}^n}(\mathbb{D},dA)$ onto $L_a^{2,\mathbb{C}^n}(\mathbb{D},dA)$ and $\widetilde{J}: L^{2,\mathbb{C}^n}(\mathbb{D},dA) \to L^{2,\mathbb{C}^n}(\mathbb{D},dA)$ is defined by $\widetilde{J}F(z) = F(\overline{z})$ and $(\Phi f)(z) = \Phi(z)f(z)$.
Let $\Phi \in L^\infty_{M_n}(\mathbb{D})$ and

$$\Phi = \begin{pmatrix} \phi_{11} & 0 & \cdots & 0 \\ 0 & \phi_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_{nn} \end{pmatrix}.$$

Then each entry ϕ_{ij} of Φ is in $L^\infty(\mathbb{D})$ and

$$\mathbb{S}_\Phi = \begin{pmatrix} S_{\phi_{11}} & 0 & \cdots & 0 \\ 0 & S_{\phi_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_{\phi_{nn}} \end{pmatrix}.$$

This is so as $L_a^{2,\mathbb{C}^n}(\mathbb{D},dA) = \underbrace{L_a^2(\mathbb{D}) \oplus L_a^2(\mathbb{D}) \oplus \cdots L_a^2(\mathbb{D})}_{n-times}$.

Let

$$\begin{aligned} L^{2,\mathbb{C}^n}(0,\infty) &= L^2(0,\infty) \otimes \mathbb{C}^n \\ &= L^2(0,\infty) \oplus L^2(0,\infty) \oplus \cdots \oplus L^2(0,\infty). \end{aligned}$$

For $F, G \in L^{2,\mathbb{C}^n}(0,\infty)$, the norm is defined by

$$\|F\|_{L^{2,\mathbb{C}^n}} = \left(\int_0^\infty \|F(x)\|^2_{\mathbb{C}^n} dx \right)^{\frac{1}{2}}$$

and the inner product is defined by

$$\langle F, G \rangle = \int_0^\infty \langle F(x), G(x) \rangle_{\mathbb{C}^n} dx.$$

With the above inner product $L^{2,\mathbb{C}^n}(0,\infty)$ is a Hilbert space. For details, see [2]. Let

$$H(x+y) = \begin{pmatrix} \frac{\sqrt{x}\sqrt{y}}{x+y}\frac{e^{-(x+y)}}{x+y} & 0 & \cdots & 0 \\ 0 & \frac{\sqrt{x}\sqrt{y}}{x+y}\frac{e^{-(x+y)}}{x+y} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\sqrt{x}\sqrt{y}}{x+y}\frac{e^{-(x+y)}}{x+y} \end{pmatrix}_{n\times n}.$$

Define $B_H : L^{2,\mathbb{C}^n}(0,\infty) \to L^{2,\mathbb{C}^n}(0,\infty)$ by

$$(B_H F)(x) = \int_0^\infty H(x+y)F(y)dy.$$

The map B_H is well-defined, linear and for $G \in L^{2,\mathbb{C}^n}(0,\infty)$,

$$\langle B_H F, G \rangle = \int_0^\infty \int_0^\infty G^*(x)H(x+y)F(y)dydx,$$

where $G^*(x)$ denotes the adjoint of $G(x)$. Notice that

$$B_H = \begin{pmatrix} K_{\widetilde{h}_{11}} & 0 & \cdots & 0 \\ 0 & K_{\widetilde{h}_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K_{\widetilde{h}_{nn}} \end{pmatrix}$$

where $(K_{\widetilde{h}}f)(x) = \int_0^\infty \frac{\sqrt{x}\sqrt{y}}{x+y}\widetilde{h}(x+y)f(y)dy, f \in L^2(0,\infty), \widetilde{h}(x) = \frac{e^{-x}}{x}$ and $\widetilde{h}_{ij}(x) = \widetilde{h}(x)$, for all $i, j = 1, 2, \cdots n$.

Lemma 5.2. *The operator $B_H : L^{2,\mathbb{C}^n}(0,\infty) \to L^{2,\mathbb{C}^n}(0,\infty)$ is a bounded linear operator and* $\|B_H\| = 1$.

Proof. Let $F = (f_1, f_2, \cdots f_n)^T$, where $f_i \in L^2(0,\infty)$ for all $i = 1, 2, \cdots, n$. Then $G = B_H F = (g_1, g_2, \cdots g_n)^T$ and $g_i \in L^2(0,\infty)$ for all $i = 1, 2, \cdots, n$. Now

$$\begin{aligned} \|B_H F\|^2 &= \int_0^\infty \|(B_H F)(x)\|^2_{\mathbb{C}^n} dx = \int_0^\infty \|G(x)\|^2_{\mathbb{C}^n} dx \\ &= \int_0^\infty \left(\sum_{j=1}^n |g_j(x)|^2 \right) dx = \sum_{j=1}^n \int_0^\infty |g_j(x)|^2 dx \\ &= \sum_{j=1}^n \int_0^\infty |(K_{\widetilde{h}_{jj}} f_j)(x)|^2 dx = \sum_{j=1}^n \|K_{\widetilde{h}_{jj}} f_j\|^2 \\ &\le \sum_{j=1}^n \|K_{\widetilde{h}_{jj}}\|^2 \|f_j\|^2 \le \sum_{j=1}^n \|f_j\|^2 \\ &= \sum_{j=1}^n \int_0^\infty |f_j(x)|^2 dx = \int_0^\infty \left(\sum_{j=1}^n |f_j(x)|^2 \right) dx \end{aligned}$$

$$\begin{aligned} &= \int_0^\infty \|F(x)\|^2_{\mathbb{C}^n} dx \\ &= \|F\|^2_{L^{2,\mathbb{C}^n}}. \end{aligned}$$

Thus $\|B_H\| \le 1$. Now it remains to show that $\|B_H\| \ge 1$. Let $f \in L^2(0,\infty)$ and $F = (f, 0, 0, \cdots)^T$. Then $\|F\| = \|f\|$. So,

$$|\langle K_{\widetilde{h}_{11}} f, f \rangle| = |\langle B_H F, F \rangle| \le \|B_H\| \, \|F\|^2 = \|B_H\| \, \|f\|^2$$

gives $1 = \|K_{\widetilde{h}_{11}}\| \le \|B_H\|$ as $K_{\widetilde{h}_{11}}$ is self-adjoint. Hence $\|B_H\| = 1$. □

Now we generalize Theorem 3.1, for the case $p = q = 2$, to vector-valued functions.

Theorem 5.3. *If $F,G \in L^{2,\mathbb{C}^n}(0,\infty)$, then*
$\left|\int_0^\infty \int_0^\infty G^*(x)H(x+y)F(y)dxdy\right| \leq \left(\int_0^\infty \|F(x)\|^2_{\mathbb{C}^n}dx\right)^{\frac{1}{2}} \left(\int_0^\infty \|G(y)\|^2_{\mathbb{C}^n}dy\right)^{\frac{1}{2}}$, *where the constant factor* 1 *is the best possible.*

Proof. Since $\|B_H\| = 1$, so the result follows from the fact that

$$|\langle B_H F, G\rangle| \leq \|F\|_{L^{2,\mathbb{C}^n}} \|G\|_{L^{2,\mathbb{C}^n}},$$

for all $F,G \in L^{2,\mathbb{C}^n}(0,\infty)$. □

Now let $\widetilde{\phi}_{lj}(e^{i\theta}) = -i(\pi-\theta)e^{i\theta}, 0 \leq \theta < 2\pi, 1 \leq l,j \leq n$ and $\phi_{lj}(z)$ be the harmonic extension of $\widetilde{\phi}_{lj}$ into $\mathbb{D}$.

$$\Phi = \begin{pmatrix} \phi_{11} & 0 & \cdots & 0 \\ 0 & \phi_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_{nn} \end{pmatrix}.$$

It is not difficult to see that

$$\mathbb{S}_\Phi = \begin{pmatrix} S_{\phi_{11}} & 0 & \cdots & 0 \\ 0 & S_{\phi_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_{\phi_{nn}} \end{pmatrix},$$

is unitarily equivalent to

$$B_H = \begin{pmatrix} K_{\widetilde{h}_{11}} & 0 & \cdots & 0 \\ 0 & K_{\widetilde{h}_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K_{\widetilde{h}_{nn}} \end{pmatrix},$$

where $\widetilde{h}_{ij}(x) = \frac{e^{-x}}{x}, 1 \leq i,j \leq n$. Hence $\|\mathbb{S}_\Phi\| = 1$.

Let $u_k = (0,0,\cdots,0,1,0,\cdots,0)$ with 1 in the k^{th} place and $\gamma_{kl} = e_l \otimes u_k, k = 1,2,\cdots n, l = 0,1,2,\cdots$. Then $\{u_k\}_{k=1}^n$ from an orthonormal basis for $\mathbb{C}^n$ and $\{\gamma_{kl}\}, k = 1,2,\cdots,n; l = 0,1,\cdots$ form an orthonormal basis for $L_a^{2,\mathbb{C}^n}(\mathbb{D},dA) = L_a^2(\mathbb{D}) \otimes \mathbb{C}^n$.

Theorem 5.4. *Let $\widetilde{F} = f \otimes x \in L_a^{2,\mathbb{C}^n}(\mathbb{D},dA)$ and $\widetilde{G} = g \otimes y \in L_a^{2,\mathbb{C}^n}(\mathbb{D},dA)$. Then*

$$\left|\sum_{l,l'=0}^{\infty}\sum_{k=1}^{n} \frac{\sqrt{l+1}\sqrt{l'+1}\langle f\otimes x, e_l \otimes u_k\rangle \overline{\langle g \otimes y, e_{l'} \otimes u_k\rangle}}{(l+l'+2)^2}\right| \leq \|f \otimes x\| \, \|g \otimes y\|.$$

Proof. Notice that

$$\langle \widetilde{F}, \gamma_{kl}\rangle = \langle f \otimes x, e_l \otimes u_k\rangle = \langle f, e_l\rangle \langle x, u_k\rangle$$

and

$$\langle \widetilde{G}, \gamma_{ml'}\rangle = \langle g \otimes y, e_{l'} \otimes u_m\rangle = \langle g, e_{l'}\rangle \langle y, u_m\rangle.$$

Hence

$$\begin{aligned}
\langle \mathbb{S}_\Phi \widetilde{F}, \widetilde{G}\rangle &= \sum_{k,m=1}^{n}\sum_{l,l'=0}^{\infty} \langle \widetilde{F}, \gamma_{kl}\rangle \overline{\langle \widetilde{G}, \gamma_{ml'}\rangle} \langle \mathbb{S}_\Phi(\gamma_{kl}), \gamma_{ml'}\rangle \\
&= \sum_{k,m=1}^{n}\sum_{l,l'=0}^{\infty} \langle \widetilde{F}, \gamma_{kl}\rangle \overline{\langle \widetilde{G}, \gamma_{ml'}\rangle} \langle (S_\phi \otimes I_{\mathbb{C}^n})(e_l \otimes u_k), e_{l'} \otimes u_m\rangle \\
&= \sum_{k,m=1}^{n}\sum_{l,l'=0}^{\infty} \langle f, e_l\rangle \langle x, u_k\rangle \overline{\langle g, e_{l'}\rangle}\, \overline{\langle y, u_m\rangle} \langle S_\phi e_l \otimes u_k, e_{l'} \otimes u_m\rangle \\
&= \sum_{k,m=1}^{n}\sum_{l,l'=0}^{\infty} \langle f, e_l\rangle \langle x, u_k\rangle \overline{\langle g, e_{l'}\rangle}\, \overline{\langle y, u_m\rangle} \langle S_\phi e_l, e_{l'}\rangle \langle u_k, u_m\rangle \\
&= \sum_{k=1}^{n}\sum_{l,l'=0}^{\infty} \langle f, e_l\rangle \langle x, u_k\rangle \overline{\langle g, e_{l'}\rangle}\, \overline{\langle y, u_k\rangle} \langle S_\phi e_l, e_{l'}\rangle.
\end{aligned}$$

Thus

$$|\langle \mathbb{S}_\Phi \widetilde{F}, \widetilde{G}\rangle| = \left| \sum_{l,l'=0}^{\infty}\sum_{k=1}^{n} \frac{\sqrt{l+1}\,\sqrt{l'+1}\langle f\otimes x, e_l \otimes u_k\rangle \overline{\langle g\otimes y, e_{l'}\otimes u_k\rangle}}{(l+l'+2)^2} \right|$$

and since $\mathbb{S}_\Phi$ is a bounded linear operator in $L_a^{2,\mathbb{C}^n}(\mathbb{D}, dA)$ and $\|\mathbb{S}_\Phi\| = 1$, we obtain

$$|\langle \mathbb{S}_\Phi \widetilde{F}, \widetilde{G}\rangle| \leq \|\widetilde{F}\|_{L_a^{2,\mathbb{C}^n}(\mathbb{D},dA)} \|\widetilde{G}\|_{L_a^{2,\mathbb{C}^n}(\mathbb{D},dA)} = \|f\otimes x\| \;\; \|g \otimes y\|.$$

The result follows. □

Corollary 5.5. *If* $\sum_{k=1}^{n}\sum_{l=0}^{\infty} |a_{kl}|^2 < \infty$ *and* $\sum_{k=1}^{n}\sum_{l'=0}^{\infty} |b_{kl'}|^2 < \infty$, *then*

$$\left| \sum_{l,l'=0}^{\infty}\sum_{k=1}^{n} \frac{\sqrt{l+1}\,\sqrt{l'+1}a_{kl}\overline{b}_{kl'}}{(l+l'+2)^2} \right| \leq \left(\sum_{k=1}^{n}\sum_{l=0}^{\infty} |a_{kl}|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{n}\sum_{l'=0}^{\infty} |b_{kl'}|^2 \right)^{\frac{1}{2}}$$

and the constant factor 1 *is the best possible.*

Proof. It is possible to find $x_k,\ y_k,\ k = 1, 2, \cdots, n$, and sequences $(c_l)_{l=0}^{\infty}, (c_{l'})_{l'=0}^{\infty}$ such that $a_{kl} = x_k c_l, b_{kl'} = y_k c_{l'}, \sum_{l=0}^{\infty} |c_l|^2 < \infty$ and $\sum_{l'=0}^{\infty} |c_{l'}|^2 < \infty$. Let $f(z) = \sum_{l=0}^{\infty} c_l e_l$ and $g(z) = \sum_{l'=0}^{\infty} c_{l'} e_{l'}$. Then $f, g \in L_a^2(\mathbb{D})$. So, for $x = (x_k)_{k=1}^{n}, y = (y_k)_{k=1}^{n} \in \mathbb{C}^n$, we have $f \otimes x, g\otimes y \in L_a^{2,\mathbb{C}^n}(\mathbb{D}, dA)$. Now

$$\begin{aligned}
\|f\otimes x\|^2 = \|f\|^2\|x\|^2 &= \sum_{l=0}^{\infty} |c_l|^2 \sum_{k=1}^{n} |x_k|^2 \\
&= \sum_{k=1}^{n}\sum_{l=0}^{\infty} |c_l|^2 |x_k|^2 \\
&= \sum_{k=1}^{n}\sum_{l=0}^{\infty} |a_{kl}|^2.
\end{aligned}$$

Similarly,

$$\|g\otimes y\|^2=\sum_{k=1}^{n}\sum_{l'=0}^{\infty}|b_{kl'}|^2.$$

On the other hand,

$$\begin{aligned}\langle f\otimes x, e_l\otimes u_k\rangle &= \langle f, e_l\rangle\langle x, u_k\rangle\\ &= x_k c_l\\ &= a_{kl}\end{aligned}$$

and

$$\begin{aligned}\langle g\otimes y, e_{l'}\otimes u_k\rangle &= \langle g, e_{l'}\rangle\langle y, u_k\rangle\\ &= y_k c_{l'}\\ &= b_{kl'}.\end{aligned}$$

Hence the results follows from Theorem 5.4. Since $\|\mathbb{S}_\Phi\|=1$, the constant factor 1 is the best possible. □

6 Hankel operators with operator valued symbols

In this section we generalize the inequality (4.2) for Hilbert space valued functions. In this case the integral operator involved have kernels that are matrix-valued (infinite matrix) functions. Let Ξ be a separable infinite dimensional Hilbert space. The space $L^{2,\Xi}(\mathbb{D})$ is defined to be the set of all (equivalence classes of) measurable, norm-square integrable, $\Xi-$valued functions defined on $\mathbb{D}$. When endowed with the inner product defined by the equation

$$\langle f,g\rangle=\int_{\mathbb{D}}\langle f(z),g(z)\rangle_\Xi\,dA,\ f,g\in L^{2,\Xi}(\mathbb{D}),$$

the space $L^{2,\Xi}(\mathbb{D})$ becomes a separable Hilbert space. Let $L_a^{2,\Xi}(\mathbb{D})$ be the corresponding Bergman space. A function Φ from $\mathbb{D}$ into $\mathcal{L}(\Xi)$ is called weakly measurable in case the complex valued function $z\to\langle\Phi(z)x,y\rangle$ is Lebesgue measurable for every x and y in Ξ. If Φ is weakly measurable then the real valued function $z\to\|\Phi(z)\|$ is measurable and the space of all (equivalence classes of) weakly measurable, essentially bounded, $\mathcal{L}(\Xi)-$valued functions on $\mathbb{D}$ will be denoted by $L^\infty_{\mathcal{L}(\Xi)}(\mathbb{D})$. The space $L^\infty_{\mathcal{L}(\Xi)}(\mathbb{D})$ is a C^*-algebra with the algebraic operations defined pointwise and norm defined by the equation

$$\|\Phi\|_\infty=ess\sup_{z\in\mathbb{D}}\|\Phi(z)\|,\Phi\in L^\infty_{\mathcal{L}(\Xi)}(\mathbb{D}),$$

where $\|\Phi(z)\|=\sup_n\sup_m|\langle\Phi(z)u_n,u_m\rangle|, z\in\mathbb{D},\{u_n\}_{n=0}^\infty$ is the orthonormal basis for Ξ and involution is defined by the equation $\Phi^*(z)=(\Phi(z))^*$. The mapping $\zeta\to\Phi(\zeta)f,\zeta\in\mathbb{D}$ are measurable for $f\in\Xi$. This follows from the Pettis Theorem (see [2]) as Ξ is separable. Let $H^\infty_{\mathcal{L}(\Xi)}(\mathbb{D})=H^\infty(\mathbb{D})\otimes\mathcal{L}(\Xi)$. For $\Phi\in L^\infty_{\mathcal{L}(\Xi)}(\mathbb{D})$, we define the Hankel operator $\mathbf{S}_\Phi$ from $L_a^{2,\Xi}(\mathbb{D})$ into itself as $\mathbf{S}_\Phi f=Q(\mathfrak{I}(\Phi f))$, where Q is the orthogonal projection from $L^{2,\Xi}(\mathbb{D})$

onto $L_a^{2,\Xi}(\mathbb{D})$ and the symbol Φf denote the function on $\mathbb{D}$ defined by $(\Phi f)(z) = \Phi(z)f(z)$ and $\mathfrak{J} : L^{2,\Xi}(\mathbb{D}) \to L^{2,\Xi}(\mathbb{D})$ is defined by $\mathfrak{J}F(z) = F(\overline{z})$. In the following theorem we extend Theorem 5.3 for $\Xi-$valued functions.

Theorem 6.1. *Let $H(x) = \frac{e^{-x}}{x} \otimes I_\Xi$, where I_Ξ is the identity operator from the Hilbert space Ξ into itself. Let $L^{2,\Xi}(0,\infty) = L^2(0,\infty) \otimes \Xi$ and define $K_H : L^{2,\Xi}(0,\infty) \to L^{2,\Xi}(0,\infty)$ by*

$$(K_H F)(x) = \int_0^\infty H(x+y)F(y)dy.$$

Then for $F, G \in L^{2,\Xi}(0,\infty)$,

$$\left| \int_0^\infty \langle (K_H F)(x), G(x) \rangle_\Xi dx \right| \leq \|F\|_{L^{2,\Xi}(0,\infty)} \|G\|_{L^{2,\Xi}(0,\infty)}.$$

Proof. Let $\widetilde{h}(x) = \frac{e^{-x}}{x}$ and define $K_{\widetilde{h}} \in \mathcal{L}(L^2(0,\infty))$ by

$$(K_{\widetilde{h}} f)(x) = \int_0^\infty \frac{\sqrt{x}\sqrt{y}}{x+y} \frac{e^{-(x+y)}}{x+y} f(y)dy.$$

It is not difficulties to see that the operator K_H is well-defined and since $L^{2,\Xi}(0,\infty) = L^2(0,\infty) \otimes \Xi$, we have $K_H = \sum_{n=0}^{\infty} \oplus K_{\widetilde{h}} = K_{\widetilde{h}} \otimes I_\Xi$, where $(K_{\widetilde{h}} \otimes I_\Xi)(f \otimes z) = K_{\widetilde{h}} f \otimes z$ if $f \in L^2(0,\infty)$ and $z \in \Xi$. Now $\|K_H\| = \left\| \sum_{n=0}^{\infty} \oplus K_{\widetilde{h}} \right\| = \|K_{\widetilde{h}}\| = 1$. Thus by Cauchy-Schwarz inequality it follows that

$$\begin{aligned} |\langle K_H F, G \rangle| &\leq \|K_H\| \, \|F\|_{L^{2,\Xi}(0,\infty)} \|G\|_{L^{2,\Xi}(0,\infty)} \\ &= \|F\|_{L^{2,\Xi}(0,\infty)} \|G\|_{L^{2,\Xi}(0,\infty)}. \end{aligned}$$

Hence

$$\left| \int_0^\infty \langle (K_H F)(x), G(x) \rangle_\Xi dx \right| \leq \|F\|_{L^{2,\Xi}(0,\infty)} \|G\|_{L^{2,\Xi}(0,\infty)}.$$

□

Theorem 6.2. *If $\widetilde{F} = f \otimes x, \widetilde{G} = g \otimes y \in L_a^{2,\Xi}(\mathbb{D}) = L_a^2(\mathbb{D}) \otimes \Xi$, then*

$$\left| \sum_{l,l'=0}^{\infty} \sum_{k=0}^{\infty} \frac{\sqrt{l+1}\,\sqrt{l'+1} \langle f \otimes x, e_l \otimes u_k \rangle \overline{\langle g \otimes y, e_{l'} \otimes u_k \rangle}}{(l+l'+2)^2} \right| \leq \|f \otimes x\| \, \|g \otimes y\|.$$

Proof. Let $\widetilde{\phi}(e^{i\theta}) = -i(\pi - \theta)e^{i\theta}, 0 \leq \theta \leq 2\pi$ and ϕ be the harmonic extension of $\widetilde{\phi}$ to $\mathbb{D}$. Let $\Phi = \phi \otimes I_\Xi$. Then $\Phi \in L^\infty_{\mathcal{L}(\Xi)}(\mathbb{D})$. Let $\mathbf{S}_\Phi$ be the Hankel operator from $L_a^{2,\Xi}(\mathbb{D})$ into itself with symbol Φ. Notice that since $L_a^{2,\Xi}(\mathbb{D}) = L_a^2(\mathbb{D}) \otimes \Xi$, we have $\mathbf{S}_\Phi = S_\phi \otimes I_\Xi$. Thus $\|\mathbf{S}_\Phi\| = \|S_\phi\| = 1$.

Let $\Upsilon_{kl} = e_l \otimes u_k, k = 0,1,2,\cdots$ and $l = 0,1,2,\cdots$. The sequence $\{\Upsilon_{kl}\}$ from an orthonormal basis for $L_a^{2,\Xi}(\mathbb{D})$. Then

$$|\langle \mathbf{S}_\Phi \widetilde{F}, \widetilde{G} \rangle| = \sum_{l,l'=0}^{\infty} \sum_{k=0}^{\infty} \frac{\sqrt{l+1}\,\sqrt{l'+1} \langle f \otimes x, e_l \otimes u_k \rangle \overline{\langle g \otimes y, e_{l'} \otimes u_k \rangle}}{(l+l'+2)^2}.$$

Since

$$|\langle \mathbf{S}_\Phi \widetilde{F}, \widetilde{G}\rangle| \leq \|\mathbf{S}_\Phi\| \, \|\widetilde{F}\| \, \|\widetilde{G}\| = \|f\otimes x\| \|g\otimes y\|,$$

the result follows. □

Corollary 6.3. *Let $\widetilde{F} = f\otimes x$ and $\widetilde{G} = g\otimes y$ where $f, g \in L^2_a(\mathbb{D})$ and $x, y \in \Xi$. Let $c_l(f)$ and $c_{l'}(g)$ denote the l^{th} and $l^{'\,th}$ Fourier coefficients of f and g respectively. Then*

$$\left| \sum_{l,l'=0}^{\infty} \frac{\sqrt{l+1}\,\sqrt{l'+1}\langle c_l(f)x, c_{l'}(g)y\rangle_\Xi}{(l+l'+2)^2} \right| \leq \|\widetilde{F}\|_{L^{2,\Xi}_a(\mathbb{D})} \|\widetilde{G}\|_{L^{2,\Xi}_a(\mathbb{D})}.$$

Proof. Let $\Upsilon_{kl} = e_l \otimes u_k, k = 0,1,2,\cdots$ and $l = 0,1,2,\cdots$. Then the sequence $\{\Upsilon_{kl}\}$ forms an orthonormal basis for $L^{2,\Xi}_a(\mathbb{D})$. Hence $\langle \widetilde{F}, \Upsilon_{kl}\rangle = c_l(f)\langle x, u_k\rangle$ and $\langle \widetilde{g}, \Upsilon_{kl'}\rangle = c_{l'}(g)\langle y, u_k\rangle$.

Also

$$\begin{aligned}
&\sum_{l,l'=0}^{\infty}\sum_{k=0}^{\infty} \frac{\sqrt{l+1}\,\sqrt{l'+1}\langle f\otimes x, e_l\otimes u_k\rangle \overline{\langle g\otimes y, e_{l'}\otimes u_k\rangle}}{(l+l'+2)^2} \\
&= \sum_{l,l'=0}^{\infty}\sum_{k=0}^{\infty} \frac{\sqrt{l+1}\,\sqrt{l'+1}\langle c_l(f)x, u_k\rangle \overline{\langle c_{l'}(g)y, u_k\rangle}}{(l+l'+2)^2} \\
&= \sum_{l,l'=0}^{\infty}\sum_{k=0}^{\infty} \frac{\sqrt{l+1}\,\sqrt{l'+1}\langle c_l(f)x, u_k\rangle \langle u_k, c_{l'}(g)y\rangle}{(l+l'+2)^2} \\
&= \sum_{l,l'=0}^{\infty} \frac{\sqrt{l+1}\,\sqrt{l'+1}\langle c_l(f)x, c_{l'}(g)y\rangle_\Xi}{(l+l'+2)^2}.
\end{aligned}$$

Now the result follows from Theorem 6.2. □

Corollary 6.4. *If $\sum_{l,k=0}^{\infty} |a_{kl}|^2 < \infty$ and $\sum_{l',k=0}^{\infty} |b_{kl'}| < \infty$, then*

$$\left| \sum_{k,l,l'=0}^{\infty} \frac{\sqrt{l+1}\,\sqrt{l'+1} a_{kl}\overline{b}_{kl'}}{(l+l'+2)^2} \right| = \left(\sum_{k,l=0}^{\infty} |a_{kl}|^2 \right)^{\frac{1}{2}} \left(\sum_{k,l'=0}^{\infty} |b_{kl'}|^2 \right)^{\frac{1}{2}}$$

and the constant factor 1 *is sharp.*

Proof. The proof is similar to the proof of Corollary 5.5. □

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