Communications in Mathematical Analysis
Volume 17, Number 2, pp. 295–310 (2014) www.math-res-pub.org/cma
ISSN 1938-9787

Application of the SPPS Method to the One-dimensional Quantum Scattering

V. Rabinovich*
National Polytechnic Institute
ESIME Zacatenco
D. F. México, CP 07738, México

F. Urbano-Altamirano†
National Polytechnic Institute
UPIITA
D. F. México, CP 07340, México

(Communicated by Vladimir Rabinovich)

Abstract

The transmission and reflection coefficients for the scattering of a particle on one-dimensional potential are calculated by means of Spectral Parameter Power Series (SPPS). The results were compared with known results.

AMS Subject Classification: 81U05, 34L16.

Keywords: potential barrier, scattering, Schrödinger equation, Spectral Parameter Power Series (SPPS)

1 Introduction

We consider the scattering of a one-dimensional particle of complete energy E on the potential barrier. The scattering process is described by a Schrödinger equation

$$\mathcal{H}\psi(x) = \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\psi(x) = E\psi(x), x \in \mathbb{R} \tag{1.1}$$

where $\hbar$ is the Planck constant, $m > 0$ is an effective mass of the particle, V is an electric potential of an external field, ψ is a wave function.

*E-mail address: vladimir.rabinovich@gmail.com
†E-mail address: edurbano06@gmail.com

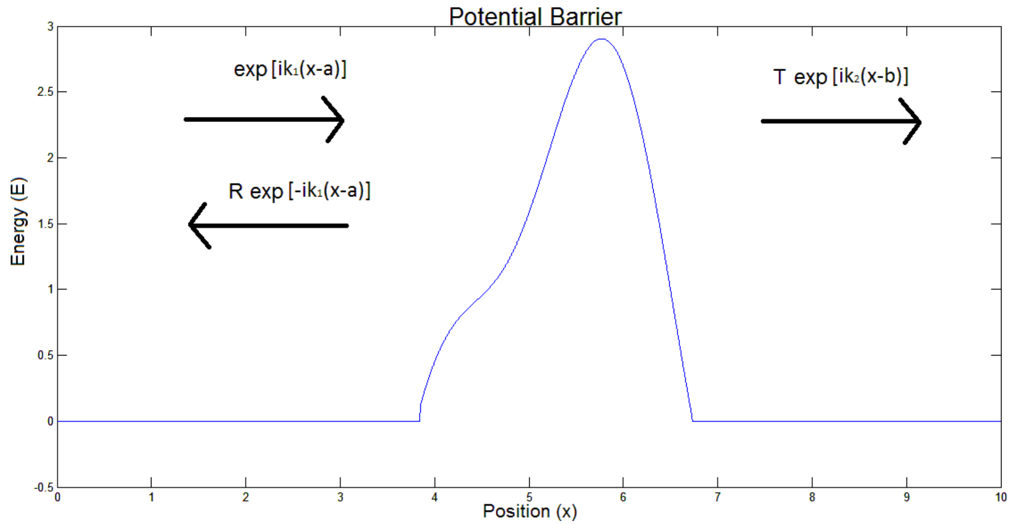

Figure 1. (Potential barrier and input-output wave functions. a=3.85, b=6.73

We suppose that V is a real piecewise continuous function on $\mathbb{R}$, such that

$$V(x) = \begin{matrix} V_1, x \in (-\infty, a) \\ V_0(x), x \in [a,b] \\ V_2, x \in (b, +\infty) \end{matrix},$$

We suppose for the definiteness, that $V_1 \leq V_2$. It yields that $\mathcal{H}$ is a self-adjoint operator in $L^2(\mathbb{R})$ with a domain $H^2(\mathbb{R})$ with a continuous spectrum $[V_1, \infty)$.

We rewrite equation (1.1) as

$$\mathcal{H}_1\psi(x) = \left(-\frac{d^2}{dx^2} + q(x)\right)\psi(x) = \lambda\psi(x), \lambda \in \mathbb{R}, x \in \mathbb{R}, \tag{1.2}$$

where

$$q(x) = \begin{matrix} q_1, x \in (-\infty, a) \\ q_0(x), x \in [a,b] \\ q_2, x \in (b, +\infty) \end{matrix},$$

$$q_0(x) = \frac{2mV_0(x)}{\hbar^2}, q_j = \frac{2mV_j}{\hbar^2}, j = 1,2, \lambda = \frac{2mE}{\hbar^2}.$$

Let the complete energy of the particle $E \in (V_2, \infty)$. It implies that $\lambda > q_2 \geq q_1$. We set

$$k_j = \sqrt{\lambda - q_j} > 0, j = 1,2$$

We define the solutions of the equation (1.2) of the form

$$\psi(x) = \begin{cases} e^{ik_1(x-a)} + Re^{-ik_1(x-a)}, x < a \\ c_1\psi_1(x) + c_2\psi_2(x), x \in (a,b) \\ Te^{ik_2(x-b)}, x > b \end{cases}, \tag{1.3}$$

where $\psi_1(x), \psi_2(x)$ are linearly independent solutions of equation (1.2) on (a,b). The wave function $\psi(x)$ describes the process of the scattering of a particle with energy $E > V_2$ moving from the left to right. The complex number $R = R(\lambda)$ is called the reflection coefficient, and $T = T(\lambda)$ is called the transmission coefficient. We will find the coefficients R and T using the continuity of ψ and ψ' on $\mathbb{R}$. It is well known that the coefficients R and T satisfied the relation

$$|R|^2 + \frac{k_1}{k_2}|T|^2 = 1. \tag{1.4}$$

Let ψ_1, ψ_2 be linear independent solutions of (1.1) on $[a,b]$ satisfying the conditions of the Cauchy problem

$$\psi_1(a) = 1, \psi_1'(a) = 0, \tag{1.5}$$

$$\psi_2(a) = 0, \psi_2'(a) = 1. \tag{1.6}$$

Applying (1.3),(1.5),(1.6) we obtain the equations

$$\begin{aligned} 1+R &= c_1, \\ ik_1(1-R) &= c_2, \\ T &= c_1\psi_1(b) + c_2\psi_2(b), \\ ik_2T &= c_1\psi_1'(b) + c_2\psi_2'(b). \end{aligned} \tag{1.7}$$

Equations (1.7) imply that

$$\begin{aligned} (1+R)\psi_1(b) + ik_1(1-R)\psi_2(b) &= T, \\ (1+R)\psi_1'(b) + ik_1(1-R)\psi_2'(b) &= ik_2T, \end{aligned} \tag{1.8}$$

and

$$R = \frac{\psi_1'(b) + k_2k_1\psi_2(b) + i\left[k_1\psi_2'(b) - k_2\psi_1(b)\right]}{k_2k_1\psi_2(b) - \psi_1'(b) + i\left[k_2\psi_1(b) + k_1\psi_2'(b)\right]}, \tag{1.9}$$

$$T = \left[(1+R)\psi_1(b) + ik_1(1-R)\psi_2(b)\right]. \tag{1.10}$$

It is clear that $k_j = k_j(\lambda) = \sqrt{\lambda - q_j} > 0, j = 1,2, \psi_j(b) = \psi_j(b,\lambda), j = 1,2$ and hence $R = R(\lambda), T = T(\lambda)$ depend on the spectral parameter $\lambda = \frac{2mE}{\hbar^2}$, and therefore from the energy E of the particle.

Remark 1.1. As usual the reflection and transmission coefficients are defined for wave functions ψ of the form

$$\psi(x) = \begin{cases} e^{-ik_1x} + R_1e^{ik_1x}, x < a, \\ T_1e^{ik_2x}, x > b. \end{cases}$$

Hence

$$R_1 = Re^{-2ik_1a}, T_1 = e^{-ik_2b}T,$$

and $|R| = |R_1|, |T| = |T_1|$.

There is an extensive literature devoted to finding the reflection and transmission coefficients by analytical and numerical methods (see for instance [1],[6], [7], [8], [15], [14], [16], [17], [18], etc.).

A numerical implementation of R and T are based on solutions of the Cauchy problem for the equation (1.2) on the interval (a,b). Of course one can use the canonical Runge-Kutta method or its improving for the numerical calculation of solutions ψ_1, ψ_2. But if we are interesting in the behavior of $R(\lambda), T(\lambda)$ on a large interval of the energy the Runge-Kutta method demands big machinery resources. In this paper solutions of Cauchy problem (1.5), (1.6) are sought of the form of a power series

$$\psi(z,\lambda) = \sum_{k=0}^{\infty} a_k(z)\lambda^k \tag{1.11}$$

with respect to a spectral parameter $\lambda \in \mathbb{C}$ with coefficients a_k defined by some recursive formulas (Spectral parameter power series method, abbreviated SPPS method). The SPPS method has been discovered by V.V. Kravchenko [11], [12] and has been successfully applied to different problems of Mathematical Physics which are reduced to spectral Sturm-Liouville problems [4], [5], [10], [13]. In [3] the SPPS method was applied to the analysis of electromagnetic waveguides.

The paper is organized as follows. In Section 2 we give some known analytical expressions for reflection and transmission coefficients R, T which will be used later for the comparison with R, T obtained by the SPPS method. Section 3 is devoted to the numerical calculations of R, T. We show that the results of calculations obtained by this method give a good coincidence with results obtained from analytical formulas.

2 Analytical form of the transmission and reflection coefficient

$1^{0.}$ *Rectangular barrier*. Let

$$V(x) = \begin{cases} U_0 > 0, x \in [0,b] \\ 0, x \neq [0,b] \end{cases},$$

and $E > U_0$. We set

$$k_1 = \frac{\sqrt{2mE}}{\hbar}, k_2 = \frac{\sqrt{2m(E-U_0)}}{\hbar}.$$

Then simple calculations taking into account the continuity of solutions and their derivatives of equation (1.1) implies the formulas for T and R (see for instance [18])

$$T = \left[1 + \frac{1}{4}\left(\frac{k_1^2 + k_2^2}{k_1 k_2}\right)^2 \sinh^2(k_2 b)\right]^{-1}, \tag{2.1}$$

$$R = \frac{1}{4} T \left(\frac{k_1^2 + k_2^2}{k_1 k_2}\right)^2 \sinh^2(k_2 b). \tag{2.2}$$

2^0. *Potential* $V(x) = \frac{U_0}{\cosh^2(\alpha x)}$, where $U_0 > 0, \alpha > 0$ are constants. A graph of this potential is shown in Figure 2.

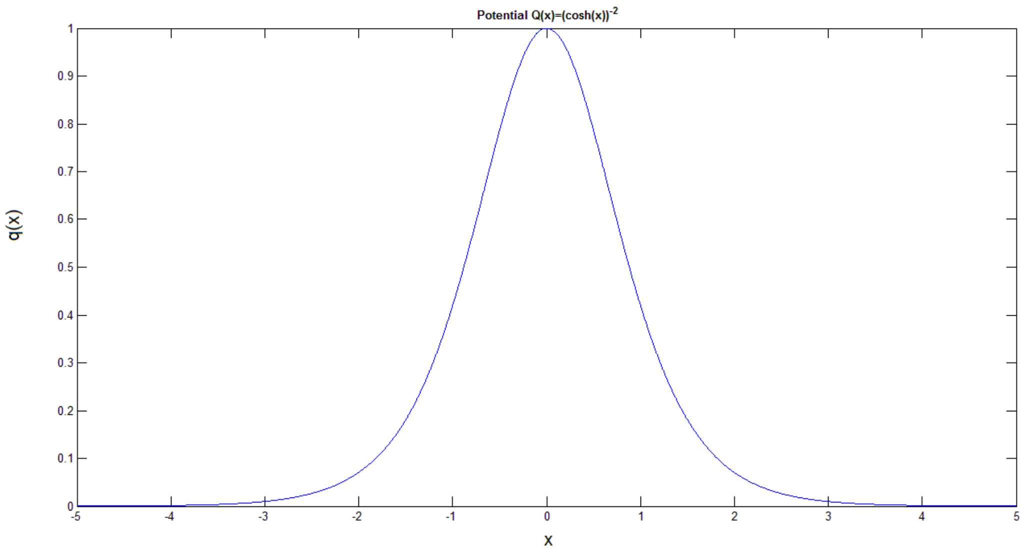

Figure 2. Graph of the potential $q(x) = \frac{U_0}{\cosh^2(\alpha x)}$ with $U_0 = 1\ eV$ and $\alpha = 1 \times 10^9\ nm^{-1}$ in the interval $x \in [-5, 5]$

This potential is not finite, but $V(x)$ is exponentially decreasing at infinity, that is

$$V(x) \leq Ce^{-2\alpha|x|}, C > 0.$$

Then we can consider the solution ψ of the equation (1.1) with the asymptotics

$$\psi(x) \sim \begin{cases} e^{ikx} + Re^{-ikx}, x \to -\infty \\ Te^{ikx}, x \to +\infty \end{cases}, \ k = \frac{\sqrt{2mE}}{\hbar},$$

where R and T are transmission and reflection coefficients. In the classical book [14] the solution $\psi(x)$ of equation (1.1) with potential $\frac{U_0}{\cosh^2(\alpha x)}$ has been obtained as

$$\psi(x) = (1-\xi^2)^{-ik/2\alpha} F\left[(-ik/\alpha) - s, (-ik/\alpha) + s + 1, (-ik/\alpha) + 1, \frac{1}{2}(1-\xi)\right] \tag{2.3}$$

where F is a hypergeometric function, $\xi = \tanh(\alpha x)$, $k = \frac{\sqrt{2mE}}{\hbar}$, and $s = \frac{1}{2}\left(-1 + \sqrt{1 - \frac{8mU_0}{\alpha^2\hbar^2}}\right)$. The asymptotics of the solution $\psi(x)$ for $x \to -\infty$ is

$$\psi(x) \sim e^{-ikx} \frac{\Gamma(ik/\alpha)\Gamma(1-(ik/\alpha))}{\Gamma(-s)\Gamma(1+s)} + e^{ikx} \frac{\Gamma(-ik/\alpha)\Gamma(1-(ik/\alpha))}{\Gamma((-ik/\alpha)-s)\Gamma((-ik/\alpha)+s+1)}. \tag{2.4}$$

Taking into account that $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin x\pi}$ one can obtain

$$|R|^2 = \frac{\cos^2\left(\frac{1}{2}\pi\sqrt{1 - \frac{8mV_0}{\hbar^2\alpha^2}}\right)}{\left[\sinh^2(\pi k/\alpha) + \cos^2\left(\frac{1}{2}\pi\sqrt{1 - \frac{8mV_0}{\hbar^2\alpha^2}}\right)\right]}, \tag{2.5}$$

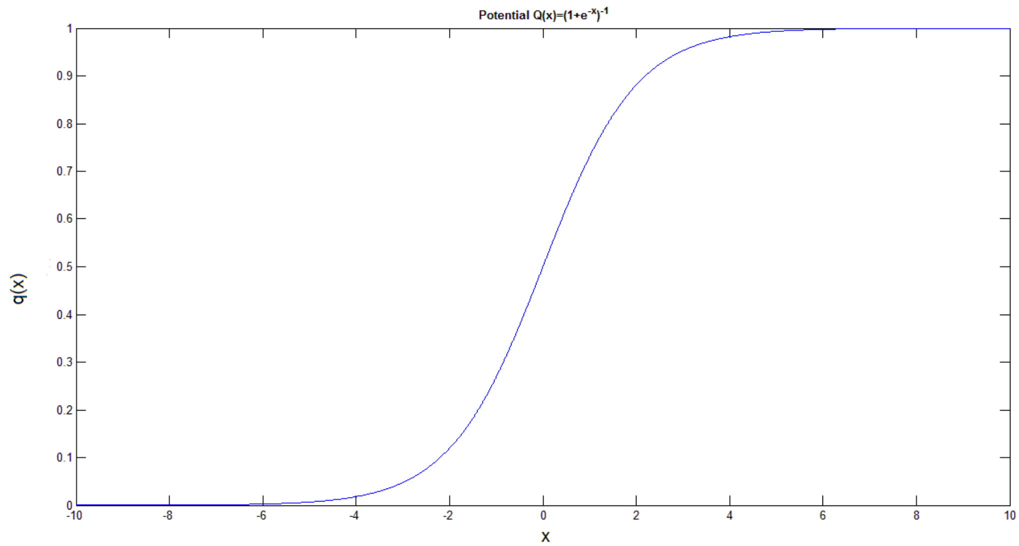

Figure 3. Graph of the potential $q(x) = \frac{U_0}{1+e^{-\alpha x}}$ with $U_0 = 1\ eV$ and $\alpha = 1 \times 10^9\ nm^{-1}$ in the interval $x \in [-10, 10]$

if $\frac{8mV_0}{\hbar^2\alpha^2} < 1$ and otherwise

$$|R|^2 = \frac{\cos^2\left(\frac{1}{2}\pi\sqrt{1-\frac{8mV_0}{\hbar^2\alpha^2}}\right)}{\left[\sinh^2(\pi k/\alpha) + \cos^2\left(\frac{1}{2}\pi\sqrt{\frac{8mV_0}{\hbar^2\alpha^2}-1}\right)\right]}. \tag{2.6}$$

Applying formula $|R|^2 + |T|^2 = 1$ we obtain that

$$|T|^2 = \frac{\sinh^2(\pi k/\alpha)}{\sinh^2(\pi k/\alpha) + \cos^2\left(\frac{1}{2}\pi\sqrt{1-\frac{8mV_0}{\hbar^2\alpha^2}}\right)} \tag{2.7}$$

if $\frac{8mV_0}{\hbar^2\alpha^2} < 1$. Otherwise

$$|T|^2 = \frac{\sinh^2(\pi k/\alpha)}{\sinh^2(\pi k/\alpha) + \cos^2\left(\frac{1}{2}\pi\sqrt{\frac{8mV_0}{\hbar^2\alpha^2}-1}\right)}. \tag{2.8}$$

3^0. *Potential* $V(x) = \frac{U_0}{1+e^{-\alpha x}}$ where $U_0 > 0$, $\alpha > 0$ are constants. A graph of an example of this kind of potential is shown in Figure 3.

One can see that $\lim_{x\to-\infty} V(x) = 0$ and $\lim_{x\to+\infty} V(x) = U_0$. As above we consider solutions of the equation (1.1) with the asymptotics

$$\psi(x) \sim \begin{cases} e^{ik_1x} + Re^{-ik_1x}, x \to -\infty \\ Te^{ik_2x}, x \to +\infty \end{cases},$$
$$k_1 = \frac{\sqrt{2mE}}{\hbar}, k_2 = \frac{\sqrt{2m(E-U_0)}}{\hbar}, E > U_0.$$

Following to [14] the solution to the Schrödinger equation (1.1) with potential $V(x) = \frac{U_0}{1+e^{-\alpha x}}$ is equal to

$$\psi(x) = F\left[i[k_1 - k_2]/\alpha, -i[k_1 + k_2]/\alpha, \frac{-2ik_2}{\alpha} + 1, \xi\right]$$

where $\xi = -e^{-\alpha x}$, $k_1 = \frac{\sqrt{2mE}}{\hbar}$ and $k_2 = \frac{\sqrt{2m(E-U_0)}}{\hbar}$. The asymptotic form of the solution $(x \to -\infty)$ is

$$u(x) \approx (-1)^{-ik_2/\alpha}\left[C_1e^{ik_1x} + C_2e^{-ik_1x}\right]$$

where

$$C_1 = \frac{\Gamma(-2ik_1/\alpha)\Gamma(-2ik_2/\alpha+1)}{\Gamma(-i(k_1+k_2)/\alpha)\Gamma(-i(k_1+k_2)/\alpha+1)}$$

$$C_2 = \frac{\Gamma(2ik_1/\alpha)\Gamma(-2ik_2/\alpha+1)}{\Gamma(i(k_1-k_2)/\alpha)\Gamma(i(k_1-k_2)/\alpha+1)}$$

and applying the formula $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$ one can find the module of the reflection coefficient

$$|R|^2 = \left|\frac{C_2}{C_1}\right|^2 = \frac{\sinh^2(\pi(k_1-k_2)/\alpha)}{\sinh^2(\pi(k_1+k_2)/\alpha)}, k_1 > k_2. \tag{2.9}$$

Applying the formula (1.4) we obtain the module of the transmission coefficient

$$|T|^2 = \frac{k_2}{k_1}(1-|R|^2). \tag{2.10}$$

3 Spectral parameter power series method (SPPS method)

We shortly describe here the SPPS method. Let

$$-\psi" + q(x)\psi = \lambda\psi, x \in (a,b) \tag{3.1}$$

be a Schrödinger equation on the interval (a,b) with a piecewise continuous potential $q(x)$ on $[a,b]$. Since $q \in L^1([a,b])$ every generalized solution ψ of the equation (3.1) belongs to $C^{(1)}([a,b])$. Following to the SPPS method the general solution ψ of (3.1) is equal to

$$\psi = c_1\psi_1 + c_2\psi_2,$$

where C_1, C_2 are arbitrary complex constant, and

$$\begin{aligned} \psi_1 &= u_0 \sum_{n=0}^{\infty} \lambda^n \widetilde{X}^{(2n)}, \\ \psi_2 &= u_0 \sum_{n=0}^{\infty} \lambda^n X^{(2n+1)}, \end{aligned} \tag{3.2}$$

where u_0 is a particular solution of equation (3.1) such that u_0 is a solution of the homogeneous equation

$$-u''(x) + q(x)u(x) = 0, \quad x \in (a,b), \tag{3.3}$$

such that $u_0^{-1} \in C([a,b])$. $\widetilde{X}^{(2n)}, X^{(2n+1)}$ are found by the recursive formulas

$$\begin{aligned} \widetilde{X}^{(0)} &\equiv 1, X^{(0)} \equiv 1, \\ \widetilde{X}^{(n)}(x) &= (-1)^{n-1} \int_a^x \widetilde{X}^{(n-1)}(s) \left(u_0^2(s)\right)^{(-1)^{n-1}} ds, \\ X^{(n)}(x) &= (-1)^n \int_a^x X^{(n-1)}(s) \left(u_0^2(s)\right)^{(-1)^n} ds. \end{aligned} \tag{3.4}$$

The solution u_0 can be find of the form

$$u_0 = y_1 + i y_2$$

where y_1 and y_2 are given by

$$y_1 = \sum_{n=0}^{\infty} \widetilde{Y}^{(2n)}, y_2 = \sum_{n=0}^{\infty} Y^{(2n+1)},$$

where

$$\begin{aligned} \widetilde{Y}^{(0)} &= 1, Y^{(0)} = 1, \\ \widetilde{Y}^{(n)}(x) &= \int_a^x \widetilde{Y}^{(n-1)}(s) (q(s))^{\frac{1+(-1)^{n-1}}{2}} ds, \\ Y^{(n)}(x) &= \int_a^x Y^{(n-1)}(s) (q(s))^{\frac{1+(-1)^n}{2}} ds. \end{aligned}$$

Applying particular solutions (3.2) of equation (3.1) we obtain the analytical expression for the reflection and transmission coefficients given by formulas (1.9), (1.10) where ψ_1, ψ_2 are defined by (3.2).

4 Numerical Implementation

We show here the results obtained using the SPPS method. For the numerical calculation of R and T we truncated the series in (3.2), that is

$$\begin{aligned} \psi_1 &= \psi_0 \sum_{n=0}^{N} \lambda^n \widetilde{X}^{(2n)}, \\ \psi_2 &= \psi_0 \sum_{n=0}^{N} \lambda^n X^{(2n+1)}. \end{aligned} \tag{4.1}$$

The calculations were performed in MATLAB and *spapi* routines for the calculations of the integrals in formulas (3.4).

In Figure 4 we present results for the reflection R and transmission T coefficients for a rectangular potential barrier which are obtained numerically by SPPS ($N = 120$) and compare with those obtained analytically. We considered a potential barrier $V_0 = 5\ eV$ and $a = 2\ nm$. The effective electron mass m is the effective mass of $GaAs$ ($m = 0.067m_0 = 6.1030 \times 10^{-32} kg$). Note that this material is widely used in the optoelectronics [1], [2], [9]. Tunneling occurs in the part of the graph corresponding to $E \in [0, 5]\ eV$.

In Figure 5 we show the graphic of absolute errors versus the analytical values for the transmission and reflection coefficients of Figure 4.

In Figure 6 the results of the transmission coefficient T for the potential $\frac{U_0}{\cosh^2(\alpha x)}$ with values $U_0 = 1\ eV$ and $\alpha = 1 \times 10^9\ nm^{-1}$ obtained analytically and with SPPS are compared. For applying the SPPS method we changed $\frac{U_0}{\cosh^2(\alpha x)}$ by its truncation on the segment $[-5, 5]$ nm, because $\frac{U_0}{\cosh^2(\alpha x)} = 0$ for $|x| > 5$ with the precision machinery . For the comparison of the SPPS method and analytical method we apply formulas (2.5),(2.6),(2.7),(2.8) on the energy interval $[0, 1]\ eV$.

In Figure 7 we show the absolute error for transmission coefficient of Figure 6.

In Figure 8 the results for the reflection coefficient R of the potential $\frac{U_0}{1+e^{-\alpha x}}$ with values $U_0 = 1\ eV$ and $\alpha = 1 \times 10^9\ nm^{-1}$ obtained analytically and with SPPS are compared. For the application of the SPPS method we change the potential $\frac{U_0}{1+e^{-\alpha x}}$ by

$$V(x) = \begin{cases} 0, x < -10 \\ \frac{U_0}{1+e^{-\alpha x}}, |x| \leq 10 \\ U_0, x > 10 \end{cases},$$

because $\frac{U_0}{1+e^{-\alpha x}}$ coincides with $V(x)$ with the precision machinery.

In Figure 9 we show the absolute error for reflection coefficient of Figure 8.

In Table 1 we present the maximum absolute errors for the tranmission and reflection coefficients that previously we showed.

Finally, we work with a double rectangular barrier potential with $V_0 = 5\ eV$ and width of the barriers $b = 2\ nm$. The graphic of the potential is shown in Figure 10 and the transmission coefficient T obtained with SPPS is shown in Figure 11.

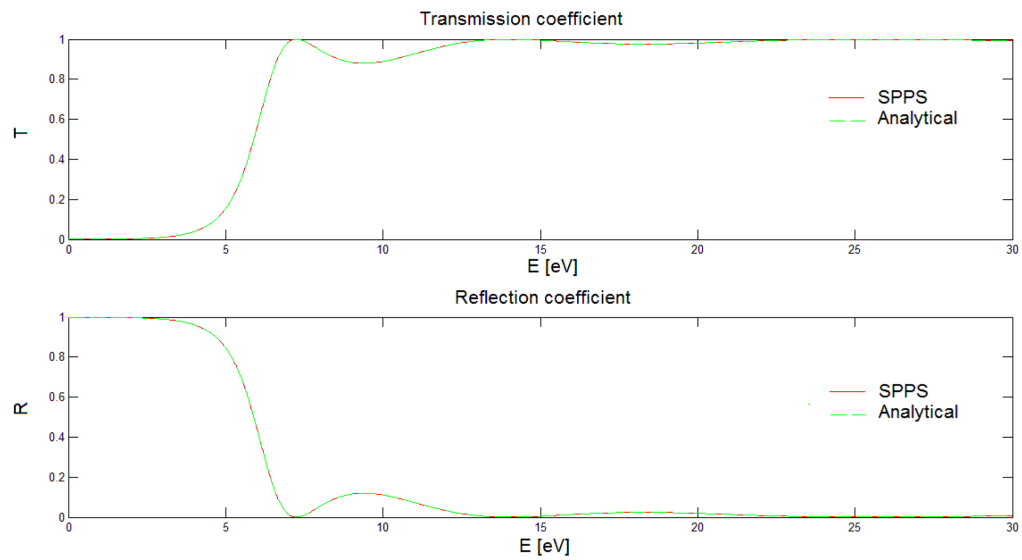

Figure 4. Comparison of the results for the reflection and transmission coefficients of a rectangular barrier potential ($V_0 = 5\ eV$ and $b = 2\ nm$) obtained analytically and with SPPS.

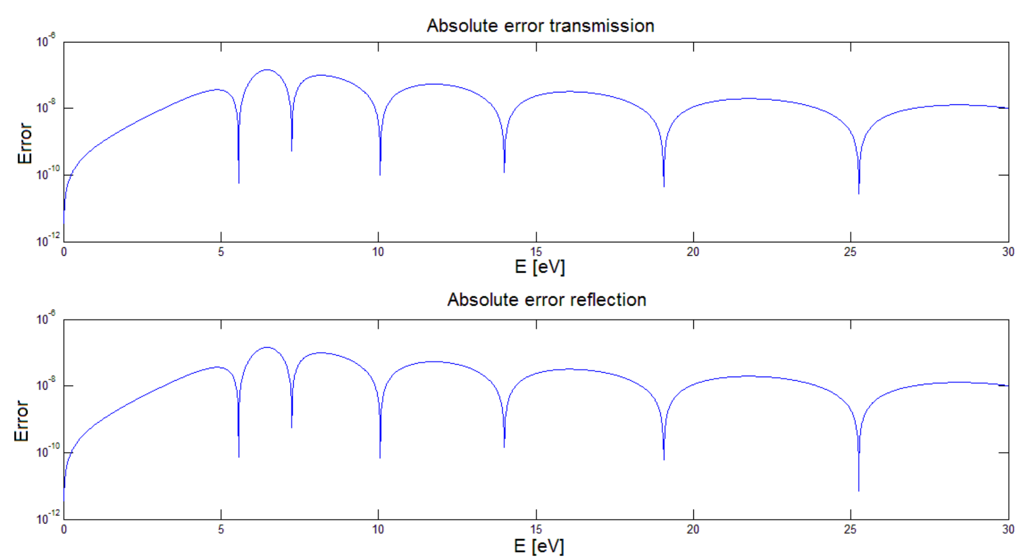

Figure 5. Absolute error for the SPPS results for coefficients of reflection R and tranmission T of a rectangular barrier ($V_0 = 5\ eV$ and $b = 2\ nm$).

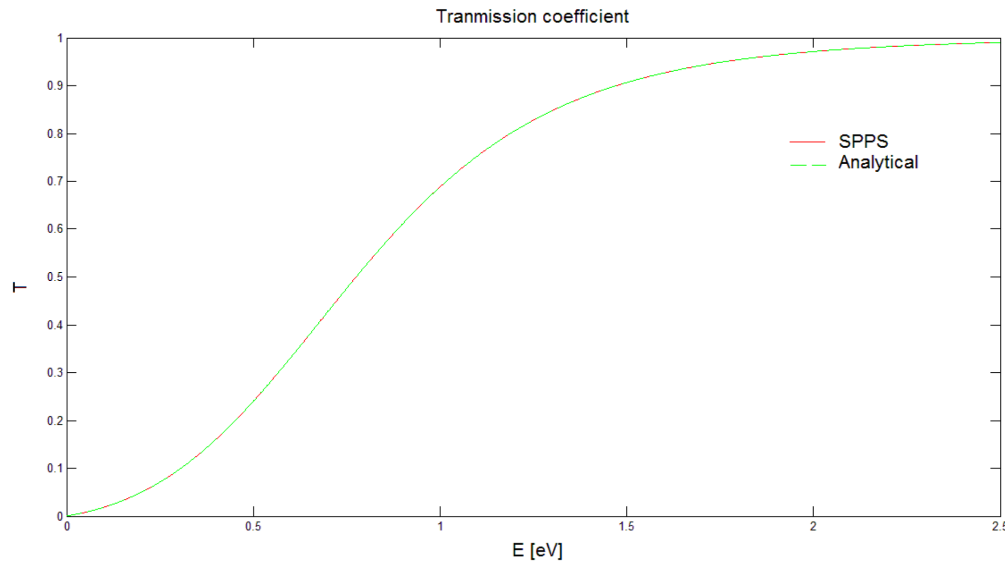

Figure 6. Comparison of the results for the transmission coefficient T for the potential $V_0(x) = \frac{U_0}{\cosh^2(\alpha x)}$ where $U_0 = 1\ eV$ and $\alpha = 1 \times 10^9\ nm^{-1}$ obtained analytically and with SPPS.

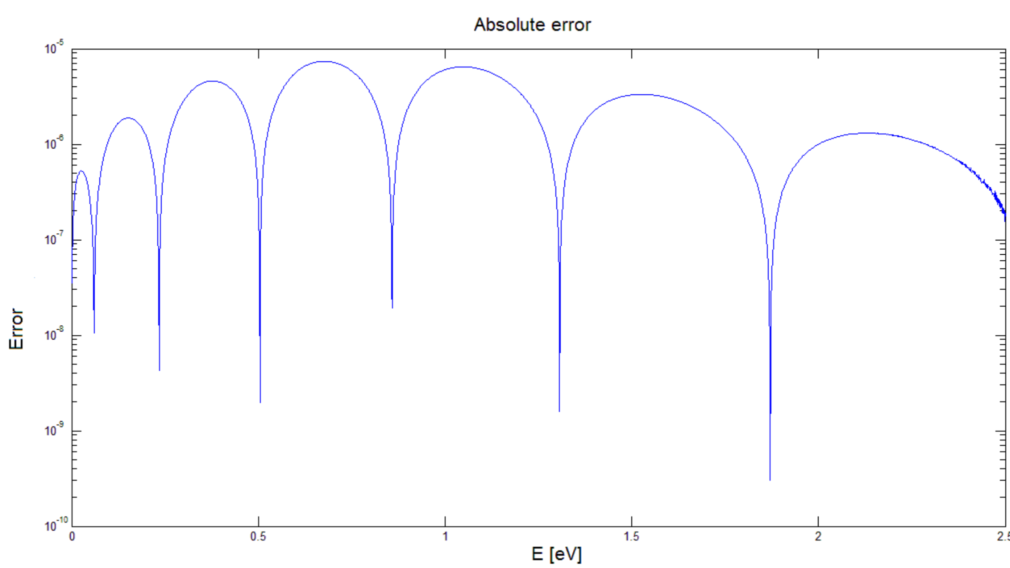

Figure 7. Absolute error for SPPS results for transmission coefficient T of the potential $V_0(x) = \frac{U_0}{\cosh^2(\alpha x)}$ where $U_0 = 1\ eV$ and $\alpha = 1 \times 10^9\ nm^{-1}$.

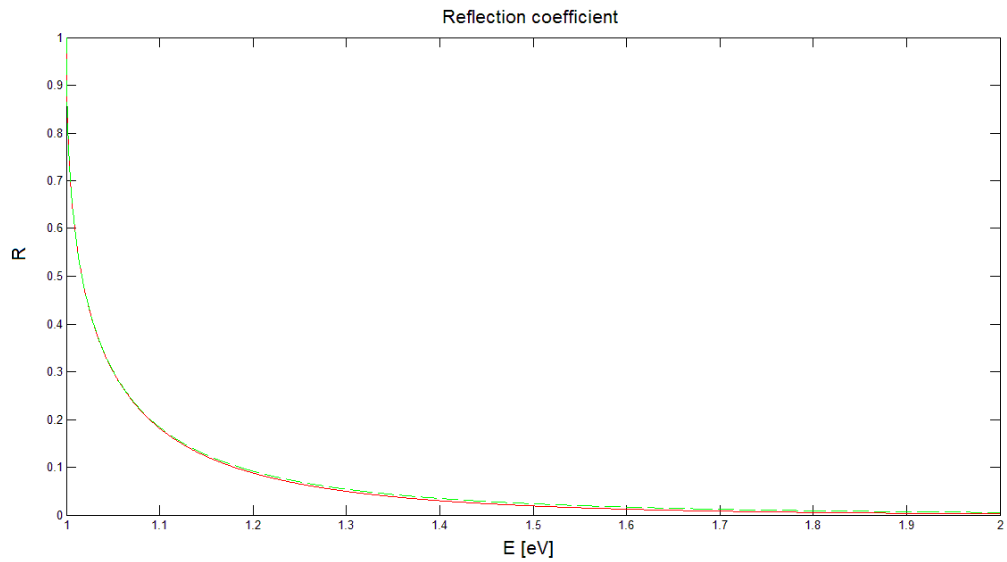

Figure 8. Comparison of the results for the reflection coefficient R for the potential $V_0(x) = \frac{U_0}{1+e^{-\alpha x}}$ where $U_0 = 1\ eV$ and $\alpha = 1 \times 10^9\ nm^{-1}$ obtained analytically and with SPPS.

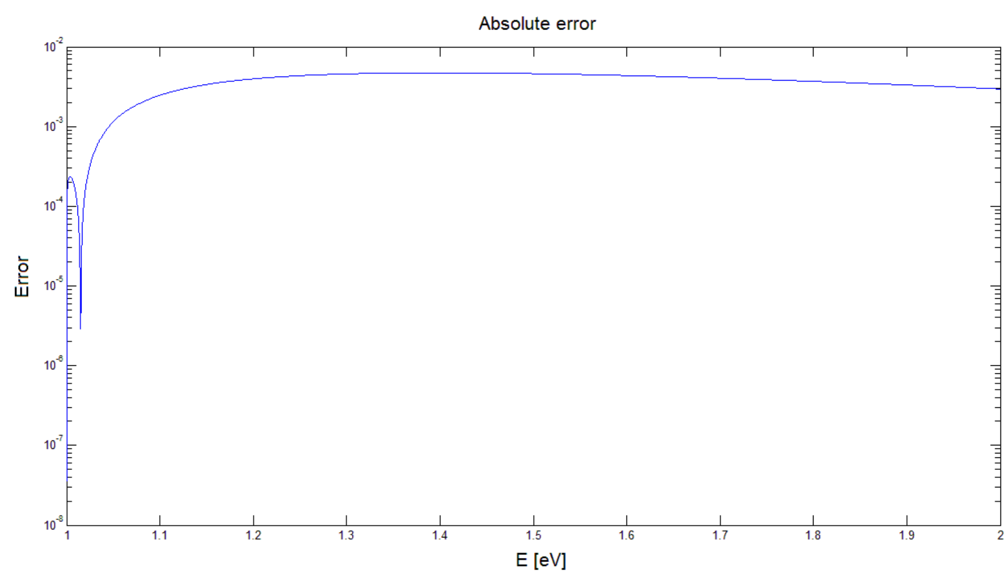

Figure 9. Absolute error of analytical and SPPS results for reflection coefficient R of the potential $V_0(x) = \frac{U_0}{1+e^{-\alpha x}}$ where $U_0 = 1\ eV$ and $\alpha = 1 \times 10^9\ nm^{-1}$.

Table 1. Absolute errors

$q(x)$	Maximum absolute error
5 eV (Reflection coefficient)	1.4451×10^{-7}
5 eV (Transmission coefficient)	1.4454×10^{-7}
$\frac{U_0}{\cosh^2(\alpha x)}$	7.3972×10^{-6}
$\frac{U_0}{1+e^{-\alpha x}}$	4.7137×10^{-3}

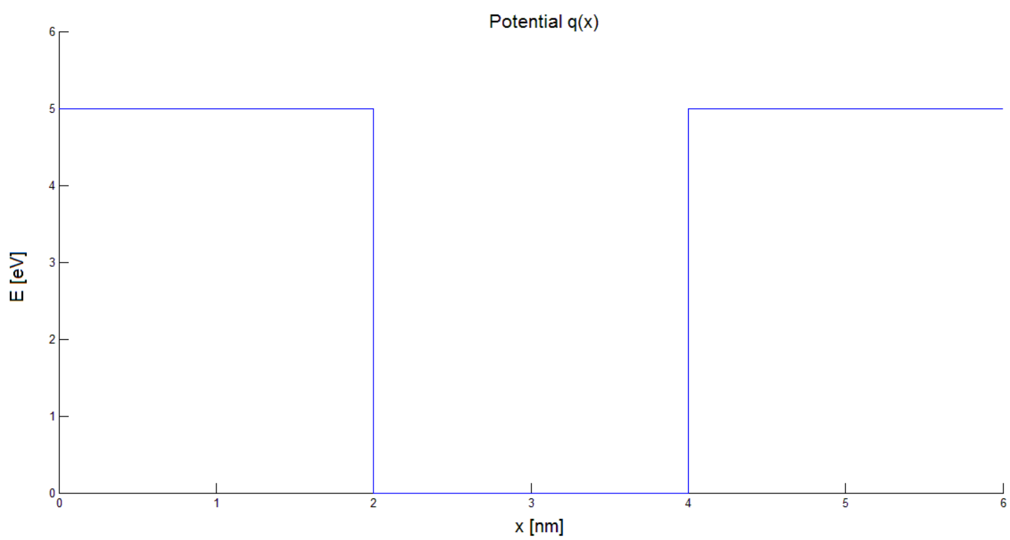

Figure 10. Graph of a double barrier potential with height $V_0 = 5\ eV$ and width of the every barrier $b = 2\ nm$.

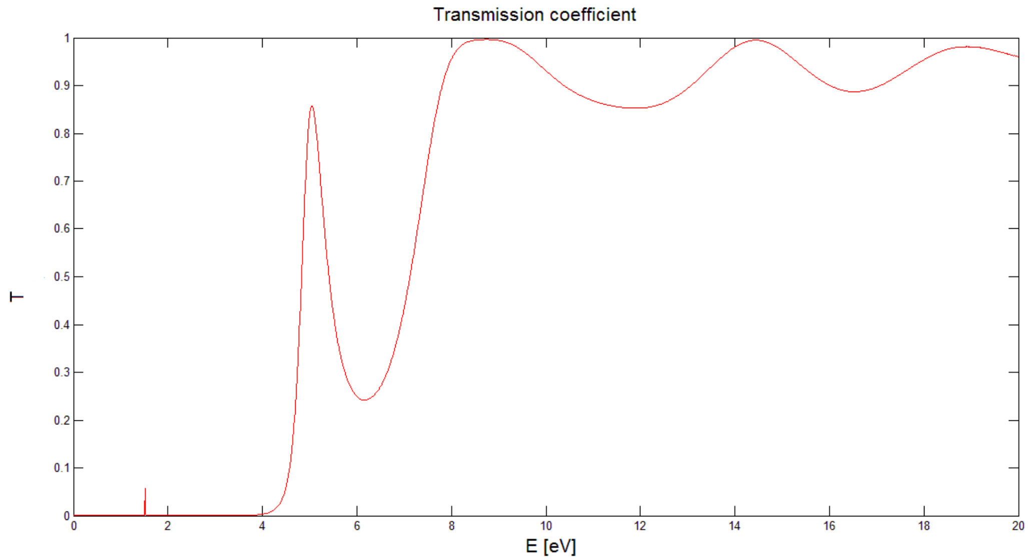

Figure 11. Transmission coefficient T for a double barrier potential obtained by means of SPPS.

5 Conclusions

We obtained the general formulas of transmission and reflection coefficients for a scattering of a particle on a potential barrier applying the SPPS method. We give a comparison of numerical results obtained by SPPS method with numerical results obtained from well known analytical formulas. The comparisons reveals a satisfactory perfomance of SPPS method for this kind of problems.

Acknowledgments

The first author grateful to the National System of Researchers of Mexico (SNI) for the support of his scientific activity, the CONACYT project 000000000179872, and the PROMEP project of the Network "Análisis, Teoría de Operadores y Física Matemática" (Mexico).

The second author wish to thank to CONACYT and the program BEIFI of the National Polytechnic Institute of Mexico (IPN) for the support of his academical and scientific activities.

References

[1] Y. Ando and T. Itoh, Calculation of transmission tunneling current across arbitrary potential barriers. *J. Appl. Phys.* **61** (1987), pp 1497-502.

[2] S. L. Ban, J. E. Hasbun, and X. X. Liang, A novel method for quantum transmission across arbitrary potential barriers. *J. Lumin.* **87** (2000), pp 369-371.

[3] V. Barrera-Figueroa, V. V. Kravchenko, and V. S. Rabinovich, Spectral parameter power series analysis of isotropic planarly layered waveguides. *Appl. Anal*, http://dx.doi.org/10.1080/00036811.2013.794940, 27 pages, 2013.

[4] R. Castillo-Pérez, V. V. Kravchenko, H. Oviedo-Galdeano, and V. S. Rabinovich, Dispersion equation and eigenvalues for quantum wells using spectral parameter power series. *J. Math. Phys.* **52** (2011), pp 1-10.

[5] R. Castillo-Pérez, K. V. Khmelnytskaya, V. V. Kravchenko, and H. Oviedo-Galdeano, Efficient calculation of the reflectance and transmittance of finite inhomogeneus layer. *J. Opt. A: Pure Appl. Opt*, **11** (2009), pp 1-6.

[6] A. Chandra and L. F Eastman, Quantum mechanical reflection at triangular 'planar-doped' potential barriers for transistors. *J. Appl. Phys.* **53** (1982), pp 9165-9169.

[7] D. N. Christodoulides, A. G. Andreou, R. I. Joseph, and C. R. Westgate, Analytical calculation of the quantum-mechanical transmission coefficient for a triangular, planar-doped potential barrier. *Solid-State Electron.* **28** (1985), pp 821-822.

[8] K. H.Gundlach, Zur Berechnung des Tunnelstroms durch eine trapezförmige Potentialstufe. *Solid-State Electron.* **9** (1966), pp 949-957.

[9] P. Harrison, *Quantum Wells, Wires and Dots, Second Edition*, Willey-Interscience, Chichister 2005, pp 459.

[10] K. V Khmelnytskaya, V. V. Kravchenko, and H. C. Rosu, Eigenvalue problems, spectral parameter power series, and modern applications, arXiv preprint arXiv:1112.1633 (2011).

[11] V. V Kravchenko, *Applied Pseudoanalytic Theory*, Birkhäuser Verlag, Basel 2009, pp 73-91.

[12] V. V Kravchenko, A representation for solutions of the Sturm-Liouville equation. *Complex Variables* **53** (2008), pp 775-789.

[13] V. V. Kravchenko and R. M. Porter, Spectral parameter power series for Sturm-Liouville problems. *Math. Meth. Appl. Sci*. **33** (2010), pp 459-468.

[14] L. D. Landau and E. M. Lisfhitz, *Quantum Mechanics, Non-relativistic theory*, Pergamon Press, 1977, pp 79-81.

[15] W. W. Lui and M. Fukuma, Exact solution of the Schrödinger equation across an arbitrary one-dimensional piecewise-linear potential barrier. *J. Appl. Phys.* **60** (1986), pp 1555-1559.

[16] A. Messiah, *Quantum Mechanics*, *Volume I*. North Holland Publishing Company, Amsterdam 1961, pp 231-242.

[17] P. Su, Z. Cao, K. Chen, C. Yin, and Q. Shen, Explicit expression for the reflection and transmission probabilities through an arbitrary potential barrier. *J. Phys. A: Math. Theor*y. **41** (2008), pp1-7.

[18] N. Zettili, *Quantum Mechanics*: concepts and applications, Willey-Interscience, Chichister 2009, pp 224-229.