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On Dispersion Decay for Discrete Wave Equations

Elena Kopylova*
Faculty of Mathematics Vienna University
and Institute for Information Transmission Problems
Russian Academy of Sciences

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Abstract

We derive dispersion estimates for solutions of the one-dimensional discrete wave equations. In particular, we weaken the conditions on the potentials of previous works.

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1 Introduction

We are concerned with the one-dimensional discrete wave equation

$$\ddot{u}(t) = -\mathrm{H}u, \quad \mathrm{H} := -\Delta_L + q, \quad t \in \mathbb{R} \tag{1.1}$$

with a real potential q. Here Δ_L is the discrete Laplacian given by

$$(\Delta_L u)_n = u_{n+1} - 2u_n + u_{n-1}, \quad n \in \mathbb{Z}.$$

In matrix form (1.1) reads

$$\mathrm{i}\dot{\mathbf{u}}(t) = \mathbf{H}\mathbf{u}(t), \quad t \in \mathbb{R}, \tag{1.2}$$

where

$$\mathbf{u}_n(t) = \big(u_n(t), \dot{u}_n(t)\big), \quad \mathbf{H} = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i}\mathrm{H} & 0 \end{pmatrix}$$

We suppose that the potential q satisfies

$$|q_n| \le C(1+|n|)^{-\beta}, \quad n \in \mathbb{Z} \tag{1.3}$$

with some $\beta > 3$. We will use the weighted spaces $\ell^2_\sigma = \ell^2_\sigma(\mathbb{Z})$ with the norm

$$\|u\|_{\ell^2_\sigma} = \|(1+|n|)^\sigma u\|_{\ell^2}, \quad \sigma \in \mathbb{R}.$$

*E-mail address: Elena.Kopylova@univie.ac.at

Denote

$$B(\sigma,\sigma')=\mathcal{L}(\ell^2_\sigma,\ell^2_{\sigma'}),\quad \mathbf{B}(\sigma,\sigma')=\mathcal{L}(\ell^2_\sigma\oplus\ell^2_\sigma,\ell^2_{\sigma'}\oplus\ell^2_{\sigma'})$$

the spaces of bounded linear operators from ℓ^2_σ to $\ell^2_{\sigma'}$ and from $\ell^2_\sigma\oplus\ell^2_\sigma$ to $\ell^2_{\sigma'}\oplus\ell^2_{\sigma'}$, respectively. We restrict ourselves to the non-singular case, when the boundary points $\lambda=0,4$ of the spectrum are not resonances for the operator $\mathrm{H}=-\Delta_L+q$.

Our main results are as follows. In the non-singular case the following asymptotics hold

$$e^{-it\mathbf{H}}P_c=\mathcal{O}(t^{-3/2}),\quad t\to\infty \tag{1.4}$$

in $\mathbf{B}(\sigma,-\sigma)$ with $\sigma>5/2$. Here P_c is the Riesz projection in $\ell^2\oplus\ell^2$ onto the (absolutely) continuous spectrum of $\mathbf{H}$.

In this respect we recall that under the condition (1.3) it is well-known that the spectrum of H consists of a purely absolutely continuous part covering $[0,4]$ plus a finite number of eigenvalues located in $\mathbb{R}\setminus[0,4]$. In addition, there could be resonances at the boundary of the continuous spectrum.

The dispersion decay of type (1.4) has been obtained for the first time in [6] for discrete Schrödinger, wave and Klein–Gordon equations with compactly supported potentials (the discrete Klein–Gordon equation corresponds to $\mathrm{H}=-\Delta_L+m^2+q$ with $m>0$ in (1.1)). The result has been generalized in [8] to discrete Schrödinger equation with non-compactly supported potentials under the decay condition (1.3) with $\beta>5$. Recently in [2] the dispersion decay was obtained under condition $\sum_{\mathbb{Z}}|n|^2|q_n|<\infty$ for discrete Schrödinger and Klein–Gordon equations and under condition

$$\sum_{n\in\mathbb{Z}}|n|^3|q_n|<\infty \tag{1.5}$$

for discrete wave equation. The result of [2] is based on generalization of the van der Corput lemma together with the novel fact that the scattering data associated with H are in the Wiener algebra.

Here we improve the result [2] for the wave equation by reducing the decay rate (1.5) to (1.3) with $\beta=3$. We adapt to the discrete casethe approach of [7], which relies on the Puiseux expansions of the resolvent at the edge points of the continuous spectrum.

2 Free equation

Here we consider the free equation (1.2) with $q=0$:

$$\mathrm{i}\dot{\mathbf{u}}(t)=\mathbf{H}_0\mathbf{u}(t),\quad t\in\mathbb{R}, \tag{2.1}$$

where

$$\mathbf{H}_0=\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i}\mathrm{H}_0 & 0\end{pmatrix},\quad \mathrm{H}_0=-\Delta_L.$$

It is well-known that H_0 is self-adjoint and the discrete Fourier transform

$$\hat{u}(\theta)=\sum_{n\in\mathbb{Z}}u_n\mathrm{e}^{\mathrm{i}\theta n},\quad \theta\in\mathbb{T}:=\mathbb{R}/2\pi\mathbb{Z}.$$

maps H_0 to the operator of multiplication by $\phi(\theta) = 2 - 2\cos\theta$:

$$-\widehat{\Delta_L u}(\theta) = \phi(\theta)\widehat{u}(\theta).$$

In particular, the spectrum $\mathrm{Spec}(H_0) = [0,4]$ is purely absolutely continuous.

We will use the notation $[K]_{n,k}$ for the kernel of an operator K, that is,

$$(Ku)_n = \sum_{k\in\mathbb{Z}} [K]_{n,k} u_k, \quad n \in \mathbb{Z},$$

The kernel of the resolvent $R_0(\omega) = (H_0 - \omega)^{-1}$ is given by

$$[R_0(\omega)]_{n,k} = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{e^{-i\theta(n-k)}}{\phi(\theta) - \omega} d\theta = \frac{e^{-i\theta(\omega)|n-k|}}{2i\sin\theta(\omega)}, \quad \omega \in \Xi := \mathbb{C} \setminus [0,4], \tag{2.2}$$

$n,k \in \mathbb{Z}$. Here $\theta(\omega)$ is the unique solution of the equation

$$2 - 2\cos\theta = \omega, \quad \theta \in \Sigma := \{-\pi \le \mathrm{Re}\,\theta \le \pi,\ \mathrm{Im}\,\theta < 0\}/2\pi\mathbb{Z}. \tag{2.3}$$

Observe that $\theta \mapsto \omega = 2 - 2\cos\omega$ is a biholomorphic map from $\Sigma \to \Xi$.

Next we collect some properties obtained in [6].

Lemma 2.1. *For* $R_0(\omega)$ *the following properties hold:*

P1 *The resolvent* $R_0(\omega)$ *is an analytic function with values in* $B(0,0)$ *for* $\omega \in \Xi$.

P2 *For* $\omega \in (0,4)$ *the limiting absorption principle holds, which is the convergence*

$$R_0(\omega \pm i\varepsilon) \to R_0(\omega \pm i0), \quad \varepsilon \to 0+ \tag{2.4}$$

in $B(\sigma, -\sigma)$ *with* $\sigma > 1/2$.

P3 *At the edge points* $\mu_- = 0$ *and* $\mu_+ = 4$ *the following asymptotics hold*

$$R_0(\omega) = A_\pm(\omega - \mu_\pm)^{-1/2} + B_\pm + O(|\omega - \mu_\pm|^{1/2}), \quad \omega \to \mu_\pm, \quad \omega \in \Xi \tag{2.5}$$

in $B(\sigma, -\sigma)$ *with* $\sigma > 5/2$. *Here* $A_\pm$, $B_\pm$ *are the operators associated with the kernels*

$$[A_\pm]_{n,k} = \frac{i}{2}(\mp 1)^{n-k+1}, \quad [B_\pm]_{n,k} = -\frac{1}{2}|n-k|(\mp 1)^{n-k+1}, \tag{2.6}$$

respectively.

P4 *The asymptotics* (2.5) *can be differentiated twice with respect to* ω:

$$\begin{aligned} R_0'(\omega) &= -\tfrac{1}{2} A_\pm (\omega - \mu_\pm)^{-3/2} + O(|\omega - \mu_\pm|^{-1/2}), \\ R_0''(\omega) &= \tfrac{3}{4} A_\pm (\omega - \mu_\pm)^{-5/2} + O(|\omega - \mu_\pm|^{-3/2}), \end{aligned} \quad \omega \to \mu_\pm, \quad \omega \in \Xi, \tag{2.7}$$

in $B(\sigma, -\sigma)$ *with* $\sigma > 5/2$.

Now we turn to the free wave equation. The resolvent $\mathbf{R}_0(\lambda) = (\mathbf{H}_0 - \lambda)^{-1}$ can be expressed in terms of R_0 (see [6]):

$$\mathbf{R}_0(\lambda) = \begin{pmatrix} \lambda \mathrm{R}_0(\lambda^2) & \mathrm{i}\mathrm{R}_0(\lambda^2) \\ -\mathrm{i}(1+\lambda^2 \mathrm{R}_0(\lambda^2)) & \lambda \mathrm{R}_0(\lambda^2) \end{pmatrix}, \quad \lambda \in \mathbb{C} \setminus [-2,2]. \tag{2.8}$$

Then properties **P1**–**P4** imply the corresponding properties of $\mathbf{R}_0$. In particular,

$$[\mathbf{R}_0]^{12}(\lambda) = iA_-\lambda^{-1} + iB_- + O(\lambda), \quad \lambda \to 0, \quad \lambda \in \mathbb{C} \setminus [-2,2]. \tag{2.9}$$

where $[\cdot]^{ij}$ denotes the ij entry of the corresponding matrix operator.

The continuous spectrum of $\mathbf{H}_0$ coincides with $[-2,2]$. For the kernel of the free propagator the following spectral representation holds

$$[\mathrm{e}^{-\mathrm{i}t\mathbf{H}_0}]_{n,k} = \frac{1}{2\pi\mathrm{i}} \int\limits_{(-2,0)\cup(0,2)} \mathrm{e}^{-\mathrm{i}t\lambda} [\mathbf{R}_0(\lambda+\mathrm{i}0) - \mathbf{R}_0(\lambda - \mathrm{i}0)]_{n,k}\, d\lambda. \tag{2.10}$$

Due to (2.9) $[\mathbf{R}_0]^{12}(\lambda+\mathrm{i}0) - [\mathbf{R}_0]^{12}(\lambda-\mathrm{i}0) \sim \lambda^{-1}$ and then the first component $u_n(t)$ of the solution of the free wave equation (2.1) does not decay as $t \to \pm\infty$.

Remark 2.2. (see [2]). Note that the first component of the solution is given by

$$u_n(t) = \sum_{m\in\mathbb{Z}} c_{n-m}(t) u_m(0) + s_{n-m}(t) \dot{u}_m(0), \tag{2.11}$$

where

$$c_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\sqrt{1-\cos\theta}\,\sqrt{2}t) \mathrm{e}^{\mathrm{i}\theta n} d\theta = J_{2|n|}(2t), \tag{2.12}$$

$$s_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(\sqrt{1-\cos\theta}\,\sqrt{2}t)}{\sqrt{1-\cos\theta}} \mathrm{e}^{\mathrm{i}\theta n} d\theta = \int_0^t c_n(s) ds$$

$$= \frac{t^{2|n|+1}}{2^{|n|}(|n|+1)!}\, {}_1F_2\Big(\frac{2|n|+1}{2}; (\frac{2|n|+3}{2}, 2|n|+1); -t^2\Big). \tag{2.13}$$

Here $J_n(x)$, ${}_pF_q(\underline{u};\underline{v};x)$ denote the Bessel and generalized hypergeometric functions, respectively. In particular, while $c_n(t) = O(t^{-1/2})$ for fixed n, we have $s_n(t) = \frac{1}{2} + O(t^{-1/2})$ for fixed n.

3 Ruiseux expansion of resolvent

Consider the resolvent $\mathrm{R}(\omega) = (\mathrm{H} - \omega)^{-1}$, $\omega \in \Xi$ of perturbed operator H. The next lemma is a consequence of Lemma 3.3 from [2] (see also [8])

Lemma 3.1. *Let q satisfies* (1.3) *with $\beta > 1$. Then the convergence*

$$\mathrm{R}(\omega \pm \mathrm{i}\varepsilon) \to \mathrm{R}(\omega \pm \mathrm{i}0), \quad \varepsilon \to 0+, \quad \omega \in (0,4) \tag{3.1}$$

holds in $B(\sigma, -\sigma)$ with $\sigma > 1/2$.

The resolvent $\mathbf{R}(\omega)=(\mathbf{H}-\omega)^{-1}$ can be expressed in terms of $\mathrm{R}(\omega)$ (see [6]):

$$\mathbf{R}(\omega)=\begin{pmatrix}\omega\mathrm{R}(\omega^2) & \mathrm{i}\mathrm{R}(\omega^2)\\ -\mathrm{i}(1+\omega^2\mathrm{R}(\omega^2)) & \omega\mathrm{R}(\omega^2)\end{pmatrix}. \tag{3.2}$$

Representation (3.2) and Lemma 3.1 imply the limiting absorption principle for the perturbed resolvent:

Lemma 3.2. *Suppose* (1.3) *with* $\beta>1$ *holds. Then for* $\lambda\in(-2,0)\cup(0,2)$ *the convergence*

$$\mathbf{R}(\lambda\pm\mathrm{i}\varepsilon)\to\mathbf{R}(\lambda\pm\mathrm{i}0),\quad \varepsilon\to 0+,$$

holds in $\mathbf{B}(\sigma,-\sigma)$ *with* $\sigma>1/2$.

Now we consider $\mathrm{R}(\omega)$ near the edge points $\mu_-=0$ and $\mu_+=4$

Definition 3.3. Any nonzero function $u\in\ell^\infty(\mathbb{Z})$ satisfying the equation $\mathrm{H}u=\mu_-u$ (*or* $\mathrm{H}u=\mu_+u$) is called a resonance function, and in this case the point μ_- (or μ_+) is called a resonance.

Below we assume that

$$\textbf{Spectral condition}\text{: The points } \mu_\pm \text{ are no resonances.} \tag{3.3}$$

The condition is equivalent to the boundedness of the resolvent $\mathrm{R}(\omega)$ at the edge points of the continuous spectrum:

Lemma 3.4. *(see [2, Lemma 4.3 and Corollary 4.4]). Let* (1.3) *with* $\beta>2$ *holds. Then condition* (3.3) *is equivalent to the boundedness of the families*

$$\{\mathrm{R}(\omega),\ |\omega-\mu_\pm|\le\varepsilon,\ \omega\in\Xi\} \tag{3.4}$$

in $B(\sigma,-\sigma)$ *with* $\sigma>3/2$ *for sufficiently small* $\varepsilon>0$.

Further we prove that this boundedness provides the asymptotics (1.4).

The Born decomposition formulas

$$\mathrm{R}(\omega)=(1+\mathrm{R}_0(\omega)\mathrm{q})^{-1}\mathrm{R}_0(\omega),\qquad \mathrm{R}(\omega)=\mathrm{R}_0(\omega)(1+\mathrm{q}\mathrm{R}_0(\omega))^{-1} \tag{3.5}$$

imply

$$(1+\mathrm{R}_0(\omega)\mathrm{q})^{-1}=1-\mathrm{R}(\omega)\mathrm{q},\qquad (1+\mathrm{q}\mathrm{R}_0(\omega))^{-1}=1-\mathrm{q}\mathrm{R}(\omega). \tag{3.6}$$

Hence, since $q\in B(\sigma,\sigma+\beta)$, we obtain from the previous lemma that for any $\sigma\in(1/2,\beta-1/2)$ the operators $(1+\mathrm{R}_0(\omega)\mathrm{q})^{-1}$ and $(1+q\mathrm{R}_0(\omega))^{-1}$ are bounded in $B(-\sigma,-\sigma)$ and $B(\sigma,\sigma)$, respectively. In particular, using the following formulas for the derivatives of R (cf. [4, 5]):

$$\mathrm{R}'=(1+\mathrm{R}_0\mathrm{q})^{-1}\mathrm{R}_0'(1+\mathrm{q}\mathrm{R}_0)^{-1},\ \mathrm{R}''=\left[(1+\mathrm{R}_0\mathrm{q})^{-1}\mathrm{R}_0''-2\mathrm{R}'\mathrm{q}\mathrm{R}_0'\right](1+\mathrm{q}\mathrm{R}_0)^{-1}. \tag{3.7}$$

for $\beta>3$ we obtain

$$\mathrm{R}'(\omega\pm\mathrm{i}\varepsilon)\to\mathrm{R}'(\omega\pm\mathrm{i}0),\quad \mathrm{R}''(\omega\pm\mathrm{i}\varepsilon)\to\mathrm{R}''(\omega\pm\mathrm{i}0),\quad \varepsilon\to0+,\quad \omega\in(0,4), \tag{3.8}$$

in $B(\sigma,-\sigma)$ with $\sigma>\frac{5}{2}$. Our next task will be to obtain asymptotics of the resolvent $\mathrm{R}(\omega)$ at the edge points $\mu_\pm$. We start with the following lemma:

Lemma 3.5. *Assume* (3.3)*, suppose* (1.3) *holds for some* $\beta > 2$*, and let* $\sigma \in (3/2, \beta - 1/2)$*. Then*

$$\|(1 + \mathrm{R}_0(\omega)\mathrm{q})^{-1}\alpha^{\pm}\|_{\ell^2_{-\sigma}} = \mathcal{O}(|\omega - \mu_{\pm}|^{1/2}), \quad \omega \to \mu_{\pm}, \quad \omega \in \Xi, \tag{3.9}$$

and

$$\sum_n \alpha_n^{\pm}[(1 + q\mathrm{R}_0(\omega))^{-1}\mathrm{f}]_{\mathrm{n}} = \mathcal{O}(|\omega - \mu_{\pm}|^{1/2}), \quad \omega \to \mu_{\pm}, \quad \omega \in \Xi, \tag{3.10}$$

for any $f \in \ell^2_{\sigma}$*, where* $\alpha_n^{\pm} = (\mp 1)^n$*.*

In particular,

$$(1 + \mathrm{R}_0(\omega)\mathrm{q})^{-1}\mathrm{A}_{\pm}(1 + \mathrm{q}\mathrm{R}_0(\omega))^{-1} = \mathcal{O}(|\omega - \mu_{\pm}|), \quad \omega \to \mu_{\pm}, \quad \omega \in \Xi, \tag{3.11}$$

in $B(\sigma, -\sigma)$*, where* $A_{\pm}$ *is given in* (2.6)*.*

Proof. The asymptotics (2.5) imply

$$\mathrm{R}(\omega) = (1 + \mathrm{R}_0(\omega)\mathrm{q})^{-1}\mathrm{R}_0(\omega) = (1 + \mathrm{R}_0(\omega)\mathrm{q})^{-1}[\mathrm{A}_{\pm}(\omega - \mu_{\pm})^{-1/2} + \mathcal{O}(1)],$$

$$\mathrm{R}(\omega) = \mathrm{R}_0(\omega)(1 + \mathrm{q}\mathrm{R}_0(\omega))^{-1} = [\mathrm{A}_{\pm}(\omega - \mu_{\pm})^{-1/2} + \mathcal{O}(1)](1 + \mathrm{q}\mathrm{R}_0(\omega))^{-1}.$$

and the claim follows from the continuity of $\mathrm{R}(\omega)$, $(1 + \mathrm{R}_0(\omega)\mathrm{q})^{-1}$, and $(1 + q\mathrm{R}_0(\omega))^{-1}$ in $B(-\sigma, -\sigma)$ and $B(\sigma, \sigma)$, respectively. The last claim follows since $A_{\pm} = \frac{1}{2\mathrm{i}}\alpha^{\pm} \otimes \alpha^{\pm}$. □

Lemma 3.6. *Suppose* (1.3) *holds for some* $\beta > 3$ *and* (3.3) *holds. Then we have the following asymptotics in* $B(\sigma, -\sigma)$ *with* $\sigma > 5/2$

$$\begin{aligned} &\mathrm{R}(\omega) = \mathrm{R}_{\pm} + \mathcal{O}(|\omega - \mu_{\pm}|^{1/2}), \\ &\mathrm{R}'(\omega) = \mathcal{O}(|\omega - \mu_{\pm}|^{-1/2}), \qquad \omega \to \mu_{\pm}, \quad \omega \in \Xi. \\ &\mathrm{R}''(\omega) = \mathcal{O}(|\omega - \mu_{\pm}|^{-3/2}), \end{aligned} \tag{3.12}$$

Proof. Asymptotics (2.5), (3.9)–(3.11), and formulas (3.7) imply

$$\mathrm{R}'(\omega) = \mathcal{O}(|\omega - \mu_{\pm}|^{-1/2}), \quad \mathrm{R}''(\omega) = \mathcal{O}(|\omega - \mu_{\pm}|^{-3/2}), \quad \omega \to \mu_{\pm}, \quad \omega \in \Xi \tag{3.13}$$

in $B(\sigma, -\sigma)$ with $\sigma > 5/2$. The asymptotics (3.13) coincide with the asymptotics (3.12) for the derivatives. Asymptotics (3.12) for $\mathrm{R}(\omega)$ can be obtained by integration of asymptotics (3.12) for the first derivative. □

Then representation (3.2) and Lemma 3.6 imply

Corollary 3.7. *Let conditions* (1.3) *and* (3.3) *hold. Then the following asymptotics hold*

$$\begin{aligned} &\mathbf{R}(\lambda) = \mathbf{R}_{\pm} + \mathcal{O}(|\lambda \mp 2|^{1/2}), \\ &\mathbf{R}'(\lambda) = \mathcal{O}(|\lambda \mp 2|^{-1/2}), \qquad \lambda \to \pm 2, \quad \lambda \in \mathbb{C} \setminus [-2, 2] \\ &\mathbf{R}''(\lambda) = \mathcal{O}(|\lambda \mp 2|^{-3/2}), \end{aligned} \tag{3.14}$$

in $\mathbf{B}(\sigma, -\sigma)$ *with* $\sigma > 5/2$*.*

Corollary 3.8. *The resolvent* $\mathbf{R}(\omega)$ *is analytic function of* ω *in* $\{|\omega| \le \delta, \pm\mathrm{Im}\,\omega \ge 0\}$ *for some small* $\delta > 0$*.*

4 Dispersion decay

Theorem 4.1. *Let conditions* (1.3) *with* $\beta > 3$ *and* (3.3) *hold. Then asymptotics* (1.4) *hold, i.e.*

$$e^{-it\mathbf{H}} P_c = \mathcal{O}(t^{-3/2}), \quad t \to \infty. \tag{4.1}$$

in $\mathbf{B}(\sigma, -\sigma)$ *with* $\sigma > 5/2$.

Proof. For the dynamical group associated with the perturbed wave equation (1.2) the spectral representation holds (cf. [6]):

$$e^{-it\mathbf{H}} P_c = \frac{1}{2\pi i} \int\limits_{[-2,2]} e^{-it\lambda}(\mathbf{R}(\lambda + i0) - \mathbf{R}(\lambda - i0))\, d\lambda = \int\limits_{[-2,2]} e^{-it\lambda} F(\lambda) d\lambda, \tag{4.2}$$

where $F(\lambda) = \dfrac{1}{\pi} \operatorname{Im} \mathbf{R}(\lambda + i0)$. The asymptotic expansion of $F(\lambda)$ at the points ± 2 can be deduced from (3.14). Thus we obtain

$$\begin{aligned} F(\lambda) &= \mathcal{O}(|\lambda \mp 2|^{1/2}), \\ F'(\lambda) &= \mathcal{O}(|\lambda \mp 2|^{-1/2}), \qquad \lambda \to \pm 2, \quad \lambda \in (-2, 2). \\ F''(\lambda) &= \mathcal{O}(|\lambda \mp 2|^{-3/2}), \end{aligned}$$

Hence the desired decay for large t follows from Lemma 4.2 below. □

The following lemma is a special case of [4, Lemma 10.2].

Lemma 4.2 ([4])**.** *Assume* $\mathcal{B}$ *is a Banach space,* $a > 0$*, and* $F \in C(0, a; \mathcal{B})$ *satisfies* $F(0) = F(a) = 0$*,* $F'' \in L^1_{loc}(0, a; \mathcal{B})$*, as well as* $F''(\lambda) = \mathcal{O}(\lambda^{-3/2})$ *and* $F''(a - \lambda) = \mathcal{O}(\lambda^{-3/2})$ *as* $\lambda \to 0+$. *Then*

$$\int\limits_0^a e^{-it\lambda} F(\lambda) d\lambda = \mathcal{O}(t^{-3/2}), \quad t \to \infty.$$

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