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BESOV SPACES ASSOCIATED WITH OPERATORS

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Abstract

Recent work of Bui, Duong and Yan in [2] defined Besov spaces associated with a certain operator L under the weak assumption that L generates an analytic semigroup e^{-tL} with Poisson kernel bounds on $L^2(\mathcal{X})$ where $\mathcal{X}$ is a (possibly non-doubling) quasi-metric space of polynomial upper bound on volume growth. This note aims to extend certain results in [2] to a more general setting when the underlying space can have different dimensions at 0 and infinity. For example, we make some extensions to the Besov norm equivalence result in Proposition 4.4 of [2], such as to more general class of functions with suitable decay at 0 and infinity, and to non-integer $k \geq 1$.

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1 Introduction

The theory of Besov spaces has been an active area of research in the last few decades because of its important role in the study of approximation of functions and regularity of solutions to partial differential equations.

Classical theory of Besov spaces, for example, can be found in [3, 4, 9, 14, 13, 16, 17]. Some of more recent results on Besov spaces are [15, 18, 10, 8].

Recent work of Bui, Duong and Yan in [2] defined Besov spaces associated with a certain operator L under the weak assumption that L generates an analytic semigroup e^{-tL} with Poisson kernel bounds on $L^2(\mathcal{X})$ where $\mathcal{X}$ is a (possibly non-doubling) quasi-metric space of polynomial upper bound on volume growth. When L is the Laplace operator $-\Delta$ or its square root $\sqrt{-\Delta}$ acting on the Euclidean space $\mathbb{R}^n$, this class of Besov spaces associated with the operator L are equivalent to the classical Besov spaces. Depending on the choice of L, the Besov spaces are natural settings for generic estimates for certain singular integral operators such as the fractional powers L^α.

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This note aims to extend certain results in [2] to a more general setting when the underlying space can have different dimensions at 0 and infinity, that is, for some $n > 0$, $N \geq 0$, and $C > 0$,

$$\mu(B(x,r)) \leq \begin{cases} Cr^n, & 0 < r \leq 1 \\ Cr^N, & 1 < r < \infty \end{cases}$$

for all balls B. Here n is the local dimension and N is the global dimension or the dimension at infinity.

An example of this case is in Lie groups of polynomial growth (see, for example, [1]). Consider when L is the Laplace operator Δ_N with Neumann boundary conditions on a bounded Lipschitz domain Ω of $\mathbb{R}^n$. See, for example, [6]. The heat kernel $p_t(x,y)$ in this case satisfies

$$\begin{aligned} 0 \leq p_t(x,y) &\leq \frac{C}{V(x, \sqrt{t})} e^{-\alpha|x-y|^2/t} \\ &= C \max\left\{\frac{1}{t^{n/2}}, 1\right\} e^{-\alpha|x-y|^2/t} \\ &= \begin{cases} \dfrac{C}{t^{n/2}} e^{-\alpha|x-y|^2/t}, & 0 < t \leq 1 \\ Ce^{-\alpha|x-y|^2/t}, & 1 < t < \infty \end{cases} \end{aligned}$$

for some positive constants C and α, where $V(x, \sqrt{t})$ denotes the volume of the ball with centre x and radius $\sqrt{t}$ in $\mathbb{R}^n$. In this case N can be chosen to be 0, so that $V(x, \sqrt{t})$ is bounded by a constant.

While many results in [2] carry over to this note, there are some difficulties with the change in dimension. We omit most of the results that carry forward with similar proofs. Instead of using Poisson kernel bounds (polynomial type), which posed some technical difficulties, we use Gaussian kernel bounds (exponential type), which is a stronger assumption.

In Proposition 4.4 of [2] it was shown that the Besov norms defined by $t^k L^k e^{-tL}$ are equivalent to one another for positive $k \geq 1$. In this note we aim to answer the open, interesting question of extending that result to more general class of functions $\Psi_t(L)$ with suitable decay at 0 and infinity. We also make some more extensions to the Besov norm equivalence result in Proposition 4.4 of [2], such as to non-integer $k \geq 1$.

The paper is organized as follows. In Section 2, we give definitions of quasi-metric spaces of polynomial upper bounds on volume growth, then some assumptions on the operator L, and define Besov norms associated with L. We also give an upper bound estimate of the Besov norm of the heat kernels.

In Section 3, we introduce the space of test functions associated with L. We then define Besov norms for linear functionals (on space of test functions) and Besov spaces associated with L.

In Section 4, we study an embedding theorem for the Besov spaces. We also study the equivalence of the Besov norms with respect to different functions of L. We extend the Besov norm equivalence to more general class of functions $\Psi_t(L)$ with suitable decay at 0 and infinity, and to non-integer $k \geq 1$.

2 Besov norms associated with operators

2.1 Spaces of polynomial upper bounds on volume growth

Assume $\mathcal{X}$ is a quasi-metric measure space satisfying

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$ for all $x,y \in \mathcal{X}$;

(iii) There exists a constant $C \in [1,\infty)$ such that for all x,y and $z \in \mathcal{X}$,

$$d(x,y) \leq C(d(x,z) + d(z,y));$$

(iv) For some $n > 0$, $N \geq 0$, and $C > 0$,

$$\mu(B(x,r)) \leq \begin{cases} Cr^n, & 0 < r \leq 1 \\ Cr^N, & 1 < r < \infty \end{cases}$$

for all balls B. Here n is the local dimension and N is the global dimension or the dimension at infinity.

Throughout the paper we will use $C, c, \ldots$ to denote positive constants; these may not be the same on any two consecutive appearances.

The following estimate will be frequently used in the paper.

Lemma 2.1. *Let $1 \leq p \leq \infty$. For every $\alpha > 0$, there exists $C > 0$ such that*

$$\int_{\mathcal{X}} \left[e^{-\alpha d(x,y)^2/t}\right]^p d\mu(x) \leq \begin{cases} Ct^{n/2}, & 0 < t \leq 1 \\ Ct^{N/2}, & 1 < t < \infty \end{cases}$$

for $y \in \mathcal{X}$.

Proof. Fix $y \in \mathcal{X}$. For $p = \infty$, we clearly have

$$\sup_x e^{-\alpha d(x,y)^2/t} \leq C, \quad 0 < t \leq 1$$
$$\sup_x e^{-\alpha d(x,y)^2/t} \leq C, \quad 1 < t < \infty$$

Next suppose $1 \le p < \infty$. For $0 < t \le 1$ we have

$$\begin{aligned}
&\int_X \left[e^{-\alpha d(x,y)^2/t}\right]^p \, d\mu(x) \\
&\quad \le \int_{B(y,\sqrt{t})} \left[e^{-\alpha d(x,y)^2/t}\right]^p \, d\mu(x) \\
&\qquad + \sum_{k=1}^{\infty} \int_{2^{k-1}\sqrt{t} \le d(x,y) < 2^k \sqrt{t}} \left[e^{-\alpha d(x,y)^2/t}\right]^p \, d\mu(x) \\
&\quad \le C \left\{ t^{n/2} + \sum_{k=1}^{\infty} \left[e^{-\alpha (2^{2k}) t}\right]^p (2^k \sqrt{t})^N \right\} \\
&\quad \le C \left\{ t^{n/2} + \sum_{k=1}^{\infty} \left[e^{-\alpha (2^{2k}) t}\right]^p 2^{kN} t^{n/2} \right\} \quad \text{for } n < N \\
&\quad \le C t^{n/2}.
\end{aligned}$$

For $1 < t < \infty$ we have

$$\begin{aligned}
&\int_X \left[e^{-\alpha d(x,y)^2/t}\right]^p \, d\mu(x) \\
&\quad \le \int_{B(y,\sqrt{t})} \left[e^{-\alpha d(x,y)^2/t}\right]^p \, d\mu(x) \\
&\qquad + \sum_{k=1}^{\infty} \int_{2^{k-1}\sqrt{t} \le d(x,y) < 2^k \sqrt{t}} \left[e^{-\alpha d(x,y)^2/t}\right]^p \, d\mu(x) \\
&\quad \le C \left\{ t^{N/2} + \sum_{k=1}^{\infty} \left[e^{-\alpha (2^{2k}) t}\right]^p (2^k \sqrt{t})^N \right\} \\
&\quad \le C t^{N/2}.
\end{aligned}$$

Hence the inequalities follow. □

2.2 Assumptions on operators

Assume L is densely-defined on $L^2(\mathcal{X})$ and satisfies

(S) L generates a holomorphic semigroup e^{-zL} for $z = t + is$ with $t > 0$ and $|\arg z| < \rho$ for some $\rho > 0$,

(K) the heat kernel of L satisfies bounds of Gaussian type, i.e. the kernel $p_t(x, y)$ of e^{-tL} satisfies

$$|p_t(x,y)| \le \begin{cases} \dfrac{C}{t^{n/2}} e^{-\alpha d(x,y)^2/t}, & 0 < t \le 1 \\ \dfrac{C}{t^{N/2}} e^{-\alpha d(x,y)^2/t}, & 1 < t < \infty \end{cases}$$

for some $C > 0$ and for all $x, y \in \mathcal{X}$.

(H) the kernels $p_t(x,y)$ of e^{-tL} satisfy the Hölder continuity estimates

$$|p_t(x,y) - p_t(x,y')| \leq \begin{cases} \dfrac{Cd(y,y')}{t^{n/2+1}} e^{-\alpha d(x,y)^2/t}, & 0 < t \leq 1 \\ \dfrac{Cd(y,y')}{t^{N/2+1}} e^{-\alpha d(x,y)^2/t}, & 1 < t < \infty \end{cases}$$

whenever $d(y,y') \leq d(x,y)/2$.

(C) L satisfies the conservation property $e^{-tL}1 = 1$. This is equivalent to

$$\int_X p_t(x,y)\, d\mu(y) = 1.$$

The following are some useful properties related to our assumptions.

Proposition 2.2. *For $k = 1,2,\ldots$, let $p_{k,t}(x,y)$ denote the kernel of the operator $t^k L^k e^{-tL}$.*

(a) Suppose L satisfies (S) and (K). Then $p_{k,t}(x,y)$ satisfies the size estimate (DK), i.e. for every $k \in \mathbb{N}$, there is a constant c_k satisfying

$$|p_{k,t}(x,y)| \leq \begin{cases} \dfrac{c_k}{t^{n/2}} e^{-\alpha_k d(x,y)^2/t}, & 0 < t \leq 1 \\ \dfrac{c_k}{t^{N/2}} e^{-\alpha_k d(x,y)^2/t}, & 1 < t < \infty \end{cases}$$

for all $x,y \in \mathcal{X}$.

(b) Suppose L satisfies (S), (K) and (H). Then $p_{k,t}(x,y)$ satisfies the Hölder estimate (DH), i.e. there is a constant c_k satisfying

$$|p_{k,t}(x,y) - p_{k,t}(x,y')| \leq \begin{cases} \dfrac{c_k d(y,y')}{t^{n/2+1}} e^{-\alpha_k d(x,y)^2/t}, & 0 < t \leq 1 \\ \dfrac{c_k d(y,y')}{t^{N/2+1}} e^{-\alpha_k d(x,y)^2/t}, & 1 < t < \infty \end{cases}$$

whenever $d(y,y') \leq d(x,y)/2$.

(c) Suppose L satisfies (C). Then we have

$$\int_X p_{k,t}(x,y)\, d\mu(y) = 0$$

for every $x \in \mathcal{X}$.

Proof. The proof is a modification of that in Proposition 2.2 in [2].

For a proof of part (a), we refer the reader to Theorem 3 in [7].

To show part (b), we first observe that

$$t^k L^k e^{-tL} = (-2)^k \left(\frac{d^k}{dt^k} e^{-\frac{t}{2}L} \right) e^{-\frac{t}{2}L}.$$

Next, by using assumption (H) and (DK), we obtain, for $0 < t \leq 1$,

$$\begin{aligned}
&|p_{k,t}(x,y) - p_{k,t}(x,y')| \\
&\quad = 2^k \left| \int_{\mathcal{X}} p_{k,\frac{t}{2}}(x,z)(p_{\frac{t}{2}}(z,y) - p_{\frac{t}{2}}(z,y'))d\mu(z) \right| \\
&\quad \leq c_k \int_{\mathcal{X}} \frac{1}{t^{n/2}} e^{-\alpha_k d(x,z)^2/t} \frac{d(y,y')}{t^{n/2+1}} e^{-\alpha_k d(z,y)^2/t} \, d\mu(z) \\
&\quad \leq \frac{c_k d(y,y')}{t^{n/2+1}} e^{-\alpha_k d(x,y)^2/t}.
\end{aligned}$$

Similarly, for $1 < t < \infty$, we have

$$\begin{aligned}
&|p_{k,t}(x,y) - p_{k,t}(x,y')| \\
&\quad = 2^k \left| \int_{\mathcal{X}} p_{k,\frac{t}{2}}(x,z)(p_{\frac{t}{2}}(z,y) - p_{\frac{t}{2}}(z,y'))d\mu(z) \right| \\
&\quad \leq c_k \int_{\mathcal{X}} \frac{1}{t^{N/2}} e^{-\alpha_k d(x,z)^2/t} \frac{d(y,y')}{t^{N/2+1}} e^{-\alpha_k d(z,y)^2/t} \, d\mu(z) \\
&\quad \leq \frac{c_k d(y,y')}{t^{N/2+1}} e^{-\alpha_k d(x,y)^2/t}.
\end{aligned}$$

Hence we have shown (b).

To show part (c), we just use assumption (C) and also

$$t^k L^k e^{-tL} = (-1)^k \left(\frac{d}{dt}\right)^k e^{-tL}.$$

Thus the proof of the proposition is finished. □

2.3 Besov norms associated with operators

Assume L satisfies (S) and (K). Let $k_t(x,y) = p_{1,t}(x,y)$ be the kernel of $\Psi_t(L) = tLe^{-tL}$. By Proposition 2.2, $k_t(x,y)$ satisfies

$$|k_t(x,y)| \leq \begin{cases} \frac{c}{t^{n/2}} e^{-\alpha d(x,y)^2/t}, & 0 < t \leq 1 \\ \frac{c}{t^{N/2}} e^{-\alpha d(x,y)^2/t}, & 1 < t < \infty. \end{cases}$$

Let f be a complex valued measurable function on $\mathcal{X}$ satisfying the following growth condition (G):

$$\int_{\mathcal{X}} |f(x)| e^{-\alpha d(x,y_0)^2} \, d\mu(x) < \infty$$

for some $y_0 \in \mathcal{X}$. Then we have that

$$\Psi_t(L) f(x) = \int_{\mathcal{X}} k_t(x,y) f(y) \, d\mu(y)$$

is defined for all $x \in \mathcal{X}$.

Definition 2.3. Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$ and $1 \leq p,q \leq \infty$. For any f satisfying (G), we define its $\dot{B}^{\alpha,L}_{p,q}$-norm by

$$\|f\|_{\dot{B}^{\alpha,L}_{p,q}} = \left\{ \int_0^\infty (t^{-\alpha}\|\Psi_t(L)f\|_p)^q \frac{dt}{t} \right\}^{1/q}$$

for $q < \infty$ and

$$\|f\|_{\dot{B}^{\alpha,L}_{p,q}} = \sup_{t>0} t^{-\alpha}\|\Psi_t(L)f\|_p$$

for $q = \infty$, whenever these are finite.

There exists functions with finite Besov norm but not necessarily smooth. In the following proposition we give an upper bound estimate of the Besov norm of the heat kernels. For any $k \in \mathbb{N}$, we denote $\Psi_{k,t}(L) = t^k L^k e^{-tL}$ to be the operator whose kernel is $p_{k,t}$; so $\Psi_{1,t}(L) = \Psi_t(L)$.

Proposition 2.4. *Let $-1 < \alpha < 1$ and $1 \le p,q \le \infty$. Suppose that f satisfies (S) and (K). Then for $k \in \mathbb{N}$ and $z \in \mathcal{X}$,*

$$\|p_{k,s}(\cdot,z)\|_{\dot{B}^{\alpha,L}_{p,q}} \le \begin{cases} C_n s^{-\alpha-n/2p'}, & 0 < s \le 1 \\ C_N s^{-\alpha-N/2p'}, & 1 < s < \infty \end{cases}$$

where $C_n > 0$ depends on α, n, k, p and q, and $C_N > 0$ depends on α, N, k, p and q.

Proof. The proof is a modification of that in Proposition 2.7 in [2].

Fix $k \in \mathbb{N}$. Using (DK) in Proposition 2.2, the kernel of the operator $\Psi_t(L)\Psi_{k,s}(L)$ is

$$\mathcal{K}_{t,s}(x,z) = \int_{\mathcal{X}} k_t(x,y) p_{k,s}(y,z)\, d\mu(y).$$

Let $\tilde{K}_t$ be a kernel satisfying

$$|\tilde{K}_t(x,z)| \le \begin{cases} \dfrac{c_k}{t^{n/2}} e^{-\alpha_k d(x,z)^2/t}, & 0 < t \le 1 \\ \dfrac{c_k}{t^{N/2}} e^{-\alpha_k d(x,z)^2/t}, & 1 < t < \infty \end{cases}$$

for all $x, z \in \mathcal{X}$. Then $\mathcal{K}_{t,s}$ satisfies the size estimate

$$|\mathcal{K}_{t,s}(x,z)| \le C \min\left\{\frac{s}{t}, \frac{t}{s}\right\} |\tilde{K}_{t+s}(x,z)|.$$

Put $\phi(y) = p_{k,s}(y,z)$, $y \in \mathcal{X}$. We then have that

$$\begin{aligned} &|\Psi_t(L)\phi(x)| \\ &\quad = |\mathcal{K}_{t,s}(x,z)| \\ &\quad \le \begin{cases} C \min\left\{\dfrac{s}{t}, \dfrac{t}{s}\right\} \dfrac{e^{-\alpha_k d(x,z)^2/(t+s)}}{(t+s)^{n/2}}, & 0 < t+s \le 1 \\ C \min\left\{\dfrac{s}{t}, \dfrac{t}{s}\right\} \dfrac{e^{-\alpha_k d(x,z)^2/(t+s)}}{(t+s)^{N/2}}, & 1 < t+s < \infty \end{cases} \end{aligned}$$

Hence, using Lemma 2.1,

$$\|\Psi_t(L)\phi\|_p \le \begin{cases} C\min\left\{\frac{s}{t},\frac{t}{s}\right\}(t+s)^{-n/2p'}, & 0<t+s\le 1 \\ C\min\left\{\frac{s}{t},\frac{t}{s}\right\}(t+s)^{-N/2p'}, & 1<t+s<\infty \end{cases}$$

Therefore, for $0<s\le 1$,

$$\begin{aligned} &\|\phi\|_{\dot{B}^{\alpha,L}_{p,q}} \\ &= \left\{\int_0^\infty (t^{-\alpha}\|\Psi_t(L)(\phi)\|_p)^q \frac{dt}{t}\right\}^{1/q} \\ &\le C\left\{\int_0^s \left(\frac{t^{1-\alpha}}{s(t+s)^{n/2p'}}\right)^q \frac{dt}{t} + \int_s^\infty \left(\frac{t^{-1-\alpha}s}{(t+s)^{n/2p'}}\right)^q \frac{dt}{t}\right\}^{1/q} \\ &\le Cs^{-\alpha-n/2p'}\left\{\int_0^1 \left(\frac{t^{1-\alpha}}{(1+t)^{n/2p'}}\right)^q \frac{dt}{t} + \int_1^\infty \left(\frac{t^{-1-\alpha}}{(1+t)^{n/2p'}}\right)^q \frac{dt}{t}\right\}^{1/q} \\ &\le Cs^{-\alpha-n/2p'}, \end{aligned}$$

where the constant C in the final inequality depends on α, n, k, p and q.

For $1<s<\infty$,

$$\begin{aligned} &\|\phi\|_{\dot{B}^{\alpha,L}_{p,q}} \\ &= \left\{\int_0^\infty (t^{-\alpha}\|\Psi_t(L)(\phi)\|_p)^q \frac{dt}{t}\right\}^{1/q} \\ &\le C\left\{\int_0^s \left(\frac{t^{1-\alpha}}{s(t+s)^{N/2p'}}\right)^q \frac{dt}{t} + \int_s^\infty \left(\frac{t^{-1-\alpha}s}{(t+s)^{N/2p'}}\right)^q \frac{dt}{t}\right\}^{1/q} \\ &\le Cs^{-\alpha-N/2p'}\left\{\int_0^1 \left(\frac{t^{1-\alpha}}{(1+t)^{N/2p'}}\right)^q \frac{dt}{t} + \int_1^\infty \left(\frac{t^{-1-\alpha}}{(1+t)^{N/2p'}}\right)^q \frac{dt}{t}\right\}^{1/q} \\ &\le Cs^{-\alpha-N/2p'}, \end{aligned}$$

where the constant C in the final inequality depends on α, N, k, p and q. □

3 Besov spaces associated with operators

3.1 Definitions of Besov spaces

Firstly, we use a similar approach as in [2] to define a "space of test functions".

Definition 3.1. Suppose L satisfies (S) and (K). Let $-1<\alpha<1$ and $1\le p,q\le\infty$. A function f is in the space of test functions $\mathcal{M}^{\alpha,L}_{p,q}$ if $f=Lg$ for some g, and the following are satisfied:

(i) $\|f\|_{\dot{B}^{\alpha,L}_{p,q}}<\infty$;

(ii) There is a $C>0$ such that

$$|f(x)|+|g(x)| \le Ce^{-\alpha d(x,x_0)^2} \tag{3.1}$$

for some $x_0\in\mathcal{X}$, and for every $x\in\mathcal{X}$.

For $q = \infty$, we assume, in addition, that

$$\|t^{-\alpha}\Psi_t(L)f\|_p \to 0 \text{ as } t \to 0 \text{ or } t \to \infty,$$

and when $p = \infty$, we assume that

$$e^{-sL}f \to f \text{ in } \dot{B}^{\alpha,L}_{\infty,q} \text{ as } s \to 0.$$

Let $p_{k,t}(\cdot,y)$ be the kernel of $\Psi_{k,t}(L) = t^k L^k e^{-tL}$. Proposition 3.1 in [2] implies that for any $t > 0$ and $x \in \mathcal{X}$,

$$k_t(x,\cdot) = k_t^*(\cdot,x) = p^*_{1,t}(\cdot,x) \in \mathcal{M}^{\alpha,L^*}_{p,q},$$

$-1 < \alpha < 1$ and $1 \leq p,q \leq \infty$. Thus for any $f \in (\mathcal{M}^{-\alpha,L^*}_{p',q'})'$, and for each $t > 0$ and $x \in \mathcal{X}$,

$$\Psi_t(L)f(x) = (f, k_t(x,\cdot)) = \int_{\mathcal{X}} f(x)k_t(x,\cdot)\,d\mu(x)$$

is well-defined, where $(\cdot,\cdot)$ denotes the pairing between a linear functional and a test function.

Definition 3.2. Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$ and $1 \leq p,q \leq \infty$. We define the Besov space $\dot{B}^{\alpha,L}_{p,q}$ associated to an operator L by

$$\dot{B}^{\alpha,L}_{p,q} = \left\{ f \in (\mathcal{M}^{-\alpha,L^*}_{p',q'})' : \|f\|_{\dot{B}^{\alpha,L}_{p,q}} < \infty \right\},$$

where

$$\|f\|_{\dot{B}^{\alpha,L}_{p,q}} = \left\{ \int_0^\infty (t^{-\alpha}\|\Psi_t(L)(f)\|_p)^q \frac{dt}{t} \right\}^{1/q}.$$

Definition 3.3. Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$, $1 \leq p,q \leq \infty$ and $s > 0$. Let $f \in (\mathcal{M}^{-\alpha,L^*}_{p',q'})'$. Define a linear functional $e^{-sL}f$ on $\mathcal{M}^{-\alpha,L^*}_{p',q'}$ by

$$(e^{-sL}f, \phi) = (f, e^{-sL^*}\phi), \quad \forall \phi \in \mathcal{M}^{-\alpha,L^*}_{p',q'}. \tag{3.2}$$

4 Properties of Besov spaces associated with operators

4.1 Embedding theorem

Theorem 4.1. *Suppose that L satisfies (S) and (K). Let $1 \leq p_1 \leq p_2 \leq \infty$, $-1 < \alpha_2 \leq \alpha_1 < 1$ and $\alpha_1 - \frac{\min(n,N)}{2p_1} = \alpha_2 - \frac{\min(n,N)}{2p_2}$. Then*

$$\dot{B}^{\alpha_1,L}_{p_1,q} \subseteq \dot{B}^{\alpha_2,L}_{p_2,q},$$

for every $1 \leq q \leq \infty$.

Proof. The proof is a modification of that in Theorem 4.1 (iv) in [2].

Suppose $f \in \dot{B}^{\alpha_1,L}_{p,q}$. Because

$$\Psi_{2t}(L)f = 2e^{-tL}\Psi_t(L)f, \quad t > 0,$$

from the kernel bound condition (K), it follows that for all $x \in \mathcal{X}$,

$$\Psi_{2t}(L)f(x) \le C \int_{\mathcal{X}} \frac{e^{-\alpha d(x,y)^2/t}}{t^{\min(n,N)/2}} |\Psi_t(L)f(y)|\, d\mu(y).$$

Let $r \ge 0$, where $\frac{1}{p_2} = \frac{1}{p_1} + \frac{1}{r} - 1$. By applying a similar argument as in the proof of Young's inequality (see e.g., Theorem 1.2.12 in [11]) and Lemma 2.1, it follows that

$$\begin{aligned}
&\|\Psi_{2t}(L)\|_{p_2} \\
&\le C\|\Psi_t(L)f\|_{p_1}\Big(\sup_y \Big\| \frac{e^{-\alpha d(\cdot,y)^2/t}}{t^{\min(n,N)/2}} \Big\|_r\Big) \\
&\le C\|\Psi_t(L)f\|_{p_1} t^{\min(n,N)(\frac{1}{p_2}-\frac{1}{p_1})/2}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\|f\|_{\dot{B}^{\alpha_2,L}_{p_2,q}} \\
&\le C\Big\{\int_0^\infty (t^{-\alpha_2}\|\Psi_{2t}(L)f\|_{p_2})^q \frac{dt}{t}\Big\}^{1/q} \\
&\le C\Big\{\int_0^\infty \Big(t^{-\alpha_2+\min(n,N)(\frac{1}{p_2}-\frac{1}{p_1})/2}\|\Psi_t(L)f\|_{p_1}\Big)^q \frac{dt}{t}\Big\}^{1/q} \\
&= C\|f\|_{\dot{B}^{\alpha_1,L}_{p_1,q}}.
\end{aligned}$$

Thus we have finished proving the Theorem. □

4.2 Norm equivalence

Proposition 4.4 in [2] gave the result that the Besov norms defined by $t^kL^ke^{-tL}$ are equivalent to each other for positive $k \ge 1$. In the next proposition we show the equivalence of Besov norms of more general class of functions $\Psi_t(L)$ with suitable decay at 0 and infinity.

Proposition 4.2. *Suppose L satisfies (S) and (K). Let $0 < \alpha < 1$ and $1 \le p,q \le \infty$. For any $f \in (\mathcal{M}^{-\alpha,L^*}_{p',q'})'$, we define a family of Besov norms by*

$$\|f\|_{\dot{B}^{\alpha,\Psi_t(L)}_{p,q}} = \Big\{\int_0^\infty (t^{-\alpha}\|\Psi_t(L)f\|_p)^q \frac{dt}{t}\Big\}^{1/q}$$

for $q < \infty$ and

$$\|f\|_{\dot{B}^{\alpha,\Psi_t(L)}_{p,q}} = \sup_{t>0} t^{-\alpha}\|\Psi_t(L)f\|_p$$

for $q = \infty$. Assume that $\Psi_t(L)$ and $\tilde{\Psi}_t(L)$ are two classes of functions of L which satisfy the following conditions:

- *(i) $\Psi(\xi)$ and $\tilde{\Psi}(\xi)$ are holomorphic functions on the positive x-axis such that $\Psi(\xi)$ and $\tilde{\Psi}(\xi)$ tend to 0 as ξ tends to 0 and as ξ tends to infinity.*
- *(ii) The operators $\Psi_t(L)$ and $\tilde{\Psi}_t(L)$ have kernel bounds (K).*

(iii) There exists $\tilde{\tilde{\Psi}}_t(L)$ *with kernel bounds* (K) *such that*

$$\tilde{\Psi}_t(L) = \Psi_t(L)\tilde{\tilde{\Psi}}_t(L).$$

(iv) The functions $\Psi(\xi)$ *and* $\tilde{\Psi}(\xi)$ *satisfy for some constant* C

$$\tilde{\Psi}_t(L) = Ct\frac{d}{dt}(\Psi_t(L)).$$

Then the Besov norms with respect to $\Psi_t(L)$ *and* $\tilde{\Psi}_t(L)$ *are equivalent to each other.*

Proof. The proof is a modification of that in Proposition 4.4 in [2].

First, it follows from condition (iii) that there exists a constant C such that

$$\begin{aligned}
&\|\tilde{\Psi}_t(L)f\|_p \\
&\quad = \|\tilde{\tilde{\Psi}}_t(L)\Psi_t(L)f\|_p \\
&\quad \le \|\tilde{\tilde{\Psi}}_t(L)\|_{p\to p}\|\Psi_t(L)f\|_p \\
&\quad \le C\|\Psi_t(L)f\|_p,
\end{aligned}$$

This then gives

$$\|f\|_{\dot{B}_{p,q}^{\alpha,\tilde{\Psi}_t(L)}} \le C\|f\|_{\dot{B}_{p,q}^{\alpha,\Psi_t(L)}}.$$

To obtain the reverse inequality, first assume $1 \le q < \infty$. Recall Hardy's inequality: For $0 < r < \infty$ and non-negative measurable function g,

$$\Big(\int_0^\infty t^{-r-1}\Big[\int_0^t g(s)\,ds\Big]^q dt\Big)^{1/q} \le \frac{q}{r}\Big(\int_0^\infty t^{-r-1}[tg(t)]^q\,dt\Big)^{1/q}.$$

(See for example, Lemma 3.14, Chapter V in [15].)

Next, it follows from condition (iv) that, for every $\phi \in \mathcal{M}_{p',q'}^{-\alpha,L^*}$,

$$\begin{aligned}
\frac{d}{ds}(\Psi_s(L)f,\phi) &= (\frac{1}{s}\tilde{\Psi}_s(L)f,\phi), \\
\int_u^t (\tilde{\Psi}_s(L)f,\phi)\frac{ds}{s} &= (\Psi_t(L)f,\phi) - (\Psi_u(L)f,\phi) \\
&= (\Psi_t(L)f,\phi) - (f,\Psi_u(L^*)\phi).
\end{aligned}$$

By condition (i) and an argument similar to the proof of Theorem 3.4 in [2], we observe that $\Psi_u(L^*)\phi \to 0$ in $\mathcal{M}_{p',q'}^{-\alpha,L^*}$ norm as $u \to 0$. It follows that

$$(\Psi_t(L)f,\phi) = \int_0^t (\tilde{\Psi}_s(L)f,\phi)\frac{ds}{s} \tag{4.1}$$

in $\mathcal{M}_{p,q}^{\alpha,L}$. This and Hardy's inequality with $g(s) = \frac{1}{s}\|\tilde{\Psi}_s(L)f\|_p$ gives

$$\begin{aligned}
&\Big\{\int_0^\infty (t^{-\alpha}\|\Psi_t(L)f\|_p)^q\frac{dt}{t}\Big\}^{1/q} \\
&\quad \le \Big\{\int_0^\infty t^{-\alpha q}\Big(\int_0^t \|\tilde{\Psi}_s(L)f\|_p\frac{ds}{s}\Big)^q\frac{dt}{t}\Big\}^{1/q} \\
&\quad \le \frac{q}{r}\Big\{\int_0^\infty t^{-\alpha q}\Big(\frac{t}{t}\|\tilde{\Psi}_t(L)f\|_p\Big)^q\frac{dt}{t}\Big\}^{1/q} \\
&\quad = \frac{q}{r}\Big\{\int_0^\infty \Big(t^{-\alpha}\|\tilde{\Psi}_t(L)f\|_p\Big)^q\frac{dt}{t}\Big\}^{1/q},
\end{aligned}$$

where $r = \alpha q > 0$; that is,

$$\|f\|_{\dot{B}^{\alpha,\Psi_t(L)}_{p,q}} \leq \frac{q}{r}\|f\|_{\dot{B}^{\alpha,\tilde{\Psi}_t(L)}_{p,q}}.$$

Finally, assume $q = \infty$. Then by (4.1) and Minkowski's inequality, it follows that

$$\begin{aligned}
&t^{-\alpha}\|\Psi_t(L)f\|_p \\
&\quad\leq t^{-\alpha}\int_0^t s^{-\alpha}\|\tilde{\Psi}_s(L)f\|_p s^{\alpha}\,\frac{ds}{s} \\
&\quad\leq \Big(\sup_{s>0} s^{-\alpha}\|\tilde{\Psi}_s(L)f\|_p\Big)t^{-\alpha}\int_0^t s^{\alpha-1}\,ds \\
&\quad= \frac{1}{\alpha}\|f\|_{\dot{B}^{\alpha,\tilde{\Psi}_t(L)}_{p,\infty}}.
\end{aligned}$$

Hence the reverse inequality for $q = \infty$ follows. Thus the proof of the proposition is complete. □

Next we look more at the equivalence of Besov norms of more general class of functions. Let $0 < \alpha < 1$ and $f \in$ domain of L^{α}. Assume L has a bounded holomorphic functional calculus on L^2. We have

$$\begin{aligned}
&\|f\|_{\dot{B}^{\alpha,L}_{p,q}} \\
&\quad= \left\{\int_0^{\infty}(t^{-\alpha}\|\Psi_t(L)f\|_p)^q\,\frac{dt}{t}\right\}^{1/q} \\
&\quad= \left\{\int_0^{\infty}(t^{-\alpha}\|tLe^{-tL}f\|_p)^q\,\frac{dt}{t}\right\}^{1/q} \\
&\quad= \left\{\int_0^{\infty}\Big(t^{-\alpha}\Big(\int_{\mathcal{X}}\big|tLe^{-tL}f\big|^p\,dx\Big)^{1/p}\Big)^q\,\frac{dt}{t}\right\}^{1/q} \\
&\quad= \left\{\int_0^{\infty}\Big(\int_{\mathcal{X}}\big|t^{-\alpha}tLe^{-tL}f\big|^p\,dx\Big)^{q/p}\,\frac{dt}{t}\right\}^{1/q},
\end{aligned}$$

If we replace tLe^{-tL} by $t^kL^ke^{-tL}$ for $k \geq 1 > \alpha$, put $\tilde{\Psi}_t(z) = (tz)^{-\alpha}\Psi_t(z)$ and $g = L^{\alpha}f$, with $g \in L^p$, then it follows that

$$\begin{aligned}
&\|f\|_{\dot{B}^{\alpha,L}_{p,q}} \\
&\quad= \left\{\int_0^{\infty}\Big(\int_{\mathcal{X}}\big|t^{-\alpha}t^kL^ke^{-tL}f\big|^p\,dx\Big)^{q/p}\,\frac{dt}{t}\right\}^{1/q} \\
&\quad= \left\{\int_0^{\infty}\Big(\int_{\mathcal{X}}\big|t^{k-\alpha}L^{k-\alpha}L^{\alpha}e^{-tL}f\big|^p\,dx\Big)^{q/p}\,\frac{dt}{t}\right\}^{1/q} \\
&\quad= \left\{\int_0^{\infty}\Big(\int_{\mathcal{X}}\big|t^{k-\alpha}L^{k-\alpha}e^{-tL}L^{\alpha}f\big|^p\,dx\Big)^{q/p}\,\frac{dt}{t}\right\}^{1/q} \\
&\quad= \left\{\int_0^{\infty}\Big(\int_{\mathcal{X}}\big|\tilde{\Psi}_t(L)g\big|^p\,dx\Big)^{q/p}\,\frac{dt}{t}\right\}^{1/q} \\
&\quad= \left\{\int_0^{\infty}\|\tilde{\Psi}_t(L)g\|_p^q\,\frac{dt}{t}\right\}^{1/q}.
\end{aligned}$$

Let $\tilde{\Psi}_t(L) = t^{-\alpha}L^{-\alpha}\Psi_t(L)$ and $\tilde{\beta}_t(L) = t^{-\alpha}L^{-\alpha}\beta_t(L)$. Then [12] gives us that the Besov norms with respect to $\tilde{\Psi}_t(L)$ and $\tilde{\beta}_t(L)$ are equivalent to each other for the case $p = q = 2$. That is,

$$\|f\|_{\dot{B}^{\alpha,\Psi_t(L)}_{2,2}} = \left\{\int_0^\infty \|\tilde{\Psi}_t(L)g\|_2^2 \frac{dt}{t}\right\}^{1/2} \approx \|f\|_{\dot{B}^{\alpha,\beta_t(L)}_{2,2}} = \left\{\int_0^\infty \|\tilde{\beta}_t(L)g\|_2^2 \frac{dt}{t}\right\}^{1/2}.$$

Furthermore, in [5] we have that, for $1 < p < \infty$,

$$\left\|\left\{\int_0^\infty |\tilde{\Psi}_t(L)g|^2 \frac{dt}{t}\right\}^{1/2}\right\|_p \approx \left\|\left\{\int_0^\infty |\tilde{\beta}_t(L)g|^2 \frac{dt}{t}\right\}^{1/2}\right\|_p.$$

When $p = 2$, it follows that

$$\begin{aligned}
&\left\|\left\{\int_0^\infty |\tilde{\Psi}_t(L)g|^2 \frac{dt}{t}\right\}^{1/2}\right\|_p \\
&= \left\{\int_{\mathcal{X}}\left(\int_0^\infty \left|\tilde{\Psi}_t(L)g\right|^2 \frac{dt}{t}\right)^{p/2} dx\right\}^{1/p} \\
&= \left\{\int_{\mathcal{X}}\left(\int_0^\infty \left|\tilde{\Psi}_t(L)g\right|^2 \frac{dt}{t}\right)^{2/2} dx\right\}^{1/2} \\
&= \left\{\int_{\mathcal{X}}\left(\int_0^\infty \left|\tilde{\Psi}_t(L)g\right|^2 \frac{dt}{t}\right) dx\right\}^{1/2} \\
&= \left\{\int_0^\infty \left(\int_{\mathcal{X}}\left|\tilde{\Psi}_t(L)g\right|^2 dx\right) \frac{dt}{t}\right\}^{1/2} \\
&= \left\{\int_0^\infty \left(\int_{\mathcal{X}}\left|\tilde{\Psi}_t(L)g\right|^2 dx\right)^{2/2} \frac{dt}{t}\right\}^{1/2} \\
&= \left\{\int_0^\infty \|\tilde{\Psi}_t(L)g\|_2^2 \frac{dt}{t}\right\}^{1/2} \\
&= \|f\|_{\dot{B}^{\alpha,\Psi_t(L)}_{2,2}}.
\end{aligned}$$

Therefore [5] also gives us that the Besov norms with respect to $\tilde{\Psi}_t(L)$ and $\tilde{\beta}_t(L)$ are equivalent to each other for the case $p = q = 2$. That is,

$$\|f\|_{\dot{B}^{\alpha,\Psi_t(L)}_{2,2}} = \left\{\int_0^\infty \|\tilde{\Psi}_t(L)g\|_2^2 \frac{dt}{t}\right\}^{1/2} \approx \|f\|_{\dot{B}^{\alpha,\beta_t(L)}_{2,2}} = \left\{\int_0^\infty \|\tilde{\beta}_t(L)g\|_2^2 \frac{dt}{t}\right\}^{1/2}.$$

The following proposition extends the Besov norm equivalence result of Proposition 4.4 in [2] to non-integer $k \geq 1$.

Proposition 4.3. *Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$ and $1 \leq p,q \leq \infty$. For any $f \in (\mathcal{M}^{-\alpha,L^*}_{p',q'})'$, and $k = 1,2,\ldots$, we define a family of Besov norms by*

$$\|f\|_{\dot{B}^{\alpha,L,k}_{p,q}} = \left\{\int_0^\infty (t^{-\alpha}\|t^kL^ke^{-tL}f\|_p)^q \frac{dt}{t}\right\}^{1/q}$$

for $q < \infty$ and

$$\|f\|_{\dot{B}^{\alpha,L,k}_{p,q}} = \sup_{t>0} t^{-\alpha}\|t^kL^ke^{-tL}f\|_p$$

for $q = \infty$, where $t^kL^ke^{-tL}f(x) = (f, p_{k,t}(x,\cdot))$. Then these norms for different values of non-integer w, for $k < w < k+1$, are equivalent to each other.

Proof. Let $w = k+\alpha$, where $0<\alpha<1$. Firstly, it can be seen that

$$\begin{aligned}
&\|t^w L^w e^{-tL} f\|_p \\
&\quad = \|t^\alpha L^\alpha e^{-tL/2} t^k L^k e^{-tL/2} f\|_p \\
&\quad \le \|t^\alpha L^\alpha e^{-tL/2}\|_{p\to p} \|t^k L^k e^{-tL/2} f\|_p \\
&\quad \le C\|t^k L^k e^{-tL/2} f\|_p,
\end{aligned}$$

where the final inequality is true because the operator norm $\|t^\alpha L^\alpha e^{-tL/2}\|_{p\to p}$ is uniformly bounded, which follows from interpolation (see [11]) between $\alpha = 0$ and $\alpha = 1$. Therefore we have

$$\|f\|_{\dot{B}^{\alpha,L,w}_{p,q}} \le C\|f\|_{\dot{B}^{\alpha,L,k}_{p,q}}$$

for any non-integer value w, where $k<w<k+1$.

To prove the reverse inequality, observe that

$$\begin{aligned}
&\|t^{k+1} L^{k+1} e^{-tL} f\|_p \\
&\quad = \|t^{1-\alpha} L^{1-\alpha} e^{-tL/2} t^{k+\alpha} L^{k+\alpha} e^{-tL/2} f\|_p \\
&\quad \le \|t^{1-\alpha} L^{1-\alpha} e^{-tL/2}\|_{p\to p} \|t^{k+\alpha} L^{k+\alpha} e^{-tL/2} f\|_p \\
&\quad \le C\|t^{k+\alpha} L^{k+\alpha} e^{-tL/2} f\|_p, \\
&\quad = C\|t^w L^w e^{-tL/2} f\|_p,
\end{aligned}$$

where the final inequality is true because the operator norm $\|t^{1-\alpha} L^{1-\alpha} e^{-tL/2}\|_{p\to p}$ is uniformly bounded, which follows from interpolation between $\alpha = 0$ and $\alpha = 1$. Therefore we have

$$\|f\|_{\dot{B}^{\alpha,L,k+1}_{p,q}} \le C\|f\|_{\dot{B}^{\alpha,L,w}_{p,q}}$$

for any non-integer value w, where $k<w<k+1$. Hence the proof of the proposition is complete. □

Next we see if we have norm equivalence when we replace the semigroup $t^k L^k e^{-tL}$ by the resolvent $t^k L^k (tL+1)^{-m}$, for $k<m$.

Let $\lambda > 0$. The Laplace transform gives

$$(\lambda I + L)^{-m} = \frac{1}{m!}\int_0^\infty s^{m-1} e^{-\lambda s} e^{-sL}\, ds.$$

We have

$$\begin{aligned}
&(tL+1)^{-m} \\
&\quad = \left[t(L+\frac{1}{t})\right]^{-m} \\
&\quad = t^{-m}(L+\frac{1}{t})^{-m} \\
&\quad = \frac{t^{-m}}{m!}\int_0^\infty s^{m-1} e^{-s/t} e^{-sL}\, ds.
\end{aligned}$$

Then for $k < m$, it follows that

$$\begin{aligned} & t^k L^k (tL+1)^{-m} \\ &\quad = \frac{t^{k-m}}{m!} \int_0^\infty s^{m-1} e^{-s/t} L^k e^{-sL}\, ds \\ &\quad = \frac{t^{-m}}{m!} \int_0^\infty s^{m-1} e^{-s/t} t^k L^k e^{-sL}\, ds \\ &\quad = \frac{t^{-m}}{m!} \int_0^\infty s^m e^{-s/t} \Big(\frac{t}{s}\Big)^k s^k L^k e^{-sL} \frac{ds}{s}. \end{aligned}$$

Therefore we have

$$\begin{aligned} & \|t^k L^k (tL+1)^{-m}\|_p \\ &\quad \le \frac{t^{-m}}{m!} \int_0^\infty s^m e^{-s/t} \Big(\frac{t}{s}\Big)^k \|s^k L^k e^{-sL}\|_p \frac{ds}{s} \\ &\quad \le C t^{-m} \int_0^\infty s^m e^{-s/t} \Big(\frac{t}{s}\Big)^k \frac{ds}{s}. \end{aligned}$$

By change of variables $s/t \to w$ it then follows that

$$\begin{aligned} & \|t^k L^k (tL+1)^{-m}\|_p \\ &\quad \le C \int_0^\infty w^{m-k} e^{-w} \frac{ds}{w} \\ &\quad \le C \int_0^\infty w^{m-k-1} e^{-w}\, ds \\ &\quad \le C. \end{aligned}$$

By using the above and also observing that

$$\begin{aligned} & \|t^k L^k e^{-tL}\|_p \\ &\quad \le \|t^k L^k (tL+1)^{-m}\|_p \|(tL+1)^m e^{-tL}\|_p \\ &\quad \le C \|t^k L^k (tL+1)^{-m}\|_p \\ &\quad \le C, \end{aligned}$$

this gives a simpler proof for Proposition 4.4 in [2] than by using Hardy's inequality.

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