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I B S S B P S S W A S

B´ A KPATA[*]
Laboratoire de Mathématiques et Informatique
UFR des Sciences Fondamentales et Appliquées
Université Nangui Abrogoua
02 BP 801 Abidjan 02, Côte d'Ivoire

I FOFANA[†]
Laboratoire de Mathématiques Fondamentales
UFR de Mathématiques et Informatique
Université Félix Houphouët-Boigny
22 BP 582 Abidjan 22, Côte d'Ivoire

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Abstract

A classical result due to A. P. Calderón states that Bessel potential spaces and Sobolev spaces defined on the same Lebesgue space and of the same integer order are isomorphic. We show that this result remains true when we replace Lebesgue spaces by some particular subspaces of Wiener amalgam space.

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1 Introduction

The classical Bessel Potential space $\mathcal{B}_\beta^{L^p(\mathbb{R}^d)}$, $\beta \geq 0$ and $1 < p < \infty$, consists of all

$$u = G_\beta * f,$$

where f belongs to $L^p(\mathbb{R}^d)$ and G_β is the Bessel kernel of order β. It is well-known (see [C], [S], [A-H]) that when β is a natural number the space $\mathcal{B}_\beta^{L^p(\mathbb{R}^d)}$ (endowed with the norm

*E-mail address: kpata_akon@yahoo.fr
†E-mail address: fofana_ib_math_ab@yahoo.fr

$G_\beta * f \mapsto \|f\|_{L^p(\mathbb{R}^d)}$) is isomorphic to the classical Sobolev space $W^{\beta,p}(\mathbb{R}^d)$. This result has been extended to the variable exponent setting, that is, when the exponent p is a measurable function $p : \mathbb{R}^d \to [p_*, p^*], 1 < p_* < p^* < \infty$ (see [A-S], [G-H-N], [Cr]).
The aim of this paper is to show that the isomorphism between Sobolev spaces and Bessel potential spaces mentioned above holds when we replace the Lebesgue space by the subspace $(L^q, l^p)^\alpha_{c,0}$, $1 < q \le \alpha < p < \infty$, of the Wiener amalgam space (L^q, l^p). (We refer to the next section for the definitions and the properties of theses spaces.) The spaces $(L^q, l^p)^\alpha_{c,0}$ have been introduced by Fofana in [Fo2]. They are closely related to the Lebesgue spaces and have attracted steadily increasing interest. Indeed, boundedness properties of Riesz transforms on $(L^q, l^p)^\alpha_{c,0}$ and Riesz potential operators between $(L^q, l^p)^\alpha_{c,0}$ spaces have been recently established in [D-F-S]. In this same paper, the Sobolev space $W^1\left((L^q, l^p)^\alpha_{c,0}\right)$ (see Section 3) was introduced and analogues of Sobolev inequality and Rellich-Kondrachov compactness theorem have been obtained in this setting. As an application, the authors have established an existence theorem for the equation div $F = f$ with $f \in (L^q, l^p)^\alpha_{c,0}$. Furthermore, it has been showed in [F-K-D] that if a locally integrable function has its Riesz potential in a given Lebesgue space, then this function belongs to a space $(L^q, l^p)^\alpha_{c,0}$.
As in [Cr], we give an application to solve some nonhomogeneous differential equations. More precisely, we show that for all non-negative integers m and for all $f \in (L^q, l^p)^\alpha_{c,0}$, $1 \le q \le \alpha \le p \le 2$, the equation

$$(I - \Delta)^m u = f \tag{1.1}$$

has a solution in a Bessel potential space defined within the framework of the spaces $(L^q, l^p)^\alpha_{c,0}$.

The remainder of this paper is organized as follows. In Section 2 we recall background notions about the spaces $(L^q, l^p)^\alpha_{c,0}$ and their connections with some classical spaces arising in Harmonic Analysis. We also give some auxiliary results which will be used throughout the text. In Section 3, we describe the Sobolev spaces $W^1\left((L^q, l^p)^\alpha_{c,0}\right)$. Our main result, Theorem 4.7, is established in Section 4 where we study the Bessel potential spaces on $(L^q, l^p)^\alpha_{c,0}$ and their connections with the Sobolev spaces defined within this framework. Section 5 is devoted to the solvability of (1.1).

2 The spaces $(L^q, l^p)^\alpha_{c,0}$

Notation 2.1. *The euclidean space $\mathbb{R}^d$ is endowed with its usual scalar product $(x, \xi) \mapsto x.\xi$ and the norm of $x \in \mathbb{R}^d$ is denoted by $|x|$.*
We denote by $L^0 = L^0(\mathbb{R}^d, \mathbb{C})$ the space of equivalence classes (modulo equality Lebesgue almost everywhere) of Lebesgue measurable complex functions on $\mathbb{R}^d$.
For $1 \le q \le \infty$, $\|\cdot\|_q$ denotes the usual norm on the classical Lebesgue space $L^q = L^q(\mathbb{R}^d, \mathbb{C})$ and q' the conjugate of q: $\frac{1}{q} + \frac{1}{q'} = 1$ with the convention $\frac{1}{\infty} = 0$.
For any real $r > 0$

- $I^r_k = \prod_{j=1}^{d} \left[k_j r, \left(k_j + 1\right) r\right), \quad k = \left(k_j\right)_{1 \le j \le d} \in \mathbb{Z}^d,$

- $J^r_x = \prod_{j=1}^{d} \left(x_j - \frac{r}{2},\, x_j + \frac{r}{2}\right), \quad x = (x_j)_{1 \le j \le d} \in \mathbb{R}^d.$

Let $1 \leq q,\ \alpha,\ p \leq \infty$.

- *For any* $f \in L^0$ *and any real number* $r > 0$

$$ {}_r\|f\|_{q,\,p} = \begin{cases} \left(\sum\limits_{k\in\mathbb{Z}^d}\left(\|f\chi_{I_k^r}\|_q\right)^p\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \sup\limits_{k\in\mathbb{Z}^d}\left\|f\chi_{I_k^r}\right\|_q & \text{if } p = \infty, \end{cases} $$

  where χ_A *denotes the characteristic function of the subset A of* $\mathbb{R}^d$.

- $(L^q,\ l^p) = (L^q,\ l^p)(\mathbb{R}^d,\ \mathbb{C}) = \left\{f \in L^0\ :\ {}_1\|f\|_{q,\,p} < \infty\right\}$.

- $\|f\|_{q,\,p,\,\alpha} = \sup\limits_{r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\ {}_r\|f\|_{q,\,p}, \qquad f \in L^0.$

- $(L^q,\ l^p)^\alpha = (L^q,\ l^p)^\alpha(\mathbb{R}^d,\ \mathbb{C}) = \left\{f \in L^0\ :\ \|f\|_{q,\,p,\,\alpha} < \infty\right\}$.

We denote by $M^1(\mathbb{R}^d)$ *the space of finite measures on* $\mathbb{R}^d$ *and by* $M^\infty(\mathbb{R}^d)$ *that of Radon measures* μ *on* $\mathbb{R}^d$ *satisfying*

$$ \sup_{x\in\mathbb{R}^d} |\mu|(J_x^1) < +\infty, $$

where $|\mu|$ *stands for the total variation of the measure* μ.

It is known (see [Fo1]) that for $1 \leq q \leq \alpha \leq p \leq \infty$, $\left((L^q,\ l^p)^\alpha,\ \|\cdot\|_{q,\,p,\,\alpha}\right)$ is a complex Banach space. It is clearly a subspace of the well-known Wiener amalgam spaces $(L^q,\ l^p)$ (see [Ho]) and closely related to the Lebesgue spaces as described below.

Proposition 2.2. *([Fo1]or [Fo2]). Suppose that* $1 \leq q \leq \alpha \leq p \leq \infty$. *We have:*

- $\|f\|_{q,\,p,\,\alpha} \leq \|f\|_\alpha, \qquad f \in L^0$
  and consequently, $L^\alpha \subset (L^q,\ l^p)^\alpha$

- $\|f\|_{\alpha,\,p,\,\alpha} = \|f\|_\alpha, \qquad f \in L^0$
  and therefore $(L^\alpha,\ l^p)^\alpha = L^\alpha$

- *there is a real constant* $C > 0$ *such that*

$$ \|f\|_{q,\,\alpha,\,\alpha} \leq \|f\|_\alpha \leq C\|f\|_{q,\,\alpha,\,\alpha}, \qquad f \in L^0 $$

  and so $(L^q,\ l^\alpha)^\alpha = L^\alpha$

- *for any* $u \in \mathbb{R}^d$,

$$ \|\tau_u f\|_{q,\,p,\,\alpha} \leq 2^{d\left(1+\frac{1}{p}\right)} 3^{\frac{d}{q}} \|f\|_{q,\,p,\,\alpha}, \qquad f \in L^0, $$

  where τ_u *denotes the translation operator with translation vector u.*

In [Fo2], Fofana has introduced some special subspaces of $(L^q,\ l^p)^\alpha$ defined as below.

Definition 2.3. For $1 \leq q \leq \alpha \leq p \leq \infty$, we define :

a) $(L^q,\ l^p)^\alpha_c = \left\{ f \in (L^q,\ l^p)^\alpha \ :\ \lim_{u\to 0} \|\tau_u f - f\|_{q,\,p,\,\alpha} = 0 \right\}$

b) $(L^q,\ l^p)^\alpha_0 = \left\{ f \in (L^q,\ l^p)^\alpha \ :\ \lim_{r\to\infty} \|f\chi_{\mathbb{R}^d\setminus J^r_0}\|_{q,\,p,\,\alpha} = 0 \right\}$

c) $(L^q,\ l^p)^\alpha_{c,\,0} = (L^q,\ l^p)^\alpha_c \cap (L^q,\ l^p)^\alpha_0$.

In [D-F-S], it was proved that $(L^q,\ l^p)^\alpha_{c,\,0}$ is closed in $(L^q,\ l^p)^\alpha$.
Let us recall that for $f \in L^0$, $g \in L^0$ and μ a Radon measure on $\mathbb{R}^d$, the convolution products $g * f$ and $\mu * f$ are given by the formulas

$$g * f(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy \quad \text{and} \quad \mu * f(x) = \int_{\mathbb{R}^d} f(x-y)d\mu(y)$$

at all points $x \in \mathbb{R}^d$ where these integrals are defined. We have the following Young's inequality.

Theorem 2.4. *Suppose that $1 \le q \le \alpha \le p < \infty$. There exists a constant $C > 0$ such that for any element $(\mu,\ f)$ of $M^1(\mathbb{R}^d) \times (L^q,\ l^p)^\alpha$ we have*

$$\|\mu * f\|_{q,\,p,\,\alpha} \le C|\mu|(\mathbb{R}^d)\|f\|_{q,\,p,\,\alpha}. \tag{2.1}$$

In particular, for any element $(g,\ f)$ of $L^1 \times (L^q,\ l^p)^\alpha$, we have

$$\|g * f\|_{q,\ p,\,\alpha} \le C\|g\|_1 \|f\|_{q,\ p,\ \alpha}.$$

Proof. For $\alpha = q$ or $\alpha = p$, the result is known since in these cases $(L^q,\ l^p)^\alpha = L^\alpha$.
We now suppose that $1 \le q < \alpha < p < \infty$.
For any $x \in \mathbb{R}^d$ we have

$$|\mu * f(x)| \le \int_{\mathbb{R}^d} |f(x-y)|\ d|\mu|(y)$$

and by Hölder's inequality

$$|\mu * f(x)| \le \left(|\mu|(\mathbb{R}^d)\right)^{\frac{1}{q'}} \left(\int_{\mathbb{R}^d} |f(x-y)|^q\ d|\mu|(y) \right)^{\frac{1}{q}}.$$

So, for any $(r, k) \in \mathbb{R}^*_+ \times \mathbb{Z}^d$, we have

$$\begin{aligned}
\int_{I^r_k} |\mu * f(x)|^q dx &\le \left(|\mu|(\mathbb{R}^d)\right)^{\frac{q}{q'}} \int_{I^r_k} \left(\int_{\mathbb{R}^d} |\tau_y f(x)|^q d|\mu|(y) \right) dx \\
&= \left(|\mu|(\mathbb{R}^d)\right)^{\frac{q}{q'}} \int_{\mathbb{R}^d} \left(\int_{I^r_k} |\tau_y f(x)|^q dx \right) d|\mu|(y) \\
&\quad \text{(by Fubini theorem)} \\
&= \left(|\mu|(\mathbb{R}^d)\right)^{\frac{q}{q'}} \int_{\mathbb{R}^d} \|\tau_y f \chi_{I^r_k}\|^q_q d|\mu|(y).
\end{aligned}$$

It follows that

$$\begin{aligned}\left(\sum_{k\in\mathbb{Z}^d}\|(\mu*f)\chi_{I_k^r}\|_q^p\right)^{\frac{1}{p}} &\leq \left(|\mu|(\mathbb{R}^d)\right)^{\frac{1}{q'}}\left(\sum_{k\in\mathbb{Z}^d}\left(\int_{\mathbb{R}^d}\|\tau_y f\chi_{I_k^r}\|_q^q d|\mu|(y)\right)^{\frac{p}{q}}\right)^{\frac{1}{p}}\\ &= \left(|\mu|(\mathbb{R}^d)\right)^{\frac{1}{q'}}\left(\sum_{k\in\mathbb{Z}^d}\left(\int_{\mathbb{R}^d}\|\tau_y f\chi_{I_k^r}\|_q^q d|\mu|(y)\right)^{\frac{p}{q}}\right)^{\frac{q}{p}\times\frac{1}{q}}.\end{aligned}$$

By using Minkowski's inequality we get

$$\begin{aligned}\left(\sum_{k\in\mathbb{Z}^d}\|(\mu*f)\chi_{I_k^r}\|_q^p\right)^{\frac{1}{p}} &\leq \left(|\mu|(\mathbb{R}^d)\right)^{\frac{1}{q'}}\left(\int_{\mathbb{R}^d}\left(\sum_{k\in\mathbb{Z}^d}\left(\|\tau_y f\chi_{I_k^r}\|_q^q\right)^{\frac{p}{q}}\right)^{\frac{q}{p}} d|\mu|(y)\right)^{\frac{1}{q}}\\ &= \left(|\mu|(\mathbb{R}^d)\right)^{\frac{1}{q'}}\left(\int_{\mathbb{R}^d}\left(\sum_{k\in\mathbb{Z}^d}\|\tau_y f\chi_{I_k^r}\|_q^p\right)^{\frac{q}{p}} d|\mu|(y)\right)^{\frac{1}{q}}.\end{aligned}$$

Hence, from Proposition 2.2 we deduce that

$$\begin{aligned}r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\left(\sum_{k\in\mathbb{Z}^d}\|(\mu*f)\chi_{I_k^r}\|_q^p\right)^{\frac{1}{p}} &\leq \left(|\mu|(\mathbb{R}^d)\right)^{\frac{1}{q'}}\left(\int_{\mathbb{R}^d}\|\tau_y f\|_{q,\,p,\,\alpha}^q\, d|\mu|(y)\right)^{\frac{1}{q}}\\ &\leq C|\mu|(\mathbb{R}^d)\|f\|_{q,\,p,\,\alpha}\end{aligned}$$

where $C = 2^{d\left(1+\frac{1}{p}\right)}3^{\frac{d}{q}}$. Therefore,

$$\|\mu*f\|_{q,\,p,\,\alpha} \leq C|\mu|(\mathbb{R}^d)\|f\|_{q,\,p,\,\alpha}.$$

The particular case follows by taking $d\mu(x) = g(x)dx$ with $g\in L^1$. □

As a consequence of Theorem 2.4, we have the following result.

Theorem 2.5. *Suppose that* $1\leq q\leq\alpha\leq p<\infty$. *We have the following assertions.*
(i) If $(\mu,\ f)\in M^1(\mathbb{R}^d)\times(L^q,\ l^p)_c^\alpha$ *then* $\mu*f\in(L^q,\ l^p)_c^\alpha$.
(ii) If $(\mu,\ f)\in M^1(\mathbb{R}^d)\times(L^q,\ l^p)_0^\alpha$ *then* $\mu*f\in(L^q,\ l^p)_0^\alpha$.

Proof. i) Assume that $(\mu,\ f)\in M^1(\mathbb{R}^d)\times(L^q,\ l^p)_c^\alpha$. Let u be an element of $\mathbb{R}^d$. Notice that $\tau_u(\mu*f)-\mu*f = (\tau_u f - f)*\mu$. According to Proposition 2.2, $\tau_u f$ belongs to $(L^q,\ l^p)^\alpha$. Then $\tau_u f - f$ belongs to $(L^q,\ l^p)^\alpha$. It follows from Theorem 2.4 that $\mu*f$ and $\tau_u(\mu*f)-\mu*f$ are two elements of $(L^q,\ l^p)^\alpha$ and

$$\|\tau_u(\mu*f)-\mu*f\|_{q,\,p,\,\alpha}\leq C|\mu|(\mathbb{R}^d)\|\tau_u f - f\|_{q,\,p,\,\alpha}.$$

From $\lim_{u\to0}\|\tau_u f - f\|_{q,\,p,\,\alpha} = 0$ we deduce that $\lim_{u\to0}\|\tau_u(\mu*f)-\mu*f\|_{q,\,p,\,\alpha} = 0$. Thus, $\mu*f$ belongs to $(L^q,\ l^p)_c^\alpha$.
ii) Assume that $(\mu,\ f)\in M^1(\mathbb{R}^d)\times(L^q,\ l^p)_0^\alpha$. Theorem 2.4 asserts that $\mu*f$ belongs to

$(L^q,\, l^p)^\alpha$.
Let $\delta > 0$ and $\eta > 0$. For $x \in \mathbb{R}^d \backslash J_0^{\eta+\delta}$ we have

$$\begin{aligned}
\mu * f(x) &= \int_{\mathbb{R}^d} f(x-y)d\mu(y) \\
&= \int_{J_0^\delta} f(x-y)d\mu(y) + \int_{\mathbb{R}^d \backslash J_0^\delta} f(x-y)d\mu(y) \\
&= \int_{J_0^\delta} f(x-y)\chi_{J_0^\eta}(x-y)d\mu(y) + \int_{J_0^\delta} f(x-y)\chi_{\mathbb{R}^d \backslash J_0^\eta}(x-y)d\mu(y) \\
&+ \int_{\mathbb{R}^d \backslash J_0^\delta} f(x-y)d\mu(y).
\end{aligned}$$

Since $x-y \in J_0^\eta$ and $y \in J_0^\delta$ imply that $x \in J_0^\eta + J_0^\delta \subset J_0^{\eta+\delta}$, we have for $x \in \mathbb{R}^d \backslash J_0^{\eta+\delta}$,

$$\mu * f(x) = \int_{J_0^\delta} f(x-y)\chi_{\mathbb{R}^d \backslash J_0^\eta}(x-y)d\mu(y) + \int_{\mathbb{R}^d \backslash J_0^\delta} f(x-y)d\mu(y).$$

Equivalently, for $x \in \mathbb{R}^d \backslash J_0^{\eta+\delta}$,

$$\mu * f(x) = (\mu \lfloor J_0^\delta) * f\chi_{\mathbb{R}^d \backslash J_0^\eta}(x) + [\mu \lfloor (\mathbb{R}^d \backslash J_0^\delta)] * f(x),$$

where $\mu \lfloor J_0^\delta$ and $\mu \lfloor (\mathbb{R}^d \backslash J_0^\delta)$ denote the restriction of μ to J_0^δ and $\mathbb{R}^d \backslash J_0^\delta$ respectively. It follows that

$$|(\mu * f)\chi_{\mathbb{R}^d \backslash J_0^{\eta+\delta}}| \leq |(\mu \lfloor J_0^\delta) * f\chi_{\mathbb{R}^d \backslash J_0^\eta}| + |[\mu \lfloor (\mathbb{R}^d \backslash J_0^\delta)] * f|.$$

Therefore,

$$\|(\mu * f)\chi_{\mathbb{R}^d \backslash J_0^{\eta+\delta}}\|_{q,\,p,\,\alpha} \leq \|(\mu \lfloor J_0^\delta) * f\chi_{\mathbb{R}^d \backslash J_0^\eta}\|_{q,\,p,\,\alpha} + \|[\mu \lfloor (\mathbb{R}^d \backslash J_0^\delta)] * f\|_{q,\,p,\,\alpha}.$$

By Theorem 2.4, we have

$$\begin{aligned}
\|(\mu \lfloor J_0^\delta) * f\chi_{\mathbb{R}^d \backslash J_0^\eta}\|_{q,\,p,\,\alpha} &\leq C|\mu \lfloor J_0^\delta|(\mathbb{R}^d)\|f\chi_{\mathbb{R}^d \backslash J_0^\eta}\|_{q,\,p,\,\alpha} \\
&\leq C|\mu|(\mathbb{R}^d)\|f\chi_{\mathbb{R}^d \backslash J_0^\eta}\|_{q,\,p,\,\alpha}
\end{aligned}$$

and

$$\|[\mu \lfloor (\mathbb{R}^d \backslash J_0^\delta)] * f\|_{q,\,p,\,\alpha} \leq C|\mu|(\mathbb{R}^d \backslash J_0^\delta)\|f\|_{q,\,p,\,\alpha}.$$

In addition,

$$\lim_{\eta\to\infty} \|f\chi_{\mathbb{R}^d \backslash J_0^\eta}\|_{q,\,p,\,\alpha} = 0 \quad \text{and} \quad \lim_{\delta\to\infty} |\mu|(\mathbb{R}^d \backslash J_0^\delta) = 0.$$

So, for $\varepsilon > 0$, there exists a real number $N > 0$ such that for $\delta > N$ and $\eta > N$, we have

$$\|(\mu * f)\chi_{\mathbb{R}^d \backslash J_0^{\eta+\delta}}\|_{q,\,p,\,\alpha} < \varepsilon.$$

This shows that $\lim_{r\to\infty} \|(\mu * f)\chi_{\mathbb{R}^d \backslash J_0^r}\|_{q,\,p,\,\alpha} = 0$. We conclude that $\mu * f$ belongs to $(L^q,\, l^p)_0^\alpha$.
□

In the sequel, we shall denote by C^∞ the class of indefinitely differentiable functions on $\mathbb{R}^d$. The Schwartz space of rapidly decreasing C^∞-functions on $\mathbb{R}^d$ will be denoted by $\mathcal{S}$. Let j be an element of $\{1, 2, ..., d\}$. It is well-known (see [G]) that the Riesz transform R_j defined by

$$R_j f(x) = \lim_{\varepsilon\to 0} \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} \int_{|y|>\varepsilon} f(x-y) \frac{y_j}{|y|^{d+1}} dy, \quad f \in \mathcal{S}, \ x = (x_1, x_2, ..., x_d) \in \mathbb{R}^d$$

extends to a linear bounded operator on L^q for $1 < q < \infty$. Recently, this boundedness property of the Riesz transforms was generalized as follows.

Proposition 2.6. *(See [D-F-S]). Suppose that* $1 < q \leq \alpha < p \leq \infty$. *Then the Riesz transforms* R_j *for* $j = 1, 2, ..., d$ *satisfy*

$$\|R_j(f)\|_{q,p,\alpha} \leq C\|f\|_{q,p,\alpha}, \quad f \in (L^q, l^p)^\alpha_{c,0},$$

where C *is a positive constant not depending on* f.

In order to recall an approximation property in $(L^q, l^p)^\alpha_{c,0}$, we introduce the following notation.

Notation 2.7. *We denote by* ρ *a fixed element of* C^∞, *non-negative, with support included in the unit ball* $\overline{B}(0, 1) = \{x \in \mathbb{R}^d \ / \ |x| \leq 1\}$ *and satisfying* $\int_{\mathbb{R}^d} \rho(x)dx = 1$.
ω *is a fixed element of* C^∞ *satisfying* $\chi_{J_0^1} \leq \omega < \chi_{J_0^2}$.
We set

$$\rho_m(x) = m^d \rho(mx) \quad and \quad \omega_m(x) = \omega\left(\frac{x}{m}\right), \quad x \in \mathbb{R}^d, \ m \in \mathbb{N}^*.$$

With Notation 2.7, we state the following result established in [D-F-S].

Proposition 2.8. *Let* $1 \leq q \leq \alpha \leq p \leq \infty$. *Then*

$$\lim_{m\to\infty} \|(f\omega_m) * \rho_m - f\|_{q,p,\alpha} = 0, \quad f \in (L^q, l^p)^\alpha_{c,0}.$$

Proposition 2.8 shows that the closure of the space $\mathcal{D}$ of infinitely differentiable functions on $\mathbb{R}^d$ with compact support in $(L^q, l^p)^\alpha$ is the space $(L^q, l^p)^\alpha_{c,0}$.

3 The Sobolev spaces $W^1\left((L^q, l^p)^\alpha_{c,0}\right)$

We fix $q, \alpha, p \in [1, \infty]$ such that $q \leq \alpha \leq p$ and $q < \infty$.

Definition 3.1. The Sobolev space $W^1\left((L^q, l^p)^\alpha_{c,0}\right)$ is defined as follows:

$$W^1\left((L^q, l^p)^\alpha_{c,0}\right) = \{f \in (L^q, l^p)^\alpha_{c,0} \ : \ \frac{\partial f}{\partial x_j} \in (L^q, l^p)^\alpha_{c,0}, \ j = 1, 2, ..., d\},$$

where $\frac{\partial f}{\partial x_j}$ stands for the partial derivative of f with respect to the j-th coordinate in the sense of distribution.

The Sobolev spaces $W^1\left((L^q,\, l^p)^\alpha_{c,0}\right)$ have been introduced in [D-F-S]. In the following proposition, we list some of their properties.

Proposition 3.2. *a) The mapping* $f \mapsto \|f\|_{W^1((L^q,\, l^p)^\alpha)}$ *where*

$$\|f\|_{W^1((L^q,\, l^p)^\alpha)} = \|f\|_{q,\, p,\, \alpha} + \sum_{j=1}^{d} \|\frac{\partial f}{\partial x_j}\|_{q,\, p,\, \alpha}$$

is a norm on $W^1\left((L^q,\, l^p)^\alpha_{c,0}\right)$.
b) $\left(W^1\left((L^q,\, l^p)^\alpha_{c,0}\right),\, \|\cdot\|_{W^1((L^q,\, l^p)^\alpha)}\right)$ *is a complex Banach space.*
c) For any $f \in W^1\left((L^q,\, l^p)^\alpha_{c,0}\right)$, *we have*

$$\lim_{m\to\infty} \|(f\omega_m) * \rho_m - f\|_{W^1((L^q,\, l^p)^\alpha)} = 0,$$

where ω_m *and* ρ_m *are defined as in Notation 2.7.*

Notice that Definition 3.1 and the assertions a) and b) of Proposition 3.2 hold if we take $(L^q,\, l^p)^\alpha$ or $(L^q,\, l^p)^\alpha_c$ in place of $(L^q,\, l^p)^\alpha_{c,0}$ (see [D-F-S]).
A consequence of Proposition 2.8 and Proposition 3.2 is the following result.

Theorem 3.3. *A function* f *belongs to* $W^1\left((L^q,\, l^p)^\alpha_{c,0}\right)$ *if and only if there exists a sequence* $(f_m)_{m\geq 1}$ *such that*

a) each $f_m \in \mathcal{D}$

b) $\|f - f_m\|_{q,\, p,\, \alpha} \to 0$

c) For each $1 \leq i \leq d$, *the sequence* $\left(\frac{\partial f_m}{\partial x_i}\right)$ *converges in* $(L^q,\, l^p)^\alpha$.

Proof. The necessary part is an immediate consequence of Proposition 3.2.
Conversely, suppose that the assertions a), b), and c) hold. Then $(f_m)_{m\geq 1}$ is a Cauchy sequence in the Banach space $W^1((L^q,\, l^p)^\alpha)$. So there exists a function $g \in W^1((L^q,\, l^p)^\alpha)$ such that $f_m \to g$ in $W^1((L^q,\, l^p)^\alpha)$. It follows from b) that $f = g$. Therefore, $f \in W^1((L^q,\, l^p)^\alpha)$. Since (f_m) and $(\frac{\partial f_m}{\partial x_j})$ are two sequences of elements of $\mathcal{D}$, their limits in $(L^q,\, l^p)^\alpha$ belong to $(L^q,\, l^p)^\alpha_{c,0}$. □

More generally, for any non-negative integer k, the Sobolev space $W^k\left((L^q,\, l^p)^\alpha_{c,0}\right)$ is defined as the space of all elements $f \in (L^q,\, l^p)^\alpha_{c,0}$ satisfying $\frac{\partial^\gamma f}{\partial x^\gamma} \in (L^q,\, l^p)^\alpha_{c,0}$ whenever $|\gamma| \leq k$. This space can be equipped with the norm

$$\|f\|_{W^k((L^q,\, l^p)^\alpha)} = \sum_{|\gamma|\leq k} \|\frac{\partial^\gamma f}{\partial x^\gamma}\|_{q,\, p,\, \alpha}, \quad \left(\frac{\partial^0 f}{\partial x^0} = f\right). \tag{3.1}$$

4 The Bessel Potential spaces $\mathcal{B}_{\beta}^{(L^q,\, l^p)^{\alpha}_{c,\,0}}$

If $\mu \in M^1(\mathbb{R}^d)$ and $f \in L^1$ then their respective Fourier transforms $\widehat{\mu}$ and $\widehat{f}$ are defined by

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{2\pi i x.\xi}\, d\mu(x), \quad \xi \in \mathbb{R}^d$$

and

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{2\pi i x.\xi} f(x)dx, \quad \xi \in \mathbb{R}^d.$$

Let $1 \leq q \leq \alpha \leq p < \infty$ and let $\beta \geq 0$. The Bessel kernel G_β can be introduced in terms of Fourier transform by

$$\widehat{G_\beta}(x) = (1 + 4\pi^2|x|^2)^{-\frac{\beta}{2}}, \quad x \in \mathbb{R}^d.$$

It is well-known (see [S]) that for all $x \in \mathbb{R}^d$, $G_\beta(x) > 0$, and $\|G_\beta\|_1 = 1$.
For $f \in (L^q,\, l^p)^{\alpha}_{c,\,0}$ we set

$$\mathcal{F}_\beta(f) = \begin{cases} G_\beta * f & \text{if } \beta > 0, \\ f & \text{if } \beta = 0. \end{cases}$$

According to Theorem 2.4, the convolution product written above is well defined. More precisely, we have $\mathcal{F}_\beta(f) \in (L^q,\, l^p)^{\alpha}_{c,\,0}$ with

$$\|\mathcal{F}_\beta(f)\|_{q,\,p,\,\alpha} \leq C\|f\|_{q,\,p,\,\alpha}, \tag{4.1}$$

where C is a positive constant not depending on f.
The Bessel potential space $\mathcal{B}_{\beta}^{(L^q,\, l^p)^{\alpha}_{c,\,0}}$ is defined as

$$\mathcal{B}_{\beta}^{(L^q,\, l^p)^{\alpha}_{c,\,0}} = \{\mathcal{F}_\beta(f) \ :\ f \in (L^q,\, l^p)^{\alpha}_{c,\,0}\}.$$

The norm of $f \in \mathcal{B}_{\beta}^{(L^q,\, l^p)^{\alpha}_{c,\,0}}$ is written as $\|f\|_{\mathcal{B}_{\beta}^{(L^q,\, l^p)^{\alpha}_{c,\,0}}}$, and is defined to be the $(L^q,\, l^p)^{\alpha}$ norm of a function g that satisfies $\mathcal{F}_\beta(g) = f$. In other words,

$$\|f\|_{\mathcal{B}_{\beta}^{(L^q,\, l^p)^{\alpha}_{c,\,0}}} = \|g\|_{q,\,p,\,\alpha}, \quad if \quad f = \mathcal{F}_\beta(g). \tag{4.2}$$

The following result ensures that relation (4.2) gives a consistent definition of $\|f\|_{\mathcal{B}_{\beta}^{(L^q,\, l^p)^{\alpha}_{c,\,0}}}$.

Theorem 4.1. *Suppose that $\beta \geq 0$ and $1 \leq q \leq \alpha \leq p < \infty$. If g_1 and g_2 are two elements of $(L^q,\, l^p)^{\alpha}_{c,\,0}$ such that $\mathcal{F}_\beta(g_1) = \mathcal{F}_\beta(g_2)$, then $g_1 = g_2$.*

For the proof of Theorem 4.1, we need the following results.

Lemma 4.2. *(See [D]). Let n be an integer satisfying $n > d$. If $\mu \in M^\infty(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}$ then*

$$\int_{\mathbb{R}^d} |\varphi(x)|\, d|\mu|(x) \leq C \sup_{x\in\mathbb{R}^d} |\mu|(J_x^1) \sup_{x\in\mathbb{R}^d} (1 + |x|)^n |\varphi(x)|,$$

where C is a constant not depending on μ and φ.

Lemma 4.3. *(See [S]). For $\beta \geq 0$, $\mathcal{F}_\beta : f \mapsto G_\beta * f$ is a mapping of $\mathcal{S}$ onto itself.*

Proof of Theorem 4.1. Let us consider $\phi \in \mathcal{S}$ and $g \in (L^q,\, l^p)^\alpha_{c,0}$. We have

$$\int_{\mathbb{R}^d} \mathcal{F}_\beta(g)(x)\phi(x)dx = \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} G_\beta(x-y)g(y)dy\right)\phi(x)dx,$$

and

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |G_\beta(x-y)|\,|g|(y)|\phi(x)|dy\,dx = \int_{\mathbb{R}^d} h(x)|\phi|(x)dx$$

where $h = G_\beta * |g|$ belongs to $(L^q,\, l^p)^\alpha_{c,0}$. Since the measure $h(x)dx$ belongs to $M^\infty(\mathbb{R}^d)$ and ϕ belongs to $\mathcal{S}$, we deduce from Lemma 4.2 that

$$\int_{\mathbb{R}^d} h(x)|\phi|(x)dx < \infty.$$

Therefore, we may apply Fubini-Tonelli theorem to get

$$\begin{aligned}
\int_{\mathbb{R}^d} \mathcal{F}_\beta(g)(x)\phi(x)dx &= \int_{\mathbb{R}^d} g(y)\left(\int_{\mathbb{R}^d} G_\beta(x-y)\phi(x)dx\right)dy \\
&= \int_{\mathbb{R}^d} g(y)\left(\int_{\mathbb{R}^d} G_\beta(y-x)\phi(x)dx\right)dy \\
&= \int_{\mathbb{R}^d} g(y)(G_\beta * \phi)(y)dy \\
&= \int_{\mathbb{R}^d} g(y)\mathcal{F}_\beta(\phi)(y)dy.
\end{aligned}$$

Therefore, if g_1 and g_2 belong to $(L^q,\, l^p)^\alpha_{c,0}$ and $\mathcal{F}_\beta(g_1) = \mathcal{F}_\beta(g_2)$ then

$$\int_{\mathbb{R}^d} (g_1 - g_2)(y)\mathcal{F}_\beta(\phi)(y)dy = 0, \quad \phi \in \mathcal{S}.$$

It follows from Lemma 4.3 that $\int_{\mathbb{R}^d}(g_1 - g_2)(y)\psi(y)dy = 0$ for all $\psi \in \mathcal{S}$ and therefore $g_1 = g_2$. □

Remark 4.4. It is an immediate consequence of the definition of the Bessel potential space and inequality (4.1) that

$$\mathcal{B}_\beta^{(L^q,\, l^p)^\alpha_{c,0}} \subset \mathcal{B}_\gamma^{(L^q,\, l^p)^\alpha_{c,0}} \quad \text{and} \quad \|f\|_{\mathcal{B}_\gamma^{(L^q,\, l^p)^\alpha_{c,0}}} \leq C\,\|f\|_{\mathcal{B}_\beta^{(L^q,\, l^p)^\alpha_{c,0}}} \quad \text{if} \quad \beta > \gamma,$$

where C is a positive constant not depending on f. Also $\mathcal{F}_\beta$ is an isomorphism of $\mathcal{B}_\gamma^{(L^q,\, l^p)^\alpha_{c,0}}$ onto $\mathcal{B}_{\gamma+\beta}^{(L^q,\, l^p)^\alpha_{c,0}}$, if $\gamma \geq 0, \beta \geq 0$.

To precise the connection between the scale of potential spaces $\mathcal{B}_\beta^{(L^q,\, l^p)^\alpha_{c,0}}$ and that of Sobolev spaces $W^k\left((L^q,\, l^p)^\alpha_{c,0}\right)$, we will use the following lemmas.

Lemma 4.5. *(See [S]). Let $\delta > 0$.*
(i) There exists a finite measure μ_δ on $\mathbb{R}^d$ such that its Fourier transform $\widehat{\mu_\delta}$ is given by

$$\widehat{\mu_\delta}(x) = \frac{(2\pi|x|)^\delta}{(1+4\pi^2|x|^2)^{\frac{\delta}{2}}}.$$

(ii) There exists a pair of finite measures ν_δ and λ_δ on $\mathbb{R}^d$ such that

$$(1+4\pi^2|x|^2)^{\frac{\delta}{2}} = \widehat{\nu_\delta}(x) + (2\pi|x|)^\delta \widehat{\lambda_\delta}(x).$$

Lemma 4.6. *Suppose that $1 < q \leq \alpha < p < \infty$ and $\beta \geq 1$. Then f belongs to $\mathcal{B}_\beta^{(L^q,\, l^p)^\alpha_{c,0}}$ if and only if for $j = 1, 2, ..., d$, $\frac{\partial f}{\partial x_j}$ and f belong to $\mathcal{B}_{\beta-1}^{(L^q,\, l^p)^\alpha_{c,0}}$. Moreover, the two norms $\|f\|_{\mathcal{B}_\beta^{(L^q,\, l^p)^\alpha_{c,0}}}$ and $\|f\|_{\mathcal{B}_{\beta-1}^{(L^q,\, l^p)^\alpha_{c,0}}} + \sum_{j=1}^{d} \|\frac{\partial f}{\partial x_j}\|_{\mathcal{B}_{\beta-1}^{(L^q,\, l^p)^\alpha_{c,0}}}$ are equivalent.*

Proof. a) For the proof of the necessary part, we will examine two cases.
Particular case. Assume that $f \in \mathcal{S}$. Then, according to Lemma 4.3, there exists $g \in \mathcal{S}$ such that $f = \mathcal{F}_\beta(g)$. Let j be an element of $\{1, 2, ..., d\}$. We have

$$\begin{aligned}
\widehat{\left(\frac{\partial f}{\partial x_j}\right)}(x) &= -2i\pi x_j \widehat{f}(x) \\
&= -2i\pi x_j \widehat{(G_\beta * g)}(x) \\
&= -2i\pi x_j \widehat{G_\beta}(x)\widehat{g}(x) \\
&= -2i\pi x_j (1+4\pi^2|x|^2)^{\frac{-\beta}{2}} \widehat{g}(x) \\
&= (1+4\pi^2|x|^2)^{\frac{-(\beta-1)}{2}} \left(-i\frac{x_j}{|x|} \frac{2\pi|x|}{(1+4\pi^2|x|^2)^{\frac{1}{2}}} \widehat{g}(x)\right).
\end{aligned}$$

According to Lemma 4.5, there exists a finite measure μ_1 on $\mathbb{R}^d$ such that $\widehat{\mu_1}(x) = \frac{2\pi|x|}{(1+4\pi^2|x|^2)^{\frac{1}{2}}}$.
It follows that

$$\begin{aligned}
\widehat{\left(\frac{\partial f}{\partial x_j}\right)}(x) &= \widehat{G_{\beta-1}}(x)\left(-i\frac{x_j}{|x|}\widehat{\mu_1}(x)\widehat{g}(x)\right) \\
&= \widehat{G_{\beta-1}}(x)\left(-i\frac{x_j}{|x|}\widehat{(\mu_1 * g)}(x)\right) \\
&= \widehat{G_{\beta-1}}(x)\widehat{(-R_j(\mu_1 * g))}(x).
\end{aligned}$$

By Theorem 2.5 and Proposition 2.6, $g^{(j)} = -R_j(\mu_1 * g)$ belongs to $(L^q,\, l^p)^\alpha_{c,0}$ and satisfies

$$\begin{aligned}
\widehat{\left(\frac{\partial f}{\partial x_j}\right)}(x) &= \widehat{G_{\beta-1}}(x)\widehat{g^{(j)}}(x) \\
&= \widehat{(G_{\beta-1} * g^{(j)})}(x) \\
&= \widehat{\left(\mathcal{F}_{\beta-1}(g^{(j)})\right)}(x).
\end{aligned}$$

Hence,

$$\frac{\partial f}{\partial x_j} = \mathcal{F}_{\beta-1}(g^{(j)}).$$

General case. Assume that $f \in \mathcal{B}_{\beta}^{(L^q,\, l^p)_{c,0}^{\alpha}}$. Then $f = \mathcal{F}_{\beta}(g)$ with $g \in (L^q,\, l^p)_{c,0}^{\alpha}$. According to Proposition 2.8, there exists a sequence $(g_m) \in \mathcal{D}$ (hence in $\mathcal{S}$) such that $g_m \to g$ in $(L^q,\, l^p)^{\alpha}$. For each integer m , let us set $f_m = \mathcal{F}_{\beta}(g_m)$. Then, from inequality (4.1), we have $f_m \to f$ in $(L^q,\, l^p)^{\alpha}$. It follows from the particular case that for each integer m and each $j = 1, 2, ..., d$,

$$\frac{\partial f_m}{\partial x_j} = \mathcal{F}_{\beta-1}(g_m^{(j)}),$$

where $g_m^{(j)} = -R_j(\mu_1 * g_m)$. Since μ_1 is a finite measure, Theorem 2.4 asserts that $k \mapsto \mu_1 * k$ is a bounded operator in $(L^q,\, l^p)^{\alpha}$and then, thanks to Theorem 2.5 $\mu_1 * k \in (L^q,\, l^p)_{c,0}^{\alpha}$. In addition, $k \mapsto R_j(\mu_1 * k)$ is a bounded operator in $(L^q,\, l^p)^{\alpha}$ since the Riesz transforms R_j are bounded in $(L^q,\, l^p)^{\alpha}$ according to Proposition 2.6. Therefore, for $j = 1, 2, ..., d$, the sequence $(g_m^{(j)})$ converges to $g^{(j)} = -R_j(\mu_1 * g)$ in $(L^q,\, l^p)^{\alpha}$. Then, from inequality (4.1), we have $\frac{\partial f_m}{\partial x_j} \to \mathcal{F}_{\beta-1}(g^{(j)})$ in $(L^q,\, l^p)^{\alpha}$. So the sequence (f_m) converges in the Banach space $W^1\,((L^q,\, l^p)^{\alpha})$. Thus $\frac{\partial f}{\partial x_j} = \mathcal{F}_{\beta-1}(g^{(j)})$ and

$$\begin{aligned} \sum_{j=1}^{d} \Big\|\frac{\partial f}{\partial x_j}\Big\|_{\mathcal{B}_{\beta-1}^{(L^q,\, l^p)_{c,0}^{\alpha}}} &= \sum_{j=1}^{d} \|g^{(j)}\|_{q,\, p,\, \alpha} \\ &\leq C\|g\|_{q,\, p,\, \alpha}, \end{aligned}$$

where C is a real constant not depending on g. It follows that

$$\sum_{j=1}^{d} \Big\|\frac{\partial f}{\partial x_j}\Big\|_{\mathcal{B}_{\beta-1}^{(L^q,\, l^p)_{c,0}^{\alpha}}} \leq C\|f\|_{\mathcal{B}_{\beta}^{(L^q,\, l^p)_{c,0}^{\alpha}}}.$$

Combining this inequality with the estimate $\|f\|_{\mathcal{B}_{\beta-1}^{(L^q,\, l^p)_{c,0}^{\alpha}}} \leq C\|f\|_{\mathcal{B}_{\beta}^{(L^q,\, l^p)_{c,0}^{\alpha}}}$, we get

$$\|f\|_{\mathcal{B}_{\beta-1}^{(L^q,\, l^p)_{c,0}^{\alpha}}} + \sum_{j=1}^{d} \Big\|\frac{\partial f}{\partial x_j}\Big\|_{\mathcal{B}_{\beta-1}^{(L^q,\, l^p)_{c,0}^{\alpha}}} \leq C\|f\|_{\mathcal{B}_{\beta}^{(L^q,\, l^p)_{c,0}^{\alpha}}}. \tag{4.3}$$

b)To prove the converse, assume that for $j = 1, 2, ..., d$, $\frac{\partial f}{\partial x_j}$ and f belong to $\mathcal{B}_{\beta-1}^{(L^q,\, l^p)_{c,0}^{\alpha}}$. Then $f = \mathcal{F}_{\beta-1}(g)$ and $\frac{\partial f}{\partial x_j} = \mathcal{F}_{\beta-1}\left(h_j\right)$ with $g,\, h_j \in (L^q,\, l^p)_{c,0}^{\alpha}$.
For any $j \in \{1, 2, ..., d\}$ and any $\psi \in \mathcal{D}$ we have

$$\begin{aligned} \int_{\mathbb{R}^d} h_j(x)\mathcal{F}_{\beta-1}(\psi)(x)dx &= \int_{\mathbb{R}^d} \mathcal{F}_{\beta-1}(h_j)(x)\psi(x)dx \\ &= \int_{\mathbb{R}^d} \frac{\partial f}{\partial x_j}(x)\psi(x)dx \\ &= -\int_{\mathbb{R}^d} f(x)\frac{\partial \psi}{\partial x_j}(x)dx \end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)\frac{\partial\psi}{\partial x_j}(x)dx &= \int_{\mathbb{R}^d} \mathcal{F}_{\beta-1}(g)(x)\frac{\partial\psi}{\partial x_j}(x)dx \\ &= \int_{\mathbb{R}^d} g(x)\mathcal{F}_{\beta-1}\left(\frac{\partial\psi}{\partial x_j}\right)(x)dx \\ &= \int_{\mathbb{R}^d} g(x)\frac{\partial}{\partial x_j}\left[\mathcal{F}_{\beta-1}(\psi)\right](x)dx \\ &= -\int_{\mathbb{R}^d} \frac{\partial g}{\partial x_j}(x)\mathcal{F}_{\beta-1}(\psi)(x)dx.\end{aligned}$$

So

$$\int_{\mathbb{R}^d} h_j(x)\mathcal{F}_{\beta-1}(\psi)(x)dx = \int_{\mathbb{R}^d} \frac{\partial g}{\partial x_j}(x)\mathcal{F}_{\beta-1}(\psi)(x)dx.$$

A simple limiting argument leads us to

$$\int_{\mathbb{R}^d} h_j(x)\mathcal{F}_{\beta-1}(\psi)(x)dx = \int_{\mathbb{R}^d} \frac{\partial g}{\partial x_j}(x)\mathcal{F}_{\beta-1}(\psi)(x)dx, \quad \psi\in\mathcal{S}, \quad j=1,2,...,d.$$

By Lemma 4.3, we have

$$\int_{\mathbb{R}^d} h_j(x)\psi(x)dx = \int_{\mathbb{R}^d} \frac{\partial g}{\partial x_j}(x)\psi(x)dx, \quad \psi\in\mathcal{S}, \quad j=1,2,...,d.$$

Therefore,

$$h_j = \frac{\partial g}{\partial x_j}(x), \quad j=1,2,...,d.$$

It follows that $g\in W^1\left((L^q, l^p)^\alpha_{c,0}\right)$. According to Theorem 3.3, there exists a sequence $(g_m)\in\mathcal{D}$ (hence in $\mathcal{S}$), such that $g_m\to g$ in $W^1((L^q, l^p)^\alpha)$. Since $\mathcal{F}_1$ is a mapping of $\mathcal{S}$ onto itself, we can write, for each integer m, $g_m=\mathcal{F}_1(h_m)$ with $h_m\in\mathcal{S}$.
Let x be an element of $\mathbb{R}^d$. According to Lemma 4.5, part (ii) , with $\delta=1$, there exists a pair of finite measures ν_1 and λ_1 on $\mathbb{R}^d$ such that

$$(1+4\pi^2|x|^2)^{\frac{1}{2}} = \widehat{\nu_1}(x)+2\pi|x|\widehat{\lambda_1}(x).$$

Then, since $\widehat{G_1}(x)=(1+4\pi^2|x|^2)^{-\frac{1}{2}}$, we have

$$\begin{aligned}\widehat{h_m}(x) &= \widehat{h_m}(x)\widehat{G_1}(x)\left(\widehat{\nu_1}(x)+2\pi|x|\widehat{\lambda_1}(x)\right) \\ &= (G_1*h_m)\widehat{\ }(x)\left(\widehat{\nu_1}(x)+2\pi|x|\widehat{\lambda_1}(x)\right) \\ &= \widehat{\mathcal{F}_1(h_m)}(x)\left(\widehat{\nu_1}(x)+2\pi|x|\widehat{\lambda_1}(x)\right) \\ &= \widehat{g_m}(x)\left(\widehat{\nu_1}(x)+2\pi|x|\widehat{\lambda_1}(x)\right) \\ &= \widehat{g_m}(x)\widehat{\nu_1}(x)+\widehat{\lambda_1}(x)\sum_{j=1}^d 2\pi\frac{x_j^2}{|x|}\widehat{g_m}(x).\end{aligned}$$

As it is well-known (see [S]) that for any $j \in \{1, 2, ..., d\}$

$$\widehat{\left(R_j\left(\frac{\partial g_m}{\partial x_j}\right)\right)}(x) = 2\pi \frac{x_j^2}{|x|}\widehat{g_m}(x),$$

we have

$$\begin{aligned}\widehat{h_m}(x) &= \widehat{g_m}(x)\widehat{\nu_1}(x) + \widehat{\lambda_1}(x)\sum_{j=1}^{d}\widehat{\left(R_j\left(\frac{\partial g_m}{\partial x_j}\right)\right)}(x)\\ &= \widehat{g_m}(x)\widehat{\nu_1}(x) + \widehat{\lambda_1}(x)\widehat{\left(\sum_{j=1}^{d} R_j\left(\frac{\partial g_m}{\partial x_j}\right)\right)}(x).\end{aligned}$$

So

$$h_m = \nu_1 * g_m + \lambda_1 * \left(\sum_{j=1}^{d} R_j\left(\frac{\partial g_m}{\partial x_j}\right)\right).$$

It follows that

$$\|h_m\|_{q,\,p,\,\alpha} \leq C\left(\|g_m\|_{q,\,p,\,\alpha} + \sum_{j=1}^{d}\|\frac{\partial g_m}{\partial x_j}\|_{q,\,p,\,\alpha}\right),$$

where C is a real constant not depending on g_m.
Let us consider the sequence (f_m) defined by $f_m = \mathcal{F}_{\beta-1}(g_m)$. Then, for each integer m, we have

$$f_m = \mathcal{F}_\beta(h_m) \quad \text{and} \quad \|f_m\|_{\mathcal{B}_\beta^{(L^q,\,l^p)^\alpha_{c,0}}} = \|h_m\|_{q,\,p,\,\alpha}.$$

Thus, for each integer m,

$$\|f_m\|_{\mathcal{B}_\beta^{(L^q,\,l^p)^\alpha_{c,0}}} \leq C\left(\|g_m\|_{q,\,p,\,\alpha} + \sum_{j=1}^{d}\|\frac{\partial g_m}{\partial x_j}\|_{q,\,p,\,\alpha}\right).$$

The same inequality holds when f_m is replaced by $f_m - f_{m'}$, and g_m is replaced by $g_m - g_{m'}$ (m and m' being any non-negative integers). This shows that the sequence (f_m) also converges in $\mathcal{B}_\beta^{(L^q,\,l^p)^\alpha_{c,0}}$. Hence, by letting $m \to \infty$, we obtain $f \in \mathcal{B}_\beta^{(L^q,\,l^p)^\alpha_{c,0}}$ and

$$\|f\|_{\mathcal{B}_\beta^{(L^q,\,l^p)^\alpha_{c,0}}} \leq C\left(\|g\|_{q,\,p,\,\alpha} + \sum_{j=1}^{d}\|\frac{\partial g}{\partial x_j}\|_{q,\,p,\,\alpha}\right).$$

It follows that

$$\|f\|_{\mathcal{B}_\beta^{(L^q,\,l^p)^\alpha_{c,0}}} \leq C\left(\|f\|_{\mathcal{B}_{\beta-1}^{(L^q,\,l^p)^\alpha_{c,0}}} + \sum_{j=1}^{d}\|\frac{\partial f}{\partial x_j}\|_{\mathcal{B}_{\beta-1}^{(L^q,\,l^p)^\alpha_{c,0}}}\right). \tag{4.4}$$

Inequality (4.4) together with inequality (4.3) end the proof. □

The following result gives the identity between the potential spaces $\mathcal{B}_\beta^{(L^q,\,l^p)^\alpha_{c,0}}$ and the Sobolev spaces $W^k\left((L^q,\,l^p)^\alpha_{c,0}\right)$.

Theorem 4.7. *Suppose that k is a positive integer and $1 < q \leq \alpha < p < \infty$. Then $\mathcal{B}_k^{(L^q,\, l^p)^\alpha_{c,0}} = W^k\left((L^q,\, l^p)^\alpha_{c,0}\right)$ in the sense that $f \in \mathcal{B}_k^{(L^q,\, l^p)^\alpha_{c,0}}$ if and only if $f \in W^k\left((L^q,\, l^p)^\alpha_{c,0}\right)$, and the two norms given respectively by (3.1) and (4.2) are equivalent.*

Proof. The identity between $W^k\left((L^q,\, l^p)^\alpha_{c,0}\right)$ and $\mathcal{B}_k^{(L^q,\, l^p)^\alpha_{c,0}}$ is complete, and obvious, when $k = 0$. However, it is clear that if $k \geq 1$, then $f \in W^k\left((L^q,\, l^p)^\alpha_{c,0}\right)$ if and only if f and $\frac{\partial f}{\partial x_j}$ belong to $W^{k-1}\left((L^q,\, l^p)^\alpha_{c,0}\right)$ for $j = 1,\, ...,\, d$. The two norms

$$\|f\|_{W^k\left((L^q,\, l^p)^\alpha_{c,0}\right)} \quad \text{and} \quad \|f\|_{W^{k-1}\left((L^q,\, l^p)^\alpha_{c,0}\right)} + \sum_{j=1}^{d} \|\frac{\partial f}{\partial x_j}\|_{W^{k-1}\left((L^q,\, l^p)^\alpha_{c,0}\right)}$$

are also obviously equivalent. Thus Lemma 4.6 extends the identity of $W^k\left((L^q,\, l^p)^\alpha_{c,0}\right)$ and $\mathcal{B}_k^{(L^q,\, l^p)^\alpha_{c,0}}$ from $k = 0$ to $k = 1,\, 2, ...$ □

5 Solvability of nonhomogeneous differential equations

Let us recall that in [Ho], the Fourier transform $f \mapsto \widehat{f}$ defined on L^1 has been extended to the spaces $(L^q,\, l^p)$. In fact, F. Holland proved that if f belongs to $(L^q,\, l^p)$, $1 \leq q, p \leq 2$, then there exists a unique element $\widehat{f} \in \left(L^{p'},\, l^{q'}\right)$ such that for any sequence $(r_n)_{n\geq 1}$ of positive real numbers increasing to ∞, the sequence $\left(\widehat{f\chi_{J_0^{r_n}}}\right)_{n\geq 1}$ converges in $\left(L^{p'},\, l^{q'}\right)$ to $\widehat{f}$. In addition,

$$\int_{\mathbb{R}^d} g(x)\widehat{f}(x)\, dx = \int_{\mathbb{R}^d} \widehat{g}(x)\, f(x)\, dx, \quad g \in (L^q,\, l^p)$$

and

$$_1\|\widehat{f}\|_{p',\, q'} \leq C\, _1\|f\|_{q,\, p}$$

where C is a real constant depending on d, q and p.
In [Fo3], I. Fofana has proved the following Hausdorff-Young inequalities.

Proposition 5.1. *Suppose that $1 \leq q \leq \alpha \leq p \leq 2$. Then there exists a positive real constant K such that we have*

$$r^{-d\left(\frac{1}{q}-\frac{1}{p}\right)}\, _r\|\widehat{f}\|_{p',\, q'} \leq K\, _{\frac{1}{r}}\|f\|_{q,\, p}, \quad f \in (L^q,\, l^p),\ r > 0$$

and

$$\|\widehat{f}\|_{p',\, q',\, \alpha'} \leq K\, \|f\|_{q,\, p,\, \alpha}, \quad f \in (L^q,\, l^p)^\alpha.$$

The following result is an existence theorem for equation (1.1).

Theorem 5.2. *Suppose that $1 \leq q \leq \alpha \leq p \leq 2$. Let f be an element of $(L^q,\, l^p)^\alpha_{c,0}$ and let m be a non-negative integer. Then the equation*

$$(I - \Delta)^m u = f$$

has a solution $u \in \mathcal{B}_{2m}^{(L^q,\, l^p)^\alpha_{c,0}}$.

Proof. It is well-known that the operator $(I-\Delta)^m$ is defined via the Fourier transform by

$$\widehat{[(I-\Delta)^m u]}(x) = (1+4\pi^2|x|^2)^m \widehat{u}(x), \qquad x \in \mathbb{R}^d.$$

Thus, equation (1.1) may be written as

$$(1+4\pi^2|x|^2)^m \widehat{u}(x) = \widehat{f}(x), \qquad x \in \mathbb{R}^d$$

that is

$$\widehat{u} = (1+4\pi^2|x|^2)^{-m}\widehat{f},$$

and therefore

$$\widehat{u} = \widehat{G_{2m}}\widehat{f}.$$

As f belongs to $(L^q, l^p)^\alpha_{c,0}$, Proposition 2.8 asserts that there exists a sequence (f_k) of elements of $\mathcal{D}$ (hence of $\mathcal{S}$) such that $f_k \to f$ in $(L^q, l^p)^\alpha$. For each integer k, let us set $u_k = G_{2m} * f_k$. Since the Fourier transform is a bounded linear operator from $(L^q, l^p)^\alpha$ into $\left(L^{p'}, l^{q'}\right)^{\alpha'}$ and the operator $T_{G_{2m}}$ defined by $T_{G_{2m}}(f) = G_{2m} * f$ is also a bounded linear operator on $(L^q, l^p)^\alpha_{c,0}$, we have on the one hand $\widehat{u_k} = \widehat{G_{2m} * f_k} \to \widehat{G_{2m} * f}$ in $\left(L^{p'}, l^{q'}\right)^{\alpha'}$ and on the other hand, $\widehat{u_k} = \widehat{G_{2m}}\widehat{f_k} \to \widehat{G_{2m}}\widehat{f}$ in $\left(L^{p'}, l^{q'}\right)^{\alpha'}$. Therefore,

$$\widehat{G_{2m} * f} = \widehat{G_{2m}}\widehat{f}.$$

It follows that

$$\widehat{u} = \widehat{G_{2m} * f}.$$

Hence $u = G_{2m} * f$ is a solution of (1.1) and u belongs to $\mathcal{B}_{2m}^{(L^q, l^p)^\alpha_{c,0}}$. □

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References

[A-H] D. R. ADAMS and L. I. HEDBERG, *Function spaces and potential theory*, Grundlehren der mathematischen Wissenschaften, vol. 314, Springer-Verlag, London-Berlin-Heidelberg-New York 1996.

[A-S] A. ALMEIDA and S. SAMKO, Characterization of Riesz and Bessel potentials on variable Lebesgue spaces. *J. Funct. Spaces Appl.* **4**, 2 (2006), 113-144.

[C] A. P. CALDERÓN, Lebesgue spaces of differentiable functions and distributions. *Proc. Sympos. Pure Math.*, **vol. IV**, pp 33-49. American Mathematical Society, Providence, R.I., 1961.

[Cr] D. I. CRUZ-BÁEZ, A characterization of Bessel potentials on $L^{p(.)}(\mathbb{R}^n)$ spaces. *Proceedings of the 14th WSEAS International Conference on Applied Mathematics*, 52-56.

[D-F-S] M. DOSSO, I. FOFANA and M. SANOGO, On some subspaces of Morrey-Sobolev spaces and boundedness of Riesz integrals. *Ann. Polon. Math.* **108**, 2 (2013), 133-153.

[D] D. DOUYON, *Intégrale fractionnaire sur* $M^{p,\alpha}(\mathbb{R}^d)$ *et applications*, Thèse de doctorat, Université de Bamako 2010.

[F-K-D] I. FOFANA, B. A. KPATA and D. DOUYON, Some properties of Radon measures having their Riesz potential in a Lebesgue space. *preprint*.

[Fo1] I. FOFANA, Étude d'une classe d'espaces de fonctions contenant les espaces de Lorentz. *Afrika Mat.* **(2)**, 1 (1988), 29-50.

[Fo2] I. FOFANA, *Espace* $(L^q, l^p)^\alpha(\mathbb{R}^d, n)$*, espace de fonctions à moyenne fractionnaire intégrable*, Thèse d'État, Université d'Abidjan-Cocody, 1995. http://greenstone.refer.bf/collect/thef/index/assoc/HASH6a39.dir/CS_02767.pdf.

[Fo3] I. FOFANA, Transformation de Fourier dans $(L^q, l^p)^\alpha$ et $M^{p,\alpha}$. *Afrika Mat.* **(3)**, 5 (1995), 53-76.

[G] L. GRAFAKOS, *Classical Fourier Analysis*, Second Edition, Graduate Texts in Mathematics (249), Springer, 2008.

[G-H-N] P. GURKA, P. HARJULEHTO and A. NEKVINDA, Bessel potential spaces with variable exponent. *Math. Inequal. Appl.* **10**, 3 (2007), 661-676.

[Ho] F. HOLLAND, Harmonic Analysis on amalgams of L^p and l^q. *J. London Math. Soc.* **(2)**, 10 (1975), 295-305.

[S] E. M. STEIN, *Singular integrals and Differentiability Properties of functions*, Princeton University Press, Princeton, New Jersey 1970.