Communications in **M**athematical **A**nalysis
Volume 15, Number 2, pp. 29–38 (2013) www.math-res-pub.org/cma
ISSN 1938-9787

A Note on Carleman Estimates and Unique Continuation Property for the Boussinesq System

Youcef Mammeri *
Laboratoire Amiénois de Mathématique Fondamentale et Appliquée
CNRS UMR 7352, Université de Picardie Jules Verne
80069 Amiens, France

(Communicated by Irena Lasiecka)

Abstract

A Carleman estimates is established to prove a unique continuation property of the solution of the Boussinesq system. We can prove that if the solution of the Boussinesq systems vanishes in an open subset, then this solution is identically equal to zero in the horizontal component of the open subset.

AMS Subject Classification: 35B60, 35Q53

Keywords: Boussinesq system, Carleman estimates, UCP, Treves' inequality.

1 Introduction

The two-way propagation of small amplitude, long wavelength, gravity waves in shallow water, described by its surface η and its velocity u, was first derived by Boussinesq [5] as a system of the form

$$\begin{aligned} \eta_t + u_x + (\eta u)_x &= 0 \\ u_t + \eta_x + uu_x - u_{xxt} &= 0. \end{aligned}$$

In this paper, we consider the following generalized regularized Boussinesq system, proposed in [2, 3], for $\rho \geq 1$ an integer,

$$\begin{aligned} \eta_t + u_x + (\eta^\rho u)_x + au_{xxx} - b\eta_{xxt} &= 0 \\ u_t + \eta_x + u^\rho u_x + c\eta_{xxx} - du_{xxt} &= 0. \end{aligned}$$

The purpose of this work is to prove a unique continuation property. More precisely, we show that if $(\eta, u) = (\eta(x,t), u(x,t))$ is solution of the system and (η, u) vanishes on an open subset Ω of $\mathbb{R}\times\mathbb{R}$, then u vanishes identically on the horizontal component $\Omega_h := \{(x,t) \in$

*E-mail address: youcef.mammeri@u-picardie.fr

$\mathbb{R}\times\mathbb{R}; \exists x_1$ with $(x_1,t)\in\Omega\}$ of Ω.
Carleman estimates can be used. These estimates are based on exponential weight for the solution of the equation. More precisely, if v is solution of $\mathcal{L}v = Vv$, with $\mathcal{L}$ a linear operator, V a well-defined potential, the Carleman estimates is written for Ψ a convex function and $\tau > 0$ to choose

$$||e^{\tau\Psi(x)}v|| \leq C||e^{\tau\Psi(x)}\mathcal{L}v||.$$

Saut and Scheurer [12] proved such a result for a general class of dispersive equations, including the Korteweg-de Vries one. An alternative approach was suggested by Bourgain [4]. The method here is based on an analytic continuation of the Fourier transform using the theorem of Paley-Wiener. We proved therefore a unique continuation property of the solution of the Boussinesq systems with $b = d = 0$ [10]. The method of Bourgain depends on the dispersion relation σ which has to satisfy the following growth property: $\forall R > 0, \exists |k| > R$ such that

$$\sigma'(k) \geq f(k) \text{ with } \lim_{|k|\to\infty} f(k) = +\infty.$$

In the case of the Boussinesq systems, we obtain

$$\sigma(k) = k\sqrt{\frac{(1-ak^2)(1-ck^2)}{(1+bk^2)(1+dk^2)}},$$

and the property holds if $b = d = 0$.
We propose here to prove a unique continuation property using a Carleman estimate. We are inspired by a work of Davila and Menzala [7] who proved a unique continuation property of the scalar one-dimensional Benjamin-Bona-Mahony equation. The first section is devoted to the local well-posedness of solution of the generalized Boussinesq system. We establish a Carleman estimates in the second section. The third section deals with the unique continuation property.

2 Initial value problem

We consider the initial value problem, for $x \in \mathbb{R}, t \in \mathbb{R}, \rho \geq 1$ an integer,

$$\begin{aligned} \eta_t + u_x + (\eta^\rho u)_x + au_{xxx} - b\eta_{xxt} &= 0 \\ u_t + \eta_x + u^\rho u_x + c\eta_{xxx} - d\eta_{xxt} &= 0 \\ \eta(x,0) = \eta_0(x),\ u(x,0) = u_0(x). \end{aligned}$$

If $\rho = 1$, it has been proved the existence and uniqueness of local in time solution [2, 3]. We denote $H^s(\mathbb{R})$ the Sobolev space of order s equipped with the norm $||u||_s$.

Theorem 2.1.
Let $a,b,c,d > 0$, $s > 1/2$ and $(\eta_0,u_0) \in H^s(\mathbb{R})\times H^s(\mathbb{R})$. There exists a constant $C_0 > 0$, depending only on s, such that for

$$T = \frac{C_0}{(||\eta_0||_s + ||u_0||_s)^\rho},$$

there exists a unique solution $(\eta, u) \in C([-T,T];H^s(\mathbb{R})) \times C([-T,T];H^s(\mathbb{R}))$ *of the Boussinesq system with* (η_0, u_0) *as initial datum.*
Moreover, for all $M > 0$ *with* $\|\eta_0\|_s + \|u_0\|_s \le M$ *and* $\|\mu_0\|_s + \|v_0\|_s \le M$*, there exists* $C_1 > 0$ *such that solutions* (η, u) *and* (μ, v)*, of initial data* (η_0, u_0) *and* (μ_0, v_0) *respectively, satisfy for* $t \in [-T,T]$*, with* $T = C_0/M^\rho$*,*

$$\|\eta(t) - \mu(t)\|_s + \|u(t) - v(t)\|_s \le C_1(\|\eta_0 - \mu_0\|_s + \|u_0 - v_0\|_s).$$

Proof. Let $T > 0$. The Duhamel's formula implies that (η, u) is the solution of the initial value problem if and only if (η, u) is the solution of the following equation, for $t \in [0,T]$,

$$(\eta,u)(t) = \Phi(\eta,u)(t) := S_t(\eta_0,u_0) - \int_0^t S_{t-\tau}\left(\frac{\partial_x(\eta^\rho u)}{1-b\partial_x^2}, \frac{\partial_x(u^{\rho+1})/(\rho+1)}{1-d\partial_x^2}\right)(\tau)\,d\tau, \tag{2.1}$$

with

$$S_t(\eta,u) \quad := \quad \int_{-\infty}^{+\infty} \mathrm{e}^{-ikA(k)t}(\eta,u)dk \ \text{ and } A(k) = \begin{pmatrix} 0 & \frac{1-ak^2}{1+bk^2} \\ \frac{1-ck^2}{1+dk^2} & 0 \end{pmatrix}. \tag{2.2}$$

The eigenvalues of $A(k)$ are

$$\sigma(k) = \pm\sqrt{\frac{(1-ak^2)}{(1+bk^2)}\frac{(1-ck^2)}{(1+dk^2)}},$$

and the matrix $\mathrm{e}^{-ikA(k)t}$ is uniformly bounded.
We aim at applying the fixed point theorem. We deduce, for $t \in [0,T]$,

$$\|\Phi(\eta,u)(t)\|_s \quad \le \quad C(\|\eta_0\|_s + \|u_0\|_s) + C\int_0^t \left\|\frac{\partial_x(\eta^\rho u)}{1-b\partial_x^2}\right\|_s + \left\|\frac{\partial_x(u^{\rho+1})}{1-d\partial_x^2})\right\|_s (\tau)\,d\tau.$$

For $b > 0$, the definition of the Sobolev norm provides

$$\left\|\frac{\partial_x f}{1-b\partial_x^2}\right\|_s \quad = \quad \left(\int_{-\infty}^{+\infty}(1+k^2)^s\left|\frac{ik}{1+bk^2}\widehat{f}(k)\right|^2\right)^{1/2} \le C\|f\|_s.$$

Since $s > 1/2$, the Sobolev space being an algebra, the Sobolev embedding implies that there exists a constant $C_s > 0$, depending only on s, such that

$$\|\Phi(\eta,u)(t)\|_s \quad \le \quad C(\|\eta_0\|_s + \|u_0\|_s) + C_s T\left(\sup_{t\in[0,T]}(\|u\|_s\|\eta\|_s^\rho + \|u(t)\|_s^{\rho+1})\right). \tag{2.3}$$

Then there exists $C_0 > 0$ such that for $T = C_0/(\|\eta_0\|_s + \|u_0\|_s)^\rho$, the closed ball $\overline{B}_T$ defined by

$$\begin{aligned} \sup_{t\in[0,T]} \|\eta(t)\|_s \quad &\le \quad 2C(\|\eta_0\|_s + \|u_0\|_s) \\ \sup_{t\in[0,T]} \|u(t)\|_s \quad &\le \quad 2C(\|\eta_0\|_s + \|u_0\|_s) \end{aligned}$$

satisfies $\Phi(\overline{B}_T) \subseteq \overline{B}_T$. Indeed, let $(\eta,u) \in \overline{B}_T$, the inequality (2.3) becomes

$$\|\Phi(\eta,u)(t)\|_s \le (\|\eta_0\|_s + \|u_0\|_s) C(1 + 2^{\rho+2} C^\rho C_s C_0),$$

and $C(\|\eta_0\|_s + \|u_0\|_s)C(1 + 2^{\rho+2}C^\rho C_s C_0) \le 2C(\|\eta_0\|_s + \|u_0\|_s)$ if $C_0 \le 1/(2^{\rho+2}C^\rho C_s)$.
Let (η,u) and (μ,v) be in $\overline{B}_T$. The Duhamel's formula (2.1) provides, for $t \in [0,T]$,

$$\|\Phi(\eta,u)(t) - \Phi(\mu,v)(t)\|_s \le C\int_0^t \left\|\frac{\partial_x}{1-b\partial_x^2}(\eta^\rho u - \mu^\rho v)\right\|_s + \left\|\frac{\partial_x}{1-d\partial_x^2}(u^{\rho+1} - v^{\rho+1})\right\|_s (\tau)\,d\tau.$$

We can note that

$$\eta^\rho u - \mu^\rho v = \eta^\rho(u-v) + v(\eta-\mu)\sum_{i=0}^{\rho-1}\eta^{\rho-1-i}\mu^i, \text{ and } u^{\rho+1} - v^{\rho+1} = (u-v)\sum_{i=0}^{\rho} u^{\rho-i}v^i.$$

This provides, thanks to the Sobolev embedding,

$$\begin{aligned}\|\Phi(\eta,u)(t) - \Phi(\mu,v)(t)\|_s &\le 2^\rho C^\rho C_s T(\|\eta_0\|_s + \|u_0\|_s)^\rho \left(\sup_{t\in[-T,T]}(\|\eta-\mu\|_s(t) + \|u-v\|_s(t))\right)\\ &= 2^\rho C^\rho C_s C_0 \left(\sup_{t\in[-T,T]}(\|\eta-\mu\|_s(t) + \|u-v\|_s(t))\right).\end{aligned}$$

For $C_0 < 1/(2^\rho C^\rho C_s)$, the map Φ is a contraction on $\overline{B}_T$. Finally, according to the fixed point theorem, there exists a unique solution (η,u) of $\Phi(\eta,u)(t) = (\eta,u)(t)$ in $\overline{B}_T$.
It remains to prove the continuity with the initial datum. Let (η,u) and (μ,v) be solutions of the initial value problem with initial datum (η_0,u_0) and (μ_0,v_0) respectively, such that $\|\eta_0\|_s + \|u_0\|_s \le M$ and $\|\mu_0\|_s + \|v_0\|_s \le M$. The Duhamel's formula (2.1) gives for $t \in [0,T]$, with $T = C_0/M^\rho$, with $C_0 \ll 1$,

$$\begin{aligned}\|(\eta,u)(t) - (\mu,v)(t)\|_s &\le C(\|\eta_0-\mu_0\|_s + \|u_0-v_0\|_s)\\ &\quad + C\int_0^t \left\|\frac{\partial_x}{1-b\partial_x^2}(\eta^\rho u - \mu^\rho v)\right\|_s + \left\|\frac{\partial_x}{1-d\partial_x^2}(u^{\rho+1} - v^{\rho+1})\right\|_s (\tau)\,d\tau\\ &\le C(\|\eta_0-\mu_0\|_s + \|u_0-v_0\|_s) + \frac{1}{2}\left(\sup_{t\in[0,T]}\|\eta-\mu\|_s(t) + \sup_{t\in[0,T]}\|u-v\|_s(t)\right),\end{aligned}$$

thus

$$\sup_{t\in[0,T]}(\|\eta-\mu\|_s + \|u-v\|_s) \le 2C(\|\eta_0-\mu_0\|_s + \|u_0-v_0\|_s).$$

□

3 Carleman estimates

The aim of this section is to find a Carleman estimates for the Boussinesq system. First we recall the Treves'inequality [13].

Theorem 3.1.
Let $P = P(\partial_x, \partial_t)$ be a differential operator of order m with constant coefficients. Then for all $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2, \delta > 0, \tau > 0, \Phi \in C_0^\infty(\mathbb{R}^2)$ and $\Psi(x,t) = (x-\delta)^2 + \delta^2 t^2$,

$$\frac{2^{2|\alpha|}\tau^{|\alpha|}\delta^{2\alpha_2}}{\alpha!}\int_{\mathbb{R}^2}|P^{(\alpha)}(D)\Phi|^2 e^{2\tau\Psi}dxdt \le C(m,\alpha)\int_{\mathbb{R}^2}|P(D)\Phi|^2 e^{2\tau\Psi}dxdt.$$

with

$$|\alpha| = \sum_{j=1}^{n}\alpha_j,\ \alpha! = \alpha_1!\dots\alpha_n! \text{ and } C(m,\alpha) = \begin{cases} \sup\limits_{|r+\alpha|\le m}\begin{pmatrix} r+\alpha \\ \alpha \end{pmatrix} & \text{if } |\alpha| \le m \\ 0 & \text{if } |\alpha| > m \end{cases}$$

This inequality is used to prove Carleman estimates of the Boussinesq system.

Theorem 3.2.
We define

$$\mathcal{L} := \begin{pmatrix} \partial_t - c_1\partial_{xxt} + c_2\partial_{xxx} + f_{1,1}(x,t)\partial_x & a\partial_{xxx} + f_{1,2}(x,t)\partial_x \\ c\partial_{xxx} + f_{2,1}(x,t)\partial_x & \partial_t - c_3\partial_{xxt} + c_4\partial_{xxx} + f_{2,2}(x,t)\partial_x \end{pmatrix},$$

where c_1, c_2, c_3, c_4 are constant in $\mathbb{R}$, $f_{i,j} \in L^\infty(\mathbb{R}^2), 1 \le i,j \le 2$. Let $\delta > 0$ and $B_\delta := \{(x,t) \in \mathbb{R}^2; x^2 + t^2 < \delta^2\}$. Then, there exists $C > 0$ such that for all $\Phi = (\Phi_1, \Phi_2) \in C_0^\infty(B_\delta) \times C_0^\infty(B_\delta)$, $\Psi(x,t) = (x-\delta)^2 + \delta^2 t^2$ and $\tau > 0$ with

$$\frac{\|f_{1,1}\|_\infty^2}{c_1\tau^2\delta^2} \le \frac{1}{4}, \frac{\|f_{2,2}\|_\infty^2}{c_3\tau^2\delta^2} \le \frac{1}{4}, \frac{\|f_{2,1}\|_\infty^2}{c^2\tau^2} \le \frac{1}{4}, \frac{\|f_{1,2}\|_\infty^2}{a^2\tau^2} \le \frac{1}{4},$$

we have

$$\tau^3\int_{B_\delta}(|\Phi_1|^2 + |\Phi_2|^2)e^{2\tau\Psi}dxdt + \tau^2\delta^2\int_{B_\delta}(|\Phi_{1,x}|^2 + |\Phi_{2,x}|^2)e^{2\tau\Psi}dxdt \le C\int_{B_\delta}|\mathcal{L}\Phi|^2 e^{2\tau\Psi}dxdt. \tag{3.1}$$

Proof. We define the differential operator

$$P := \begin{pmatrix} \partial_t - c_1\partial_{xxt} + c_2\partial_{xxx} & a\partial_{xxx} \\ c\partial_{xxx} & \partial_t - c_3\partial_{xxt} + c_4\partial_{xxx} \end{pmatrix}.$$

The Fourier transform gives for $(\xi_1, \xi_2, \tau) \in \mathbb{R}^3$

$$\widehat{P}(\xi,\tau) = i\begin{pmatrix} \tau + c_1\xi^2\tau - c_2\xi^3 & -a\xi^3 \\ -c\xi^3 & \tau + c_3\xi^2\tau - c_4\xi^3 \end{pmatrix}.$$

Lemma 3.3.
For all $\Phi = (\Phi_1, \Phi_2) \in C_0^\infty(B_\delta) \times C_0^\infty(B_\delta)$, $\Psi(x,t) = (x-\delta)^2 + \delta^2 t^2$ and $\tau > 0$, we have

$$\begin{aligned} &\tau^3(c_2^2 + c^2)\int_{B_\delta}|\Phi_1|^2 e^{2\tau\Psi}dxdt + \tau^2(c_1^2\delta^2 + c^2)\int_{B_\delta}|\Phi_{1,x}|^2 e^{2\tau\Psi}dxdt \\ &+ \tau^3(c_4^2 + a^2)\int_{B_\delta}|\Phi_2|^2 e^{2\tau\Psi}dxdt + \tau^2(c_3^2\delta^2 + a^2)\int_{B_\delta}|\Phi_{2,x}|^2 e^{2\tau\Psi}dxdt \quad \le \quad \int_{B_\delta}|P\Phi|^2 e^{2\tau\Psi}dxdt. \end{aligned}$$

Proof. With the same notations of Treves'inequality, we have for $\alpha = (3,0)$

$$\frac{\widehat{P_{1,1}}^{|\alpha|}(\xi,\tau)}{\partial\xi^3} = -i6c_2, P_{1,1}^{|\alpha|}\Phi_1 = -i6c_2\Phi_1, C(3,(3,0)) = \sup_{|r+\alpha|\le 3}\begin{pmatrix} r+\alpha \\ \alpha \end{pmatrix} = 1.$$

Applying the preceding corollary with $P_{1,1}$ and α gives

$$\frac{2^6\tau^3}{6}\int_{\mathbb{R}^2}|-i6c_2\Phi_1|^2 \mathrm{e}^{2\tau\Psi}dxdt \le \int_{\mathbb{R}^2}|P_{1,1}\Phi_1|^2\mathrm{e}^{2\tau\Psi}dxdt,$$

what implies

$$\tau^3 c_2^2\int_{\mathbb{R}^2}|\Phi_1|^2\mathrm{e}^{2\tau\Psi}dxdt \le \int_{\mathbb{R}^2}|P_{1,1}\Phi_1|^2\mathrm{e}^{2\tau\Psi}dxdt.$$

In the same way, we obtain for $\alpha = (1,1)$

$$\frac{\widehat{P_{1,1}}^{|\alpha|}(\xi,\tau)}{\partial\xi^2\partial\tau} = i2c_1\xi, P^{|\alpha|}\Phi_1 = 2c_1\Phi_{1,x}, C(3,(1,1)) = 2,$$

and the Treves'inequality provides

$$\frac{2^4\tau^2\delta^2}{1}\int_{\mathbb{R}^3}|2c_1\,\Phi_{1,x}|^2\mathrm{e}^{2\tau\Psi}dxdt \le 2\int_{\mathbb{R}^3}|P_{1,1}\Phi_1|^2\mathrm{e}^{2\tau\Psi}dxdt,$$

thus

$$\tau^2 c_1^2\int_{\mathbb{R}^2}|\Phi_{1,x}|^2\mathrm{e}^{2\tau\Psi}dxdt \le \int_{\mathbb{R}^2}|P_{1,1}\Phi_1|^2\mathrm{e}^{2\tau\Psi}dxdt.$$

On the other hand, we have for $\alpha = (3,0)$, respectively $\alpha = (2,0)$,

$$\frac{\widehat{P_{2,1}}^{|\alpha|}(\xi,\tau)}{\partial\xi^3} = -i6c, P_{2,1}^{|\alpha|}\Phi_1 = -i6c\Phi_1, C(3,(3,0)) = 1$$
$$\frac{\widehat{P_{2,1}}^{|\alpha|}(\xi,\tau)}{\partial\xi^2} = -i6c\xi, P_{2,1}^{|\alpha|}\Phi_1 = 6c\Phi_{1,x}, C(3,(2,0)) = 1.$$

The Treves'inequality given by the preceding corollary with $P_{2,1}$ and α gives

$$\begin{aligned}\tau^3c^2\int_{\mathbb{R}^2}|\Phi_1|^2\mathrm{e}^{2\tau\Psi}dxdt &\le \int_{\mathbb{R}^2}|P_{2,1}\Phi_1|^2\mathrm{e}^{2\tau\Psi}dxdt\\ \tau^2c_1^2\int_{\mathbb{R}^2}|\Phi_{1,x}|^2\mathrm{e}^{2\tau\Psi}dxdt &\le \int_{\mathbb{R}^2}|P_{2,1}\Phi_1|^2\mathrm{e}^{2\tau\Psi}dxdt.\end{aligned}$$

We obtain similar inequalities for $P_{1,2}$ and $P_{2,2}$ in the same way. □

Lemma 3.4.

We have

$$\begin{aligned}
\int_{B_\delta} |f_{1,1}(x,t)\Phi_{1,x}|^2 \mathrm{e}^{2\tau\Psi} dxdt &\leq \frac{2\|f_{1,1}\|_\infty^2}{c_1^2\tau^2\delta^2}\left(\int_{B_\delta} |\mathcal{L}_{1,1}\Phi_1|^2 \mathrm{e}^{2\tau\Psi} dxdt + \int_{B_\delta} |f_{1,1}(x,t)\Phi_{1,x}|^2 \mathrm{e}^{2\tau\Psi} dxdt\right)\\
\int_{B_\delta} |f_{2,2}(x,t)\Phi_{2,x}|^2 \mathrm{e}^{2\tau\Psi} dxdt &\leq \frac{2\|f_{2,2}\|_\infty^2}{c_3^2\tau^2\delta^2}\left(\int_{B_\delta} |\mathcal{L}_{2,2}\Phi_2|^2 \mathrm{e}^{2\tau\Psi} dxdt + \int_{B_\delta} |f_{2,2}(x,t)\Phi_{2,x}|^2 \mathrm{e}^{2\tau\Psi} dxdt\right)\\
\int_{B_\delta} |f_{2,1}(x,t)\Phi_{1,x}|^2 \mathrm{e}^{2\tau\Psi} dxdt &\leq \frac{2\|f_{2,1}\|_\infty^2}{c^2\tau^2}\left(\int_{B_\delta} |\mathcal{L}_{2,1}\Phi_1|^2 \mathrm{e}^{2\tau\Psi} dxdt + \int_{B_\delta} |f_{2,1}(x,t)\Phi_{1,x}|^2 \mathrm{e}^{2\tau\Psi} dxdt\right)\\
\int_{B_\delta} |f_{1,2}(x,t)\Phi_{2,x}|^2 \mathrm{e}^{2\tau\Psi} dxdt &\leq \frac{\|2f_{1,2}\|_\infty^2}{a^2\tau^2}\left(\int_{B_\delta} |\mathcal{L}_{1,2}\Phi_1|^2 \mathrm{e}^{2\tau\Psi} dxdt + \int_{B_\delta} |f_{1,2}(x,t)\Phi_{2,x}|^2 \mathrm{e}^{2\tau\Psi} dxdt\right).
\end{aligned}$$

Proof. We have thanks to the above lemma

$$\begin{aligned}
\int_{B_\delta} |f_{1,1}(x,t)\Phi_{1,x}|^2 \mathrm{e}^{2\tau\Psi} dxdt &\leq \|f_{1,1}\|_\infty^2 \int_{B_\delta} |\Phi_{1,x}|^2 \mathrm{e}^{2\tau\Psi} dxdt\\
&\leq \frac{\|f_{1,1}\|_\infty^2}{c_1^2\tau^2\delta^2}\int_{B_\delta} |P_{1,1}\Phi_1|^2 \mathrm{e}^{2\tau\Psi} dxdt\\
&\leq \frac{2\|f_1\|_\infty^2}{c_1^2\tau^2\delta^2}\int_{B_\delta} \left(|\mathcal{L}_{1,1}\Phi_1|^2 + |f_{1,1}(x,t)\Phi_{1,x}|^2\right) \mathrm{e}^{2\tau\Psi} dxdt.
\end{aligned}$$

In the same manner, it gets

$$\begin{aligned}
\int_{B_\delta} |f_{2,1}(x,t)\Phi_{1,x}|^2 \mathrm{e}^{2\tau\Psi} dxdt &\leq \|f_{2,1}\|_\infty^2 \int_{B_\delta} |\Phi_{1,x}|^2 \mathrm{e}^{2\tau\Psi} dxdt\\
&\leq \frac{\|f_{2,1}\|_\infty^2}{c^2\tau^2}\int_{B_\delta} |P_{2,1}\Phi_1|^2 \mathrm{e}^{2\tau\Psi} dxdt\\
&\leq \frac{2\|f_{2,1}\|_\infty^2}{c^2\tau^2}\int_{B_\delta} \left(|\mathcal{L}_{2,1}\Phi_1|^2 + |f_{2,1}(x,t)\Phi_{1,x}|^2\right) \mathrm{e}^{2\tau\Psi} dxdt.
\end{aligned}$$

We obtain similar inequalities for $f_{1,2}$ and $f_{2,2}$ in the same way. □

To conclude, it is enough to choose $\tau > 0$ large enough with

$$\frac{2\|f_{1,1}\|_\infty^2}{c_1\tau^2\delta^2} \leq \frac{1}{2}, \frac{2\|f_{2,2}\|_\infty^2}{c_3\tau^2\delta^2} \leq \frac{1}{2}, \frac{2\|f_{2,1}\|_\infty^2}{c^2\tau^2} \leq \frac{1}{2}, \frac{2\|f_{1,2}\|_\infty^2}{a^2\tau^2} \leq \frac{1}{2}.$$

□

Corollary 3.5.
Let $T > 0$. If $\Phi \in \left(C^1([-T,T];H^3(\mathbb{R}))\right)^2$, $\Phi_t \in \left(C^1([-T,T];H^2(\mathbb{R}))\right)^2$ and $\operatorname{supp}\Phi \subseteq B_\delta \times B_\delta$*, the inequality* (3.1) *remains true.*

Proof. The proof is done by regularization. □

4 Unique continuation property

The unique continuation property is now proven. The proof is similar to the scalar case of the paper of Davila and Menzala [7]. Therefore we only write sketch of the proof for easy to read.

Lemma 4.1.
Let $s \geq 5, T > 0, f_{i,j} \in L^\infty(\mathbb{R} \times [-T,T]), 1 \leq i, j \leq 2,$ *and* c_1, c_2, c_3, c_4 *be real constants. Let* $w \in \left(C^1([-T,T]; H^s(\mathbb{R}))\right)^2$ *be the solution of* $\mathcal{L}w = 0$*. Assume that* $w \equiv 0$ *when* $x < t^2$ *in a neighborhood of* $(0,0)$*. Then there exists a neighborhood of* $(0,0)$ *in which* $w \equiv 0$*.*

Remark 4.2. If $(\eta, u) \in \left(C^1([-T,T]; H^s(\mathbb{R}))\right)^2$ is solution of $\mathcal{L}(\eta, u) = 0$, since

$$\begin{aligned}(1 - c_1\partial_x^2)\eta_t &= -(au_{xxx} + c_2\eta_{xxx} + f_{1,1}\eta_x + f_{1,2}u_x)\\ (1 - c_3\partial_x^2)u_t &= -(c\eta_{xxx} + c_4u_{xxx} + f_{2,1}\eta_x + f_{2,2}u_x),\end{aligned}$$

then $(\eta_t, u_t) \in (C([-T,T]; H^r(\mathbb{R})))^2$ with $r = s$ if $(a = c = c_2 = c_4 = 0)$, $r = s-3$ if $b = d = 0$ and $r = s-1$ if not. Therefore the Carleman estimate (3.1) holds if $s \geq 2$, $s \geq 5$ and $s \geq 3$ respectively.

Proof. Let $0 < \delta < 1$, choose $\chi \in C_0^\infty(B_\delta) \times C_0^\infty(B_\delta)$ such that $\chi = 1$ in O_1 a neighborhood of $(0,0)$ and define $\Phi := \chi w$. It follows that $\Phi \in C([-T,T]; H^s(\mathbb{R})) \times C([-T,T]; H^s(\mathbb{R}))$ and $\operatorname{supp}\Phi \subseteq B_\delta \times B_\delta$. We deduce thanks to the preceding corollary, for $\tau > 0$ large enough,

$$\tau^3 \int_{B_\delta} (|\Phi_1|^2 + |\Phi_2|^2)e^{2\tau\Psi}dxdt + \tau^2\delta^2 \int_{B_\delta} (|\Phi_{1,x}|^2 + |\Phi_{2,x}|^2)e^{2\tau\Psi}dxdt \leq C \int_{B_\delta} |\mathcal{L}\Phi|^2 e^{2\tau\Psi}dxdt. \tag{4.1}$$

The right hand side integral holds on $B_\delta \backslash O_1$, since $\mathcal{L}\Phi = 0$ in O_1.
For $(x,t) \neq 0$ in supp Φ, we have

$$\Psi(x,t) = (x-\delta)^2 + \delta^2 t^2 < \delta^2 \text{ and } \Psi(0,0) = \delta^2.$$

Then for $(x,t) \in \operatorname{supp}\mathcal{L}\Phi \subseteq B_\delta \times B_\delta$, there exists $0 < \varepsilon < \delta^2$ such that $\Psi(x,t) \leq \delta^2 - \varepsilon$. On the other hand, we can choose O_2 a neighborhood of $(0,0)$ with $\Psi(x,t) > \delta^2 - \varepsilon$. in O_2. The inequality (4.1) is then written for all $\tau > 0$

$$C\tau^3 e^{2\tau(\delta^2-\varepsilon)} \int_{O_2} |\Phi_1|^2 + |\Phi_2|^2 dxdt \leq e^{2\tau(\delta^2-\varepsilon)} \int_{B_\delta \backslash O_1} |\mathcal{L}\Phi|^2 dxdt.$$

Tending τ to infinity implies Φ vanishes in O_2. However $w = \Phi$ in $O_2 \subseteq O_1$ and $w \equiv 0$ in O_2. □

Corollary 4.3.
Let $s \geq 5, T > 0, A, B, C \in L^\infty(\mathbb{R} \times [-T,T])$*, and* c_1, c_2, c_3, c_4 *be real constants. Let* $w = (\eta, u) \in \left(C^1([-T,T]; H^s(\mathbb{R}))\right)^2$ *be the solution of*

$$\begin{pmatrix} \partial_t - b\partial_{xxt} + A(x,t)\partial_x & a\partial_{xxx} + \partial_x + B(x,t)\partial_x \\ c\partial_{xxx} + \partial_x & \partial_t - d\partial_{xxt} + C(x,t)\partial_x \end{pmatrix} w = 0.$$

We consider the curve $x = \mu(t), \mu(0) = 0, \mu$ *a continuously differential function in a neighborhood of* $(0,0)$*. Assume that* $w \equiv 0$ *when* $x < \mu(t)$ *in a neighborhood of* $(0,0)$*. Then there exists a neighborhood of* $(0,0)$ *in which* $w \equiv 0$*.*

Proof. We consider the Holmgren's transformation $(x,t) \longrightarrow (X,T)$ with

$$\begin{array}{rcl} X & = & x-\mu(t)+t^2 \\ T & = & t. \end{array}$$

This change of variables provides $W = W(X,T)$ satisfying $W \equiv 0$ when $X < T^2$ in a neighborhood of $(0,0)$ and $\mathcal{L}W = 0$ with

$$\mathcal{L} := \begin{pmatrix} \partial_T + (-\mu'(T)+2T)\partial_X - b\partial_{XXT} - b(-\mu'(T)+2T)\partial_{XXX} + A\partial_X & a\partial_{XXX} + \partial_X + B\partial_X \\ c\partial_{XXX} + \partial_X & \partial_T + (-\mu'(T)+2T)\partial_X - d\partial_{XXT} - d(-\mu'(T)+2T)\partial_{XXX} + C\partial_X \end{pmatrix}.$$

□

Theorem 4.4.
Let $s \geq 5$ *and* $T > 0$ *and* $(\eta,u) \in C([-T,T];H^s(\mathbb{R})) \times C([-T,T];H^s(\mathbb{R}))$ *solution of the Boussinesq system. If* $(\eta,u) \equiv 0$ *in an open subset* $\Omega \subseteq (\mathbb{R} \times [-T,T])^2$*, then* $(\eta,u) \equiv 0$ *in the horizontal component of* Ω*.*

Proof. The proof follows [11, 9] or [7] applying the preceding corollary with $A = \rho\eta^{\rho-1}u, B = \eta^\rho$ and $C = u^\rho$. Since $s > 1/2$, the functions A,B,C belong to $L^\infty(\mathbb{R} \times [-T,T])$ thanks to the Sobolev embedding. □

Remark 4.5. These results can be generalized to higher dimensionnal Boussinesq systems:

$$\begin{array}{rcl} \eta_t + \nabla.U + \nabla.(\eta^\rho u) + a\Delta\nabla.U - b\Delta\eta_t & = & 0 \\ U_t + \nabla\eta + \dfrac{1}{\rho+1}\nabla|U|^{\rho+1} + c\Delta\nabla\eta - d\Delta U_t & = & 0, \end{array}$$

where $\eta = \eta(x_1,...,x_n,t)$, $U = (U_1,...,U_n)(x_1,...,x_n,t)$. We obtain the following Carleman estimates.

Theorem 4.6.
We define

$$\mathcal{L} := \begin{pmatrix} \partial_t - c_1\Delta\partial_t + c_2\Delta\nabla. + f_{1,1}(x,t)\nabla. & a\Delta\nabla. + f_{1,2}(x,t)\nabla. \\ c\Delta\nabla + f_{2,1}(x,t)\nabla & \partial_t - c_3\Delta\partial_t + c_4\Delta\nabla. + f_{2,2}(x,t)\nabla. \end{pmatrix},$$

where c_1,c_2,c_3,c_4 *are constant in* $\mathbb{R}$*,* $f_{i,j} \in L^\infty(\mathbb{R}^{n+1}), 1 \leq i,j \leq 2$*. Let* $\delta > 0$ *and* $B_\delta := \{(x,t) \in \mathbb{R}^2; \sum_{i=1}^n x_i^2 + t^2 < \delta^2\}$*. Then, there exists* $C > 0$ *such that for all* $\Phi = (\Phi_1,...,\Phi_{n+1}) \in \left(C_0^\infty(B_\delta)\right)^{n+1}$*,* $\Psi(x,t) = \sum_{i=1}^n (x_i-\delta)^2/n + \delta^2 t^2$ *and* $\tau > 0$ *with*

$$\frac{\|f_{i,j}\|_\infty^2}{\tau^2} \ll 1 \text{ for } 1 \leq i,j \leq 2,$$

we have

$$\tau^3 \int_{B_\delta} \left(\sum_{i=1}^{n+1} |\Phi_i|^2 \right) e^{2\tau\Psi} dxdt + \tau^2 \int_{B_\delta} \left(\sum_{i=1}^{n+1} |\nabla.\Phi_i|^2 \right) e^{2\tau\Psi} dxdt \le C \int_{B_\delta} |\mathcal{L}\Phi|^2 e^{2\tau\Psi} dxdt.$$

References

[1] T. B. Benjamin, J. L. Bona, and J. J. Mahony, Model equations for long waves in nonlinear dispersive systems. *Philos. Trans. Roy. Soc. London Ser. A* **272** no. 1220 (1972), pp 47–78.

[2] J.L. Bona, M. Chen, and J.-C. Saut, Boussinesq equations and other systems for small amplitude long waves in nonlinear dispersive media. I. Derivation and linear theory. *J. Nonlinear Sci.* **12** no. 4 (2002), pp 283–318.

[3] J.L. Bona, M. Chen, and J.-C. Saut, Boussinesq equations and other systems for small amplitude long waves in nonlinear dispersive media. II. The nonlinear theory. *Nonlinearity* **17** no. 3 (2004), pp 925–952.

[4] J. Bourgain, On the compactness of the support of solutions of dispersive equations. *Internat. Math. Res. Notices*, **9** (1997), pp 437–447.

[5] J. V. Boussinesq, Théorie générale des mouvements qui sont propagés dans un canal rectangulaire horizontal. *C.R. Acad. Sci. Paris* **73** (1871), pp 256–260.

[6] T. Carleman, Sur les systemes linéaires aux dérivées partielles du premier ordre à deux variables. *C. R. Acad. Sci. Paris*, **197** (1933), pp 471–474.

[7] M. Davilla and G. Perla Menzala, Unique continuation for the Benjamin-Bona-Mahony and Boussinesq's equations. *Nonlinear differ. equ. appl.* **5** (1998), pp 367–382.

[8] L. Hormander, *Linear Partial Differential Operators*, Springer.Verlag, Berlin/Heidelberg/New York, 1969.

[9] A. Kozakevicius, O. Vera, On the unique continuation property for a nonlinear dispersive system. *EJQTDE*, **14** (2005), pp 1–23.

[10] Y. Mammeri, Unique continuation property for Boussinesq-type systems. *Commun. Math. Anal.* **9** no. 1 (2010), pp 121–127.

[11] L. Nirenberg, Uniqueness of Cauchy problems for differential equations with constant leading coefficient. *Comm. Pure Appl. Math.*, **10** (1957), pp 89-105.

[12] J.-C. Saut, B. Scheurer, Unique continuation for some evolution equations. *J. Diff. Eqs.*, **66** (1987), pp 118-139.

[13] F. Treves, *Linear Partial Differential Equations with Constant Coefficients*, Gordon and Breach, N. York, London, Paris, 1966.