Estimates for Dirichlet Heat Kernels, Intrinsic Ultracontractivity and Expected Exit Time on Lipschitz Domains

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(Communicated by Irena Lasiecka)

Abstract

We prove pointwise estimates for the heat kernel of a second-order elliptic operator with Dirichlet boundary conditions on a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 1$. Applications to obtain estimates for intrinsic ultracontractivity of the heat semigroup and expected exit time of a Brownian motion are given.

AMS Subject Classification: 31B25, 35K05, 47D06

Keywords: Boundary behavior, heat kernel, Dirichlet boundary conditions, elliptic operator, Green function, Lipschitz domain, intrinsic ultracontractivity, exit time.

1 Introduction

Let D be a bounded Lipschitz domain (by a domain we mean a connected open set) in $\mathbb{R}^n$, $n \geq 1$, and $L = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})$ be a second order elliptic partial differential operator with Dirichlet boundary conditions on D, where the symmetric real matrix $(a_{ij}(x))_{i,j}$ is assumed to be Hölder continuous on D with an exponent $\varepsilon \in]0,1]$, and uniformly elliptic, i.e. $\mu^{-1}\|\xi\|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \mu\|\xi\|^2$ for all $x \in D$, all $\xi \in \mathbb{R}^n$ and some constant $\mu \geq 1$. It is well known that the spectrum of L on D is discrete, $\sigma(L) = \{\lambda_i, i \in \mathbb{N}\}$ with $0 < \lambda_0 < \lambda_1 < ...$, and each λ_i is an eigenvalue with finite multiplicity. By [17, Proposition 1.4.3] the first eigenvalue λ_0 is simple and its eigenfunction φ_0 can be chosen positive and normalized by $\|\varphi_0\|_{L^2} = 1$ (the ground state eigenfunction). The semigroup e^{-tL} associated with L on D is intrinsically ultracontractive. This is even true for more general domains such as Hölder domains of order 0 (see [8]). Let P_t be the heat kernel of L on D (the integral kernel of e^{-tL}). One of the aims of this paper is to establish pointwise lower and upper bounds for P_t

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which describe its behavior near the boundary ∂D for a long and a short time. For $n = 1, D$ means a bounded interval. A bounded domain D in $\mathbb{R}^n$, $n \geq 2$, is called a *Lipschitz domain* if there exist positive constants k and r_0 such that for every $Q \in \partial D$, there exist a function $f_Q : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying $|f_Q(x') - f_Q(y')| \leq k|x' - y'|$ for all $x', y' \in \mathbb{R}^{n-1}$ and an orthonormal coordinate system CS_Q with origin Q such that if $y = (y', y_n)$ in the CS_Q coordinate, then

$$D \cap B(Q, r_0) = B(Q, r_0) \cap \{y = (y', y_n) : y_n > f_Q(y')\}$$

and

$$\partial D \cap B(Q, r_0) = B(Q, r_0) \cap \{y = (y', y_n) : y_n = f_Q(y')\}.$$

We will call k the *Lipschitz constant* of D and r_0 the *localization radius* of D. It is easy to see that for each $Q \in \partial D$, the truncated right circular cone with vertex at Q, $\{x : x_n < -k|x'|\} \cap B(Q, r_0) \subset D^c$. Thus D satisfies the exterior cone condition with aperture $\psi = \arctan(1/k)$. In particular it is uniformly Δ-regular in the sense of [5], and so there is a strong barrier for L on D (see [5, Remark 5.2]). It also satisfies the interior cone condition with the same aperture since the truncated right circular cone with vertex at Q, $\{x : x_n > k|x'|\} \cap B(Q, r_0) \subset D$. Let g be the Green function of L with Dirichlet boundary conditions on D and x_0 be a fixed point in D. We know the existence of constants $C > 1, 0 < \beta \leq 1 \leq \alpha$ depending only on n, μ, ε, k and r_0 such that $C^{-1}d^\alpha(x) \leq g(x, x_0) \wedge 1 \leq Cd^\beta(x)$ for all $x \in D$, where $d(x)$ means the euclidean distance from x to the boundary ∂D. Such estimates are even valid (see [3, p.3]) on NTA-domains introduced in [22]. By [6, Theorem 9.3] and [4, Remarque 5] we can take $\alpha = \beta = 1$, when D is $C^{1,\gamma}$, $\gamma > 0$. We recall that in the special case when $L = -\Delta$, $\alpha = \alpha_n(\psi)$ is the maximal order of barriers and $\beta = \alpha_n(\pi - \psi)$ (see [1, p.110-111] and [24, Proposition 2]). By [16, Theorem 5], we have

$$C^{-1}\varphi_0(x) \leq g(x, x_0) \wedge 1 \leq C\varphi_0(x), \tag{1.1}$$

for all $x \in D$. Thus

$$C^{-1}d^\alpha(x) \leq \varphi_0(x) \leq Cd^\beta(x), \tag{1.2}$$

for all $x \in D$. Throughout the paper, we will use the letter C to denote a positive constant whose value is unimportant and may change from line to line. We shall say that two positive functions f_1 and f_2 are comparable, written $f_1 \sim f_2$, if and only if there exists a constant $C \geq 1$ such that $C^{-1}f_1 \leq f_2 \leq Cf_1$. We shall also use the notation $f_1 \wedge f_2$ and $f_1 \vee f_2$ to mean $\min(f_1, f_2)$ and $\max(f_1, f_2)$, respectively.

In Section 1, we prove two-sided pointwise estimates for the Dirichlet heat kernel P_t on the bounded Lipschitz domain D. These estimates extend those recently proved on bounded $C^{1,1}$-domains in [28, Corollary 6.3] (see also [27] for short time behavior, and [32] and [33] for $L = -\Delta$). Recall also that less explicit short time estimates on Lipschitz domains were established in [30, 20]. In Section 2, we use our estimates to prove estimates for intrinsic ultracontractivity on Lipschitz domains which are recently proved (see [25]) in the special cases where D is a $C^{1,\gamma}$-domain, $\gamma > 0$ and $L = -\Delta$. In Section 3, we prove estimates for the expected exit time of a Brownian motion with drift term which extend the ones estalished in [7].

2   Estimates for the Dirichlet heat kernel

Our main result in this section is the following.

Theorem 2.1. *There exist positive constants C, c_1 and c_2 depending only on n, μ, ε and D such that for all $x, y \in D$ and all $t > 0$,*

$$C^{-1} \min(1, \frac{\varphi_0(x)}{1 \wedge t^{\beta/2}}) \min(1, \frac{\varphi_0(y)}{1 \wedge t^{\beta/2}}) e^{-\lambda_0 t} \frac{e^{-c_2 \frac{|x-y|^2}{t}}}{1 \wedge t^{n/2}} \leq P_t(x, y)$$

$$\leq C \min(1, \frac{\varphi_0(x)}{1 \wedge t^{\alpha/2}}) \min(1, \frac{\varphi_0(y)}{1 \wedge t^{\alpha/2}}) e^{-\lambda_0 t} \frac{e^{-c_1 \frac{|x-y|^2}{t}}}{1 \wedge t^{n/2}}.$$

Proof. We prove the upper bound. By [16, Theorem 3], there are positive constants C and c_1 depending only on n, μ, ε and D such that

$$P_t(x, y) \leq C \frac{\varphi_0(x)\varphi_0(y)}{1 \wedge t^{n/2+\alpha}} e^{-\lambda_0 t} e^{-c_1 \frac{|x-y|^2}{t}}, \tag{2.1}$$

for all $x, y \in D$ and $t > 0$. Furthermore by [17, Corollary 3.2.8], there are positive constants C and c_1 depending only on n, μ, ε and D such that

$$P_t(x, y) \leq \frac{C}{t^{n/2}} e^{-c_1 \frac{|x-y|^2}{t}}, \tag{2.2}$$

for all $x, y \in D$ and $t > 0$. To complete the proof let us consider a point $Q \in \partial D$. The functions $(x, t) \to P_t(x, y)$ and $(x, t) \to e^{-\lambda_0 t} \varphi_0(x)$ are two positive solutions of the parabolic equation $\partial u / \partial t + Lu = 0$ on $D \times (0, \infty)$ continuously vanishing on the lateral boundary $\partial D \times (0, \infty)$. By the local comparison theorem [19, Theorem 1.6], there exists a constant $C = C(n, \mu, \varepsilon, k, r_0) > 0$ such that

$$\frac{P_t(x, y)}{e^{-\lambda_0 t} \varphi_0(x)} \leq C \frac{P_{\frac{3t}{2}}((0, \sqrt{t}/2), y)}{e^{-\lambda_0 \frac{t}{2}} \varphi_0((0, \frac{\sqrt{t}}{2}))},$$

for all $x, y \in D : |x - Q| \leq \frac{\sqrt{t}}{16}, t \in]0, r_0^2]$. Here the point $(0, \sqrt{t}/2)$ is represented in the orthonormal coordinate system CS_Q with origin Q. By (1.2), $\varphi_0((0, \frac{\sqrt{t}}{2})) \geq C^{-1} t^{\alpha/2}$ and by (2.2),

$$P_{\frac{3t}{2}}((0, \sqrt{t}/2), y) \leq \frac{C}{t^{n/2}} e^{-2c_1 \frac{|(0, \sqrt{t}/2)-y|^2}{3t}}$$

with

$$|(0, \sqrt{t}/2) - y|^2 \geq |x - y|^2/2 - |(0, \sqrt{t}/2) - x|^2 \geq |x - y|^2/2 - t.$$

It follows that

$$P_t(x, y) \leq C \frac{\varphi_0(x)}{t^{n/2+\alpha/2}} e^{-c_1 \frac{|x-y|^2}{t}}, \tag{2.3}$$

for all $x, y \in D : |x - Q| \leq \frac{\sqrt{t}}{16}$, $t \in]0, r_0^2]$. By taking $Q \in \partial D : d(x) = |x - Q|$ and applying (2.3), we deduce that

$$P_t(x, y) \leq C \frac{\varphi_0(x)}{t^{n/2 + \alpha/2}} e^{-c_1 \frac{|x-y|^2}{t}}, \tag{2.4}$$

for all $x, y \in D : d(x) \leq \frac{\sqrt{t}}{16}$, $t \in]0, r_0^2]$. By recalling that for $d(x) \geq \frac{\sqrt{t}}{16}$, $\varphi_0(x) \geq C^{-1} t^{\alpha/2}$ and combining (2.2) and (2.4) we get

$$P_t(x, y) \leq C \frac{\varphi_0(x)}{t^{n/2 + \alpha/2}} e^{-c_1 \frac{|x-y|^2}{t}}, \tag{2.5}$$

for all $x, y \in D$ and $t \in]0, r_0^2]$. Exchanging the roles of x and y in (2.5) and combining it with (2.1), (2.2) and (2.5), we get the upper bound. We prove the lower bound. By [16, Proposition 4] there is a constant $C > 0$ such that

$$P_t(x, y) \geq C^{-1} \varphi_0(x) \varphi_0(y) e^{-\lambda_0 t}, \tag{2.6}$$

for all $x, y \in D$ and $t > T$, for some T large enough. In particular the lower bound holds for $t > T$. We next prove the lower bound for $t \in]0, r_0^2]$. To this end we need to distinguish three cases. Let $a, b \in]0, 1[$ be fixed.

Case 1: $d(x) \geq a \sqrt{t}$ and $d(y) \geq b \sqrt{t}$ with $t \in]0, r_0^2]$.

D is a Lipschitz domain, so it is an NTA-domain (see [22]) and then a uniform domain (see [21]). In particular by [21, (3.1)], there is a constant $c = c(D) > 0$ such that x and y can be connected by a rectifiable curve $l \subset D$ with length $|l| \leq c|x - y|$ and $d(l, \partial D) > \frac{a \wedge b}{2c} \sqrt{t}$. Thus by the same proof given in [27, p.386], we obtain

$$P_t(x, y) \geq \frac{C^{-1}}{t^{n/2}} e^{-c_2 \frac{|x-y|^2}{t}}, \tag{2.7}$$

where C and c_2 are positive constants depending only on $n, \mu, \varepsilon, D, a$ and b.

Case 2: $d(x) \leq a \sqrt{t}$ and $d(y) \geq b \sqrt{t}$ with $t \in]0, r_0^2]$.

Let $Q \in \partial D$. The functions $(x, t) \to P_t(x, y)$ and $(x, t) \to e^{-\lambda_0 t} \varphi_0(x)$ are two positive solutions of the parabolic equation $\partial u / \partial t + L u = 0$ on $D \times (0, \infty)$ continuously vanishing on the lateral boundary $\partial D \times (0, \infty)$. By the local comparison theorem [19, Theorem 1.6], there exists a constant $C = C(n, \mu, \varepsilon, k, r_0) > 0$ such that

$$\frac{P_t(x, y)}{e^{-\lambda_0 t} \varphi_0(x)} \geq C^{-1} \frac{P_{\frac{t}{2}}((0, \sqrt{t}/2), y)}{e^{-\lambda_0 \frac{3t}{2}} \varphi_0((0, \sqrt{t}/2))}, \tag{2.8}$$

for all $x, y \in D : |x - Q| \leq \frac{\sqrt{t}}{16}$ with $t \in]0, r_0^2]$. Here the point $(0, \sqrt{t}/2)$ is represented in the orthonormal coordinate system CS_Q with origin Q. If furthermore we take $d(y) \geq b \sqrt{t}$ we have, by (2.7),

$$P_{\frac{t}{2}}((0, \sqrt{t}/2), y) \geq \frac{C^{-1}}{t^{n/2}} e^{-c_2 \frac{|(0, \sqrt{t}/2) - y|^2}{t/2}} \tag{2.9}$$

with

$$|(0, \sqrt{t}/2) - y|^2 \leq 2|(0, \sqrt{t}/2) - x|^2 + 2|x - y|^2 \leq t + 2|x - y|^2. \tag{2.10}$$

Combining (2.8)-(2.10) and recalling that by (1.2), $\varphi_0((0, \sqrt{t}/2)) \leq Ct^{\beta/2}$, we obtain

$$P_t(x,y) \geq C^{-1} \frac{\varphi_0(x)}{t^{(n+\beta)/2}} e^{-4c_2 \frac{|x-y|^2}{t}}, \qquad (2.11)$$

for all $x, y \in D : |x - Q| \leq \frac{\sqrt{t}}{16}, d(y) \geq b\sqrt{t}$ with $t \in]0, r_0^2]$. By taking $Q \in \partial D : d(x) = |x - Q|$ and applying (2.11), we deduce that

$$P_t(x,y) \geq C^{-1} \frac{\varphi_0(x)}{t^{(n+\beta)/2}} e^{-4c_2 \frac{|x-y|^2}{t}}, \qquad (2.12)$$

for all $x, y \in D : d(x) \leq \frac{\sqrt{t}}{16}, d(y) \geq b\sqrt{t}$ with $t \in]0, r_0^2]$. Combining (2.12) and (2.7), we get

$$P_t(x,y) \geq C^{-1} \frac{\varphi_0(x)}{t^{(n+\beta)/2}} e^{-4c_2 \frac{|x-y|^2}{t}}, \qquad (2.13)$$

for all $x, y \in D : d(x) \leq \sqrt{t}, d(y) \geq b\sqrt{t}$ with $t \in]0, r_0^2]$.

Case 3: $d(x) \leq a\sqrt{t}$ and $d(y) \leq b\sqrt{t}$ with $t \in]0, r_0^2]$.

Let $Q \in \partial D : d(x) = |x - Q|$. In the local coordinate system CS_Q let $z = (z', z_n) = (0, \frac{\sqrt{t}}{2})$. We have $d(z) \geq c\sqrt{t}$, where $c = \frac{1}{2\sqrt{1+k^2}}$. Hence for $\xi \in B(z, \frac{c\sqrt{t}}{2})$, $d(\xi) \geq \frac{c\sqrt{t}}{2}$ and $|x - \xi|^2 \leq 4t$. So, by the semigroup identity and Case 2, we have

$$
\begin{aligned}
P_t(x,y) &= \int_D P_{\frac{t}{2}}(x,\xi) P_{\frac{t}{2}}(\xi,y) d\xi \\
&\geq \int_{B(z,\frac{c\sqrt{t}}{2})} P_{\frac{t}{2}}(x,\xi) P_{\frac{t}{2}}(\xi,y) d\xi \\
&\geq C^{-1} \int_{B(z,\frac{c\sqrt{t}}{2})} \frac{\varphi_0(x)}{t^{(n+\beta)/2}} e^{-4c_2 \frac{|x-\xi|^2}{t}} \frac{\varphi_0(y)}{t^{(n+\beta)/2}} e^{-4c_2 \frac{|\xi-y|^2}{t}} d\xi \\
&= C^{-1} \frac{\varphi_0(x)\varphi_0(y)}{t^{n+\beta}} \int_{B(z,\frac{c\sqrt{t}}{2})} e^{-4c_2 \frac{(|x-\xi|^2+|\xi-y|^2)}{t}} d\xi \\
&\geq C^{-1} \frac{\varphi_0(x)\varphi_0(y)}{t^{n+\beta}} \int_{B(z,\frac{c\sqrt{t}}{2})} e^{-4c_2 \frac{(3|x-\xi|^2+2|x-y|^2)}{t}} d\xi \\
&\geq C^{-1} \frac{\varphi_0(x)\varphi_0(y)}{t^{n+\beta}} e^{-8c_2 \frac{|x-y|^2}{t}} \int_{B(z,\frac{c\sqrt{t}}{2})} d\xi \\
&= C^{-1} \frac{\varphi_0(x)\varphi_0(y)}{t^{n/2+\beta}} e^{-8c_2 \frac{|x-y|^2}{t}}. \qquad (2.14)
\end{aligned}
$$

Combining (2.7), (2.13) and (2.14), we get

$$P_t(x,y) \geq C^{-1} \min(1, \frac{\varphi_0(x)}{t^{\beta/2}}) \min(1, \frac{\varphi_0(y)}{t^{\beta/2}}) \frac{e^{-8c_2 \frac{|x-y|^2}{t}}}{t^{n/2}}, \qquad (2.15)$$

for all $x, y \in D$ and all $t \in]0, r_0^2]$. Clearly by dividing $]0, T]$ into intervals of length $r_0^2/2$, using the semigroup identity and (2.15), we obtain

$$P_t(x,y) \geq C^{-1} \varphi_0(x)\varphi_0(y), \qquad (2.16)$$

for all $x, y \in D$ and all $t \in]r_0^2, T]$, where $C = C(n, \mu, \varepsilon, D, T/r_0^2) > 0$. Hence the lower bound in Theorem 2.1 follows from (2.6), (2.15) and (2.16) and the proof is completed. $\qquad \square$

We also obtain the following estimates:

Theorem 2.2. *There exist positive constants C, c_1 and c_2 depending only on n, μ, ε and D such that for all $x, y \in D$ and all $t > 0$,*

$$C^{-1}\min(1, \frac{\varphi_0(x)\varphi_0(y)}{1 \wedge t^\beta})e^{-\lambda_0 t}\frac{e^{-2c_2\frac{|x-y|^2}{t}}}{1 \wedge t^{n/2}} \quad \leq \quad P_t(x,y)$$

$$\leq \quad C\min(1, \frac{\varphi_0(x)\varphi_0(y)}{1 \wedge t^\alpha})e^{-\lambda_0 t}\frac{e^{-c_1\frac{|x-y|^2}{t}}}{1 \wedge t^{n/2}}.$$

Proof. The upper bound is trivial from Theorem 2.1. For the lower bound it suffices, in view of Theorem 2.1, to investigate the case $\varphi_0(x) \geq t^{\beta/2}$ and $\varphi_0(y) \leq t^{\beta/2}$, $t \in]0,1]$.
If $|x-y|^\beta \geq \varphi_0(x)/4^\beta$, then

$$\min(1, \frac{\varphi_0(x)\varphi_0(y)}{t^\beta})e^{-c_2\frac{|x-y|^2}{t}} \quad \leq \quad C\min(1, \frac{\varphi_0(y)}{t^{\beta/2}}\frac{|x-y|^\beta}{t^{\beta/2}})e^{-c_2\frac{|x-y|^2}{t}}$$

$$\leq \quad C\min(1, \frac{\varphi_0(y)}{t^{\beta/2}})$$

$$\leq \quad C\min(1, \frac{\varphi_0(x)}{t^{\beta/2}})\min(1, \frac{\varphi_0(y)}{t^{\beta/2}}),$$

where we have used in the second inequality, $\theta^{\beta/2}e^{-c_2\theta} \leq C$ for all $\theta \geq 0$.
If $|x-y|^\beta \leq \varphi_0(x)/4^\beta$, then by (1.2), $|x-y| \leq d(x)/4$ which yields $3d(x)/4 \leq d(y) \leq 5d(x)/4$. By (1.1) and the Harnack's principle [14, Theorem 5.1] applied in the case $d(x) < r_0$, and (1.2) applied in the case $d(x) \geq r_0$, we obtain $\varphi_0(x) \leq C\varphi_0(y)$ with $C = C(n, \mu, D) > 0$. Hence $C^{-1}\varphi_0(x) \leq \varphi_0(y) \leq t^{\beta/2} \leq \varphi_0(x)$ which yields

$$\min(1, \frac{\varphi_0(x)\varphi_0(y)}{t^\beta}) \sim C \sim \min(1, \frac{\varphi_0(x)}{t^{\beta/2}})\min(1, \frac{\varphi_0(y)}{t^{\beta/2}}),$$

and the proof is completed.   □

Recall that $g(x,y) = \int_0^\infty P_t(x,y)dt$ is the Dirichlet L-Green function on D. By integrating with respect to time the estimates in Theorem 2.2, we obtain the following Green function estimates which extend to Lipschitz domains those proved for $C^{1,\gamma}$-domains, $\gamma > 0$, in $\mathbb{R}^n$, $n \geq 3$ (see [26, Corollary 4.5]), and for Dini-smooth Jordan domains in the euclidean plane $\mathbb{R}^2$ (see [29, Proposition 4]). We recall that in the euclidean plane a $C^{1,\gamma}$-domain is a Dini-smooth Jordan domain and a Dini-smooth Jordan domain is a Lipschitz domain.

Corollary 2.3. *(1) For $n \geq 3$ there is a constant $C = C(n, \mu, \varepsilon, D) > 0$ such that for all $x, y \in D$,*

$$C^{-1}\min(1, \frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{2\beta}})|x-y|^{2-n} \leq g(x,y) \leq C\min(1, \frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{2\alpha}})|x-y|^{2-n}.$$

(2) For $n = 2$ there is a constant $C = C(\mu, \varepsilon, D) > 0$ such that for all $x, y \in D$,

$$C^{-1}\ln(1 + \frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{2\beta}}) \leq g(x,y) \leq C\ln(1 + \frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{2\alpha}}).$$

Proof. (1) Case: $n \geq 3$. By using the estimates in Theorem 2.2 and the change of variable $s = \frac{|x-y|^2}{t}$, we have

$$C^{-1} \int_0^\infty \min(1, s^\beta \frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{2\beta}}) \frac{s^{n/2-2}e^{-c_2 s}}{|x-y|^{n-2}} ds \leq g(x,y)$$

$$\leq C \int_0^\infty \min(1, s^\alpha \frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{2\alpha}}) \frac{s^{n/2-2}e^{-c_1 s}}{|x-y|^{n-2}} ds$$

which yields

$$C^{-1} \int_0^\infty \min(1, s^\beta) s^{n/2-2} e^{-c_2 s} ds \min(1, \frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{2\beta}}) |x-y|^{2-n} \leq g(x,y)$$

$$\leq C \int_0^\infty \max(1, s^\alpha) s^{n/2-2} e^{-c_1 s} ds \min(1, \frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{2\alpha}}) |x-y|^{2-n},$$

and so the estimates hold.

(2) Case: $n = 2$. From the lower estimate in Theorem 2.2 and the change of variable $s = \frac{|x-y|^2}{t}$, we have

$$\begin{aligned}
g(x,y) &\geq C^{-1} \int_0^1 P_t(x,y) dt \\
&\geq C^{-1} \int_0^1 \min(1, \frac{\varphi_0(x)\varphi_0(y)}{t^\beta}) \frac{e^{-c_2 \frac{|x-y|^2}{t}}}{t} dt \\
&\geq C^{-1} \int_{|x-y|^2}^\infty \min(1, s^\beta \frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{2\beta}}) \frac{e^{-c_2 s}}{s} ds \qquad (2.17)
\end{aligned}$$

If $\frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{2\beta}} \leq 1$, then from (2.17), we have

$$\begin{aligned}
g(x,y) &\geq C^{-1} \frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{2\beta}} \int_{d^2(D)}^\infty \min(1, s^\beta) \frac{e^{-c_2 s}}{s} ds \\
&\geq C^{-1} \frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{2\beta}} \\
&\geq C^{-1} \ln(1 + \frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{2\beta}}).
\end{aligned}$$

If $\frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{2\beta}} \geq 1$, then by letting $c_0 = [2\|\varphi_0\|_\infty^2]^{1/\beta}$, we obtain, from (2.17),

$$\begin{aligned}
g(x,y) &\geq C^{-1} \int_{c_0|x-y|^2/[2\varphi_0(x)\varphi_0(y)]^{1/\beta}}^{c_0} \frac{e^{-c_2 s}}{s} ds \\
&\geq C^{-1} e^{-c_2 c_0} \int_{c_0|x-y|^2/[2\varphi_0(x)\varphi_0(y)]^{1/\beta}}^{c_0} \frac{ds}{s} \\
&= \beta^{-1} C^{-1} e^{-c_2 c_0} \ln(\frac{2\varphi_0(x)\varphi_0(y)}{|x-y|^{2\beta}}) \\
&\geq C^{-1} \ln(1 + \frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{2\beta}})
\end{aligned}$$

We now prove the upper estimate. From the upper estimate in Theorem 2.2 we have

$$
\begin{aligned}
g(x,y) &= \int_0^\infty P_t(x,y)dt \\
&\leq C\Big[\int_0^1 \min(1,\frac{\varphi_0(x)\varphi_0(y)}{t^\alpha})\frac{e^{-c_1\frac{|x-y|^2}{t}}}{t}dt + \varphi_0(x)\varphi_0(y)\int_1^\infty e^{-\lambda_0 t}dt\Big] \\
&\leq C\int_0^1 \min(1,\frac{\varphi_0(x)\varphi_0(y)}{t^\alpha})\frac{e^{-c_1\frac{|x-y|^2}{t}}}{t}dt + C\varphi_0(x)\varphi_0(y). \qquad (2.18)
\end{aligned}
$$

By the inequality $\frac{t}{1+t} \leq \ln(1+t)$ for all $t \geq 0$, we have

$$
\varphi_0(x)\varphi_0(y) \leq C\ln(1+\frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{2\alpha}}). \qquad (2.19)
$$

By the change of variable $s = \frac{|x-y|^2}{t}$, we have

$$
\int_0^1 \min(1,\frac{\varphi_0(x)\varphi_0(y)}{t^\alpha})\frac{e^{-c_1\frac{|x-y|^2}{t}}}{t}dt = \int_{|x-y|^2}^\infty \min(1,s^\alpha\frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{2\alpha}})\frac{e^{-c_1 s}}{s}ds. \qquad (2.20)
$$

If $\frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{2\alpha}} \leq 1$, then we obtain

$$
\begin{aligned}
\int_{|x-y|^2}^\infty \min(1,s^\alpha\frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{2\alpha}})\frac{e^{-c_1 s}}{s}ds &\leq \frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{2\alpha}}\int_0^\infty s^{\alpha-1}e^{-c_1 s}ds \\
&\leq C\ln(1+\frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{2\alpha}}). \qquad (2.21)
\end{aligned}
$$

If $\frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{2\alpha}} \geq 1$, then by letting $c_0 = \|\varphi_0\|_\infty^{2/\alpha}$, we obtain

$$
\begin{aligned}
&\int_{|x-y|^2}^\infty \min(1,s^\alpha\frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{2\alpha}})\frac{e^{-c_1 s}}{s}ds \\
&\leq C\Big[\int_{|x-y|^2}^{c_0\frac{|x-y|^2}{[\varphi_0(x)\varphi_0(y)]^{1/\alpha}}} s^{\alpha-1}ds\frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{2\alpha}} + \int_{c_0\frac{|x-y|^2}{[\varphi_0(x)\varphi_0(y)]^{1/\alpha}}}^{c_0}\frac{ds}{s} + \int_{c_0}^\infty\frac{e^{-c_1 s}}{s}ds\Big] \\
&\leq C\Big[\frac{c_0^\alpha}{\alpha} + \frac{1}{\alpha}\ln(1+\frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{2\alpha}}) + \frac{e^{-c_1 c_0}}{c_1 c_0}\Big] \\
&\leq C\ln(1+\frac{\varphi_0(x)\varphi_0(y)}{|x-y|^{2\alpha}}). \qquad (2.22)
\end{aligned}
$$

The upper estimate holds by combining (2.18)-(2.22).   □

Remark. By using the estimates of Theorem 2.1 and the same proof as of Corollary 2.3, we also obtain the following estimates for the Green function:
For $n \geq 3$ there is a constant $C = C(n,\mu,\varepsilon,D) > 0$ such that for all $x,y \in D$,

$$
C^{-1}\min(1,\frac{\varphi_0(x)}{|x-y|^\beta})\min(1,\frac{\varphi_0(y)}{|x-y|^\beta})|x-y|^{2-n} \leq g(x,y)
$$

$$
\leq C\min(1,\frac{\varphi_0(x)}{|x-y|^\alpha})\min(1,\frac{\varphi_0(y)}{|x-y|^\alpha})|x-y|^{2-n}.
$$

For $n = 2$ there is a constant $C = C(\mu, \varepsilon, D) > 0$ such that for all $x, y \in D$,

$$g(x,y) \leq C \ln(1 + \frac{\varphi_0(x) \wedge \varphi_0(y)}{|x-y|^\alpha}).$$

However, for $n = 2$, a similar lower estimate by the member $\ln(1 + \frac{\varphi_0(x) \wedge \varphi_0(y)}{|x-y|^\beta})$ does not hold. This is clear by letting $\varphi_0(x) \sim \varphi_0(y)$ go to zero with $|x-y| > c > 0$ and using the estimates in Corollary 2.3.

It is well known (see [14, Theorem 5.5]) that the Martin boundary with respect to L is homeomorphic to the euclidean boundary ∂D and its point $Q \in \partial D$ is minimal. The Martin kernel is given by $K(x,Q) = \lim_{m \to \infty} \frac{g(x,y_m)}{g(x_0,y_m)}$, where $(y_m)_m$ is a sequence convergent to Q and $x \in D$. We derive the following Martin kernel estimates on Lipschitz domains which extend those known for $C^{1,\gamma}$-domains, $\gamma > 0$, in $\mathbb{R}^n$, $n \geq 3$ and for Dini-smooth Jordan domains in the euclidean plane $\mathbb{R}^2$.

Corollary 2.4. *For $n \geq 2$ there is a constant $C = C(n, \mu, \varepsilon, D, x_0) > 0$ such that for all $x \in D$ and all $Q \in \partial D$,*

$$C^{-1} \frac{\varphi_0(x)}{|x-Q|^{n-2+2\beta}} \leq K(x,Q) \leq C \frac{\varphi_0(x)}{|x-Q|^{n-2+2\alpha}}$$

Remarks. 1. As is mentioned in the introduction, if D is a bounded $C^{1,\gamma}$-domain, $\gamma > 0$, then the heat kernel estimates in Thoerem 2.1 hold on D with $\alpha = \beta = 1$.

2. If the coefficients $a_{ij}(x)$ are assumed to be only bounded and measurable or even bounded and continuous on D, then the Green function estimates in Corollary 2.3 (and so the heat kernel estimates in Theorem 2.1 and Theorem 2.2) may fail to hold. This follows from the paper [10]. In fact in [10], Caffarelli et al. constructed an operator L with bounded measurable coefficients (and in the dimension $n = 2$ with bounded continuous coefficients) on the unit ball $B(0,1)$ for which the L-harmonic measure is completely singular to the surface measure. If we assume that our estimates are valid for such operator on $B(0,1)$ with $\alpha = \beta = 1$, we deduce that g is comparable to $g_{-\Delta}$, the Dirichlet Green function of $-\Delta$ on $B(0,1)$, and this in turn implies the equivalence of the L-harmonic measure and the $-\Delta$-harmonic measure on $\partial B(0,1)$; and so by the Dahlberg's result [15] it follows that the L-harmonic measure and the surface measure are equivalent on $\partial B(0,1)$ which is a contradiction.

In what follows we may even provide, by the use of a result in [9], an example of an operator L with coefficients $a_{ij}(x)$ in $C^\infty(D) \cap C(\overline{D})$ for which the estimates in Corollary 2.3 are not true. Let B_r be the ball with center the origin and radius r in $\mathbb{R}^2$, $D = B_6 \cap \{x = (x_1, x_2) : x_2 > 0\}$, $p_0 = (0, \frac{11}{2})$ and $x_0 = (0,3)$. By [9, Theorem 1.1] there exists a function $h \in C^\infty(D) \cap C(\overline{D})$ such that the L-harmonic measure ω_{p_0} at p_0 and the surface measure σ are singular on $B_1 \cap \partial D$, where

$$L = \frac{\partial^2}{\partial x_1^2} + \frac{\partial}{\partial x_2}(h(x_1, x_2) \frac{\partial}{\partial x_2}).$$

Let g and $g_{-\Delta}$ be the Dirichlet Green functions on D of L and $-\Delta$, respectively. Let $\omega_x^{-\Delta}$ be the $-\Delta$-harmonic measure at x. By the Dahlberg's result [15], $\omega_x^{-\Delta} \sim \sigma$ on ∂D. Assume

that g satisfies the estimates in Corollary 2.3 on D and let $Q = (0, 6)$. By the smoothness of $\partial D \cap B(Q, 1)$, $\varphi_0 \sim d$ on $D \cap B(Q, 1)$, it follows that $g(., x_0) \sim d(.) \sim g_{-\Delta}(., x_0)$ on $D \cap B(Q, 1)$. On the other hand by the Harnack's principle [14, Theorem 5.1] and the boundary Harnack principle [14, Theorem 5.2], there is a constant $C > 0$ such that for any Borel subset $A \subset B_1 \cap \partial D$,

$$C^{-1} g(x, x_0) \le \omega_x(A) \le C g(x, x_0)$$

and

$$C^{-1} g_{-\Delta}(x, x_0) \le \omega_x^{-\Delta}(A) \le C g_{-\Delta}(x, x_0)$$

for all $x \in D \cap B(Q, 1)$. We deduce that $\omega_{p_0} \sim \sigma$ on $B_1 \cap \partial D$ which is a contradiction.
3. Consider the non-divergence form elliptic operator

$$\mathcal{L} = -\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x),$$

where $(a_{ij}(x))_{i,j}$ is uniformly elliptic and Hölder continuous on the bounded Lipschitz domain D and the functions b_i and c satisfy

$$d(x) \sum_{i=1}^{n} |b_i(x)| + d^2(x)|c(x)| \le \theta(d(x)),$$

where θ is an increasing function on $]0, \infty[$ such that $\int_0^1 \frac{\theta(t)}{t} dt < \infty$. By [6, Theorem 9.3], the Dirichlet $\mathcal{L}$-Green function $\mathcal{G}$ is comparable to the Dirichlet L-Green function g on D, where $L = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x) \frac{\partial}{\partial x_j})$. Thus $\mathcal{G}$ satisfies the estimates in Corollary 2.3 as well.

3 Estimates for intrinsic ultracontractivity

In this section, we apply the estimates in Theorem 2.2 to obtain estimates for intrinsic ultracontractivity (IU) of the heat semigroup on Lipschitz domains extending those recently proved on $C^{1,\gamma}$-domains in [25]. The IU which was first introduced by Davies and Simon in [18] is investigated by several authors. In [8, 11, 12, 17, 31], some properties and criteria for IU are given in various classes of domains. Recall that in [25, Theorem 1.1], the estimates for IU on $C^{1,\gamma}$-domains, $\gamma > 0$, are proved by first establishing some criteria formulated in terms of superpoincaré inequalities and the behavior of the ground state of Dirichlet forms. Here, we derive the estimates on the Lipschitz domain by a simple use of the bounds given in Theorem 2.2. Let $\xi_t = e^{\lambda_0 t} \sup_{x,y \in D} \frac{P_t(x,y)}{\varphi_0(x)\varphi_0(y)}$. We have the following.

Theorem 3.1. *(1) There exists a constant $C = C(n, \mu, \varepsilon, D) > 0$ such that for all $t > 0$,*

$$\max\{1, C^{-1} t^{-(\frac{n}{2}+\beta)}\} \le \xi_t \le 1 + C(1 \wedge t)^{-(\frac{n}{2}+\alpha)} e^{-(\lambda_1-\lambda_0)t}$$

(2) Let $L = -\Delta$ and $n \ge 2$. For any $\delta > 0$, there exists a bounded Lipschitz domain $D \subset \mathbb{R}^n$ such that

$$\limsup_{t \to 0} t^\delta \xi_t = +\infty$$

Proof. (1) From the upper bound in Theorem 2.2, we have

$$\xi_t \leq C(1 \wedge t)^{-(\frac{n}{2}+\alpha)}, \quad t > 0,$$

which trivially yields

$$\xi_t \leq C t^{-(\frac{n}{2}+\alpha)} e^{-(\lambda_1-\lambda_0)t}, \quad 0 < t \leq 1. \tag{3.1}$$

On the other hand, by [8, p.184], there is $C > 0$ such that for all $t > 1$,

$$\xi_t \leq 1 + C e^{-(\lambda_1-\lambda_0)t}, \tag{3.2}$$

Combining (3.1) and (3.2), we obtain the upper estimate in Theorem 3.1. Now we prove the lower estimate. By the lower bound in Theorem 2.2, we have

$$\frac{C^{-1}}{\max\{t^\beta, \varphi_0(x)\varphi_0(y)\}} \frac{e^{-c_2 \frac{|x-y|^2}{t}}}{t^{n/2}} \leq e^{\lambda_0 t} \frac{P_t(x,y)}{\varphi_0(x)\varphi_0(y)}$$

for all $x, y \in D$ and all $t \in]0, 1]$, which yields

$$\frac{C^{-1}}{\max\{t^\beta, \varphi_0^2(x)\}} \frac{1}{t^{n/2}} \leq \xi_t$$

for all $x \in D$ and all $t \in]0, 1]$, and so

$$\frac{C^{-1}}{t^{n/2+\beta}} \leq \xi_t \tag{3.3}$$

for all $t \in]0, 1]$. On the other hand by renumbering the eigenvalues in increasing order so that each λ_n is repeated according to its multiplicity and using the series expansions of the heat kernel

$$P_t(x,y) = \sum_{n \geq 0} e^{-\lambda_n t} \varphi_n(x)\varphi_n(y),$$

where φ_n is the eigenfunction associated to the eigenvalue λ_n normalized by $\|\varphi_n\|_{L^2} = 1$, we have

$$e^{-\lambda_0 t}\varphi_0^2(x) \leq P_t(x,x), \quad t > 0$$

which yields

$$1 \leq \xi_t, \quad t > 0. \tag{3.4}$$

Combining (3.3) and (3.4), we obtain the lower estimate in Theorem 3.1.

(2) Let $0 < \psi < \frac{\pi}{2}$ and let $T_\psi := \{x \in \mathbb{R}^n : x_n > |x|\cos\psi\}$ be the right circular cone with vertex at the origin and aperture ψ. Let $D = T_\psi \cap B(0, 1)$ be the truncated circular cone. It is well known that there is a positive harmonic function u_ψ on T_ψ such that $u_\psi = 0$ on ∂T_ψ. Such a function u_ψ is unique up to a multiplicative constant and is homogeneous of degree $\alpha = \alpha_n(\psi) > 0$, i.e. $u_\psi(rx) = r^\alpha u_\psi(x)$ for $r > 0$. It is known that α_n is strictly decreasing; $\alpha_n(\frac{\pi}{2}) = 1$; $\lim_{\psi \to 0} \alpha_n(\psi) = +\infty$; $\lim_{\psi \to \pi} \alpha_n(\psi) = 0$ (for $n \geq 3$); $\alpha_2(\psi) = \frac{\pi}{2\psi}$; $\alpha_4(\psi) = \frac{\pi}{\psi}$ (see [1, p.110-111] and [24]). In particular for $0 < r \leq r_0$, r_0 small and $x \in T_{\psi/2} \cap B(0, r)$

we have $u_\psi(x) \sim |x|^\alpha \sim d(x)^\alpha$ and by the boundary Harnack principle [22, Theorem 5.1], we have $u_\psi \sim g(.,x_0) \wedge 1$ on $T_\psi \cap B(0,r)$. Since $g(.,x_0) \wedge 1 \sim \varphi_0$, we have $\varphi_0(x) \sim d(x)^\alpha$ for $x \in T_{\psi/2} \cap B(0,r)$. By the same proof as in Theorem 2.1 (based on the local comparison theorem [19, Theorem 1.6]), we obtain the lower bound,

$$C^{-1}\min(1,\frac{\varphi_0(x)}{1 \wedge t^{\alpha/2}})\min(1,\frac{\varphi_0(y)}{1 \wedge t^{\alpha/2}})e^{-\lambda_0 t}\frac{e^{-c_2\frac{|x-y|^2}{t}}}{1 \wedge t^{n/2}} \leq P_t(x,y) \tag{3.5}$$

for all $x,y \in T_{\psi/2} \cap B(0,r)$ and all $t > 0$. By taking $x = y$ in (3.5), we get

$$C^{-1}\frac{1}{\max\{\varphi_0^2(x),t^\alpha\}}\frac{1}{t^{n/2}} \leq e^{\lambda_0 t}\frac{P_t(x,x)}{\varphi_0^2(x)}$$

for all $x \in T_{\psi/2} \cap B(0,r)$ and all $t \in]0,1]$. This yields

$$\frac{C^{-1}}{t^{\alpha+n/2}} \leq \xi_t$$

for all $t \in]0,1]$. Let $\delta > 0$. By recalling that $\lim_{\psi \to 0} \alpha_n(\psi) = +\infty$ and choosing ψ small enough so that $\alpha > \delta - n/2$, we obtain $\lim_{t \to 0} t^\delta \xi_t = +\infty$.   □

Remark. Note that part (2) in Theorem 3.1 is shown by a unified proof for all $n \geq 2$. In [25, Theorem 1.1 (c)], it is proved for $n = 2$ ($D \subset \mathbb{R}^2$) and could be extended to $n \geq 2$ by taking for example $D \times (0,1)^{n-2}$.

4   Estimates of the expected exit time

In this section we prove the equivalence of the expected exit time of a Brownian motion with drift term and the ground state eigenfunction φ_0 on some Lipschitz domains. This result extends [7, Theorem 1.1] and it is proved by a short and a more elementary proof. Let X_t be the diffusion process with generator $\mathcal{L} = -\Delta + b(x)\nabla_x$ on $\mathbb{R}^n$ with b being a vector-valued function such that $|b|$ is in K_{n+1}, where K_n is the well known Kato class. Let $\tau_D := \inf\{t : X_t \notin D\}$ the first exit time from D by X_t and $\mathcal{G}$ be the $\mathcal{L}$-Green function with Dirichlet boundary conditions on D. We have (see [23, (3.1)]) $E_x[\tau_D] = \int_D \mathcal{G}(x,y)dy$, $x \in D$. If we assume, furthermore, that $|b|^2$ is in K_n, we know, by [13], that $\mathcal{G}$ is comparable to $g_{-\Delta}$, the Green function of $-\Delta$ with Dirichlet boundary conditions on D. In particular $\mathcal{G}$ satisfies the same estimates given in Corollary 2.3. We have the following.

Theorem 4.1. *Let D be a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 2$ satisfying the interior cone condition with common angle $\psi \in]\cos^{-1}(\frac{1}{\sqrt{n}}),\pi[$. Then there is a constant $C = C(n,\psi,\mathrm{diam}(D)) > 1$ such that for all $x \in D$,*

$$C^{-1}\varphi_0(x) \leq E_x[\tau_D] \leq C\varphi_0(x).$$

Proof. From the Green function estimates in Corollary 2.3, for all $x,y \in D$,

$$C^{-1}\varphi_0(x)\varphi_0(y) \leq \mathcal{G}(x,y).$$

Thus for all $x \in D$,

$$C^{-1} \int_D \varphi_0(y)dy\, \varphi_0(x) \le \int_D \mathcal{G}(x,y)dy = E_x[\tau_D],$$

which implies the lower bound. As we see the lower bound is valid for a Lipschitz domain without the restriction in the theorem. We prove the upper bound. We note that for $\psi \in\,]\cos^{-1}(\frac{1}{\sqrt{n}}), \pi[$, $\alpha = \alpha_n(\psi) < \alpha_n(\cos^{-1}(\frac{1}{\sqrt{n}})) = 2$ (see [1, p.112] or [2, p.316]) and by the remark following Corollary 2.3, for all $x, y \in D$,

$$\mathcal{G}(x,y) \le C\frac{\varphi_0(x)}{|x-y|^{n-2+\alpha}},$$

which implies

$$\begin{aligned}
E_x[\tau_D] &= \int_D \mathcal{G}(x,y)dy \\
&\le C\int_D \frac{dy}{|x-y|^{n-2+\alpha}}\varphi_0(x) \\
&\le C\int_0^{d(D)} r^{1-\alpha}dr\, \varphi_0(x) \\
&= \frac{C(d(D))^{2-\alpha}}{2-\alpha}\varphi_0(x).
\end{aligned}$$

$\square$

Remark. As is stated in [7], if D is a bounded Lipschitz domain with Lipschitz constant $k < \frac{1}{\sqrt{n-1}}$, then it satisfies the interior cone condition with common angle $\psi = \tan^{-1}(\frac{1}{k}) > \tan^{-1}(\sqrt{n-1}) = \cos^{-1}(\frac{1}{\sqrt{n}})$, and so Theorem 4.1 is valid for such domains.

The following result obtained for $\mathcal{L} = -\Delta$ improves Theorem 3.3 in [7] and it shows that Theorem 4.1 is sharp.

Theorem 4.2. *Let $n \ge 2$, $T_\psi := \{x \in \mathbb{R}^n : x_n > |x|\cos\psi\}$ be the right circular cone with vertex at the origin and angle $\psi < \cos^{-1}(\frac{1}{\sqrt{n}})$, $\alpha = \alpha_n(\psi) > 2$. Let $D = T_\psi \cap B(0,1)$ be the truncated circular cone. Then there is a constant $C = C(n,\psi) > 1$ such that for all $x \in T_{\frac{\psi}{2}} \cap B(0,\frac{1}{2})$,*

$$E_x[\tau_D] \ge C^{-1}(d(x))^{2-\alpha}\varphi_0(x).$$

Proof. By integrating (3.5) with respect to time we obtain for all $x, y \in T_{\frac{\psi}{2}} \cap B(0,r)$ with $0 < r \le r_0$, r_0 small,

$$C^{-1}\min(1, \frac{\varphi_0(x)}{|x-y|^\alpha})\min(1, \frac{\varphi_0(y)}{|x-y|^\alpha})|x-y|^{2-n} \le g_{-\Delta}(x,y).$$

By noting that for $x \in T_{\frac{\psi}{2}} \cap \{B(0,r) \setminus B(0,\frac{r}{2})\}$ and $y \in T_{\frac{\psi}{2}} \cap \{B(0,\frac{r}{3}) \setminus B(0,\frac{r}{4})\}$, we have $|x-y| \sim r \sim d(x) \sim d(y)$ and $\varphi_0(x) \sim (d(x))^\alpha \sim \varphi_0(y)$, it follows that for $x \in T_{\frac{\psi}{2}} \cap \{B(0,r) \setminus B(0,\frac{r}{2})\}$,

$$E_x[\tau_D] = \int_D g_{-\Delta}(x,y)dy \ge \frac{C^{-1}}{r^{n-2}}\int_{T_{\frac{\psi}{2}} \cap \{B(0,\frac{r}{3})\setminus B(0,\frac{r}{4})\}} dy$$

$$\geq \quad C^{-1}r^2$$
$$\geq \quad C^{-1}(d(x))^{2-\alpha}\varphi_0(x),$$

with C independant of $r \leq r_0$. We deduce that

$$E_x[\tau_D] \geq C^{-1}(d(x))^{2-\alpha}\varphi_0(x)$$

for all $x \in T_{\frac{\psi}{2}} \cap B(0, r_0)$. Since $E_x[\tau_D] \sim$ constant $\sim (d(x))^{2-\alpha}\varphi_0(x)$ on the compact subset $\overline{T}_{\frac{\psi}{2}} \cap \{\overline{B}(0, \frac{1}{2}) \setminus B(0, r_0)\}$, it follows that

$$E_x[\tau_D] \geq C^{-1}(d(x))^{2-\alpha}\varphi_0(x)$$

for all $x \in T_{\frac{\psi}{2}} \cap B(0, \frac{1}{2})$. $\square$

Acknowledgments

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