Communications in Mathematical Analysis
Volume 14, Number 1, pp. 72–84 (2013) www.math-res-pub.org/cma
ISSN 1938-9787

Finite Time Blow-Up of Solutions for Damped Wave Equation with Nonlinear Memory

Berbiche Mohamed*
Laboratoire d'Informatique EEDIS
Université de Sidi-Bel-Abbes
Sidi-Bel-Abbes Algérie.

Hakem Ali†
Laboratoire d'Informatique EEDIS
Université de Sidi-Bel-Abbes
Sidi-Bel-Abbes Algérie.

(Communicated by Irena Lasiecka)

Abstract

We consider the Cauchy problem for a semi-linear wave equation with nonlinear mixed damping term and time nonlocal nonlinearity in multi-dimensional space $\mathbb{R}^N$, we prove the local existence and nonexistence of global solutions theorems.

AMS Subject Classification: 35L15, 35L70, 35B33.

Keywords: Damped wave equation, Local existence, Blow-up.

1 Introduction

The main purpose of this paper is to present results concerning the local existence and the blow-up of solutions for the following Cauchy problem

$$u_{tt} - \Delta u + |u|^{m-1} u_t = \int_0^t (t-s)^{-\gamma} |u(s,.)|^p ds, t > 0, x \in \mathbb{R}^N \tag{1.1}$$

subjected to the initial data

$$u(0,x) = u_0(x), u_t(0,x) = u_1(x), x \in \mathbb{R}^N, \tag{1.2}$$

where the unknown function u is real-valued, $N \geq 1, 0 < \gamma < 1, m > 1$ and $p > 1$.

*E-mail address: berbichemed@yahoo.fr
†E-mail address: hakemali@yahoo.com

Our article is motivated by the recent paper by Cazenave et *al.* [2] which deals with the global existence and blow-up for the parabolic equation with non-local in time non-linearity

$$\partial_t u(t,x) - \Delta u(t,x) = \int_0^t (t-\tau)^{-\gamma} |u|^{p-1} u(\tau,x) d\tau, \tag{1.3}$$

where $0 \leq \gamma < 1$, $p > 1$ and $u_0 \in C_0(\mathbb{R}^N)$. They proved that, if

$$p_\gamma = 1 + \frac{2(2-\gamma)}{(N-2+2\gamma)_+}$$

and $p_* = \max\left\{\frac{1}{\gamma}, p_\gamma\right\} \in (0,+\infty]$, where $(u)_+ = \max(u,0)$, then:

(i) If $\gamma \neq 0$, $p \leq p_*$, and $u_0 \geq 0$, $u_0 \not\equiv 0$, then u blows up in finite time.

(ii) If $\gamma \neq 0$, $p > p_*$, and $u_0 \in L_{q_{sc}}(\mathbb{R}^N)$ (where $q_{sc} = N(p-1)/(4-2\gamma)$) with $\|u_0\|_{L_{q_{sc}}}$ sufficiently small, then u exists globally.

Their study reveals the surprising fact that for Eq. (1.3) the critical exponent in Fujita's sense p_* is not the one predicted by scaling. Eq. (1.3) is a pseudo-parabolic equation and as it is well known scaling is efficient for detecting the Fujita exponent only for equations of parabolic type. Needless to say that the equation considered by Cazenave et *al.* [2] is a genuine extension of the one considered by Fujita in his pioneering work [5].

In the case eq. (1.1) with linear damping, it reads

$$\partial_{tt} u(t,x) - \Delta u(t,x) + \partial_t u(t,x) = \int_0^t (t-\tau)^{-\gamma} |u(\tau,x)|^p d\tau. \tag{1.4}$$

Fino [4], addressed the global existence and blow-up of Eq.(1.4). The main tool used in [4] is the weighted energy method with a weight similar to the one introduced in [15]. It was shown that the solution of (1.4) behaves as that of the corresponding diffusive equation. More precisely, he proved that

(i) if $p > 1$, $\gamma \in (1/2,1)$ for $N = 1,2$ and $\gamma \in (11/16,1)$ for $N = 3$. If $p_N < p$, where
$$p_1 = 1 + \frac{2(3-2\gamma)}{(N-2+2\gamma)_+}, p_2 = 1 + \frac{4(3-2\gamma)}{(N-4+4\gamma)_+},\ p_3 = 1 + \frac{N+2(5-4\gamma)}{(N-2+4\gamma)_+}.$$
Then problem (1.4) admits a unique global mild solution with small data. While

(ii) if $1 < p \leq N/(N-2)$ for $N = 3$, and $\alpha \in (1,\infty)$ for $N = 1,2$, $(N-2)/N < \gamma < 1$ and $(u_0,u_1) \in H^1\left(\mathbb{R}^N\right) \times L^2\left(\mathbb{R}^N\right)$ such that $\int_{\mathbb{R}^N} u_i(x)\,dx > 0, i = 0,1$. If $p \leq p_*$, then the mild solution of the problem (1.4) blows up in finite time.

Moreover, he showed that in the case of $p > p_*$, the support of the solution of (1.4) is strongly suppressed by the damping, so that the solution is concentrated in a ball much smaller than $|x| < t+K$, namely $\|u\|_E = O(e^{-N/4-1/2})$, as $t \to \infty$, outside every ball $B(t^{1/2+\delta})$, $\delta > 0$. Furthermore, he proved that the total energy of the solutions decays at the rate of the linear equation, namely

$$\|Du(t,.)\|_{L^2(\mathbb{R}^n)} = \begin{cases} O(t^{-N/4+1/2-\gamma}), & t \to \infty,\ for\ N = 1, \\ O(t^{1/2-\gamma}), & t \to \infty,\ for\ N = 2, \\ O(t^{-\gamma}), & t \to \infty,\ for\ N = 3. \end{cases}$$

In the present paper, we would like to investigate the nonexistence of global nontrivial solutions for Eq.(1.1). The method used to prove the blow-up result is the test function method considered by Mitidieri and Pohozaev [10],[11], Pohozaev and Tesei [9] Zhang [16], Fino [4].

Before closing this section, we define some notation to be used below. $L^p\left(\mathbb{R}^N\right)(1\leq p\leq\infty)$ is the usual Lebesgue space with the norm $\|.\|_{L^p\left(\mathbb{R}^N\right)}$.

Let s be a nonnegative integer. Then $H^s=H^s\left(\mathbb{R}^N\right)$ denotes the Sobolev space of L^2 functions equipped with the norm $\|.\|_{H^s\left(\mathbb{R}^N\right)}$. Also, $C^k(I;H^s\left(\mathbb{R}^N\right))$ denotes the space of k-times continuously differentiable functions on the interval I with values in the Sobolev space $H^s\left(\mathbb{R}^N\right)$. Finally, positive constants will be denoted by C and will change from line to line.

2 Some preliminary results

In this section, we prepare several lemmas for the proof of Theorem 3.1 and Theorem 4.2.

If $AC[0,T]$ is the space of all functions which are absolutely continuous on $[0,T]$ with $0<T<\infty$, then, for $f\in AC[0,T]$, the left-handed and right-handed Riemann-Liouville fractional derivatives $D^{\alpha}_{0|t}f(t)$ and $D^{\alpha}_{t|T}f(t)$ of order $\alpha\in(0,1)$ are defined by

$$D^{\alpha}_{0|t}f(t)=\partial_t J^{1-\alpha}_{0|t}f(t) \text{ and } D^{\alpha}_{t|T}f(t)=\frac{1}{\Gamma(1-\alpha)}\partial_t\int_t^T(s-t)^{-\alpha}f(s)ds,\ t\in[0,T], \tag{2.1}$$

where

$$J^{\alpha}_{0|t}g(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{\alpha-1}g(s)ds, \tag{2.2}$$

is the Riemann–Liouville fractional integral, for all $g\in L^q(0,T)(1\leq q\leq\infty)$. We refer the reader to [12] for the definitions above.

Furthermore, for every $f,g\in C([0,T])$ such that $D^{\alpha}_{0|t}f(t)$, $D^{\alpha}_{t|T}f(t)$ exist and are continuous, for all $t\in[0,T], 0<\alpha<1$, we have the formula of integration by parts (see (2.64) p. 46 in [12])

$$\int_0^T\left(D^{\alpha}_{0|t}f\right)(t)g(t)dt=\int_0^T f(t)\left(D^{\alpha}_{t|T}g\right)(t)dt. \tag{2.3}$$

Note also that, for all $f\in AC^{n+1}[0,T]$ and all integers $n\geq 0$, we have
(see (2.2.30) in [12])

$$(-1)^n\,\partial_t^n.D^{\alpha}_{t|T}f(t)=D^{n+\alpha}_{t|T}f(t), \tag{2.4}$$

where $AC^{n+1}[0,T]:=\{f:[0,T]\rightarrow\mathbb{R},\text{ and }\partial_t^n f\in AC[0,T]\}$ and ∂_t^n is the usual n times derivative. Moreover, for all $1\leq q\leq\infty$, the following formula
(see [7], Lemma 2.4 p. 74)

$$D^{\alpha}_{0|t}J^{\alpha}_{0|t}:=Id_{L^q(0,T)}, \tag{2.5}$$

holds almost everywhere on $[0,T]$.

In the proof of Theorem 4.2, the following results are useful: if $w_1(t) = (1 - t/T)_+^\sigma, t \geq 0, T > 0, \sigma >> 1$, then $D_{t|T}^\alpha w_1(t) = CT^{-\sigma}(T - t/)_+^{\sigma-\alpha}$, $D_{t|T}^{\alpha+1} w_1(t) = CT^{-\sigma}(T - t)_+^{\sigma-\alpha-1}$, $D_{t|T}^{\alpha+2} w_1(t) = CT^{-\sigma}(T - t)_+^{\sigma-\alpha-2}$, for all $\alpha \in (0, 1)$; so

$$\left(D_{t|T}^\alpha w_1\right)(T) = 0, \left(D_{t|T}^\alpha w_1\right)(0) = T^{-\alpha}, \left(D_{t|T}^{\alpha+1} w_1\right)(T) = 0, \text{and } \left(D_{t|T}^{\alpha+1} w_1\right)(0) = T^{-\alpha-1}. \quad (2.6)$$

For the proof of this results, see [4].

The following Lemmas will be used in the proof of Theorems 3.1.

Lemma 2.1. *(See [14], Proposition 2.4, p. 5) If $s > \frac{N}{2}$, then*

$$H^s\left(\mathbb{R}^N\right) \subset C\left(\mathbb{R}^N\right) \cap L^\infty\left(\mathbb{R}^N\right),$$

where the inclusion is continuous. In fact

$$\|u\|_{L^\infty} \leq C \|u\|_{H^s}$$

The next Lemma is consequence of lemma 2.1 and Proposition 3.7, p. 10 in [14]

Lemma 2.2. *(See [14]) Assume that $s_1, s_2 \geq s > N/2$, then for $u \in H^{s_1}, v \in H^{s_2}$, we have the estimates*

$$\|uv\|_{H^s} \leq C \|u\|_{H^{s_1}} \|v\|_{H^{s_2}}$$

where C is constant independent of u and v.

The last Lemma uses the equivalent norm of $\|u\|_{H^{s-1}}$(see [1, Theorem 7.48, p. 214]). We omit their proof since it can be found in (see [3])

Lemma 2.3. *For any $s \in (1, 2) \cup \mathbb{N}^*$ and $p \in (1, +\infty) \cap (s - 1, +\infty)$ we have for a nonnegative function $f \in L^\infty\left(\mathbb{R}^N\right) \cap H^{s-1}\left(\mathbb{R}^N\right)$, $f^p \in H^{s-1}\left(\mathbb{R}^N\right)$ and there exists a positive constant C such that*

$$\left\|f^p\right\|_{H^{s-1}(\mathbb{R}^N)} \leq C \|f\|_{L^\infty(\mathbb{R}^N)}^{p-1} \|f\|_{H^{s-1}(\mathbb{R}^N)}.$$

Now, proceed with the following linear wave equation:

$$u_{tt} - \Delta u = h(t, x), \quad x \in \mathbb{R}^n, t > 0 \quad (2.7)$$

with initial value condition (1.2). Let us give some results which will be used in the following.

Lemma 2.4. *(See [13].) Let $s \in \mathbb{R}$. Let $(u_0, u_1) \in H^s \times H^{s-1}$ and $h(x, t) \in L^1\left([0, T], H^{s-1}\right)$. Then for every $T > 0$, there is a unique solution $u \in C([0, T], H^s) \cap C^1\left([0, T], H^{s-1}\right)$ of cauchy problem of (2.7) and (1.2). Moreover, u satisfies*

$$\|u\|_{H^s} + \|u_t\|_{H^{s-1}} \leq C(1 + T)\left(\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} + \int_0^t \|h(\tau, .)\|_{H^{s-1}} d\tau\right)$$

for all $0 \leq t \leq T$, where C only depends on s.

3 Well-Posedness

In this section, we will prove local existence and uniqueness of weak solution to the problem (1.1)-(1.2).

Theorem 3.1. *Let $N \geq 1$, $s > N/2$ and $m, p \in (1, +\infty) \cap (s-1, +\infty)$. Then for any $u_0 \in H^s\left(\mathbb{R}^N\right)$ and $u_1 \in H^{s-1}\left(\mathbb{R}^N\right)$, (1.1)-(1.2) admits a unique solution*

$$u \in C\Big([0,T]; H^s\left(\mathbb{R}^N\right)\Big) \cap C^1\Big([0,T]; H^{s-1}\left(\mathbb{R}^N\right)\Big)$$

with some positive T, which depends only on $\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}$.

Proof. First we define

$$\begin{aligned}
X_T &:= C\Big([0,T]; H^s\left(\mathbb{R}^N\right)\Big) \cap C^1\Big([0,T]; H^{s-1}\left(\mathbb{R}^N\right)\Big),\\
Y_T &:= L^\infty\Big([0,T]; H^s\left(\mathbb{R}^N\right)\Big) \cap W^{1,\infty}\Big([0,T]; H^{s-1}\left(\mathbb{R}^N\right)\Big)\\
Y_{T,M} &:= \left\{u \in Y_T;\ \sup_{0\leq t\leq T} \left(\|u(t,.)\|_{H^s} + \|u_t(t,.)\|_{H^{s-1}}\right) \leq M\right\}.
\end{aligned}$$

We also define $X_{T,M} = Y_{T,M} \cap X_T$. Obviously $X_T \subset Y_T$ and $X_{T,M} \subset Y_{T,M}$.

Set $G(u,u_t) = -|u|^{m-1} u_t + \int_0^t (t-s)^{-\gamma} |u|^p (s)\, ds$. For any $v \in Y_T$, define $\Phi[v] = u$, where $u \in X_T$ is a solution to

$$\begin{cases} u_{tt} - \Delta u = G(v, v_t) & \text{in } (0,T) \times \mathbb{R}^N,\\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & x \in \mathbb{R}^N. \end{cases} \tag{3.1}$$

Since we have $G(v,v_t) \in L^\infty\left(0,T; H^{s-1}\left(\mathbb{R}^N\right)\right)$ for any $v \in Y_T$ by Sobolev's embedding theorem, existence and uniqueness of such $u \in X_T$ is guaranteed by the theory of mixed Cauchy problems for linear wave equations.

Let $M = 4C\left(\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}\right)$. We first claim that $v \in Y_{T,M}$ implies that $\Phi[v] \in X_{T,M}$ for sufficiently small $T > 0$.

Set $\Phi[v] = u$. From the Lemma 2.4, we have

$$\|u(t,.)\|_{H^s} + \|u_t(t,.)\|_{H^{s-1}} \leq C(1+T)\Big(\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} + \int_0^t \|G(v,v_t)(\tau,.)\|_{H^{s-1}}\, d\tau\Big). \tag{3.2}$$

By using the identity

$$\begin{aligned}
\int_0^t G(v,v_t)(\tau)d\tau &= -\tfrac{1}{m}\int_0^t \frac{d}{dt}\left(|v|^{m-1} v\right)(\tau)d\tau + \int_0^t \int_0^s (s-\tau)^{-\gamma} |v(\tau)|^p\, d\tau ds\\
&= -\tfrac{1}{m}|v|^{m-1} v(t) + \tfrac{1}{m}|v(0)|^{m-1} v(0) + \tfrac{1}{1-\gamma}\int_0^t (t-s)^{1-\gamma} |v(s)|^p\, ds,
\end{aligned}$$

we find

$$\begin{aligned}
&\int_0^t \|G(v,v_t)(\tau,.)\|_{H^{s-1}}\, d\tau \leq \frac{1}{m}\left\||v|^{m-1} v\right\|_{H^{s-1}} + \frac{1}{m}\left\||v(0,.)|^{m-1} v(0,.)\right\|_{H^{s-1}} +\\
&\tfrac{1}{1-\gamma}\int_0^t (t-s)^{1-\gamma}\left\||u|^p\right\|_{H^{s-1}}(s)\, ds\\
&\leq C\|v\|_{L^\infty}^{m-1}\|v\|_{H^{s-1}} + C\left\||v(0,.)|^{m-1} v(0,.)\right\|_{H^{s-1}} + C\int_0^t (t-s)^{1-\gamma}\|v\|_{L^\infty}^{p-1}\|v\|_{H^{s-1}}(s)\, ds,
\end{aligned}$$

where we have used the Lemma 2.3.

Then By Sobolev's embedding theorem and the Lemma 2.1, we find

$$
\begin{aligned}
&\int_0^t \|G(v,v_t)(\tau,.)\|_{H^{s-1}}\,d\tau \le C\|v\|_{H^s}^{m-1}\|v\|_{H^s} + C\|v(0,.)\|_{H^s}^{m-1}\|v(0,.)\|_{H^s} + CT^{2-\gamma}\sup_{0\le t\le T}\|u\|_{H^s}^p \\
&\le CM^m + C\sup_{0\le t\le T}\|v(t,.)\|_{H^s}^{m-1}\|v(t,.)\|_{H^s} + CT^{2-\gamma}M^p \\
&\le CM^m + CT^{2-\gamma}M^p,
\end{aligned}
\tag{3.3}
$$

From (3.2) and (3.3), we get

$$
\begin{aligned}
\sup_{0\le t\le T}\left(\|u(t,.)\|_{H^s} + \|u_t(t,.)\|_{H^{s-1}}\right) \le C(1+T)\Big(\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} \\
+ CTM^m + CT^{2-\gamma}M^p\Big).
\end{aligned}
\tag{3.4}
$$

By (3.4) we arrive at

$$
\sup_{0\le t\le T}\left(\|u(t,.)\|_{H^s} + \|u_t(t,.)\|_{H^{s-1}}\right) \le C_{T,M}M,
$$

where $C_{T,M} = C(1+T)\left(\frac{1}{4C} + TM^{m-1} + CT^{2-\gamma}M^{p-1}\right)$. Since we can find $T_1 > 0$ such that $C_{T,M} \le 1$ for any $T \in (0,T_1]$, this implies the claim.

We next prove that Φ is a contraction mapping in $X_{T,M}$ for small T by using the Lemma 2.4 and the mean value theorem. When $2 \le m < \infty$, we can apply the mean value theorem directly to the nonlinear term, and it is easy to prove the claim. But because the function $|u|^{m-1}v$ is not Lipschitz continuous with respect to $(u,v) \in \mathbb{R}\times\mathbb{R}$ for $1 < m < 2$, we modify the argument in the following manner, using the fact that $m|u|^{m-1}u_t = \partial_t\left(|u|^{m-1}u\right)$, where $|u|^{m-1}u$ is Lipschitz continuous. We note that this approach was used in Lions and Strauss [8] and recently by Katayama et *al.* [6].

Suppose that $v_1, v_2 \in Y_{T,M}$, then we have $\Phi[v_1], \Phi[v_2] \in X_{T,M}$. Let w_i and $\tilde{w}_i\,(i = 1,2)$ be solutions to the following problems

$$
\begin{cases}
(w_i)_{tt} - \Delta w_i = \displaystyle\int_0^t (t-s)^{-\gamma}|v_i|^p(s)\,ds & \text{in } (0,T)\times\mathbb{R}^N, \\
w_i(0,x) = u_0, (w_i)_t(0,x) = u_1 + \frac{1}{m}|u_0|^{m-1}u_0 \text{ in } \mathbb{R}^N,
\end{cases}
\tag{3.5}
$$

and

$$
\begin{cases}
(\tilde{w}_i)_{tt} - \Delta\tilde{w}_i = -\frac{1}{m}|v_i|^{m-1}v_i & \text{in } (0,T)\times\mathbb{R}^N, \\
\tilde{w}_i(0,x) = (\tilde{w}_i)_t(0,x) = 0 & \text{in } \mathbb{R}^N,
\end{cases}
\tag{3.6}
$$

for $i = 1,2$, respectively.

Since $v_i \in Y_{T,M}$ implies that $\displaystyle\int_0^t (t-s)^{-\gamma}|v_i|^p(s)\,ds, |v_i|^{m-1}v_i$ and $\partial_t\left(|v_i|^{m-1}v_i\right) = m|v_i|^{m-1}(v_i)_t$ are functions in $L^\infty\left(0,T;H^{s-1}\right)$ by Sobolev's embedding theorem, we have $w_i \in X_T$ and

$$
\tilde{w}_i \in C\left([0,T];H^{s+1}\left(\mathbb{R}^N\right)\right) \cap C^1\left([0,T];H^s\left(\mathbb{R}^N\right)\right) \cap C^2\left([0,T];H^{s-1}\left(\mathbb{R}^N\right)\right)
$$

$(i = 1,2)$. From the uniqueness of solutions to linear wave equations, we have

$$
\Phi[v_i] = w_i + (\tilde{w}_i)_t \ \ (i = 1,2). \tag{3.7}
$$

Note that since $|v|^{l-1} v$ with $l > 1$ is a C^1 function, the mean value theorem implies

$$\left||v_1|^{l-1} v_1 - |v_2|^{l-1} v_2\right| \leq C\left(|v_1|^{l-1} + |v_2|^{l-1}\right)|v_1 - v_2|. \tag{3.8}$$

By (3.5), (3.6) and (3.8), the energy inequality and Sobolev's inequality $\|u\|_{L^\infty} \leq C\|u\|_{H^s}$ imply

$$\begin{aligned}
&\|(w_1 - w_2)_t(t,.)\|_{H^{s-1}} + \|(w_1 - w_2)(t,.)\|_{H^s} \\
&\leq C(1+T)\int_0^t \int_0^s (s-\tau)^{-\gamma}\left\|(|v_1|^p - |v_2|^p)(\tau,.)\right\|_{H^{s-1}} d\tau ds \\
&\leq C(1+T)\int_0^t (t-\tau)^{1-\gamma}\left(\|v_1\|_{H^s}^{p-1} + \|v_2\|_{H^s}^{p-1}\right)\|(v_1 - v_2)(\tau,.)\|_{H^s} \\
&\leq C(1+T)T^{2-\gamma}M^{p-1}\sup_{0\leq\tau\leq T}\|(v_1 - v_2)(\tau,.)\|_{H^s} \text{ for } 0 \leq t \leq T,
\end{aligned} \tag{3.9}$$

and similarly

$$\begin{aligned}
&\|(\tilde{w}_1 - \tilde{w}_2)_t(t,.)\|_{H^{s-1}} + \|(\tilde{w}_1 - \tilde{w}_2)(t,.)\|_{H^s} \\
&= C(1+T)\int_0^t \left\|\left(|v_1|^m v_1 - |v_2|^{m-1} v_2\right)(s,.)\right\|_{H^{s-1}} d\tau ds \\
&\leq C(1+T)TM^{m-1}\sup_{0\leq\tau\leq T}\|(v_1 - v_2)(\tau,.)\|_{H^s} \text{ for } 0 \leq t \leq T.
\end{aligned} \tag{3.10}$$

Then, (3.9) and (3.10) lead to

$$\begin{aligned}
&\sup_{0\leq t\leq T}\|\Phi[v_1] - \Phi[v_2](t,.)\|_{H^s} \leq C(1+T)\left(T^{3-\gamma}M^{p-1} + TM^{m-1}\right) \\
&\times \sup_{0\leq\tau\leq T}\|(v_1 - v_2)(\tau,.)\|_{H^s}.
\end{aligned} \tag{3.11}$$

In the following, we fix $T \in (0, T_1]$ which is small enough to have $C\left(T^{3-\gamma}M^{p-1} + TM^{m-1}\right) < \frac{1}{2}$. Then we get

$$\sup_{0\leq t\leq T}\|\Phi[v_1] - \Phi[v_2](t,.)\|_{H^s} \leq \frac{1}{2}\sup_{0\leq\tau\leq T}\|(v_1 - v_2)(\tau,.)\|_{H^s} \tag{3.12}$$

for such T.

Finally, define

$$\begin{cases} u^{(0)}(t,x) = u_0(x), \\ u^{(n)} = \Phi\left[u^{(n-1)}\right] \quad (n = 1, 2, 3, \ldots). \end{cases}$$

By (3.12), there exists some $u \in C([0,T]; H^s)$ such that $u^{(n)} \to u$ in $C([0,T]; H^s)$ as $n \to \infty$. Now, we will show that this u belongs to X_T and is solution to (1.1)-(1.2). Since $u^{(n)} \in X_{T,M}$, $\left\{u^{(n)}\right\}$ (resp. $\left\{u_t^{(n)}\right\}$) has a weak-* convergent subsequence in $L^\infty(0,T;H^s)$ $\left(\text{resp. in } L^\infty\left(0,T;H^{s-1}\right)\right)$. Since $u^{(n)} \to u$ in $C([0,T]; H^s)$, the above subsequence of $\left\{u^{(n)}\right\}$ (resp. $\left\{u_t^{(n)}\right\}$) converges weakly-* to u (resp. to u_t) in $L^\infty(0,T;H^s)$ (resp. in $L^\infty\left(0,T;H^{s-1}\right)$), and consequently we see that $u \in L^\infty(0,T;H^s)$ and $u_t \in L^\infty\left(0,T;H^{s-1}\right)$. Then we can see that $u \in Y_{T,M}$, and then we get $\Phi[u] \in X_{T,M}$. Hence we can apply (3.12) to have

$$\sup_{0\leq t\leq T}\left\|\Phi[u] - \Phi\left[u^{(n)}\right](t,.)\right\|_{H^s} \leq \frac{1}{2}\sup_{0\leq t\leq T}\left\|\left(u - u^{(n)}\right)(t,.)\right\|_{H^s}. \tag{3.13}$$

Since the right-hand side of (3.13) tends to 0 as $n \to \infty$, we get $\Phi\left[u^{(n)}\right] \to \Phi[u]$ in $C([0,T];H^s)$. Since we have showed that $u^{(n)} \to u$ in $C([0,T];H^s)$, passing to the limit in $u^{(n)} = \Phi\left[u^{(n-1)}\right]$, we obtain $u = \Phi[u] \in X_{T,M}$. This u apparently the desired solution. The uniqueness of weak solutions in $X_{T,M}$ follows immediately from (3.12). This completes the proof of theorem 3.1. □

4 Blow-up results

This section is devoted to the blow-up of solutions of the problem (1.1)-(1.2). We start by introducing the definition of the weak solution of (1.1)-(1.2).

Definition 4.1. *Let* $T > 0$, $0 < \gamma < 1$ and $u_0 \in L^1_{loc}\left(\mathbb{R}^N\right) \cap L^m_{loc}\left(\mathbb{R}^N\right), u_1 \in L^1_{loc}\left(\mathbb{R}^N\right)$. *We say that u is a weak solution if* $u \in L^p\left((0,T), L^p_{loc}\left(\mathbb{R}^N\right)\right) \cap L^m\left((0,T), L^m_{loc}\left(\mathbb{R}^N\right)\right)$ *and satisfies*

$$\begin{aligned}
&\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^N} J^{\alpha}_{0|t}\left(|u|^p\right)\varphi dxdt + \int_{\mathbb{R}^N} u_1(x)\varphi(0,x)\,dx - \int_{\mathbb{R}^N} u_0(x)\varphi_t(0,x)\,dx + \\
&\tfrac{1}{m}\int_{\mathbb{R}^N} |u_0|^{m-1} u_0(x)\varphi(0,x)\,dx = \int_0^T\int_{\mathbb{R}^N} u\varphi_{tt}dxdt - \frac{1}{m}\int_0^T\int_{\mathbb{R}^N} |u|^{m-1} u\varphi_t(t,x)\,dxdt \\
&-\int_0^T\int_{\mathbb{R}^N} u\Delta\varphi dxdt,
\end{aligned} \tag{4.1}$$

f*or all nonnegative test function* $\varphi \in C^2\left([0,T]\times\mathbb{R}^N\right)$ *such that*
$\varphi(T,.) = \varphi_t(T,.) = 0$, where $\alpha = 1-\gamma$.

Theorem 4.2. *Let* $0 < \gamma < 1$ *such that* $\frac{N-2}{N} < \gamma < 1$ *and let* p,m *such that* $p > m > 1$. *Assume that the initial data* (u_0,u_1) *satisfy*

$$\int_{\mathbb{R}^N} u_0(x)\,dx > 0, \quad \int_{\mathbb{R}^N} |u_0|^{m-1} u_0(x) > 0, \text{ and} \int_{\mathbb{R}^N} u_1(x)\,dx > 0.$$

Then if

$$N \le \min\left\{\frac{2(m+(1-\gamma)p)}{(p-1+(1-\gamma)(m-1))}, \frac{2(1+(2-\gamma)p)}{\left(\frac{(p-1)(2-\gamma)}{(p-m)}+\gamma-1\right)(p-1)}\right\} \tag{4.2}$$

or $p \le \frac{1}{\gamma}$. *Then the solution of problem (1.1)-(1.2) does not exist globally in time.*

Proof. The proof proceeds by contradiction. Let u be a nontrivial global weak solution of the problem (1.1)-(1.2) that is u defined on $(0,T)$ for all $T > 0$ and φ be a smooth test function which will be specified later. From the definition of weak solution we have

$$\begin{aligned}
&\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^N} J^{\alpha}_{0|t}\left(|u|^p\right)\varphi dxdt + \int_{\mathbb{R}^N} u_1(x)\varphi(0,x)\,dx - \int_{\mathbb{R}^N} u_0(x)\varphi_t(0,x)\,dx + \\
&\tfrac{1}{m}\int_{\mathbb{R}^N} |u_0|^{m-1} u_0(x)\varphi(0,x)\,dx = \int_0^T\int_{\mathbb{R}^N} u\varphi_{tt}dxdt - \frac{1}{m}\int_0^T\int_{\mathbb{R}^N} |u|^{m-1} u\varphi_t(t,x)\,dxdt \\
&-\int_0^T\int_{\mathbb{R}^N} u\Delta\varphi dxdt.
\end{aligned} \tag{4.3}$$

Let $\varphi(x,t) := D^{\alpha}_{t|T}\xi(t,x) = \varphi_1^l(x) D^{\alpha}_{t|T}\varphi_2(t)$ with

$$\varphi_1(x) := \Phi\left(\frac{|x|^2}{T^{\theta}}\right), \varphi_2(t) := \left(1-\frac{t}{T}\right)_+^{\eta},$$

where $D^{\alpha}_{t|T}$ is given by (2.1), $l,\eta > 1$ and $\Phi \in C^{\infty}(\mathbb{R}_+)$ be a cut-off nonincreasing function such that

$$\Phi(z) = \begin{cases} 1 & \text{if } 0 \le z \le 1 \\ 0 & \text{if } z \ge 2, \end{cases}$$

and $0 \le \Phi \le 1$, for all $z > 0$.

Then the weak formulation (4.3) reads

$$\begin{aligned}
&\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^N} J^{\alpha}_{0|t}(|u|^p) D^{\alpha}_{t|T}\xi(t,x)\,dxdt + \int_{\mathbb{R}^N} u_1(x) D^{\alpha}_{t|T}\xi(0,x)\,dx \\
&+\int_{\mathbb{R}^N} |u_0|^{m-1} u_0(x) D^{\alpha}_{t|T}\xi(0,x)\,dx - \int_{\mathbb{R}^N} u_0(x)\partial_t D^{\alpha}_{t|T}\xi(0,x)\,dx \\
&= \int_0^T\int_{\mathbb{R}^N} u\partial_t^2 D^{\alpha}_{t|T}\xi\,dxdt - \frac{1}{m}\int_0^T\int_{\mathbb{R}^N} |u|^{m-1} u\partial_t D^{\alpha}_{t|T}\xi(t,x)\,dxdt \\
&-\int_0^T\int_{\mathbb{R}^N} u\Delta D^{\alpha}_{t|T}\xi(t,x)\,dxdt.
\end{aligned} \tag{4.4}$$

Moreover, from (4.4), (2.3) and (2.5) we may write

$$\begin{aligned}
&\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^N} D^{\alpha}_{0|t} J^{\alpha}_{0|t}(|u|^p)\xi(t,x)\,dxdt + T^{\gamma-1}\int_{\mathbb{R}^N} u_1(x)\varphi_1^l(x)\,dx \\
&+T^{\gamma-1}\int_{\mathbb{R}^N} |u_0|^{m-1} u_0(x)\varphi_1^l(x)\,dx + T^{\gamma-2}\int_{\mathbb{R}^N} u_0(x)\varphi_1^l(x)\,dx \\
&= \int_0^T\int_{\mathbb{R}^N} u\varphi_1^l(x) D^{3-\gamma}_{t|T}\varphi_2(t)\,dxdt + \frac{1}{m}\int_0^T\int_{\mathbb{R}^N} |u|^{m-1} u\varphi_1^l(x) D^{2-\gamma}_{t|T}\varphi_2(t)\,dxdt \\
&-\int_0^T\int_{\mathbb{R}^N} u\Delta\varphi_1^l(x) D^{\alpha}_{t|T}\varphi_2(t)\,dxdt.
\end{aligned} \tag{4.5}$$

So, the formula $\Delta\varphi_1^l(x) = l\varphi_1^{l-1}\Delta\varphi_1 + l(l-1)\varphi_1^{l-2}|\nabla\varphi_1|^2$ and $\varphi_1 \le 1$ will allow us to write

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^N} |u|^p\xi(t,x)\,dxdt + CT^{\gamma-1}\int_{\mathbb{R}^N} u_1(x)\varphi_1^l(x)\,dx \\
&+CT^{\gamma-1}\int_{\mathbb{R}^N} |u_0|^{m-1} u_0(x)\varphi_1^l(x)\,dx + CT^{\gamma-2}\int_{\mathbb{R}^N} u_0(x)\varphi_1^l(x)\,dx \\
&\le C\int_0^T\int_{\mathbb{R}^N} |u|\varphi_1^l(x)\left|D^{3-\gamma}_{t|T}\varphi_2(t)\right|dxdt + C\int_0^T\int_{\mathbb{R}^N} |u|^m\varphi_1^l(x)\left|D^{2-\gamma}_{t|T}\varphi_2(t)\right|dxdt \\
&+C\int_0^T\int_{\mathbb{R}^N} |u|\varphi_1^{l-2}\left(|\Delta\varphi_1| + |\nabla\varphi_1|^2\right)\left|D^{\alpha}_{t|T}\varphi_2(t)\right|dxdt.
\end{aligned} \tag{4.6}$$

By applying ε-Young's inequality

$$XY \le \varepsilon X^p + C(\varepsilon)Y^{p'},\ p + p' = pp', X \ge 0, Y \ge 0,$$

to the right-hand side of (4.6), we may write

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^N}|u|\varphi_1^l(x)\left|D_{t|T}^{3-\gamma}\varphi_2(t)\right|dxdt&=\int_0^T\int_{\mathbb{R}^N}|u|\xi^{\frac{1}{p}}\xi^{-\frac{1}{p}}\varphi_1^l(x)\left|D_{t|T}^{3-\gamma}\varphi_2(t)\right|dxdt\\ \leq\varepsilon\int_0^T\int_{\mathbb{R}^N}|u|^p\xi dxdt+C(\varepsilon)\int_0^T\int_{\mathbb{R}^N}&\varphi_1^l(x)\varphi_2^{-\frac{1}{p-1}}(t)\left|D_{t|T}^{3-\gamma}\varphi_2(t)\right|^{\frac{p}{p-1}}dxdt\end{aligned}\tag{4.7}$$

for some $\varepsilon>0$. Likewise, we have the estimate

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^N}|u|^m\varphi_1^l(x)\left|D_{t|T}^{2-\gamma}\varphi_2(t)\right|dxdt&=\int_0^T\int_{\mathbb{R}^N}|u|^m\xi^{\frac{m}{p}}\xi^{-\frac{m}{p}}\varphi_1^l(x)\left|D_{t|T}^{2-\gamma}\varphi_2(t)\right|dxdt\\ \leq\varepsilon\int_0^T\int_{\mathbb{R}^N}|u|^p\xi+C(\varepsilon)\int_0^T\int_{\mathbb{R}^N}&\varphi_1^l(x)|\varphi_2(t)|^{-\frac{m}{p-m}}\left|D_{t|T}^{2-\gamma}\varphi_2(t)\right|^{\frac{p}{p-m}}dxdt\end{aligned}\tag{4.8}$$

the same is true for the third term in the right hand side of (4.6)

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^N}|u|\varphi_1^{l-2}\left(|\Delta\varphi_1|+|\nabla\varphi_1|^2\right)\left|D_{t|T}^{\alpha}\varphi_2(t)\right|dxdt\leq\varepsilon\int_0^T\int_{\mathbb{R}^N}|u|^p\xi dxdt\\ +C(\varepsilon)\int_0^T\int_{\mathbb{R}^N}\varphi_1^{l-2p'}\varphi_2^{-\frac{1}{p-1}}\left(|\Delta\varphi_1|^{p'}+|\nabla\varphi_1|^{2p'}\right)\left|D_{t|T}^{\alpha}\varphi_2(t)\right|^{p'}dxdt.\end{aligned}\tag{4.9}$$

Gathering up, (4.7), (4.8) and (4.9), with ε small enough, we infer that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^N}|u|^p\xi dxdt\leq C\Bigg(\int_0^T\int_{\mathbb{R}^N}\varphi_1^l(x)\varphi_2^{-\frac{1}{p-1}}(t)\left|D_{t|T}^{3-\gamma}\varphi_2(t)\right|^{\frac{p}{p-1}}dxdt\\ +\int_0^T\int_{\mathbb{R}^N}\varphi_1^l(x)|\varphi_2(t)|^{-\frac{m}{p-m}}\left|D_{t|T}^{2-\gamma}\varphi_2(t)\right|^{\frac{p}{p-m}}dxdt+\\ \int_0^T\int_{\mathbb{R}^N}\varphi_1^{l-2p'}\varphi_2^{-\frac{1}{p-1}}\left(|\Delta\varphi_1|^{\frac{p}{p-1}}+|\nabla\varphi_1|^{2\frac{p}{p-1}}\right)\left|D_{t|T}^{\alpha}\varphi_2(t)\right|^{\frac{p}{p-1}}dxdt\Bigg),\end{aligned}\tag{4.10}$$

for some positive constant C. At this stage, we introduce the scaled variables

$$x=T^{\frac{\theta}{2}}y,t=T\tau.$$

So, we have

$$\int_0^T\int_{\Omega_T}|u|^p\xi dxdt\leq C(T^{(\gamma-3)\frac{p}{p-1}+\frac{\theta N}{2}+1}+T^{(\gamma-2)\frac{p}{p-m}+\frac{\theta N}{2}+1}+T^{(\gamma-1-\theta)\frac{p}{p-1}+\frac{\theta N}{2}+1}),$$

where

$$\Omega_T:=\left\{x\in\mathbb{R}^N,|x|^2\leq 2T^{\theta}\right\}.$$

Now, we choose θ so that $(\gamma-2)\frac{p}{p-m}=(\gamma-1-\theta)\frac{p}{p-1}$, then $\theta=\frac{(p-1)(2-\gamma)}{(p-m)}+\gamma-1>0$ because $p>1$. Then

$$\int_0^T\int_{\Omega_T}|u|^p\xi dxdt\leq C(T^{(\gamma-3)\frac{p}{p-1}+\frac{\theta N}{2}+1}+T^{(\gamma-2)\frac{p}{p-m}+\frac{\theta N}{2}+1})\equiv T^{\delta},\tag{4.11}$$

with $\delta=\max\left\{(\gamma-3)\frac{p}{p-1}+\frac{\theta N}{2}+1,(\gamma-2)\frac{p}{p-m}+\frac{\theta N}{2}+1\right\}$.

We have to distinguish two cases:

The case $\delta < 0$: we pass to the limit in (4.11) as T goes to ∞; we get

$$\lim_{T\to\infty}\int_0^T\int_{\Omega_T}|u|^p\,\xi(x,t)\,dxdt = 0.$$

Using the Lebesgue dominated convergence theorem, the continuity in time and space of u and the fact $\lim_{T\to\infty}\xi(x,t) = 1$ as $T\to\infty$, we infer that

$$\int_0^\infty\int_{\mathbb{R}^N}|u|^p\,(x,t)\,dxdt = 0 \Rightarrow u\equiv 0.$$

The case $\delta = 0$: Using inequality (4.11) with $T\to\infty$, taking into account the fact that $\delta = 0$, we have

$$u\in L^p\Big((0,\infty);L^p\big(\mathbb{R}^N\big)\Big),$$

which implies that

$$\lim_{T\to\infty}\int_0^T\int_{\sum_T}|u|^p\,\xi dxdt = 0,$$

where

$$\sum\nolimits_T := \Big\{x\in\mathbb{R}^N, T^\theta\le|x|^2\le 2T^\theta\Big\}.$$

On the other hand, using Hölder's inequality instead of Young's one to the term

$$\int_0^T\int_{\mathbb{R}^N}|u|\varphi_1^{l-2}\Big(|\Delta\varphi_1|+|\nabla\varphi_1|^2\Big)\Big|D^\alpha_{t|T}\varphi_2(t)\Big|\,dxdt$$

we find

$$\int_0^T\int_{\mathbb{R}^N}|u|\varphi_1^{l-2}\Big(|\Delta\varphi_1|+|\nabla\varphi_1|^2\Big)\Big|D^\alpha_{t|T}\varphi_2(t)\Big|\,dxdt \le \left(\int_0^T\int_{\sum_{TR^{-1}}}|u|^p\,\xi dxdt\right)^{\frac{1}{p}}\times$$
$$\left(\int_0^T\int_{\sum_{TR^{-1}}}\varphi_1^{l-2p'}\varphi_2^{-p'}\Big(|\Delta\varphi_1|^{p'}+|\nabla\varphi_1|^{2p'}\Big)\Big|D^\alpha_{t|T}\varphi_2(t)\Big|^{p'}\,dxdt\right)^{\frac{1}{p'}}.$$

We repeat the same calculation as above by taking in this time $\varphi_1(x) := \Phi\left(\frac{|x|^2}{R^{-\theta}T^\theta}\right)$where R is fixed number such that $1 < R < T$.
Using the change of variables $x = R^{-\frac{\theta}{2}}T^{\frac{\theta}{2}}y$ and $t = T\tau$ by taking into account that $\delta = 0$, we get

$$\int_0^T\int_{\left\{|x|\le\sqrt{2}R^{-\frac{\theta}{2}}T^{\frac{\theta}{2}}\right\}}|u|^p\,\xi dxdt \le CR^{-\frac{\theta N}{2}} + CR^{\theta-\frac{\theta N}{2p'}}\left(\int_0^T\int_{\left\{R^{-\frac{\theta}{2}}T^{\frac{\theta}{2}}\le|x|\le\sqrt{2}R^{-\frac{\theta}{2}}T^{\frac{\theta}{2}}\right\}}|u|^p\,\xi dxdt\right)^{1/p}. \tag{4.12}$$

Thus, passing to the limit in (4.12) as $T\to\infty$, we get

$$\int_0^\infty\int_{\mathbb{R}^N}|u|^p\,dxdt \le CR^{-\frac{\theta N}{2}}$$

and then $R\to\infty$ which give a contradiction.

The case $p \leq 1/\gamma$ we consider $\varphi_1(x) := \Phi\left(\frac{|x|^2}{R^\theta}\right), \varphi_2(t) := \left(1 - \frac{t}{T}\right)_+^\eta$, then by taking the scaled variables $x = R^{\frac{\theta}{2}} y, t = \tau T$, it follows from (4.10) that

$$\int_0^T \int_{\mathbb{R}^N} |u|^p \xi dxdt \leq C\Bigg(\int_0^T \int_{\mathbb{R}^N} \varphi_1^l(x) \varphi_2^{-\frac{1}{p-1}}(t) \left|D_{t|T}^{3-\gamma}\varphi_2(t)\right|^{\frac{p}{p-1}} dxdt$$
$$+ \int_0^T \int_{\mathbb{R}^N} \varphi_1^l(x) |\varphi_2(t)|^{-\frac{m}{p-m}} \left|D_{t|T}^{2-\gamma}\varphi_2(t)\right|^{\frac{p}{p-m}} dxdt + R^{\theta\left(\frac{N}{2} - \frac{p}{p-1}\right)} T^{(\gamma-1)\frac{p}{p-1}+1}$$
$$\int_0^T \int_{\mathbb{R}^N} \varphi_1^{l-2p'} \varphi_2^{-\frac{1}{p-1}} \left(|\Delta\varphi_1|^{\frac{p}{p-1}} + |\nabla\varphi_1|^{2\frac{p}{p-1}}\right) \left|D_{t|T}^{\alpha}\varphi_2(t)\right|^{\frac{p}{p-1}} dxdt\Bigg),$$

so, we have

$$\int_0^T \int_{\Omega_R} |u|^p \xi dxdt \leq CR^{\frac{\theta N}{2}} (T^{(\gamma-3)p'+1} + T^{(\gamma-2)\frac{p}{p-m}+1}) +$$
$$CR^{\theta\left(\frac{N}{2} - \frac{p}{p-1}\right)} T^{(\gamma-1)\frac{p}{p-1}+1}).$$

Taking the limit as $T \to \infty$, we infer, as $p < \frac{1}{\gamma}$ that

$$\int_0^T \int_{\Omega_R} |u|^p \xi dxdt = 0.$$

Finally, by taking $R \to \infty$, we get contradiction. Precisely, in the case $p = \frac{1}{\gamma}$, we have to use condition $\frac{N}{2} - \frac{p}{p-1} < 0$ which equivalently $\gamma > (N-2)/2$ to obtained the desired convergence. This completes the proof. □

Remark 4.3. When $m = 1$, we recover the case studied by Fino [4].

Acknowledgments

The authors thank the referee for careful reading of the manuscript and insightful comments.

References

[1] R. Adams, Sobolev spaces. Academic Press , New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[2] T. Cazenave, F. Dickstein and F. D. Weissler, An equation whose Fujita critical exponent is not given by scaling, *Nonlinear Anal.* **68** (2008) pp 862-874.

[3] H. Faour, A. Fino and M. Jazar, Local existence and uniqueness for a semilinear accretive wave equation, *J. Math. Anal. Appl.* **377** (2011) pp 534-539.

[4] A. Fino, Critical exponent for damped wave equations with nonlinear memory, *Nonlinear Anal. TMA.* **74** (2011) pp 5495-5505.

[5] H. Fujita, On the blowing up of solutions of the problem for $u_t = \Delta u + u^{1+\alpha}$, *J. Fac. Sci. Univ. Tokyo* **13** (1966) pp 109-124.

[6] S. Katayama, Md. A. Sheikh and S. Tarama, The Cauchy and mixed problems for semilinear wave equations with damping terms, *Math. Japonica* **50** (3) (2000), pp 459-466.

[7] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, 2006.

[8] J. L. Lions and W. A. Strauss, Some nonlinear evolution equations, *Bull. Soc. Math. France* **93** (1965), pp 43-96.

[9] S. I. Pohozaev and A. Tesei, Blow-up of nonnegative solutions to quasilinear parabolic inequalities, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei ,* **9** Mat. Appl. 11, No. 2, pp 99-109 (2000).

[10] E. Mitidieri and S. I. Pohozaev, A priori estimates and blow-up of solutions to nonlinear partial differential equations and inequalities, *Proc. Steklov. Inst. Math.* **234** (2001) pp 1-383.

[11] E. Mitidieri and S. I. Pohozaev, Nonexistence of weak solutions for some degenerate elliptic and parabolic problems on R^N, *J. Evol. Equations* **1** (2001) 189-220.

[12] S. G. Samko , A. A. Kilbas and O. I. Marichev, Fractional Integrals and Derivatives, Theory and Applications, Gordon and Breach Science Publishers, 1987.

[13] S. Selberg, Lecture Notes Mat., 632, PDE, http:// www.math.ntnu.no/~sselberg, 2001.

[14] M.E. Taylor, Partial differential equations III, in: Nonlinear Equations, Springer, New York, 1996.

[15] G. Todorova and B. Yordanov, Critical exponent for a nonlinear wave equation with damping, *J. Differential Equations* **174** (2001) pp 464-489.

[16] Qi S. Zhang, A blow up result for a nonlinear wave equation with damping: the critical case, *C. R. Acad. Sci. Paris* **333** (2) (2001) pp 109-114.