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Existence of Mild Solutions for Nonlocal Cauchy Problem for Fractional Neutral Integro-Differential Equation with Unbounded Delay

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Abstract

In this article, we study the existence of mild solutions for the nonlocal Cauchy problem for a class of abstract fractional neutral integro-differential equations with infinite delay. The results are obtained by using the theory of resolvent operators. Finally, an application is given to illustrate the theory.

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1 Introduction

In this article, we study the existence of mild solutions for the nonlocal Cauchy problem for a class of abstract fractional neutral integro-differential equations with infinite delay

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modeled in the form

$$D_t^{\alpha}(x(t)+f(t,x_t)) = Ax(t)+\int_0^t B(t-s)x(s)ds+g(t,x_t), \quad t\in[0,b] \tag{1.1}$$

$$x_0 = \varphi+q(x_{t_1},x_{t_2},x_{t_3},\cdots,x_{t_n})\in\mathcal{B}, \quad x'(0)=x_1, \tag{1.2}$$

where $\alpha\in(1,2)$; $A,(B(t))_{t\geq 0}$ are closed linear operators defined on a common domain which is dense in a Banach space X, $D_t^{\alpha}h(t)$ represent the Caputo derivative of order $\alpha>0$ of h defined by

$$D_t^{\alpha}h(t)=\int_0^t g_{n-\alpha}(t-s)\frac{d^n}{ds^n}h(s)ds,$$

where n is the smallest integer greater than or equal to α and $g_{\beta}(t):=\frac{t^{\beta-1}}{\Gamma(\beta)}$, $t>0,\beta\geq 0$. The history $x_t:(-\infty,0]\to X$ given by $x_t(\theta)=x(t+\theta)$ belongs to some abstract phase space $\mathcal{B}$ defined axiomatically, $0<t_1<t_2<t_3<\cdots<t_n\leq b$, $q:\mathcal{B}^n\to\mathcal{B}$ and $f,g:[0,b]\times\mathcal{B}\to X$ are appropriate functions.

There exist an extensive literature of differential equations with nonlocal conditions. Motivated by physical applications, Byszewski studied in [6] the existence of mild, strong and classical solutions for the nonlocal problem for a semi-linear evolution equation. The nonlocal Cauchy problem for functional differential equations with delay is also studied by Byszewski, in the paper [7], Byszewski discuss the existence, uniqueness and continuous dependence on initial data of solutions for this type of Cauchy problem. On the other hand, Hernandez [20], study the existence of mild, strong and classical solutions for the nonlocal neutral partial functional differential equation with unbounded delay.

We observe that the fractional order can be complex from the viewpoint of pure mathematics and there is much interest in developing the theoretical analysis and numerical methods of fractional equations, because they have recently proved to be valuable in various fields of sciences and engineering [8, 18]. For details, including some applications and recent results, see the monographs of Ahn and MacVinisch [4], Gorenglo and Mainardi [19], Hilfer [22], Miller and Ross [25], and the papers of Agarwal et al. [1, 2], Cuevas et al. [11, 12, 13, 14, 17], Lakshmikantham [24] (see also [5, 9, 10] and references therein), Zhou et al. [27] and Dos Santos et. al [3, 15, 16, 17]. Our purpose in this paper is to establish the existence of mild solutions for a nonlocal fractional neutral integro-differential equations with unbounded delay.

2 Preliminaries

In what follows we recall some definitions, notations and results that we need in the sequel. Throughout this paper, $(X,\|\cdot\|)$ is a Banach space and A, $B(t)$, for $t\geq 0$, are closed linear operators defined on a common domain $\mathcal{D}=D(A)$ which is dense in X. The notation $[D(A)]$ represents the domain of A endowed with the graph norm. Let $(Z,\|\cdot\|_Z)$ and $(W,\|\cdot\|_W)$ be Banach spaces. In this paper, the notation $\mathcal{L}(Z,W)$ stands for the Banach space of bounded linear operators from Z into W endowed with the uniform operator topology and we abbreviate this notation to $\mathcal{L}(Z)$ when $Z=W$. Furthermore, for appropriate functions $K:[0,\infty)\to Z$ the notation $\widehat{K}$ denotes the Laplace transform of K. The notation, $B_r[x,Z]$

stands for the closed ball with center at x and radius $r > 0$ in Z. On the other hand, for a bounded function $x : [0,a] \to Z$ and $b \in [0,a]$, the notation $\|x\|_{Z,b}$ is defined by

$$\|x\|_{Z,b} = \sup\{\|x(s)\|_Z : s \in [0,b]\},$$

and we simplify this notation to $\|x\|_b$ when no confusion about the space Z arises.

To obtain our results, we assume that the abstract fractional integro-differential problem

$$D_t^\alpha x(t) = Ax(t) + \int_0^t B(t-s)x(s)ds, \tag{2.1}$$

$$x(0) = z \in X, \quad x'(0) = 0, \tag{2.2}$$

has an associated α-resolvent operator of bounded linear operators $(\mathcal{R}_\alpha(t))_{t\geq 0}$ on X.

Definition 2.1. A one-parameter family of bounded linear operators $(\mathcal{R}_\alpha(t))_{t\geq 0}$ on X is called an α-resolvent operator of (2.1)-(2.2) if the following conditions are verified.

(a) The function $\mathcal{R}_\alpha(\cdot) : [0,\infty) \to \mathcal{L}(X)$ is strongly continuous and $\mathcal{R}_\alpha(0)x = x$ for all $x \in X$ and $\alpha \in (1,2)$.

(b) For $x \in D(A)$, $\mathcal{R}_\alpha(\cdot)x \in C([0,\infty),[D(A)]) \bigcap C^1([0,\infty),X)$, and

$$D_t^\alpha \mathcal{R}_\alpha(t)x = A\mathcal{R}_\alpha(t)x + \int_0^t B(t-s)\mathcal{R}_\alpha(s)xds, \tag{2.3}$$

$$D_t^\alpha \mathcal{R}_\alpha(t)x = \mathcal{R}_\alpha(t)Ax + \int_0^t \mathcal{R}_\alpha(t-s)B(s)xds, \tag{2.4}$$

for every $t \geq 0$.

The existence of an α-resolvent operator for problem (2.1)-(2.2) was studied in [16]. In this work we have considered the same conditions in [15, 16].

(P$_1$) The operator $A : D(A) \subseteq X \to X$ is a closed linear operator with $[D(A)]$ dense in X. Let $\alpha \in (1,2)$. For some $\phi_0 \in (0, \frac{\pi}{2}]$, for each $\phi < \phi_0$ there is a positive constant $C_0 = C_0(\phi)$ such that $\lambda \in \rho(A)$ for each $\lambda \in \sum_{0,\alpha\vartheta} = \{\lambda \in \mathbb{C} : \lambda \neq 0, |arg(\lambda)| < \alpha\vartheta\}$, where $\vartheta = \phi + \frac{\pi}{2}$ and $\|R(\lambda,A)\| \leq \frac{C_0}{|\lambda|}$ for all $\lambda \in \sum_{0,\alpha\vartheta}$.

(P$_2$) For all $t \geq 0$, $B(t) : D(B(t)) \subseteq X \to X$ is a closed linear operator, $D(A) \subseteq D(B(t))$ and $B(\cdot)x$ is strongly measurable on $(0,\infty)$ for each $x \in D(A)$. There exist $b(\cdot) \in L^1_{loc}(\mathbb{R}^+)$ such that $\widehat{b}(\lambda)$ exists for $Re(\lambda) > 0$ and $\|B(t)x\| \leq b(t)\|x\|_1$ for all $t > 0$ and $x \in D(A)$. Moreover, the operator valued function $\widehat{B} : \sum_{0,\frac{\pi}{2}} \to \mathcal{L}([D(A)],X)$ has an analytical extension (still denoted by $\widehat{B}$) to $\sum_{0,\vartheta}$ such that $\|\widehat{B}(\lambda)x\| \leq \|\widehat{B}(\lambda)\|\|x\|_1$ for all $x \in D(A)$, and $\|\widehat{B}(\lambda)\| = O(\frac{1}{|\lambda|})$, as $|\lambda| \to \infty$.

(P$_3$) There exists a subspace $D \subseteq D(A)$ dense in $[D(A)]$ and a positive constant C_1 such that $A(D) \subseteq D(A)$, $\widehat{B}(\lambda)(D) \subseteq D(A)$, and $\|A\widehat{B}(\lambda)x\| \leq C_1\|x\|$ for every $x \in D$ and all $\lambda \in \sum_{0,\vartheta}$.

In the sequel, for $r > 0$ and $\theta \in (\frac{\pi}{2}, \vartheta)$, $\sum_{r,\theta} = \{\lambda \in \mathbb{C} : \lambda \neq 0, |\lambda| > r, |arg(\lambda)| < \theta\}$, for $\Gamma_{r,\theta}$, $\Gamma^i_{r,\theta}$, $i = 1,2,3$, are the paths $\Gamma^1_{r,\theta} = \{te^{i\theta} : t \geq r\}, \Gamma^2_{r,\theta} = \{re^{i\xi} : -\theta \leq \xi \leq \theta\}, \Gamma^3_{r,\theta} = \{te^{-i\theta} : t \geq r\}$, and $\Gamma_{r,\theta} = \bigcup_{i=1}^3 \Gamma^i_{r,\theta}$, oriented counterclockwise. In addition, $\rho_\alpha(G_\alpha)$ are the sets

$$\rho_\alpha(G_\alpha) = \{\lambda \in C : G_\alpha(\lambda) := \lambda^{\alpha-1}(\lambda^\alpha I - A - \widehat{B}(\lambda))^{-1} \in \mathcal{L}(X)\}.$$

We now define the operator family $(\mathcal{R}_\alpha(t))_{t\geq 0}$ by

$$\mathcal{R}_\alpha(t) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\Gamma_{r,\theta}} e^{\lambda t} G_\alpha(\lambda) d\lambda, & t > 0, \\ I, & t = 0. \end{cases} \tag{2.5}$$

The following result has been established in [3, Theorem 2.1].

Theorem 2.2. *Assume that conditions* $(\mathbf{P1}) - (\mathbf{P3})$ *are fulfilled. Then there exists a unique α-resolvent operator for problem (2.1)-(2.2).*

Theorem 2.3. *[3, Lemma 2.5] The function $\mathcal{R}_\alpha : [0, \infty) \to \mathcal{L}(X)$ is strongly continuous and $\mathcal{R}_\alpha : (0, \infty) \to \mathcal{L}(X)$ is uniformly continuous.*

In what follows, we always assume that the conditions $(\mathbf{P1}) - (\mathbf{P3})$ are verified.
We consider now the non-homogeneous problem

$$D^\alpha_t x(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + f(t), \quad t \in [0, a], \tag{2.6}$$

$$x(0) = x_0, \quad x'(0) = 0, \tag{2.7}$$

where $\alpha \in (1,2)$ and $f \in L^1([0,a], X)$. In the sequel, $\mathcal{R}_\alpha(\cdot)$ is the operator function defined by (2.5). We begin by introducing the following concept of a classical solution.

Definition 2.4. A function $x : [0, a] \to X$, $0 < a$, is called a classical solution of (2.6)-(2.7) on $[0, a]$ if $x \in C([0, a], [D(A)]) \bigcap C([0, a], X)$, $g_{n-\alpha} * x \in C^1([0, a], X)$, $n = 1, 2$, the condition (2.7) holds and Eq. (2.6) is verified on $[0, a]$.

Definition 2.5. Let $\alpha \in (1,2)$; we define the family $(\mathcal{S}_\alpha(t))_{t\geq 0}$ by

$$\mathcal{S}_\alpha(t)x := \int_0^t g_{\alpha-1}(t-s)\mathcal{R}_\alpha(s)xds,$$

for each $t \geq 0$.

Lemma 2.6. *[3, Lemma 2.3] The function $\mathcal{R}_\alpha(\cdot)$ is exponentially bounded in $\mathcal{L}(X)$.*

Lemma 2.7. *[3, Lemma 3.9] If the function $\mathcal{R}_\alpha(\cdot)$ is exponentially bounded in $\mathcal{L}(X)$, then $\mathcal{S}_\alpha(\cdot)$ is exponentially bounded in $\mathcal{L}(X)$.*

Lemma 2.8. *[3, Lemma 3.10] If the function $\mathcal{R}_\alpha(\cdot)$ is exponentially bounded in $\mathcal{L}(D(A))$, then $\mathcal{S}_\alpha(\cdot)$ is exponentially bounded in $\mathcal{L}(D(A))$.*

We now establish a variation of constants formula for the solutions of (2.6)-(2.7).

Theorem 2.9. *[3, Theorem 3.2] Let $z \in D(A)$. Assume that $f \in C([0,a],X)$ and $x(\cdot)$ is a classical solution of (2.6)-(2.7) on $[0,a]$. Then*

$$x(t) = \mathcal{R}_\alpha(t)z + \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds, \quad t \in [0,a]. \tag{2.8}$$

It is clear from the preceding definition that $\mathcal{R}_\alpha(\cdot)z$ is a solution of problem (2.1)-(2.2) on $(0,\infty)$ for $z \in D(A)$.

Definition 2.10. Let $f \in L^1([0,a],X)$. A function $x \in C([0,a],X)$ is called a mild solution of (2.6)-(2.7) if

$$x(t) = \mathcal{R}_\alpha(t)z + \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds, \quad t \in [0,a].$$

The following results are proved in [3, 16].

Theorem 2.11. *[3, Theorem 3.3] Let $z \in D(A)$ and $f \in C([0,a],X)$. If $f \in L^1([0,a],[D(A)])$, then the mild solution of (2.6)-(2.7) is a classical solution.*

Theorem 2.12. *[3, Theorem 3.4] Let $z \in D(A)$ and $f \in C([0,a],X)$. If $f \in W^{1,1}([0,a],X)$, then the mild solution of (2.6)-(2.7) is a classical solution.*

In the next result we denote by $(-A)^\vartheta$ the power of the operator $-A$, (see [26] for details). From [26, Lemma 6.3], there exists a constant C such that $\|(-A)^\vartheta\| \leq C$ for $0 \leq \vartheta \leq 1$.

Lemma 2.13. *[15, Lemma 3.1] Suppose that the conditions* (**P1**)-(**P3**) *are satisfied. Let $\alpha \in (1,2)$ and $\vartheta \in (0,1)$ such that $\alpha\vartheta \in (0,1)$, then there exists positive number C such that*

$$\|(-A)^\vartheta \mathcal{R}_\alpha(t)\| \leq Ce^{rt}t^{-\alpha\vartheta}, \tag{2.9}$$

$$\|(-A)^\vartheta \mathcal{S}_\alpha(t)\| \leq Ce^{rt}t^{\alpha(1-\vartheta)-1}, \tag{2.10}$$

for all $t > 0$.

Remark 2.14. [15, Remark 3.2] If $\widehat{B}(\lambda)(-A)^{-\vartheta}y = (-A)^{-\vartheta}\widehat{B}(\lambda)y$ for $y \in [D(A)]$. We can see that for $\vartheta \in (0,1)$ and $x \in [D((-A)^\vartheta)]$ that

$$(-A)^\vartheta \mathcal{R}_\alpha(t)x = \mathcal{R}_\alpha(t)(-A)^\vartheta x \quad \text{and} \quad (-A)^\vartheta \mathcal{S}_\alpha(t)x = \mathcal{S}_\alpha(t)(-A)^\vartheta x,$$

if $x \in [D((-A)^\vartheta)]$.

We will herein define the phase space $\mathcal{B}$ axiomatically, using ideas and notation developed in [23]. More precisely, $\mathcal{B}$ will denote the vector space of functions defined from $(-\infty,0]$ into X endowed with a seminorm denoted as $\|\cdot\|_\mathcal{B}$ and such that the following axioms hold:

(**A**) If $x : (-\infty,\sigma+b) \to X$, $b > 0$, $\sigma \in \mathbb{R}$ is continuous on $[\sigma,\sigma+b)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma,\sigma+b)$ the following conditions hold:

- (i) x_t is in $\mathcal{B}$.
- (ii) $\|x(t)\| \leq H\|x_t\|_\mathcal{B}$.

(*iii*) $\|x_t\|_{\mathcal{B}} \le K(t-\sigma)\sup\{\|x(s)\| : \sigma \le s \le t\} + M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$,
where $H > 0$ is a constant; $K, M : [0,\infty) \to [1,\infty)$, $K(\cdot)$ is continuous, $M(\cdot)$ is locally bounded, and H, K, M are independent of $x(\cdot)$.

(**A1**) For the function $x(\cdot)$ in (**A**), the function $t \to x_t$ is continuous from $[\sigma, \sigma+b)$ into $\mathcal{B}$.

(**B**) The space $\mathcal{B}$ is complete.

Example 2.15. The Phase Space $C_r \times L^p(g, X)$.
Let $r \ge 0$, $1 \le p < \infty$ and $g : (-\infty, -r] \to \mathbb{R}$ be a non-negative, measurable function which satisfies the conditions $(g-5)-(g-6)$ in the terminology of [23]. Briefly, this means that g is locally integrable and there exists a non-negative, locally bounded function $\eta(\cdot)$ on $(-\infty, 0]$ such that $g(\xi+\theta) \le \eta(\xi)g(\theta)$ for all $\xi \le 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. The space $C_r \times L^p(g, X)$ consists of all classes of functions $\varphi : (-\infty, 0] \to X$ such that φ is continuous on $[-r, 0]$ and is Lebesgue measurable, and $g\|\varphi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $C_r \times L^p(g, X)$ defined by

$$\|\varphi\|_{\mathcal{B}} := \sup\{\|\varphi(\theta)\| : -r \le \theta \le 0]\} + \left(\int_{-\infty}^{-r} g(\theta)\|\varphi(\theta)\|^p d\theta\right)^{1/p}.$$

The space $\mathcal{B} = C_r \times L^p(g, X)$ satisfies the axioms (**A**), (**A$_1$**) and (**B**). Moreover, when $r = 0$ and $p = 2$, we can take $H = 1$, $K(t) = 1 + \left(\int_{-t}^{0} g(\theta)d\theta\right)^{1/2}$ and $M(t) = \eta(-t)^{1/2}$, for $t \ge 0$ (see [23, Theorem 1.3.8] for details).

For additional details concerning phase space we refer the reader to [23].

3 Existence Results

In this section we study the existence of mild solutions of the abstract fractional integro-differential equations (1.1)-(1.2). Motivated by Definition (2.10), we consider the following concept of mild solution.

Definition 3.1. A function $u : (-\infty, b] \to X$, is called a mild solution of (1.1)-(1.2) on $[0, b]$, if $u_0 = \varphi$; $u|_{[0,b]} \in C([0, b] : X)$; the function $\tau \to A\mathcal{S}_\alpha(t-\tau)f(\tau, u_\tau)$ and $\tau \to \int_0^\tau B(\tau-\xi)\mathcal{S}_\alpha(t-\tau)f(\xi, u_\xi)d\xi$ is integrable on $[0, t)$ for all $t \in (0, b]$ and for $t \in [0, b]$,

$$\begin{aligned} u(t) &= \mathcal{R}_\alpha(t)(\varphi(0) + f(0,\varphi) + q(u_{t_1}, u_{t_2}, u_{t_3}, \cdots, u_{t_n})(0)) - f(t, u_t) - \int_0^t A\mathcal{S}_\alpha(t-s)f(s, u_s)ds \\ &\quad - \int_0^t \int_0^s B(s-\xi)\mathcal{S}_\alpha(t-s)f(\xi, u_\xi)d\xi ds + \int_0^t \mathcal{S}_\alpha(t-s)g(s, u_s)ds. \end{aligned}$$

In the sequel we introduce the following assumptions.

(**H$_1$**) The following conditions are satisfied.

(a) $B(\cdot)x \in C(I, X)$ for every $x \in [D((-A)^{1-\vartheta})]$.

(b) There is function $\mu(\cdot) \in L^1(I;\mathbb{R}^+)$, such that

$$\|B(s)\mathcal{S}_\alpha(t)\|_{\mathcal{L}([D((-A)^\vartheta)],X)} \le M\mu(s)t^{\alpha\vartheta-1}, \quad 0 \le s < t \le b.$$

$(\mathbf{H_2})$ The function $q : \mathcal{B}^n \to \mathcal{B}$ is continuous and exist positive constants $L_i(q)$ such that

$$\|q(\psi_1,\psi_2,\psi_3,\cdots,\psi_n) - q(\varphi_1,\varphi_2,\varphi_3,\cdots,\varphi_n)\| \le \sum_{i=1}^{n} L_i(q)\|\psi_i - \varphi_i\|_{\mathcal{B}},$$

for every $\psi_i, \varphi_i \in B_r[0,\mathcal{B}]$.

$(\mathbf{H_3})$ The function $f(\cdot)$ is $(-A)^\vartheta$-valued, $f : I\times\mathcal{B} \to [D((-A)^{-\vartheta})]$, the function $g(\cdot)$ is defined on $g : I\times\mathcal{B} \to X$, and there exist positive constants L_f and L_g such that for all $(t_i,\psi_j) \in I\times\mathcal{B}$,

$$\|(-A)^\vartheta f(t_1,\psi_1) - (-A)^\vartheta f(t_2,\psi_2)\| \le L_f(|t_1 - t_2| + \|\psi_1 - \psi_2\|_{\mathcal{B}}),$$
$$\|g(t_1,\psi_1) - g(t_2,\psi_2)\| \le L_g(|t_1 - t_2| + \|\psi_1 - \psi_2\|_{\mathcal{B}}).$$

Remark 3.2. We can assume there exists $M > 0$ such that $\|\mathcal{R}_\alpha(t)\| \le M$ and $\|\mathcal{S}_\alpha(t)\| \le M$ for all $t \in [0,b]$. In the rest of this section, M_b and K_b are the constants $M_b = \sup_{s\in[0,b]} M(s)$, $K_b = \sup_{s\in[0,b]} K(s)$, $N_q = \sup\{\|q(\psi_1,\psi_{t_2},\psi_{t_3},\cdots,\psi_{t_n})\| : \psi_i \in B_r[0,\mathcal{B}]\}$ and $N_{(-A)^\vartheta f}$, N_f, N_g represent the supreme of the functions $(-A)^\vartheta f$, f and g on $[0,b]\times B_r[0,\mathcal{B}]$.

Theorem 3.3. *Let conditions* $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and* $(\mathbf{H_3})$ *be hold. If*

$$\begin{aligned}\rho \;=\; & (M_b + K_bMH) \parallel \varphi \parallel_{\mathcal{B}} + (M_b + K_bM)N_q + K_b(M+1)N_f \\ & + K_bN_{(-A)^\vartheta f}M\frac{b^{\alpha\vartheta}}{\alpha\vartheta}\left(1 + \int_0^b \mu(\xi)d\xi\right) + K_bN_gMb < r\end{aligned}$$

and

$$\Lambda = \max\left\{M_b\left(M_b\sum_{i=1}^{n} L_i(q) + K_b\theta\right), K_b\left(M_b\sum_{i=1}^{n} L_i(q) + K_b\theta\right)\right\} < 1,$$

where

$$\theta = \left(M\sum_{i=1}^{n} L_i(q) + L_f\left(\|(-A)^{-\vartheta}\| + \frac{Mb^{\alpha\vartheta}}{\alpha\vartheta} + \frac{Mb^{\alpha\vartheta}}{\alpha\vartheta}\int_0^b \mu(\xi)d\xi\right) + ML_gb\right).$$

Then there exists a mild solution of (1.1)-(1.2) on $[0,b]$.

Proof. Consider the space $S(b) = \{x : (-\infty,b] \to X : x_0 \in \mathcal{B}; x \in C([0,b] : X)\}$ endowed with the norm

$$\parallel x \parallel_{S(b)} := M_b \parallel x_0 \parallel_{\mathcal{B}} + K_b \parallel x \parallel_b .$$

Let $Y = B_r[0,S(b)]$, we define the operator $\Gamma : Y \to S(b)$ by

$$\begin{aligned}\Gamma x(t) &= \mathcal{R}_\alpha(t)(\varphi(0) + f(0,\varphi) + q(x_{t_1},x_{t_2},x_{t_3},\cdots,x_{t_n})(0)) - f(t,x_t) \\ &\quad - \int_0^t A\mathcal{S}_\alpha(t-s)f(s,x_s)ds - \int_0^t\int_0^s B(s-\xi)\mathcal{S}_\alpha(t-s)f(\xi,x_\xi)d\xi ds \\ &\quad + \int_0^t \mathcal{S}_\alpha(t-s)g(s,x_s)ds, \text{ for } t \in [0,b], \\ (\Gamma x)_0 &= \varphi + q(x_{t_1},x_{t_2},x_{t_3},\cdots,x_{t_n}).\end{aligned}$$

Using an similar argument on the proof of Theorem 3.1 in [21], we will prove the Γ is well defined. Next we will prove that $\Gamma(Y) \subset Y$.

Let $x \in Y$ and $t \in [0, b]$, we observe from axiom (**A**) of the phase space, we obtain that $\| x_t \|_{\mathcal{B}} \le K_b \| x \|_b + M_b \| x_0 \|_{\mathcal{B}} \le r$ this implies that $x_t \in B_r[0, \mathcal{B}]$, and this case

$$\begin{aligned}
\|\Gamma x(t)\| &\le \|\mathcal{R}_\alpha(t)\|(\|\varphi(0)\| + \|f(0,\varphi)\| + \|q(x_{t_1}, x_{t_2}, x_{t_3}, \cdots, x_{t_n})(0)\|) + \|f(t, x_t)\| \\
&\quad + \int_0^t M(t-s)^{\alpha\vartheta-1}\|(-A)^\vartheta f(s, x_s)\| ds \\
&\quad + \int_0^t \int_0^s \mu(s-\xi) M(t-s)^{\alpha\vartheta-1}\|(-A)^\vartheta f(\xi, x_\xi)\| d\xi ds \\
&\quad + \int_0^t M\|g(s, x_s)\| ds \\
&\le M(H\|\varphi\|_{\mathcal{B}} + N_f + N_q) + N_f \\
&\quad + N_{(-A)^\vartheta f} \int_0^t M(t-s)^{\alpha\vartheta-1} ds \\
&\quad + N_{(-A)^\vartheta f} \int_0^t \int_0^s \mu(s-\xi) M(t-s)^{\alpha\vartheta-1} d\xi ds \\
&\quad + MN_g \int_0^t ds \\
&\le M(H\|\varphi\|_{\mathcal{B}} + N_f + N_q) + N_f + N_{(-A)^\vartheta f} M \frac{b^{\alpha\vartheta}}{\alpha\vartheta} \\
&\quad + N_{(-A)^\vartheta f} M \frac{b^{\alpha\vartheta}}{\alpha\vartheta} \int_0^b \mu(\xi) d\xi + N_g M b. \qquad (3.1)
\end{aligned}$$

and

$$\|(\Gamma x)_0\| \le \|\varphi\|_{\mathcal{B}} + N_q. \qquad (3.2)$$

From (3.1)-(3.2), we have that

$$\begin{aligned}
\| \Gamma x \|_{S(b)} &\le M_b \| (\Gamma x)_0 \|_{\mathcal{B}} + K_b \| \Gamma x \|_b \\
&\le (M_b + K_b M H) \| \varphi \|_{\mathcal{B}} + (M_b + K_b M) N_q + K_b (M+1) N_f \\
&\quad + K_b N_{(-A)^\vartheta f} M \frac{b^{\alpha\vartheta}}{\alpha\vartheta} \left(1 + \int_0^b \mu(\xi) d\xi\right) + K_b N_g M b \\
&= \rho < r. \qquad (3.3)
\end{aligned}$$

which prove that $\Gamma x \in Y$.

In order to prove that Γ satisfies a Lipschitz condition, $u, v \in Y$. If $t \in [0, b]$ we see that

$$
\begin{aligned}
&\|\Gamma u(t) - \Gamma v(t)\| \\
&\quad \leq \|\mathcal{R}_\alpha(t)(q(u_{t_1}, u_{t_2}, u_{t_3}, \cdots, u_{t_n})(0) - q(v_{t_1}, v_{t_2}, v_{t_3}, \cdots, v_{t_n})(0))\| \\
&\qquad + \|(-A)^{-\vartheta}\|\|(-A)^{\vartheta} f(t, u_t) - (-A)^{\vartheta} f(t, v_t)\| \\
&\qquad + \int_0^t \|(-A)^{1-\vartheta}\mathcal{S}_\alpha(t-s)\|\|(-A)^{\vartheta} f(s, u_s) - (-A)^{\vartheta} f(s, v_s)\| ds \\
&\qquad + \int_0^t \int_0^s \|B(s-\xi)\mathcal{S}_\alpha(t-s) f(\xi, u_\xi) - f(\xi, v_\xi)\| d\xi ds \\
&\qquad + \int_0^t \|\mathcal{S}_\alpha(t-s)\|\|g(s, u_s) - g(s, v_s)\| ds \\
&\quad \leq M \sum_{i=1}^n L_i(q)\|u_{t_i} - v_{t_i}\|_{\mathcal{B}} + \|(-A)^{-\vartheta}\| L_f(K_b\|u-v\|_b + M_b\|u_0 - v_0\|_{\mathcal{B}}) \\
&\qquad + L_f(K_b\|u-v\|_b + M_b\|u_0 - v_0\|_{\mathcal{B}}) \int_0^t M(t-s)^{\alpha\upsilon - 1} ds \\
&\qquad + L_f(K_b\|u-v\|_b + M_b\|u_0 - v_0\|_{\mathcal{B}}) \int_0^t \int_0^s \mu(s-\xi) M(t-s)^{\alpha\vartheta - 1} d\xi ds \\
&\qquad + ML_g(K_b\|u-v\|_b + M_b\|u_0 - v_0\|_{\mathcal{B}}) \int_0^t ds \\
&\quad \leq M_b\left(M \sum_{i=1}^n L_i(q) + L_f\left(\|(-A)^{-\vartheta}\| + \frac{Mb^{\alpha\vartheta}}{\alpha\vartheta} + \frac{Mb^{\alpha\vartheta}}{\alpha\vartheta}\int_0^b \mu(\xi) d\xi\right) + ML_g b\right)\|u_0 - v_0\|_{\mathcal{B}} \\
&\qquad + K_b\left(M \sum_{i=1}^n L_i(q) + L_f\left(\|(-A)^{-\vartheta}\| + \frac{Mb^{\alpha\vartheta}}{\alpha\vartheta} + \frac{Mb^{\alpha\vartheta}}{\alpha\vartheta}\int_0^b \mu(\xi) d\xi\right) + ML_g b\right)\|u-v\|_b \\
&\quad \leq M_b\theta\|u_0 - v_0\|_{\mathcal{B}} + K_b\theta\|u-v\|_b.
\end{aligned}
$$

On the other hand, a simple calculus prove that

$$
\|(\Gamma u)_0 - (\Gamma v)_0\| \quad \leq \quad \sum_{i=1}^n L_i(q)(M_b\|u_0 - v_0\|_{\mathcal{B}} + K_b\|u-v\|_b).
$$

Finally we see that

$$
\begin{aligned}
\|\Gamma u - \Gamma v\|_{S(b)} \quad &\leq \quad M_b \parallel (\Gamma u)_0 - (\Gamma v)_0 \parallel_{\mathcal{B}} + K_b \parallel \Gamma u - \Gamma v \parallel_b \\
&\leq \quad M_b\left(M_b \sum_{i=1}^n L_i(q) + K_b\theta\right)\|u_0 - v_0\| + K_b\left(M_b \sum_{i=1}^n L_i(q) + K_b\theta\right)\|u-v\|_{\mathcal{B}} \\
&\leq \quad \Lambda\|u-v\|_{S(b)},
\end{aligned}
$$

which infer that Γ is a contraction on Y. Clearly a fixed point of Γ is the unique mild solution of the nonlocal problem (1.1)-(1.2). The proof is complete. □

4 Applications

To complete this work, we study the existence of solutions for the partial integro-differential system with nonlocal conditions. In the sequel, $X = L^2([0,\pi])$, $\mathcal{B} = C_0 \times L^p(g,X)$ is the space introduced in Example 2.15 and $A : D(A) \subseteq X \to X$ is the operator defined by $Ax = x''$, with domain $D(A) = \{x \in X : x'' \in X, x(0) = x(\pi) = 0\}$. The operator A is the infinitesimal generator of an analytic semigroup, $\rho(A) = \mathbb{C} \setminus \{-n^2 : n \in \mathbb{N}\}$ and for $\alpha \in (0,1)$ and $\alpha\vartheta \in (\pi/2,\pi)$ there exists $M_{\alpha\vartheta} > 0$ such that $\| R(\lambda, A) \| \leq M_{\alpha\vartheta} \mid \lambda \mid^{-1}$ for all $\lambda \in \Sigma_{\alpha\vartheta}$ and the fractional power $(-A)^\nu : D((-A)^\nu) \subset X \to X$ of A is given by $(-A)^\nu x = \sum_{n=1}^{\infty} n^{2\nu} \langle x, z_n \rangle z_n$, where $D((-A)^\nu) = \{x \in X : (-A)^\nu x \in X\}$. Hence, A is sectorial of type and the properties $(\mathbf{P_1})$ hold. We also consider the operator $B(t) : D(A) \subseteq X \to X$, $t \geq 0$, $B(t)x = t^\delta e^{-\gamma t} Ax$ for $x \in D(A)$. Moreover, it is easy to see that conditions $(\mathbf{P_2})$ and $(\mathbf{P_3})$ in Section 2 are satisfied with $b(t) = t^\delta e^{-\gamma t}$ and $D = C_0^\infty([0,\pi])$, where $C_0^\infty([0,\pi])$ is the space of infinitely differentiable functions that vanish at $\xi = 0$ and $\xi = \pi$. Therefore, (2.1)-(2.2), has an associated α-resolvent operators $(\mathcal{R}_\alpha(t))_{t \geq 0}$ on X.

Consider the delayed fractional partial neutral integro-differential equation with nonlocal conditions

$$\begin{aligned} \frac{\partial^\alpha}{\partial t^\alpha}\left(u(t,\xi) + \int_{-\infty}^{t} \int_0^\pi b(t-s,\eta,\xi) u(s,\eta) d\eta ds\right) \\ = \frac{\partial^2}{\partial \xi^2} u(t,\xi) + \int_0^t (t-s)^\delta e^{-\gamma(t-s)} \frac{\partial^2}{\partial \xi^2} u(s,\xi) ds \\ + \int_{-\infty}^{t} a_0(s-t) u(s,\xi) ds, \quad (t,\xi) \in I \times [0,\pi], \end{aligned} \tag{4.1}$$

$$u(t,0) = u(t,\pi) = 0, \quad t \in [0,b], \quad u(\theta,\xi) = \phi(\theta,\xi) + \sum_{i=0}^{n} L_i u(t_i + \xi), \theta \leq 0, \ \xi \in [0,\pi]. \tag{4.2}$$

where $0 < t_i < b$, $L_i, i = 1,2,\ldots n$, are fixed numbers and $\frac{\partial^\alpha}{\partial t^\alpha} = D_t^\alpha$, $\alpha \in (1,2)$.

In the sequel, we assume that $\varphi(\theta)(\xi) = \phi(\theta,\xi)$ is a function in $\mathcal{B}$ and that the following conditions are verified.

(i) The functions $a_0 : \mathbb{R} \to \mathbb{R}$ are continuous and $L_g := \left(\int_{-\infty}^{0} \frac{(a_0(s))^2}{g(s)} ds\right)^{1/2} < \infty$.

(ii) The functions $b(s,\eta,\xi)$, $\frac{\partial b(s,\eta,\xi)}{\partial \xi}$ are measurable, $b(s,\eta,\pi) = b(s,\eta,0) = 0$ for all (s,η) and

$$L_f := \max\left\{\left(\int_0^\pi \int_{-\infty}^{0} \int_0^\pi g^{-1}(\theta)\Big(\frac{\partial^i}{\partial \xi^i} b(\theta,\eta,\xi)\Big)^2 d\eta d\theta d\xi\right)^{1/2} : i = 0,1\right\} < \infty.$$

Defining the operators $f, g : I \times \mathcal{B} \to X$ and $q : \mathcal{B}^n \to \mathcal{B}$ by

$$\begin{aligned} f(\psi)(\xi) &= \int_{-\infty}^{0} \int_0^\pi b(s,\eta,\xi) \psi(s,\eta) d\eta ds, \\ g(\psi)(\xi) &= \int_{-\infty}^{0} a_0(s) \psi(s,\xi) ds, \\ q(u_{t_1}, u_{t_2}, \ldots, u_{t_n})(\xi) &= \sum_{i=0}^{n} L_i u(t_i + \xi). \end{aligned}$$

Under the above conditions we can represent the system (4.1)-(4.2) into the abstract system (1.1)-(1.2). Moreover, f, g are bounded linear operators with $\|f(\cdot)\|_{\mathcal{L}(\mathcal{B},X)} \leq L_f$, $\|g(\cdot)\|_{\mathcal{L}(\mathcal{B},X)} \leq L_g$ and $\|q(\cdot)\|_{\mathcal{L}(\mathcal{B},X)} \leq N_q$. Moreover, a straightforward estimation using (ii) shows that $f(I\times\mathcal{B}) \subset D((-A)^{1/2})$ and $\|(-A)^{1/2}f\|_{\mathcal{L}(\mathcal{B},X)} \leq L_f$. By Lemma 2.13, there exists $C_\eta > 0$ such that

$$\|b(t)A\mathcal{S}_\alpha(t)\|_{\mathcal{L}([D((-A)^\eta)],X)} \leq \frac{C_\eta}{t^{\alpha(1-\eta)-1}},$$

for $\eta \in (0,1)$ and $t > 0$. Therefore, $(\mathbf{H_1}), (\mathbf{H_2})$ and $(\mathbf{H_3})$ are fulfill. The next result is a direct consequence of Theorem 3.3.

Proposition 4.1. *For b sufficiently small there exists a mild solution for the partial neutral fractional integro-differential system with nonlocal condition (4.1)-(4.2).*

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