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Atomic Decomposition of Weighted Weak Hardy Spaces on Spaces of Homogeneous Type

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Abstract

In this paper, we establish an atomic decomposition characterization of weighted weak Hardy spaces $H^{p,\infty}_{\omega}$ on spaces of homogeneous type. As an application, we prove a interpolation theorem in $H^{p,\infty}_{\omega}$.

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1 Introduction and main results

The theory of weak Hardy spaces is very important in harmonic analysis since it can sharpen the endpoint weak type estimate for variant important operators (see, for example, [5]). The weak Hardy spaces were first studies in [4] as special Hardy-Lorentz spaces which are the intermediate spaces between two Hardy spaces. Fefferman and Soria [5] established an atomic decomposition of the weak Hardy space $H^{1,\infty}(\mathbb{R}^n)$. The atomic decompositions of the weak Hardy spaces $H^{p,\infty}$ on homogeneous groups were given by Liu in [11]. Ding and Lan [2] developed the theory of weak Hardy spaces associated to expansive dilations on $\mathbb{R}^n$. The weak Hardy spaces on spaces of homogeneous type was recently studied in [3] and [17].

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The purpose of this paper is to study the theory of weighted weak Hardy spaces $H_w^{p,\infty}$ on space of homogeneous type. More precisely, we will establish atomic decomposition characterizations of weighted weak Hardy spaces on space of homogeneous type. As an application, we prove an $H^{p,\infty}$ interpolation theorem. We remark that our theory is so general that it can be applied to more variant different settings such as Euclidean spaces with A_∞-weights, Ahlfors n-regular metric measure spaces (see, for example, [9]), Lie groups of polynomial growth (see, for instance, [16]) and Carnot-Carathéodory spaces with doubling measure (see [13]).

Before giving the main results, let us recall some definitions and notions first. The following notion of spaces of homogeneous type was introduced by Coifman and Weiss in [1].

Definition 1.1. Let $(\mathbb{X},d,\mu)$ be a quasi-metric space with a regular Borel measure μ such that all balls defined by d have finite and positive measures. The quasi-metric satisfies the following triangle inequality,

$$d(x,z) \le \tau(d(x,y)+d(y,z)). \tag{1.1}$$

For any $x \in \mathbb{X}$ and $r > 0$, set $B(x,r) = \{y \in \mathbb{X} : d(x,y) < r\}$. $(\mathbb{X},d,\mu)$ is called a space of homogeneous type if there exists a constant $C \ge 1$ such that for all $x \in \mathbb{X}$ and $r > 0$,

$$\mu(B(x,2r)) \le C\mu(B(x,r)). \tag{1.2}$$

Throughout this paper, we also assume that d has the following regularity property:

$$|d(x,y)-d(x',y)| \le Cd(x,x')^{\vartheta}[d(x,y)+d(x',y)]^{1-\vartheta}, \tag{1.3}$$

where the constant ϑ is called the regularity exponent on $\mathbb{X}$.

It can be shown from (1.2) that there exist constants $1 < C, D < \infty$ such that

$$\mu(B(x,sr)) \le Cs^D\mu(B(x,r)). \tag{1.4}$$

The least possible value of D in (1.4) is called the dimension of $\mathbb{X}$. In what follows, we use D to denote the dimension of $\mathbb{X}$. Let $\mathcal{M}$ denote the Hardy-Littlewood maximal function on $\mathbb{X}$.

Definition 1.2. Let $\omega \in L^1_{loc}(\mathbb{X})$ be a nonnegative function in $\mathbb{X}$. If there exists a constant $C > 0$ such that for every ball $B \subset \mathbb{X}$,

$$\left[\frac{1}{\mu(B)}\int_B \omega(x)d\mu(x)\right]\left[\frac{1}{\mu(B)}\int_B \omega(x)^{-\frac{1}{p-1}}d\mu(x)\right]^{p-1} \le C, \quad \text{if } 1 < p < \infty,$$
$$\mathcal{M}(\omega)(x) \le C\omega(x), \quad \text{if } p = 1,$$

then we say ω is an $A_p(\mathbb{X})$ weight and write $\omega \in A_p(\mathbb{X})$. Define $A_\infty(\mathbb{X}) \equiv \bigcup_{1\le p<\infty} A_p(\mathbb{X})$. Let $q_\omega \equiv \inf\{q : \ \omega \in A_q(\mathbb{X})\}$ denote the critical index of ω.

For every ball B, denote $\omega(B) = \int_B w(x)d\mu(x)$. It is well known that if $\omega \in A_\infty(\mathbb{X})$, then there exists a constant $C \geq 1$ such that for all $x \in \mathbb{X}$ and $r > 0$,

$$\omega(B(x,2r)) \leq C\omega(B(x,r)). \tag{1.5}$$

Denote $V(x,y) = \mu(B(x,d(x,y)))$, $W(x,y) = \omega(B(x,d(x,y)))$.

The following approximation to the identity was constructed by Han, Li and Lu in [8].

Definition 1.3. A sequence $\{S_k\}_{k\in\mathbb{Z}}$ of operators is said to be an approximation to the identity if there exists constant $C > 0$ such that for all $k \in \mathbb{Z}$ and all $x, x', y, y' \in \mathbb{X}$, $S_k(x,y)$, the kernel of S_k satisfy the following conditions:

(i) $S_k(x,y) = 0$ if $d(x,y) \geq C2^{-k}$ and $|S_k(x,y)| \leq C\frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)}$;

(ii) $|S_k(x,y) - S_k(x',y)| \leq C2^{k\vartheta}d(x,x')^\vartheta \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)}$;

(iii) Property (ii) holds with x and y interchanged;

(iv) $|[S_k(x,y) - S_k(x,y')] - [S_k(x',y) - S_k(x',y')]| \leq C2^{2k\vartheta}d(x,x')^\vartheta d(y,y')^\vartheta \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)}$;

(v) $\int_{\mathbb{X}} S_k(x,y)d\mu(y) = \int_{\mathbb{X}} S_k(x,y)d\mu(x) = 1$.

We recall the definition of test functions in [8].

Definition 1.4. Let $0 < \beta, \gamma \leq \vartheta$ where ϑ is the regularity exponent on $\mathbb{X}$ given in and $r > 0$. A function φ on $\mathbb{X}$ is said to be a test function of type (x_0, r, β, γ) if f satisfies the following conditions:

(i) $|\varphi(x)| \leq C\frac{1}{V_r(x_0)+V(x,x_0)}\left(\frac{r}{r+d(x,x_0)}\right)^\gamma$;

(ii) $|\varphi(x) - \varphi(y)| \leq C\left(\frac{d(x,y)}{r+d(x,x_0)}\right)^\beta \frac{1}{V_r(x_0)+V(x,x_0)}\left(\frac{r}{r+d(x,x_0)}\right)^\gamma$ for all $x, y \in \mathbb{X}$ with $d(x,y) \leq (r + d(x,x_0))/2\tau$.

We denote by $\mathcal{G}(x_1, r, \beta, \gamma)$ the set of all test functions of type (x_1, r, β, γ). If $\varphi \in \mathcal{G}(x_1, r, \beta, \gamma)$ we define its norm by $\|\varphi\|_{\mathcal{G}(x_1,r,\beta,\gamma)} \equiv \inf\{C : (i) \text{ and } (ii) \text{ hold}\}$. Now fix $x_0 \in \mathbb{X}$ we denote $\mathcal{G}(\beta,\gamma) = \mathcal{G}(x_0, 1, \beta, \gamma)$ and by $\mathcal{G}_0(\beta,\gamma)$ the collection of all test functions in $\mathcal{G}(\beta,\gamma)$ with $\int_{\mathbb{X}} f(x)dx = 0$. It is easy to check that $\mathcal{G}(x_1, r, \beta, \gamma) = \mathcal{G}(\beta,\gamma)$ with equivalent norms for all $x_1 \in \mathbb{X}$ and $r > 0$. Furthermore, it is also easy to see that $\mathcal{G}(\beta,\gamma)$ is a Banach space with respect to the norm in $\mathcal{G}(\beta,\gamma)$.

Let $\overset{\circ}{\mathcal{G}}_\vartheta(\beta,\gamma)$ be the completion of the space $\mathcal{G}_0(\vartheta,\vartheta)$ in the norm of $\mathcal{G}(\beta,\gamma)$ when $0 < \beta, \gamma < \vartheta$. If $f \in \overset{\circ}{\mathcal{G}}_\vartheta(\beta,\gamma)$, we then define $\|f\|_{\overset{\circ}{\mathcal{G}}_\vartheta(\beta,\gamma)} = \|f\|_{\mathcal{G}(\beta,\gamma)}$. $(\overset{\circ}{\mathcal{G}}_\vartheta(\beta,\gamma))'$, the distribution space, is defined to be the set of all linear functionals L from $\overset{\circ}{\mathcal{G}}_\vartheta(\beta,\gamma)$ to $\mathbb{C}$ with the property that there exists $C \geq 0$ such that for all $f \in \overset{\circ}{\mathcal{G}}_\vartheta(\beta,\gamma)$,

$$|L(f)| \leq C\|f\|_{\overset{\circ}{\mathcal{G}}_\vartheta(\beta,\gamma)}.$$

We give the definition of non-tangential maximal functions on $\mathbb{X}$. Let $\{S_k\}$ be an approximation to the identity with regularity exponent ϑ. For $f \in (\overset{\circ}{\mathcal{G}}_\vartheta(\beta,\gamma))'$ with $\beta, \gamma \in (0,\vartheta)$, The radial maximal operator $\mathcal{M}_0$ is defined by

$$\mathcal{M}_0 f(x) \equiv \sup_{k\in\mathbb{Z}} |S_k(f)(x)|.$$

The grand maximal function is defined by

$$f^*(x) \equiv \sup\left\{|\langle f,\varphi\rangle| : \varphi \in \overset{\circ}{\mathcal{G}}_\vartheta(\beta,\gamma), \|\varphi\|_{\mathcal{G}(x,r,\beta,\gamma)} \le 1 \text{ for some } r>0\right\}.$$

Now we give the definition of weighed weak Hardy spaces $H^{p,\infty}_\omega(\mathbb{X})$.

Definition 1.5. Let $\{S_k\}$ be an approximation to the identity with regularity exponent ϑ. Let $\omega \in A_\infty(\mathbb{X})$ with $q_\omega < 1 + \frac{\vartheta}{D}$ and $p \in (q_\omega/(1+\vartheta/D), 1]$, $\sigma \in (0,\infty)$ and $\beta, \gamma \in (0,\vartheta)$. The *weighed weak Hardy space* $H^{p,\infty}_\omega(\mathbb{X})$ is defined by

$$H^{p,\infty}_\omega(\mathbb{X}) \equiv \{f \in (\overset{\circ}{\mathcal{G}}_\vartheta(\beta,\gamma))' : \mathcal{M}_0 f \in L^{p,\infty}_\omega(\mathbb{X})\}.$$

The $H^{p,\infty}_\omega$ quasi-norm of f is defined by $\|f\|_{H^{p,\infty}_\omega(\mathbb{X})} \equiv \|\mathcal{M}_0 f\|_{L^{p,\infty}_\omega(\mathbb{X})}$.

The main result of this paper is as follows.

Theorem 1.6. *Let $\omega \in A_\infty(\mathbb{X})$ with $q_\omega < 1+\frac{\vartheta}{D}$ and $p \in (q_\omega/(1+\vartheta/D), 1]$. Given $f \in H^{p,\infty}_\omega(\mathbb{X})$, there exists a sequence of bounded functions $\{f_k\}_{k=-\infty}^{\infty}$ such that*

(a) *$f - \sum_{|k|\le N} f_k \to 0$ in the sense of distributions;*

(b) *each f_k may be further decomposed as $f_k = \sum_{i=1}^{\infty} h_i^k$ in the sense of distribution, where each h_i^k satisfies:*

   (i) *h_i^k is supported in a ball B_i^k with $\{B_i^k\}$ having bounded overlapping for each k;*

   (ii) *$\int_{B_i^k} h_i^k = 0$;*

   (iii) *$\|h_i^k\|_{L^\infty} \le C2^k$ and $\sum_i \omega(B_i^k) \le C_1 2^{-kp}$ Moreover, C_1 is (up to an absolute constant) less than $\|f\|^p_{H^{p,\infty}_\omega(\mathbb{X})}$.*

Conversely, if f is a distribution satisfying (a) *and* (b) (i)-(iii)*, then $f \in H^{p,\infty}_\omega(\mathbb{X})$ and $\|f\|_{H^{p,\infty}_\omega(\mathbb{X})} \le cC_1$ (where c is some absolute constant).*

As an application of the atomic decomposition, we prove an interpolation theorem, which generalizes the result in [2].

Theorem 1.7. *Let $D/(D+\vartheta) < q < p \le 1 < p_0 < \infty$ and $\omega \in A_{p_0}(\mathbb{X})$. Suppose that T is a subadditive operator. If T is bounded both on $L^{p_0}_\omega(\mathbb{X})$ and on the weighted Hardy space $H^q_\omega(\mathbb{X})$, then T is bounded on $H^{p,\infty}_\omega(\mathbb{X})$.*

Remark 1.8. (i) Let $(\mathbb{X},d,\mu)$ be a space of homogeneous type only satisfying (1.1) and (1.2) (in the sense of Coifman and Weiss [1]). In [12], it has been shown that there exists a quasi-metric $\tilde{d}$ on $\mathbb{X}$, equivalent to d and satisfying (1.3) and

$$\mu(\tilde{B}(x,r)) \sim r^n, \text{ for some fixed } n, \tag{1.6}$$

where $\tilde{B}(x,r)=\{y\in\mathbb{X}:\tilde{d}(x,y)<r\}$. In the current paper, we only need (1.3) and a condition like (1.6) is not required.
(ii) As in the unweighted case in [3, 17], both the weighted Hardy spaces $H^p_\omega(\mathbb{X})$ and the weak Hardy spaces $H^{p,\infty}_\omega(\mathbb{X})$ can equivalently be defined via Littlewood-Paley functions, radial maximal functions, non-tangential maximal functions and grand maximal functions. Details will appear elsewhere.

2 Some lemmas

The following result was independently founded by Stein-Taibleson-Weiss [15] and by Kalton [10].

Lemma 2.1. *Let g_k be a sequence of measurable functions and let $0<r<1$. Assume that $\omega(\{|g_k|>\lambda\})\le C/\lambda^r$ with C independent of k and λ. Then, for every numerical sequence $\{c_k\}$ in l^r we have*

$$\omega\left(\left\{x:\Big|\sum_k c_k g_k\Big|>\lambda\right\}\right)\le\frac{2-r}{1-r}\frac{C}{\lambda^r}\sum_k|c_k|^r.$$

The following lemma is the Whitney decomposition theorem on space of homogeneous type $\mathbb{X}$ (see [14, 17]).

Lemma 2.2. *Let Ω be an open proper subset of $\mathbb{X}$ and let $d(x)=\inf\{d(x,y):\ y\notin\Omega\}$. Let $r(x)=d(x)/30$. Then there exist a positive number L depending on τ,n, but independent of Ω, and a sequence $\{x_k\}_k$ such that if we denote $r(x_k)$ by r_k, then*

(i) $B(x_k,r_k/4)$ *are pairwise disjoint;*

(ii) $\cup_k B(x_k,r_k)=\Omega$*;*

(iii) for every given k, $B(x_k,15r_k)\subset\Omega$;

(iv) for every given k, $x\in B(x_k,15r_k)$ implies that $15r_k<d(x)<45r_k$;

(v) for every given k, there exists a $y_k\notin\Omega$ such that $d(x_k,y_k)<45r_k$;

(vi) $\{B(x_k,13\tau^2r_k)\}_{k=1}^\infty$ have bounded overlap, that is, for every given k, the number of balls $B(x_i,13\tau^2r_i)$ whose intersections with the ball $B(x_k,13\tau^2r_k)$ are non-empty is at most L.

The following lemma is the *partition of unity* on space of homogeneous type $\mathbb{X}$ (see [17, Lemma 2.3]).

Lemma 2.3. *Let Ω be an open subset of $\mathbb{X}$ with finite measure. Consider the sequence $\{x_k\}_k$ and $\{r_k\}_k$ given in Lemma 2.2. Then there exist non-negative functions $\{\varphi_k\}_k$ satisfying:*

(i) for any given k, $0 \le \varphi_k \le 1$*, supp* $\varphi_k \subset B(x_k, 2r_k)$ *and* $\sum_k \varphi_k = \chi_\Omega$*;*

(ii) for any given k and $x \in B(x_k, r_k)$*,* $\varphi_k(x) \ge 1/C$*, where C is a positive constant independent of* Ω*;*

(iii) there exists a positive constant C independent of Ω *such that for all k and all* $\vartheta \in (0, 1]$*,* $\|\varphi_k\|_{\mathcal{G}(x_k, r_k, \vartheta, \vartheta)} \le C\mu(B(x_k, r_k))$*.*

In this case, we say that $\{\varphi_k\}_k$ *are "bump functions" associated with* $\{B_k\}_k$*.*

The following lemma can be proved as in the classical case, see [14, 7].

Lemma 2.4. *Suppose* $\omega \in A_\infty(\mathbb{X})$ *and* $q > q_\omega$*. Then there exists* $0 < \delta < \infty$ *such that for all balls B and all measurable subsets A of B,*

$$\left(\frac{|A|}{|B|}\right)^q \lesssim \frac{w(A)}{w(B)} \lesssim \left(\frac{|A|}{|B|}\right)^\delta.$$

3 Proof of Theorems 1.6

For $k \in \mathbb{Z}$, we set $\Omega_k = \{x \in \mathbb{X} : f^*(x) > 2^k\}$. Then for any $k \in \mathbb{Z}$, Ω_k is a proper open subset of $\mathbb{X}$ with $\omega(\Omega_k) \le C2^{-kp}\|f\|^p_{H^{p,\infty}_\omega(\mathbb{X})} < \infty$. Let $\{B_i^k\}_{i=1}^\infty = \{B(x_i^k, r_i^k)\}_{i=1}^\infty$ be the Whitney decomposition of Ω_k, and let φ_i^k be the "bump functions" associated to B_i^k in the sense of Lemmas 2.2 and 2.3. For each $k \in \mathbb{Z}$, define $d_k(x) = \inf\{d(x, y) : \ y \notin \Omega_k\}$. Denote $m_i^k = \frac{1}{\int_{\mathbb{X}} \varphi_i^k} \int_{\mathbb{X}} f\varphi_i^k$. We decompose f as

$$f(x) = \left(f(x)\chi_{\Omega_k^c}(x) + \sum_{i=1}^\infty m_i^k \varphi_i^k(x)\right) + \sum_{i=1}^\infty (f(x) - m_i^k)\varphi_i^k(x),$$

where and in what follows, we use A^c to denote the complement of the set A in $\mathbb{X}$. Denote

$$g_k(x) \equiv \left(f(x)\chi_{\Omega_k^c}(x) + \sum_{i=1}^\infty m_i^k \varphi_i^k(x)\right).$$

Clearly,

$$|f(x)\chi_{\Omega_k^c}(x)| \le Cf^*(x)\chi_{\Omega_k^c}(x) \le C2^k. \tag{3.1}$$

By (v) in Lemma 2.2, there exist $y_k \in \Omega_k^c$ such that

$$|m_i^k| \le Cf^*(y_k) \le C2^k. \tag{3.2}$$

Thus $|g_k(x)| \le C2^k$ for all $x \in \mathbb{X}$. Therefore, we have the uniform convergence

$$\lim_{k \to -\infty} g_k(x) = 0. \tag{3.3}$$

On the other hand, noticing that $\mu(\Omega_k) = O(2^{-kp}) \to 0$, as $k \to \infty$, we obtain

$$\lim_{k \to \infty} g_k(x) = f(x), \ a.e.. \tag{3.4}$$

By (3.3) and (3.4), we can write $f = \sum_{k=-\infty}^{\infty} g_{k+1} - g_k \equiv \sum_{k=-\infty}^{\infty} f_k$, a.e.. One can check

$$f_k = \sum_{i=1}^{\infty} [(f - m_i^k)\varphi_i^k - \sum_{j=1}^{\infty} (f - m_{ij}^{k+1})\varphi_i^k \varphi_j^{k+1}] + \sum_{j=1}^{\infty} [\sum_{i=1}^{\infty} (f - m_{ij}^{k+1})\varphi_i^k \varphi_j^{k+1} - (f - m_j^{k+1})\varphi_j^{k+1}],$$

where all the series converges in $(\mathcal{G}_0^{\vartheta}(\beta,\gamma))'$ and $m_{ij}^{k+1} = \frac{1}{\int \varphi_i^k \varphi_j^{k+1}} \int f \varphi_i^k \varphi_j^{k+1}$. Let $\beta_i^k = (f - m_i^k)\varphi_i^k - \sum_{j=1}^{\infty} (f - m_{ij}^{k+1})\varphi_i^k \varphi_j^{k+1}$ and $\gamma_j^{k+1} = \sum_{i=1}^{\infty} (f - m_{ij}^{k+1})\varphi_i^k \varphi_j^{k+1} - (f - m_j^{k+1})\varphi_j^{k+1}$. Denote $\widetilde{B}_i^k \equiv B(x_i^k, 13\tau^2 r_i^k)$, where τ is the constant appearing in the triangle inequality (1.1). Then by Lemma 2.2 (vi), we know that, for each $k \in \mathbb{Z}$, $\{\widetilde{B}_i^k\}_i$ has bounded overlap. Clearly, $\operatorname{supp}\beta_i^k \subset B(x_i^k, 2r_i^k) \subset \widetilde{B}_i^k$. Now we claim that for each $j \in \mathbb{Z}$, there exists an $i \in \mathbb{Z}$ such that $\operatorname{supp}\gamma_j^k \subset \widetilde{B}_i^k$. Indeed, $B(x_j^{k+1}, 2r_j^{k+1}) \subset \Omega_{k+1} \subset \Omega_k = \bigcup_{k=1}^{\infty} B(x_i^k, r_i^k)$. Thus there exists $B(x_i^k, r_i^k) = B(x_{i_j}^k, r_{i_j}^k)$ such that $B(x_i^k, r_i^k) \cap B(x_j^{k+1}, 2r_j^{k+1}) \neq \emptyset$. Then for any $x \in B(x_j^{k+1}, 2r_j^{k+1})$ and any $y \in B(x_i^k, r_i^k) \cap B(x_j^{k+1}, 2r_j^{k+1})$, by Lemma 2.2 (iv) and $d_{k+1}(y) \le d_k(y)$,

$$d(x, x_i^k) \le \tau^2 [d(x, x_j^{k+1}) + d(x_j^{k+1}, y) + d(y, x_i^k)] \le \tau^2 [(4/15)d_k(y) + r_i^k] \le 13\tau^2 r_i^k.$$

Therefore $\operatorname{supp}\gamma_j^k \subset B(x_j^{k+1}, 2r_j^{k+1}) \subset \widetilde{B}_i^k$, which verifies the claim. Denote $\widetilde{\gamma}_i^k = \gamma_j^k$ so that $\operatorname{supp}\widetilde{\gamma}_i^k \subset \widetilde{B}_i^k$.

Next, by (3.1), (3.2) and noticing that $\{\widetilde{B}_j^{k+1}\}_{j=1}^{\infty}$ have bounded overlap, we have

$$|\beta_i^k| = |(f - m_i^k)\varphi_i^k - \sum_{j=1}^{\infty} (f - m_{ij}^{k+1})\varphi_i^k \varphi_j^{k+1}| \le |f \varphi_i^k \chi_{\Omega_{k+1}^c}| + |m_i^k| \varphi_i^k + \sum_{j=1}^{\infty} |m_{ij}^{k+1}| \varphi_i^k \varphi_j^{k+1} \le C 2^k.$$

Similarly, $|\widetilde{\gamma}_j^k| \le C 2^k$. Obviously, $\int_{\mathbb{X}} \beta_i^k(x)dx = 0 = \int_{\mathbb{X}} \widetilde{\gamma}_i^k(x)dx$. Define $h_i^k = \beta_i^k + \widetilde{\gamma}_i^k$, then $f_k = \sum_{i=1}^{\infty} h_i^k$ and the convergence in $(\mathcal{G}_0^{\vartheta}(\beta,\gamma))'$ can be verified as in [2].

Finally, since $f \in H_\omega^{p,\infty}$ and $\{B_i^k\}$ have the bounded overlap, by (1.2),

$$\sum_{i=1}^{\infty} \omega(\widetilde{B}_i^k) \lesssim \sum_{i=1}^{\infty} \omega(B_i^k) \lesssim \omega(\Omega_k) \lesssim 2^{-kp} \|f\|_{H_\omega^{p,\infty}(\mathbb{X})}^p,$$

which verifies (iii) of (b). Thus we finish the construction of the atomic decomposition.

For the converse, we fix $\alpha > 0$, and choose k_0 so that $2^{k_0} \le \alpha < 2^{k_0+1}$. Write

$$f = \sum_{k=-\infty}^{k_0-1} f_k + \sum_{k=k_0}^{\infty} f_k = F_1 + F_2.$$

Now since

$$\mathcal{M}_0(F_1)(x) \le \sum_{k=-\infty}^{k_0-1} \mathcal{M}_0(f_k)(x) \le C \sum_{k=-\infty}^{k_0-1} 2^k \le C_3 \alpha,$$

and $\omega(\{x \in \mathbb{X} : \mathcal{M}_0(F_1)(x) > C_3\alpha\}) = 0$, we have

$$\omega(\{x \in \mathbb{X} : \mathcal{M}_0(f)(x) > (C_3+1)\alpha\}) \le \omega(\{x \in \mathbb{X} : \mathcal{M}_0(F_2)(x) > \alpha\}).$$

Set $A_{k_0} = \bigcup_{k=k_0}^{\infty} \bigcup_{i\ge1} 3\tau B_i^k$, where $3\tau B_i^k$ denotes the ball centered at x_i^k with radius $3r_i^k$. By (1.2), $\omega(A_{k_0}) \le C_0(3\tau)^D 2^{-k_0} \le C/\alpha^p$. Therefore it suffices to verify

$$I = \omega(\{x \notin A_{k_0} : \mathcal{M}_0(F_2)(x) > \alpha\}) \le C/\alpha^p. \tag{3.5}$$

Note that for $x \notin 3\tau B_i^k$ and $y \in B_i^k$, $d(x,y) \ge \frac{1}{\tau} d(x,x_i^k) - d(y,x_i^k) \ge 2d(y,x_i^k)$. Hence by the cancellation condition of h_{i_k},

$$\begin{aligned}\mathcal{M}_0(h_i^k)(x) =& \sup_j \left| \int [S_j(x,y) - S_j(x,x_i^k)] h_i^k(y) dy \right| \\ \le& C2^k \frac{\mu(B_i^k) d(y,x_i^k)^{\vartheta}}{V(x,y) d(x,y)^{\vartheta}} \le C2^k \frac{\mu(B_i^k)(r_i^k)^{\vartheta}}{\mu(B(x_i^k, d(x,x_i^k))) d(x,x_i^k)^{\vartheta}}.\end{aligned}$$

By (1.4),

$$\mu(B(x_i^k, d(x,x_i^k))) \lesssim \left(\frac{d(x,x_i^k)}{r_i^k} \right)^D \mu(B_i^k).$$

Then by Lemma 2.4, for $q \in (q_\omega, p(1+\frac{\vartheta}{D}))$

$$\mathcal{M}_0(h_i^k)(x) \lesssim 2^k \frac{\mu(B_i^k)^{1+\frac{\vartheta}{D}}}{V(x,x_i^k)^{1+\frac{\vartheta}{D}}} \lesssim 2^k \frac{\omega(B_i^k)^{(1+\frac{\vartheta}{D})/q}}{W(x,x_i^k)^{(1+\frac{\vartheta}{D})/q}}.$$

Now applying lemma 2.1 with $g_{ki} = W(x,x_i^k)^{-(1+\frac{\vartheta}{D})/q}$, $r = [(1+\frac{\vartheta}{D})/q]^{-1}$, and $c_{ki} = 2^k \cdot \omega(B_i^k)^{(1+\frac{\vartheta}{D})/q}$, we obtain

$$I \lesssim \frac{1}{\alpha^r} \sum_{k\ge k_0} \sum_i 2^{kr} \omega(B_i^k) \lesssim \frac{1}{\alpha^r} \sum_{k \ge k_0} 2^{-k(p-r)}.$$

Now since $p > r$, the last series converges and bounded by $C_0 \frac{1}{\alpha^r} 2^{-k_0(p-r)} = C/\alpha^p$, where C is independent of α. This proves (3.5) and hence Theorem 1.6 follows. □

4 Proof of Theorem 1.7

For every $f \in H_\omega^{p,\infty}(\mathbb{X})$ and $\lambda > 0$, we need to prove that

$$\omega(\{x \in \mathbb{X} : (Tf)^*(x) > \lambda\}) \lesssim C\lambda^{-p} \|f\|^p_{H_\omega^{p,\infty}(\mathbb{X})}.$$

Pick $k_0 \in \mathbb{Z}$ such that $2^{k_0} \le \lambda < 2^{k_0+1}$. By the atomic decomposition of $H_\omega^{p,\infty}(\mathbb{X})$, write f as $f = \sum_{k=-\infty}^{k_0} f_k + \sum_{k=k_0+1}^{\infty} f_k \equiv F_1 + F_2$. Noticing that $p_0 > 1$, we have

$$\begin{aligned}\|F_1\|_{L_\omega^{p_0}(\mathbb{X})} \le& C \sum_{k=-\infty}^{k_0} \|f_k\|_{L_\omega^{p_0}(\mathbb{X})} \le C \sum_{k=-\infty}^{k_0} 2^k \left(\sum_i \omega(B_i^k) \right)^{1/p_0} \\ \le& C\|f\|^{p/p_0}_{H_\omega^{p,\infty}(\mathbb{X})} \sum_{k=-\infty}^{k_0} 2^{k(1-p/p_0)} \le C\|f\|^{p/p_0}_{H_\omega^{p,\infty}(\mathbb{X})} 2^{k_0(1-p/p_0)}.\end{aligned}$$

This together with the $L^{p_0}_\omega(\mathbb{X})$ boundedness of grand maximal operator and T yields

$$\begin{aligned}\omega(\{x\in\mathbb{X}:(TF_1)^*(x)>\lambda\}) &\le \lambda^{-p_0}\|(TF_1)^*\|^{p_0}_{L^{p_0}_\omega(\mathbb{X})} \le C\lambda^{-p_0}\|TF_1\|^{p_0}_{L^{p_0}_\omega(\mathbb{X})}\\ &\le C\lambda^{-p_0}\|F_1\|^{p_0}_{L^{p_0}_\omega(\mathbb{X})} \le C\lambda^{-p_0}\|f\|^p_{H^{p,\infty}_\omega(\mathbb{X})}2^{k_0(p_0-p)}\\ &\le C\lambda^{-p}\|f\|^p_{H^{p,\infty}_\omega(\mathbb{X})}.\end{aligned}$$

Thus, to finish the proof of Theorem 1.7, it suffices to show that

$$\omega(\{x\in\mathbb{X}:(TF_2)^*(x)>\lambda\}) \le C\lambda^{-p}\|f\|^p_{H^{p,\infty}_\omega(\mathbb{X})}. \tag{4.1}$$

It is easy to see that for some constant C, $C^{-1}2^{-k}\omega(B^k_i)^{-1/q}h^k_i$ is an $H^{q,\infty}_\omega$ atom (see [1]). Then $f_k\in H^q_\omega(\mathbb{X})$ and

$$\|f_k\|^q_{H^q_\omega(\mathbb{X})} \le C\sum_i 2^{kq}\omega(B^k_i) \le C2^{k(q-p)}\|f\|^p_{H^{p,\infty}_\omega(\mathbb{X})}.$$

Since T is bounded on $H^q_\omega(\mathbb{X})$,

$$\omega(\{x\in\mathbb{X}:(Tf_k)^*(x)>\lambda\}) \le C\lambda^{-q}\|Tf_k\|^q_{H^q_\omega(\mathbb{X})} \le C\lambda^{-q}\|f_k\|^q_{H^q_\omega(\mathbb{X})}.$$

Consequently,

$$\omega(\{x\in\mathbb{X}:\left[T(f_k/\|f_k\|_{H^q_\omega(\mathbb{X})})\right]^*(x)>\lambda\}) \le C\lambda^{-q}.$$

Noting that $(TF_2)^*(x)\le\sum_{k=k_0+1}^\infty (Tf_k)^*(x)$. Then applying Lemma 2.1, we obtain

$$\begin{aligned}&\omega(\{x\in\mathbb{X}:(TF_2)^*(x)>\lambda\})\\ &\le \omega(\{x\in\mathbb{X}:\sum_{k=k_0+1}^\infty \|f_k\|_{H^q_\omega(\mathbb{X})}\cdot[T(f_k/\|f_k\|_{H^q_\omega(\mathbb{X})})]^*(x)>\lambda\}) \le \frac{2-q}{1-q}\frac{1}{\lambda^q}\sum_{k=k_0+1}^\infty \|f_k\|^q_{H^q_\omega(\mathbb{X})}\\ &\le \frac{C\|f\|^p_{H^{p,\infty}_\omega(\mathbb{X})}}{\lambda^q}\sum_{k=k_0}^\infty 2^{k(q-p)} \le C2^{k_0(q-p)}\|f\|^p_{H^{p,\infty}_\omega(\mathbb{X})}/\lambda^q \le C\lambda^{-p}\|f\|^p_{H^{p,\infty}_\omega(\mathbb{X})},\end{aligned}$$

which verifies (4.1). This completes the proof of Theorem 1.7. □

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