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Fixed Point Theorems on a Metric Space Endowed With an Arbitrary Binary Relation and Applications

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Abstract

In this paper, we establish fixed point theorems for contractive mappings on a metric space endowed with an amorphous binary relation. The presented theorems extend and generalize many existing results on metric and ordered metric spaces. We apply also our main result to derive fixed point theorems for cyclical contractive mappings.

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1 Introduction and preliminaries

A fundamental result in fixed point theory is the Banach contraction principle, asserting that, if (X,d) is a complete metric space, then any map $T : X \to X$ satisfying

$$d(Tx,Ty) \le kd(x,y),$$

for all $x,y \in X$, where $k \in [0,1)$ is a constant, has a unique fixed point.

Recently, there have been so many exciting developments in the field of existence of fixed point in partially ordered metric spaces. For more details, we refer the reader to the

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papers by Turinici [20], Ran and Reurings [16], Nieto and López [13], Agarwal et al.[1], Ćirić et al.[7], Harjani and Sadarangani [9], Jachymski [10], Bhaskar and Lakshmikantham [3], Samet et al.[18, 19], and the references therein.

In this work, some fixed point theorems for contractive mappings are established on metric spaces endowed with arbitrary binary relations. The presented theorems extend and generalize many existing results obtained on metric and ordered metric spaces. Moreover, we will show that some fixed point theorems for cyclical contractive mappings can be deduced from our main result.

Before presenting our main result, we need a few preliminaries.

Let (X,d) be a metric space, and $\mathcal{R}$ be a binary relation over X. Denote $\mathcal{S} = \mathcal{R} \cup \mathcal{R}^{-1}$; this is the symmetric relation attached to $\mathcal{R}$. Clearly,

$$x, y \in X, \quad x\mathcal{S}y \Longleftrightarrow x\mathcal{R}y \text{ or } y\mathcal{R}x.$$

Definition 1.1. We say that the subset D of X is $\mathcal{S}$-directed if for every $x, y \in D$, there exists $z \in X$ such that $x\mathcal{S}z$ and $y\mathcal{S}z$.

Definition 1.2. We say that $(X, d, \mathcal{S})$ is regular if the following condition holds: if the sequence $\{x_n\}$ in X and the point $x \in X$ are such that

$$x_n\mathcal{S}x_{n+1} \text{ for all } n \text{ and } \lim_{n\to\infty} d(x_n, x) = 0, \tag{1.1}$$

then there exists a subsequence $\{x_{n(p)}\}$ of $\{x_n\}$ such that $x_{n(p)}\mathcal{S}x$ for all p.

Definition 1.3. We say that $T : X \to X$ is a comparative mapping if T maps comparable elements into comparable elements, that is,

$$x, y \in X, \quad x\mathcal{S}y \Longrightarrow Tx\mathcal{S}Ty.$$

Let Φ be the set of functions $\varphi : [0,\infty) \to [0,\infty)$ satisfying:

(P_1) φ is nondecreasing;

(P_2) $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for each $t > 0$, where φ^n is the n-th iterate of φ.

Lemma 1.4. *Let $\varphi \in \Phi$. We have $\varphi(t) < t$ for all $t > 0$.*

Proof. Let $\varphi \in \Phi$. From property (P_2), we have

$$\lim_{n\to\infty} \varphi^n(t) = 0, \quad \text{for all } t > 0. \tag{1.2}$$

Suppose that there exists $t > 0$ such that $\varphi(t) \geq t$. Since from property (P_1), φ is nondecreasing, we obtain that

$$\varphi^n(t) \geq t, \quad \text{for all } n \geq 1.$$

Letting $n \to \infty$ in the above inequality, it follows from (1.2) that $t = 0$, which is a contradiction. Then, for all $t > 0$, we have $\varphi(t) < t$. □

Let $T : X \to X$ be a mapping. Denote, for $x, y \in X$,

$$M_T(x,y) = \max\left\{d(x,y), \frac{1}{2}\left[d(x,Tx)+d(y,Ty)\right], \frac{1}{2}\left[d(x,Ty)+d(y,Tx)\right]\right\}.$$

Lemma 1.5. *We have, for each $x \in X$,*

$$M_T(x,Tx) \le \max\left\{d(x,Tx), d(Tx,T^2x)\right\}.$$

Proof. Let $x \in X$ be arbitrary fixed. We have, successively,

$$\begin{aligned}
&M_T(x,Tx) = \max\left\{d(x,Tx), \tfrac{1}{2}\left[d(x,Tx)+d(Tx,T^2x)\right], \tfrac{1}{2}\left[d(x,T^2x)+d(Tx,Tx)\right]\right\}\\
&\le \max\left\{d(x,Tx), \tfrac{1}{2}\left[d(x,Tx)+d(Tx,T^2x)\right], \tfrac{1}{2}\left[d(x,Tx)+d(Tx,T^2x)\right]\right\}\\
&\le \max\left\{d(x,Tx), \max\left\{d(x,Tx), d(Tx,T^2x)\right\}\right\} = \max\left\{d(x,Tx), d(Tx,T^2x)\right\}.
\end{aligned}$$

Hence the conclusion. □

2 Main result

Let (X,d) be a complete metric space, and $T : X \to X$ be a mapping. Denote

$$\text{Fix}(T) = \{z \in X : z = Tz\}.$$

Our main result is the following.

Theorem 2.1. *Assume that T is a comparative map, and*

$$x, y \in X, x\mathcal{S}y \Longrightarrow d(Tx,Ty) \le \varphi(M_T(x,y)), \tag{2.1}$$

where $\varphi \in \Phi$. Suppose also that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0\mathcal{S}Tx_0$;

(ii) $(X,d,\mathcal{S})$ is regular.

Then T has a fixed point $x^* \in X$. Moreover, if in addition, $D :=$ Fix(T) is $\mathcal{S}$-directed, then x^* is the unique fixed point of T in X.

Proof. From condition (i), there exists $x_0 \in X$ such that $x_0\mathcal{S}Tx_0$. Define the sequence $\{x_n\}$ in X by:

$$x_{n+1} = Tx_n, \text{ for all } n \ge 0.$$

It follows from the property of T that $\{x_n\}$ is a sequence in X whose consecutive terms are comparable ($x_n\mathcal{S}x_{n+1}$ for all n). Now, applying (2.1), for all $n \ge 1$, we have

$$d(x_{n+1},x_n) = d(Tx_n,Tx_{n-1}) \le \varphi(M_T(x_n,x_{n-1})). \tag{2.2}$$

On the other hand, for all $n \geq 1$, we have, by Lemma 1.5

$$M_T(x_n, x_{n-1}) \leq \max\Big\{d(x_n, x_{n-1}), d(x_{n+1}, x_n)\Big\}.$$

Thus, from (2.2) and (P_1), for all $n \geq 1$, we have

$$d(x_{n+1}, x_n) \leq \varphi\Big(\max\Big\{d(x_n, x_{n-1}), d(x_{n+1}, x_n)\Big\}\Big). \tag{2.3}$$

Now, we shall prove that $\{x_n\}$ is a Cauchy sequence in (X,d). If for some $n \geq 0$, we have $x_n = x_{n+1}$, the result follows immediately. So, we can suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. Suppose that for some $n \geq 1$, we have $d(x_n, x_{n-1}) \leq d(x_{n+1}, x_n)$. From (2.3) and using Lemma 1.4, we get

$$d(x_{n+1}, x_n) \leq \varphi\Big(d(x_{n+1}, x_n)\Big) < d(x_{n+1}, x_n),$$

which is a contradiction. Thus, for all $n \geq 1$, we have $d(x_n, x_{n-1}) > d(x_{n+1}, x_n)$. Then, from (2.3), for all $n \geq 1$, we have

$$d(x_{n+1}, x_n) \leq \varphi\Big(d(x_n, x_{n-1})\Big).$$

Using (P_1), by induction, we get

$$d(x_{n+1}, x_n) \leq \varphi^n\Big(d(x_1, x_0)\Big), \text{ for all } n \geq 0. \tag{2.4}$$

Fix $\varepsilon > 0$ and let $h = h(\varepsilon)$ be a positive integer (given by (P_2)) such that

$$\sum_{n \geq h} \varphi^n\Big(d(x_1, x_0)\Big) < \varepsilon.$$

Let $m > n > h$; using the triangular inequality and (2.4), we obtain

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \varphi^k\Big(d(x_1, x_0)\Big) \leq \sum_{n \geq h} \varphi^n\Big(d(x_1, x_0)\Big) < \varepsilon.$$

Thus we proved that $\{x_n\}$ is a Cauchy sequence in the metric space (X,d).

Since the metric space (X,d) is complete, there exists $x^* \in X$ such that

$$\lim_{n \to \infty} d(x_n, x^*) = 0. \tag{2.5}$$

From condition (iii), there exists a subsequence $\{x_{n(p)}\}$ of $\{x_n\}$ such that $x_{n(p)}$ and x^* are comparable for all p. Using again (2.1), for all p, we have

$$d(x_{n(p)+1}, Tx^*) = d(Tx_{n(p)}, Tx^*) \leq \varphi\Big(M_T(x_{n(p)}, x^*)\Big). \tag{2.6}$$

Denote $\rho = d(x^*, Tx^*)$. For all p, we have

$$M_T(x_{n(p)}, x^*) =$$
$$\max\Big\{d(x_{n(p)}, x^*), \frac{1}{2}\Big[d(x_{n(p)}, x_{n(p)+1}) + \rho\Big], \frac{1}{2}\Big[d(x_{n(p)}, Tx^*) + d(x^*, x_{n(p)+1})\Big]\Big\}.$$

Letting $p \to \infty$ and using (2.5), we get

$$\lim_{p\to\infty} M_T(x_{n(p)}, x^*) = \frac{1}{2}\rho. \tag{2.7}$$

Suppose that $\rho > 0$. By definition, there exists $k = k(\rho)$ such that

$$M_T(x_{n(p)}, x^*) \le \frac{3}{4}\rho,\ \forall p \ge k.$$

Replacing into (2.6) yields (as φ is nondecreasing)

$$d(x_{n(p)+1}, Tx^*) \le \varphi\Big(\frac{3}{4}\rho\Big),\ \forall p \ge k.$$

So, passing to limit as p tends to infinity, one derives from Lemma 1.4,

$$\rho \le \varphi\Big(\frac{3}{4}\rho\Big) < \frac{3}{4}\rho < \rho,$$

which is a contradiction. Then, $\rho = 0$, i.e., $d(x^*, Tx^*) = 0$, hence $x^* \in X$ is a fixed point of T.

Now, suppose that $D := \mathrm{Fix}(T)$ is $\mathcal{S}$-directed. We shall prove that x^* is the unique fixed point of T in X. Suppose that $y^* \in D$ is another fixed point of T. Then, there exists $z \in X$ such that $x^*\mathcal{S}z$ and $y^*\mathcal{S}z$. Define the sequence $\{z_n\}$ in X by $z_0 = z$ and $z_{n+1} = Tz_n$ for all $n \ge 0$. From the property of T, for all $n \ge 0$, we have $x^*\mathcal{S}z_n$ and $y^*\mathcal{S}z_n$. Applying (2.1), for all $n \ge 0$, we have

$$d(z_{n+1}, x^*) = d(Tz_n, Tx^*) \le \varphi\Big(M_T(z_n, x^*)\Big). \tag{2.8}$$

On the other hand, we have

$$\begin{aligned}
M_T(z_n, x^*) &= \max\Big\{d(z_n, x^*), \frac{1}{2}[d(z_n, z_{n+1}) + d(x^*, x^*)], \frac{1}{2}[d(z_n, x^*) + d(x^*, z_{n+1})]\Big\}\\
&= \max\Big\{d(z_n, x^*), \frac{1}{2}d(z_n, z_{n+1}), \frac{1}{2}[d(z_n, x^*) + d(x^*, z_{n+1})]\Big\}\\
&\le \max\Big\{d(z_n, x^*), \frac{1}{2}[d(z_n, x^*) + d(x^*, z_{n+1})], \frac{1}{2}[d(z_n, x^*) + d(x^*, z_{n+1})]\Big\}\\
&= \max\Big\{d(z_n, x^*), \frac{1}{2}[d(z_n, x^*) + d(x^*, z_{n+1})]\Big\}\\
&\le \max\Big\{d(z_n, x^*), d(x^*, z_{n+1})\Big\}.
\end{aligned}$$

Using (2.8) and (P_1), for all $n \ge 0$, we obtain that

$$d(z_{n+1}, x^*) \le \varphi\Big(\max\Big\{d(z_n, x^*), d(x^*, z_{n+1})\Big\}\Big). \tag{2.9}$$

Now, we shall prove that

$$\lim_{n\to\infty} d(z_n, x^*) = 0. \tag{2.10}$$

Without restriction to the generality, we can suppose that $d(z_n, x^*) > 0$ for all n. Suppose that for some n, we have $d(z_n, x^*) \leq d(x^*, z_{n+1})$. From (2.9) and using Lemma 1.4, we get

$$d(z_{n+1}, x^*) \leq \varphi\Big(d(x^*, z_{n+1})\Big) < d(x^*, z_{n+1}),$$

which is a contradiction. Then, for all $n \geq 0$, we have $d(z_n, x^*) > d(x^*, z_{n+1})$. Using this information in (2.9), we get

$$d(z_{n+1}, x^*) \leq \varphi(d(z_n, x^*)), \text{ for all } n.$$

By induction we then derive

$$d(z_n, x^*) \leq \varphi^n(d(z_0, x^*)), \text{ for all } n.$$

This, along with (1.2) proves (2.10). Similarly, we can prove that

$$\lim_{n\to\infty} d(z_n, y^*) = 0. \tag{2.11}$$

It follows from (2.10) and (2.11) that $x^* = y^*$. This makes end to the proof. □

The following example shows that continuity of T is not needed in Theorem 2.1.

Example 2.2. Let $X = [0, \infty)$ and d be the standard metric $d(x,y) = |x-y|$, $x, y \in X$. Note that (X, d) is a complete metric space. Put $C = [0,1]$ and define the binary relation $\mathcal{R}$ over X by $\mathcal{R} = C \times C$, that is,

$$x, y \in X, \quad x\mathcal{R}y \iff (x,y) \in C \times C.$$

Clearly, $\mathcal{R} = \mathcal{S}$. We claim that $(X, d, \mathcal{S})$ is regular. Let $\{x_n\}$ be a sequence in X and $x \in X$ such that (1.1) is satisfied. This implies that $x_n \in C$ for all n. Since C is closed, we have $x \in C$. Then, we have $x_n\mathcal{S}x$ for all n. This proves our claim. Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} \dfrac{x}{2} & \text{if } \quad 0 \leq x \leq 1, \\ \\ 2x & \text{if } \quad x > 1. \end{cases}$$

Since $T([0,1]) \subset [0,1]$, it follows that T is a comparative mapping. Moreover, taking $x_0 = 1$, we have $x_0\mathcal{R}Tx_0$. On the other hand, for all $x, y \in X$ with $x\mathcal{S}y$, we have

$$d(Tx, Ty) \leq \varphi(M_T(x,y)),$$

where $\varphi(t) = \dfrac{t}{2}$ for all $t \geq 0$. By the first half of Theorem 2.1, Fix(T) is not empty. Moreover, by definition, Fix$(T) \subseteq C$, and C is $\mathcal{S}$-directed. So, by the second part of Theorem 2.1, Fix(T) is a singleton, $\{x^*\}$; in fact, $x^* = 0$. Note that in this case, the mapping T is not continuous. Also, the binary relation $\mathcal{R}$ is not a partial order on X.

Example 2.3. Let $X = \{0,1,2,3\}$ and the Euclidean metric $d(x,y) = |x-y|$, $\forall x,y \in X$. Let $\mathcal{R}$ be the binary relation over X given by

$$\mathcal{R} = \{(0,1),(0,2),(2,3),(0,0),(1,1),(2,2),(3,3)\}.$$

We claim that $(X,d,\mathcal{S})$ is regular, where $\mathcal{S} = \mathcal{R} \cup \mathcal{R}^{-1}$. Indeed, if $\{x_n\}$ is a sequence in X such that $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(p)}\}$ of $\{x_n\}$ such that $x_{n(p)} = x$ for all p. This implies that $(x_{n(p)}, x) = (x,x) \in \mathcal{R}$ for all p. Thus we have $x_{n(p)}\mathcal{S}x$ for all p. This proves our claim. Now, consider the mapping $T : X \to X$ defined by

$$Tx = \begin{cases} 0 & \text{if} \quad x \in \{0,1\}, \\ 1 & \text{if} \quad x \in \{2,3\}. \end{cases}$$

It is easy to show that T is a comparative map and condition (2.1) is satisfied with $\varphi(t) = \dfrac{t}{2}$ for all $t \geq 0$. It follows from the first part of Theorem 2.1 that T has at least one fixed point. On the other hand, by definition, $\mathrm{Fix}(T) \subseteq D := \{0,1\}$, and D is $\mathcal{S}$-directed. So, by the second part of Theorem 2.1, $\mathrm{Fix}(T)$ is a singleton, $\{x^*\}$; in fact, $x^* = 0$.

Now, we will show that for any $\varphi \in \Phi$, the condition:

$$d(Tx,Ty) \leq \varphi(M_T(x,y)), \text{ for all } x,y \in X$$

is not satisfied. Indeed, suppose that there exists some $\varphi \in \Phi$ such that the above inequality holds. Taking $x = 1$ and $y = 2$, we obtain that

$$1 = d(T1,T2) \leq \varphi(M_T(1,2)) = \varphi(1) < 1,$$

which is a contradiction.

We note also that $\mathcal{R}$ is not a partial order on X since $(0,2),(2,3) \in \mathcal{R}$ but $(0,3) \notin \mathcal{R}$.

The following result is an immediate consequence of Theorem 2.1.

Corollary 2.4. *Suppose that T is a comparative map and*

$$x,y \in X, x\mathcal{S}y \Longrightarrow d(Tx,Ty) \leq \varphi(d(x,y)),$$

where $\varphi \in \Phi$. Suppose also that the following conditions hold:

(ii) there exists $x_0 \in X$ such that $x_0\mathcal{S}Tx_0$;

(iii) $(X,d,\mathcal{S})$ is regular.

Then T has a fixed point $x^* \in X$. Moreover, if in addition, $D := \mathrm{Fix}(T)$ is $\mathcal{S}$-directed, then x^* is the unique fixed point of T in X.

Remark 2.5. Taking $\mathcal{R} = X \times X$ in Corollary 2.4, we obtain a similar result to Boyd and Wong [4].

Denote by Λ the set of nondecreasing continuous functions $\varphi : [0,\infty) \to [0,\infty)$ satisfying:

(Q_1) $0 < \varphi(t) < t$, for all $t > 0$;

(Q_2) $g(t) = \dfrac{t}{t-\varphi(t)}$ is a decreasing function on $(0,\infty)$;

(Q_3) $\displaystyle\int_0^r g(t)\,dt < \infty$, for all $r > 0$.

In [2], Altman established the following result.

Lemma 2.6. *We have* $\Lambda \subset \Phi$.

An immediate consequence from Theorem 2.1 and Lemma 2.6 is the following result.

Corollary 2.7. *Assume that T is a comparative map, and*

$$x,y \in X, x\mathcal{S}y \Longrightarrow d(Tx,Ty) \le \varphi(M_T(x,y)),$$

where $\varphi \in \Lambda$. Suppose also that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0\mathcal{S}Tx_0$;

(ii) $(X,d,\mathcal{S})$ is regular.

Then T has a fixed point $x^* \in X$. Moreover, if in addition, $D :=$ Fix(T) is $\mathcal{S}$-directed, then x^* is the unique fixed point of T in X.

Taking $\varphi(t) = kt$ in Theorem 2.1, where $k \in (0,1)$ is a constant, we obtain the following result.

Corollary 2.8. *Let T be a comparative map, and*

$$d(Tx,Ty) \le k\max\left\{d(x,y), \frac{1}{2}\left[d(x,Tx)+d(y,Ty)\right], \frac{1}{2}\left[d(x,Ty)+d(y,Tx)\right]\right\},$$

for all $x,y \in X$ with $x\mathcal{S}y$, where $k \in (0,1)$ is a constant. Suppose also that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0\mathcal{S}Tx_0$;

(ii) $(X,d,\mathcal{S})$ is regular.

Then T has a fixed point $x^* \in X$. Moreover, if in addition, $D :=$ Fix(T) is $\mathcal{S}$-directed, then x^* is the unique fixed point of T in X.

Remark 2.9. Taking $\mathcal{R} = X \times X$ in Corollary 2.8, we obtain Ćirić's fixed point theorem [6].

The following results follow immediately from Corollary 2.8.

Corollary 2.10. *Let T be a comparative map, and*

$$d(Tx,Ty) \le ad(x,y) + b\left[d(x,Tx)+d(y,Ty)\right] + c\left[d(x,Ty)+d(y,Tx)\right],$$

for all $x,y \in X$ with $x\mathcal{S}y$, where $a,b,c > 0$ and $a+2b+2c < 1$. Suppose also that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \mathcal{S} T x_0$;

(ii) $(X,d,\mathcal{S})$ is regular.

Then T has a fixed point $x^* \in X$. Moreover, if in addition, $D := \mathrm{Fix}(T)$ is $\mathcal{S}$-directed, then x^* is the unique fixed point of T in X.

Remark 2.11. Taking $\mathcal{R} = X \times X$ in Corollary 2.10, we obtain the Hardy-Rogers fixed point theorem [8].

Corollary 2.12. *Let T be a comparative map, and*

$$d(Tx,Ty) \leq k\,d(x,y),$$

for all $x,y \in X$ with $x\mathcal{S}y$, where $k \in (0,1)$ is a constant. Suppose also that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \mathcal{S} T x_0$;

(ii) $(X,d,\mathcal{S})$ is regular.

Then T has a fixed point $x^* \in X$. Moreover, if in addition, $D := \mathrm{Fix}(T)$ is $\mathcal{S}$-directed, then x^* is the unique fixed point of T in X.

Remark 2.13. Taking $\mathcal{R} = X \times X$ in Corollary 2.12, we obtain the Banach contraction principle.

Corollary 2.14. *Let T be a comparative map, and*

$$d(Tx,Ty) \leq k\left[d(x,Tx) + d(y,Ty)\right],$$

for all $x,y \in X$ with $x\mathcal{S}y$, where $k \in (0,1/2)$ is a constant. Suppose also that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \mathcal{S} T x_0$;

(ii) $(X,d,\mathcal{S})$ is regular.

Then T has a fixed point $x^* \in X$. Moreover, if in addition, $D := \mathrm{Fix}(T)$ is $\mathcal{S}$-directed, then x^* is the unique fixed point of T in X.

Remark 2.15. Taking $\mathcal{R} = X \times X$ in Corollary 2.14, we obtain Kannan's fixed point theorem [11].

Corollary 2.16. *Let T be comparative, and*

$$d(Tx,Ty) \leq k\left[d(x,Ty) + d(y,Tx)\right],$$

for all $x,y \in X$ with $x\mathcal{S}y$, where $k \in (0,1/2)$ is a constant. Suppose also that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \mathcal{S} T x_0$;

(ii) $(X,d,\mathcal{S})$ is regular.

Then T has a fixed point $x^* \in X$. Moreover, if in addition, $D := \mathrm{Fix}(T)$ is $\mathcal{S}$-directed, then x^* is the unique fixed point of T in X.

Remark 2.17. Taking $\mathcal{R} = X \times X$ in Corollary 2.16, we obtain Chatterjea's fixed point theorem [5].

3 Some consequences

3.1 Fixed point results on ordered metric spaces

Let (X,d) be a complete metric space and $\preceq$ be an order on X. Also, let $T : X \to X$ be a mapping. The following result follows from the preceding ones.

Theorem 3.1. *Assume that T maps comparable elements into comparable elements and that for all $x,y \in X$ with x and y comparable,*

$$d(Tx,Ty) \leq \varphi(M_T(x,y)),$$

where $\varphi \in \Phi$. Suppose also that if $\{x_n\}$ is a sequence in X whose consecutive terms are comparable and $x_n \to x \in X$, then there exists a subsequence $\{x_{n(p)}\}$ of $\{x_n\}$ such that every term is comparable to the limit x. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$ or $x_0 \succeq Tx_0$, then T has a fixed point. Moreover, if for every $x,y \in X$, there exists $z \in X$ which is comparable to x and y, we have uniqueness of the fixed point.

Proof. Let us define the binary relation $\mathcal{R}$ over X by:

$$x,y \in X, \quad x\mathcal{R}y \Longleftrightarrow x \preceq y.$$

Now, the result follows immediately from Theorem 2.1. □

The following results follow immediately from Theorem 3.1.

Corollary 3.2. *Assume that T maps comparable elements into comparable elements and that for all $x,y \in X$ with x and y comparable,*

$$d(Tx,Ty) \leq \varphi(d(x,y)),$$

where $\varphi \in \Phi$. Suppose also that if $\{x_n\}$ is a sequence in X whose consecutive terms are comparable and $x_n \to x \in X$, then there exists a subsequence $\{x_{n(p)}\}$ of $\{x_n\}$ such that every term is comparable to the limit x. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$ or $x_0 \succeq Tx_0$, then T has a fixed point. Moreover, if for every $x,y \in X$, there exists $z \in X$ which is comparable to x and y, we have uniqueness of the fixed point.

Corollary 3.3. *Assume that T maps comparable elements into comparable elements and that for all $x,y \in X$ with x and y comparable,*

$$d(Tx,Ty) \leq k\max\left\{d(x,y), \frac{1}{2}\left[d(x,Tx)+d(y,Ty)\right], \frac{1}{2}\left[d(x,Ty)+d(y,Tx)\right]\right\},$$

where $k \in (0,1)$ is a constant. Suppose also that if $\{x_n\}$ is a sequence in X whose consecutive terms are comparable and $x_n \to x \in X$, then there exists a subsequence $\{x_{n(p)}\}$ of $\{x_n\}$ such that every term is comparable to the limit x. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$ or $x_0 \succeq Tx_0$, then T has a fixed point. Moreover, if for every $x,y \in X$, there exists $z \in X$ which is comparable to x and y, we have uniqueness of the fixed point.

Corollary 3.4. *Assume that T maps comparable elements into comparable elements and that for all $x,y \in X$ with x and y comparable,*

$$d(Tx,Ty) \leq ad(x,y) + b\left[d(x,Tx) + d(y,Ty)\right] + c\left[d(x,Ty) + d(y,Tx)\right],$$

where $a,b,c > 0$ and $a+2b+2c < 1$. Suppose also that if $\{x_n\}$ is a sequence in X whose consecutive terms are comparable and $x_n \to x \in X$, then there exists a subsequence $\{x_{n(p)}\}$ of $\{x_n\}$ such that every term is comparable to the limit x. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$ or $x_0 \succeq Tx_0$, then T has a fixed point. Moreover, if for every $x,y \in X$, there exists $z \in X$ which is comparable to x and y, we have uniqueness of the fixed point.

The following result was obtained by Nieto and López in [13].

Corollary 3.5. *Assume that T maps comparable elements into comparable elements and that for all $x,y \in X$ with x and y comparable,*

$$d(Tx,Ty) \leq k\,d(x,y),$$

where $k \in (0,1)$ is a constant. Suppose also that if $\{x_n\}$ is a sequence in X whose consecutive terms are comparable and $x_n \to x \in X$, then there exists a subsequence $\{x_{n(p)}\}$ of $\{x_n\}$ such that every term is comparable to the limit x. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$ or $x_0 \succeq Tx_0$, then T has a fixed point. Moreover, if for every $x,y \in X$, there exists $z \in X$ which is comparable to x and y, we have uniqueness of the fixed point.

Corollary 3.6. *Assume that T maps comparable elements into comparable elements and that for all $x,y \in X$ with x and y comparable,*

$$d(Tx,Ty) \leq k\left[d(x,Tx) + d(y,Ty)\right],$$

where $k \in (0,1/2)$ is a constant. Suppose also that if $\{x_n\}$ is a sequence in X whose consecutive terms are comparable and $x_n \to x \in X$, then there exists a subsequence $\{x_{n(p)}\}$ of $\{x_n\}$ such that every term is comparable to the limit x. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$ or $x_0 \succeq Tx_0$, then T has a fixed point. Moreover, if for every $x,y \in X$, there exists $z \in X$ which is comparable to x and y, we have uniqueness of the fixed point.

Corollary 3.7. *Assume that T maps comparable elements into comparable elements and that for all $x,y \in X$ with x and y comparable,*

$$d(Tx,Ty) \leq k\left[d(x,Ty) + d(y,Tx)\right],$$

where $k \in (0,1/2)$ is a constant. Suppose also that if $\{x_n\}$ is a sequence in X whose consecutive terms are comparable and $x_n \to x \in X$, then there exists a subsequence $\{x_{n(p)}\}$ of $\{x_n\}$ such that every term is comparable to the limit x. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$ or $x_0 \succeq Tx_0$, then T has a fixed point. Moreover, if for every $x,y \in X$, there exists $z \in X$ which is comparable to x and y, we have uniqueness of the fixed point.

3.2 Fixed point results for cyclical contractive mappings

In [12], Kirk, Srinivasan and Veeramani obtained an extension of Banach's fixed point theorem by considering a cyclical contractive condition, as given by the next theorem.

Theorem 3.8. *Let (M,d) be a complete metric space, A,B are nonempty closed subsets of M and $T : M \to M$ an operator. Suppose that*

(i) $T(A) \subseteq B$, $T(B) \subseteq A$;

(ii) there exists a constant $k \in (0,1)$ such that

$$d(Tx,Ty) \le k\,d(x,y), \text{ for all } x \in A, y \in B.$$

Then $A \cap B \neq \emptyset$ and T has a unique fixed point in $A \cap B$.

In this section, using Theorem 2.1, we obtain some fixed point results for cyclic generalized contractions.

We have the following result.

Theorem 3.9. *Let (M,d) be a complete metric space, A,B are nonempty closed subsets of M and $T : M \to M$ an operator. Suppose that*

(i) $T(A) \subseteq B$, $T(B) \subseteq A$;

(ii) there exists $\varphi \in \Phi$ such that

$$d(Tx,Ty) \le \varphi\Big(M_T(x,y)\Big), \text{ for all } x \in A, y \in B.$$

Then T has a unique fixed point x^* in $X := A \cup B$. Moreover, $x^* \in A \cap B$.

Proof. By (i), T is a selfmap of X. In addition, X is a closed part of M; hence, (X,d) is a complete metric pace. Define a binary relation $\mathcal{R}$ over X by $\mathcal{R} = A \times B$, that is,

$$x,y \in X, \quad x\mathcal{R}y \Longleftrightarrow (x,y) \in A \times B.$$

Its associated symmetric relation $\mathcal{S}$ may be represented as $\mathcal{S} = (A \times B) \cup (B \times A)$; i.e.,

$$x,y \in X, \quad x\mathcal{S}y \Longleftrightarrow (x,y) \in A \times B \text{ or } (x,y) \in B \times A.$$

We shall prove that $(X,d,\mathcal{S})$ is regular. Let $\{x_n\}$ be a sequence in X and the point $x \in X$ be such that

$$x_n\mathcal{S}x_{n+1} \text{ for all } n \text{ and } \lim_{n\to\infty} d(x_n,x) = 0.$$

Denote

$$I = \{n \in N; (x_n,x_{n+1}) \in A \times B\}, \quad J = \{n \in N; (x_n,x_{n+1}) \in B \times A\}.$$

As $I \cup J = N$, at least one of these subsets is infinite. To make a choice, assume that I is infinite; hence, it may be written as a strictly increasing sequence of ranks: $I = \{n(k); k \ge 0\}$, where $k \mapsto n(k)$ is strictly increasing (hence $\lim_{k\to\infty} n(k) = \infty$). Denote $m(k) = n(k)+1$, for $k \geq 0$; it is also a strictly increasing sequence of ranks which tends to infinity. The sequences $(x_{n(k)}; k \geq 0)$ and $(x_{m(k)}; k \geq 0)$ have the properties

$$\lim_{k\to\infty} x_{n(k)} = \lim_{k\to\infty} x_{m(k)} = x$$

$$x_{n(k)} \in A,\ x_{m(k)} \in B, \text{ for all } k \geq 0.$$

As $x \in X$, we must have either $x \in A$ or $x \in B$. In the former case, we have

$$(x_{m(k)}, x) \in B \times A \ \text{ (hence } x_{m(k)} \mathcal{S} x), \ \forall k \geq 0;$$

and, in the latter one,

$$(x_{n(k)}, x) \in A \times B \ \text{ (hence } x_{n(k)} \mathcal{S} x), \ \forall k \geq 0.$$

Hence, in any situation, we get a subsequence fulfilling the property needed in Definition 1.2; and the claim is proved. On the other hand, from (i), for every $x, y \in X$, we have

$$(x,y) \in A \times B \Rightarrow (Tx, Ty) \in B \times A \subseteq \mathcal{S};$$

$$(x,y) \in B \times A \Rightarrow (Tx, Ty) \in A \times B \subseteq \mathcal{S}.$$

This implies that T is a comparative mapping. Moreover, taking $a \in A$ (since A is nonempty), we have $Ta \in B$, that is, $a\mathcal{R}Ta$. By Theorem 2.1 (the first part), we obtain that $\mathrm{Fix}(T)$ is nonempty; in addition, $\mathrm{Fix}(T) \subseteq A \cap B$. Finally, as

$$x\mathcal{R}y, \text{ for all } x, y \in A \cap B,$$

we have that $A \cap B$ is $\mathcal{S}$-directed; so that, applying Theorem 2.1 (the second part), we obtain that T has a unique fixed point in X. □

The following result (see [12, 14]) is an immediate consequence of Theorem 3.9.

Corollary 3.10. *Let (M,d) be a complete metric space, A, B be nonempty closed subsets of M and $T : M \to M$ an operator. Suppose that*

(i) $T(A) \subseteq B$, $T(B) \subseteq A$;

(ii) there exists $\varphi \in \Phi$ such that

$$d(Tx, Ty) \leq \varphi\Big(d(x,y)\Big), \textit{ for all } x \in A, y \in B.$$

Then T has a unique fixed point x^* in $X := A \cup B$. Moreover, $x^* \in A \cap B$.

Taking in Theorem 3.9, $\varphi(t) = kt$, where $k \in (0,1)$ is a constant, we obtain the following result.

Corollary 3.11. *Let (M,d) be a complete metric space, A, B be nonempty closed subsets of M and $T : M \to M$ an operator. Suppose that* *(i)* $T(A) \subseteq B$, $T(B) \subseteq A$;

(ii) *there exists a constant* $k \in (0,1)$ *such that*

$$d(Tx,Ty) \leq k \max\left\{d(x,y), \frac{1}{2}\left[d(x,Tx)+d(y,Ty)\right], \frac{1}{2}\left[d(x,Ty)+d(y,Tx)\right]\right\},$$

for all $x \in A$, $y \in B$.

Then T *has a unique fixed point* x^* *in* $X := A \cup B$. *Moreover,* $x^* \in A \cap B$.

From Corollary 3.11, we obtain the following result (see [15]).

Corollary 3.12. *Let* (M,d) *be a complete metric space,* A,B *be nonempty closed subsets of* M *and* $T : M \to M$ *an operator. Suppose that*

(i) $T(A) \subseteq B$, $T(B) \subseteq A$;

(ii) *there exist* $a,b,c > 0$ *with* $a+2b+2c < 1$ *such that*

$$d(Tx,Ty) \leq ad(x,y)+b\left[d(x,Tx)+d(y,Ty)\right]+c\left[d(x,Ty)+d(y,Tx)\right],$$

for all $x \in A$, $y \in B$.

Then T *has a unique fixed point* x^* *in* $X := A \cup B$. *Moreover,* $x^* \in A \cap B$.

Remark 3.13. Taking $b = c = 0$ in Corollary 3.12, we obtain Theorem 3.8 [12].

Taking $a = c = 0$ in Corollary 3.12, we obtain the following result (see [17]).

Corollary 3.14. *Let* (M,d) *be a complete metric space,* A,B *be nonempty closed subsets of* M *and* $T : M \to M$ *an operator. Suppose that*

(i) $T(A) \subseteq B$, $T(B) \subseteq A$;

(ii) *there exists a constant* $k \in (0,1)$ *such that*

$$d(Tx,Ty) \leq k\left[d(x,Tx)+d(y,Ty)\right], \textit{ for all } x \in A, y \in B.$$

Then T *has a unique fixed point* x^* *in* $X := A \cup B$. *Moreover,* $x^* \in A \cap B$.

Taking $a = b = 0$ in Corollary 3.12, we obtain the following result (see [15]).

Corollary 3.15. *Let* (M,d) *be a complete metric space,* A,B *be nonempty closed subsets of* M *and* $T : M \to M$ *an operator. Suppose that*

(i) $T(A) \subseteq B$, $T(B) \subseteq A$;

(ii) *there exists a constant* $k \in (0,1)$ *such that*

$$d(Tx,Ty) \leq k\left[d(x,Ty)+d(y,Tx)\right], \textit{ for all } x \in A, y \in B.$$

Then T *has a unique fixed point* x^* *in* $X := A \cup B$. *Moreover,* $x^* \in A \cap B$.

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