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ENDPOINT ESTIMATE FOR PARAMETRIZED LITTLEWOOD-PALEY OPERATOR

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Abstract

In this paper, the authors obtain the $(L^1, L^{1,\infty})$ type boundedness for the parametrized Littlewood-Paley operator $\mu_\lambda^{*,\rho}$ with kernel satisfying the logarithmic type Lipschitz condition. Moreover, the $L^p(\mathbb{R}^n)$ boundedness of the operator $\mu_\lambda^{*,\rho}$ can be deduced, where $1 < p < \infty$.

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1 Introduction

Suppose that S^{n-1} is the unit sphere of $\mathbb{R}^n$ equipped with normalized Lebesgue measure. Let $\Omega \in L^1(S^{n-1})$ be homogeneous of degree zero and

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{1.1}$$

where $x' = \frac{x}{|x|}$ for any $x \neq 0$. The parametrized Littlewood-Paley operator $\mu_\lambda^{*,\rho}$ is defined by

$$\begin{aligned}\mu_\lambda^{*,\rho}(f)(x) &= \Big(\iint_{\mathbb{R}_+^{n+1}} (\frac{t}{t+|x-y|})^{\lambda n}|\varphi_t * f(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2}\\ &= \Big(\iint_{\mathbb{R}_+^{n+1}} (\frac{t}{t+|x-y|})^{\lambda n}\Big|\frac{1}{t^\rho}\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f(z)dz\Big|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2},\end{aligned}$$

where $\lambda > 1$, $\rho > 0$, $\varphi_t(x) = \frac{1}{t^n}\varphi(\frac{x}{t})$, $\varphi(x) = \Omega(x)|x|^{-n+\rho}\chi_B(x)$, and B is a unit ball in $\mathbb{R}^n$. Inspired by Hömander's work on the parametrized Marcinkiewicz integral [4], in 1999, Sakamoto and Yabuta [7] studied the L^p boundedness of the parametrized Littlewood-Paley g_λ^* function $\mu_\lambda^{*,\rho}$ with the kernel satisfying the Lip_α condition.

In 2002, Ding, Lu and Yabuta [2] proved the $L^p(2 \leq p < \infty)$ boundedness of the parametrized Littlewood-Paley g_λ^* function $\mu_\lambda^{*,\rho}$ with a weaker kernel condition. Let $h(|x|) \equiv 1$ in the Theorem 1 of [2], we obtain

Theorem A. ([2]) *If $\Omega \in LlogL^+(S^{n-1})$ be a homogeneous function of degree zero satisfying* (1.1)*, $\rho > 0, \lambda > 1$ and $2 \leq p < \infty$, then $\|\mu_\lambda^{*,\rho}(f)\|_{L^p} \leq \frac{C}{\sqrt{\rho}}\|f\|_{L^p}$.*

For $\Omega \in L^2(S^{n-1})$, the integral modulus $\omega_2(\delta)$ of continuity of order 2 of Ω is defined by

$$\omega_2(\delta) = \sup_{|\gamma|\leq\delta}\Big(\int_{S^{n-1}} |\Omega(\gamma x') - \Omega(x')|^2 d\sigma(x')\Big)^{1/2},$$

where γ is a rotation on S^{n-1}, $|\gamma| = \sup_{x'\in S^{n-1}} |\gamma x' - x'|$.

In 2007, Ding, Lu and Xue [1] proved that the parametrized Littlewood-Paley g_λ^* function $\mu_\lambda^{*,\rho}$ is of the type $(L^1, L^{1,\infty})$, if $\Omega(x)$ satisfies (1.1) and $\int_0^1 \frac{\omega_2(\delta)}{\delta^{1+\alpha}}d\delta < \infty (0 < \alpha \leq 1)$.

Recently, Lee and Rim [5] established the (H^1, L^1), (L^∞, BMO) and (L^p, L^p) type boundedness of Marcinkiewicz integral μ_Ω when Ω satisfies a class of logarithmic type Lipschitz conditions. The main result in [5] is the following theorem.

Theorem B. ([5]) *Let $n \geq 2$ and $\Omega \in L^\infty(S^{n-1})$ be a homogeneous function of degree zero satisfying* (1.1)*. In addition, there exist constants $C > 0$ and $\alpha > 1$ such that*

$$|\Omega(y_1) - \Omega(y_2)| \leq \frac{C}{(\log\frac{1}{|y_1-y_2|})^\alpha}, \quad for\ any\ y_1, y_2 \in S^{n-1}.$$

Then the following inequalities hold:

$$\|\mu_\Omega(f)\|_{L^1} \leq C_1\|f\|_{H^1}, \qquad f \in H^1(\mathbb{R}^n)$$

$$\|\mu_\Omega(f)\|_{BMO} \leq C_\infty\|f\|_{L^\infty}, \quad f \in L^2 \cap L^\infty$$

and

$$\|\mu_\Omega(f)\|_{L^p} \leq C_p\|f\|_{L^p}. \quad f \in L^p(\mathbb{R}^n)(1 < p < \infty).$$

Remark 1.1. *Since $0 \leq |y_1 - y_2| \leq 2$ for $y_1, y_2 \in S^{n-1}$, it is reasonable that the above logarithmic type Lipschitz condition would be*

$$|\Omega(y_1) - \Omega(y_2)| \leq \frac{C}{(\log \frac{2}{|y_1-y_2|})^\alpha}.$$

In [6] the author proved $(L^1, L^{1,\infty})$ type boundedness for the parametrized Marcinkiewicz integral with variable kernel. The special case of the result in [6] is the following theorem.

Theorem C. ([6]) *Let $n \geq 2$, $\Omega \in L^\infty(S^{n-1})$ satisfy* (1.1)*. In addition, there exist constants $C_0 > 0$ and $\alpha > 2$ such that*

$$|\Omega(y_1) - \Omega(y_2)| \leq \frac{C_0}{(\log \frac{1}{|y_1-y_2|})^\alpha}, \quad for\ any\ y_1, y_2 \in S^{n-1}.$$

Then for all $\beta > 0$ and $f \in L^1(\mathbb{R}^n)$,

$$|\{x : \mu_\Omega(f)(x) > \beta\}| \leq \frac{C}{\beta}\|f\|_{L^1}.$$

Besides, the authors in [8, 9] discuss the commutators of Marcinkiewicz integrals under the same kernel as Theorem B. Inspired by this, a question arises naturally: with the same kernel in Theorem B, whether the parametrized Littlewood-Paley g_λ^* function $\mu_\lambda^{*,\rho}$ is of the type $(L^1, L^{1,\infty})$? In this paper, we will give the affirmative answer.

2 Some Lemmas

Lemma 2.1. ([3]) *Given a function f which is integrable and non-negative, and given a positive number β, there exists a sequence $\{Q_k\}$ of disjoint dyadic cubes such that*
(1) $f < \beta$, *a.e.* $x \notin \bigcup_k Q_k$;
(2) $\left|\bigcup_k Q_k\right| < \frac{C}{\beta}\|f\|_1$;
(3) $\beta < \frac{1}{Q_k}\int_{Q_k} f \leq 2^n\beta$.

Lemma 2.2. *There exists a constant $C > 0$ such that for $y \in (4B_k)^c, z \in Q_k$,*

$$\left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\right| \leq \frac{C(1 + |\Omega(y-x_k)|)}{|y-x_k|^{n-\rho}(\log \frac{|y-x_k|}{r_k})^\alpha},$$

where Q_k is a cube with x_k and a_k are the center and side length respectively, and B_k is a ball with center at a_k and radius $r_k = \frac{\sqrt{n}}{2}a_k$ for each k.

Proof. Since $y \in (4B_k)^c, z \in Q_k$, and x_k is the center of Q_k, we have $|y-z| \sim |y-x_k|$.

Write

$$\begin{aligned}
\left|\frac{y-z}{|y-z|}-\frac{y-x_k}{|y-x_k|}\right| &= \left|\frac{(y-z)|y-x_k|-(y-x_k)|y-z|}{|y-z||y-x_k|}\right| \\
&= \left|\frac{(y-z)(|y-x_k|-|y-z|)-(z-x_k)|y-z|}{|y-z||y-x_k|}\right| \\
&\leq \frac{|y-z||y-x_k-y+z|+|z-x_k||y-z|}{|y-z||y-x_k|} \\
&= \frac{2|y-z||z-x_k|}{|y-z||y-x_k|} \\
&\leq \frac{2r_k}{|y-x_k|},
\end{aligned}$$

then

$$\begin{aligned}
|\Omega(y-z)-\Omega(y-x_k)| &= \left|\Omega(\frac{y-z}{|y-z|})-\Omega(\frac{y-x_k}{|y-x_k|})\right| \\
&\leq \frac{C_0}{\left(\log\frac{2}{|\frac{y-z}{|y-z|}-\frac{y-x_k}{|y-x_k|}|}\right)^{\alpha}} \\
&\leq \frac{C_0}{\left(\log\frac{|y-x_k|}{r_k}\right)^{\alpha}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\right| &= \left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_k)}{|y-z|^{n-\rho}}+\frac{\Omega(y-x_k)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\right| \\
&\leq \frac{|\Omega(y-z)-\Omega(y-x_k)|}{|y-z|^{n-\rho}}+|\Omega(y-x_k)|\left|\frac{1}{|y-z|^{n-\rho}}-\frac{1}{|y-x_k|^{n-\rho}}\right| \\
&\leq \frac{C}{|y-x_k|^{n-\rho}(\log\frac{|y-x_k|}{r_k})^{\alpha}}+|\Omega(y-x_k)|\frac{Cr_k}{|y-x_k|^{n-\rho+1}} \\
&\leq \frac{C(1+|\Omega(y-x_k)|)}{|y-x_k|^{n-\rho}(\log\frac{|y-x_k|}{r_k})^{\alpha}}.
\end{aligned}$$

□

Lemma 2.3. ([10]) *For* $y \in (4B_k)^c$,

$$\int_{|y-x_k|+2r_k}^{\infty}\frac{(\log\frac{t}{r_k})^{2+2\epsilon}}{t^{2\rho-n+1}}dt \leq C\frac{[\log(\frac{|y-x_k|}{r_k}+2)]^{2+2\epsilon}}{(|y-x_k|+2r_k)^{2\rho-n}},$$

where $0<\epsilon<\rho-\frac{n}{2}$.

3 Main Results

Theorem 3.1. *Let $n \geq 2$, $\Omega \in L^2(S^{n-1})$ be a homogeneous function of degree zero satisfying (1.1) and there exist constants $C_0 > 0$ and $\alpha > \frac{3}{2}$, such that*

$$|\Omega(y_1) - \Omega(y_2)| \leq \frac{C_0}{(\log \frac{2}{|y_1 - y_2|})^{\alpha}} \quad for\ any\ y_1, y_2 \in S^{n-1}.$$

.

Then for $\rho > \frac{n}{2}$ and $\lambda > 2$, there exists a constant $C > 0$, such that for all $\beta > 0$ and $f \in L^1(\mathbb{R}^n)$,

$$|\{x : \mu_\lambda^{*,\rho}(f)(x) > \beta\}| \leq \frac{C}{\beta}\|f\|_{L^1}.$$

Proof. For $f \in L^1(\mathbb{R}^n)$ and $\beta > 0$, by the Calderón-Zygmund decomposition in Lemma 2.1, we have the following conclusions:
(i) $\mathbb{R}^n = F \cup E$, with $F \cap E = \emptyset$;
(ii) $E = \bigcup_k Q_k$, where $\{Q_k\}$ is a sequence of cubes with disjoint interiors;
(iii) $|f| < \beta, \quad a.e. \quad x \in F$;
(iv) $\beta < \frac{1}{|Q_k|}\int_{Q_k} |f| dx \leq 2^n \beta$, for every k;
(v) $|E| \leq \frac{C}{\beta}\int_{\mathbb{R}^n} |f| dx$.
Denote

$$u(x) = \begin{cases} f(x), & x \in F, \\ \frac{1}{|Q_k|}\int_{Q_k} f(y)dy, & x \in Q_k, \end{cases}$$

and set $b = f - u$, then $b(x) = 0$ for $x \in F$ and $\int_{Q_k} b(x)dx = 0$, for each k.

Then we have

$$|\{x : \mu_\lambda^{*,\rho}(f)(x) > \beta\}| \leq |\{x : \mu_\lambda^{*,\rho}(u)(x) > \frac{\beta}{2}\}| + |\{x : \mu_\lambda^{*,\rho}(b)(x) > \frac{\beta}{2}\}|.$$

By the $L^2(\mathbb{R}^n)$ boundedness of $\mu_\lambda^{*,\rho}$ in Theorem B and (iii) – (iv), it is easy to see that

$$\begin{aligned} |\{x : \mu_\lambda^{*,\rho}(u)(x) > \frac{\beta}{2}\}| &\leq \frac{4}{\beta^2}\int_{\mathbb{R}^n} |\mu_\lambda^{*,\rho}(u)(x)|^2 dx \leq \frac{C}{\beta^2}\|u\|_2^2 \\ &\leq \frac{C}{\beta^2}\Big[\int_F |f|^2 dx + \sum_k \int_{Q_k}\Big(\frac{1}{|Q_k|}\int_{Q_k} f(y)dy\Big)^2 dx\Big] \leq \frac{C}{\beta}\|f\|_1. \end{aligned} \tag{3.1}$$

On the other hand, we denote by x_k and a_k the center and sidelength of Q_k respectively and let B_k be a ball with center at x_k and radius $r_k = \frac{\sqrt{n}}{2}a_k$ for each k. Then

$$\begin{aligned} |\{x : \mu_\lambda^{*,\rho}(b)(x) > \frac{\beta}{2}\}| &= |\{x : \mu_\lambda^{*,\rho}(b)(x) > \frac{\beta}{2}\} \cap [E^* \cup (E^*)^c]| \\ &= |\{x : \mu_\lambda^{*,\rho}(b)(x) > \frac{\beta}{2}\} \cap E^*| + |\{x : \mu_\lambda^{*,\rho}(b)(x) > \frac{\beta}{2}\} \cap (E^*)^c|, \end{aligned}$$

where $E^* = \bigcup_k (16B_k)$. By (ii) and (v), we have

$$|\{x : \mu_\lambda^{*,\rho}(b)(x) > \frac{\beta}{2}\} \cap E^*| \leq |E^*| \leq \sum_k |16B_k| \leq \frac{C}{\beta}\|f\|_1. \tag{3.2}$$

Note that

$$|\{x:\mu_{\lambda}^{*,\rho}(b)(x)>\frac{\beta}{2}\}\cap(E^*)^c|\leq\frac{C}{\beta}\int_{(E^*)^c}\mu_{\lambda}^{*,\rho}(b)(x)dx\quad and\int_{\mathbb{R}^n}|b(x)|dx\leq C\|f\|_1.$$

Hence by (3.1), (3.2), to complete the proof of the theorem, it remains to verify that

$$\int_{(E^*)^c}\mu_{\lambda}^{*,\rho}(b)(x)dx\leq C\|b\|_1.$$

Denote

$$b_k(x)=\begin{cases}b(x), & x\in Q_k,\\ 0, & x\notin Q_k.\end{cases}$$

It is easy to see that $b(x)=\sum_k b_k(x)$. Then by the Minkowski inequality,

$$\begin{aligned}&\int_{(E^*)^c}\mu_{\lambda}^{*,\rho}(b)(x)dx\\ =&\int_{(E^*)^c}\Big[\iint_{\mathbb{R}^{n+1}_+}(\frac{t}{t+|x-y|})^{\lambda n}\Big|\sum_k\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}b_k(z)dz\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big]^{\frac12}dx\\ \leq&\int_{(E^*)^c}\sum_k\Big[\iint_{\mathbb{R}^{n+1}_+}(\frac{t}{t+|x-y|})^{\lambda n}\Big|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}b_k(z)dz\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big]^{\frac12}dx.\end{aligned}$$

Let

$$J_1=\int_{(E^*)^c}\sum_k\Big[\iint_{|y-x|<t}(\frac{t}{t+|x-y|})^{\lambda n}\Big|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}b_k(z)dz\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big]^{\frac12}dx,$$

$$J_2=\int_{(E^*)^c}\sum_k\Big[\iint_{\substack{|y-x|\geq t\\ y\in 4B_k}}(\frac{t}{t+|x-y|})^{\lambda n}\Big|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}b_k(z)dz\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big]^{\frac12}dx,$$

and

$$J_3=\int_{(E^*)^c}\sum_k\Big[\iint_{\substack{|y-x|\geq t\\ y\in (4B_k)^c}}(\frac{t}{t+|x-y|})^{\lambda n}\Big|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}b_k(z)dz\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big]^{\frac12}dx.$$

Then

$$\int_{(E^*)^c}\mu_{\lambda}^{*,\rho}(b)(x)dx\leq J_1+J_2+J_3. \tag{3.3}$$

Below we will give the estimates of J_1,J_2,J_3 respectively. First we have

$$\begin{aligned}J_1\leq&\int_{(E^*)^c}\sum_k\Big[\iint_{|y-x|<t}\Big|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}b_k(z)dz\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big]^{\frac12}dx\\ =&\int_{(E^*)^c}\sum_k\Big[\Big(\iint_{\substack{|y-x|<t\\ y\in 4B_k}}+\iint_{\substack{|y-x|<t\\ y\in(4B_k)^c}}\Big)\Big|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}b_k(z)dz\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big]^{\frac12}dx.\end{aligned}$$

Let

$$J_{11}=\int_{(E^*)^c}\sum_k\Big[\iint_{\substack{|y-x|<t\\ y\in 4B_k}}\Big|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}b_k(z)dz\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big]^{\frac12}dx,$$

$$J_{12} = \int_{(E^*)^c}\sum_k\Big[\iint_{\substack{|y-x|<t\\ y\in(4B_k)^c}}\Big|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}b_k(z)dz\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big]^{\frac{1}{2}}dx.$$

Then $J_1 \le J_{11} + J_{12}$. By $x \in (E^*)^c$, $y \in 4B_k$ and $z \in Q_k$, we have $|x-x_k| - 4r_k \le |x-x_k| - |y-x_k| \le |x-y| < t$, $|x-x_k| - 4r_k \sim |x-x_k|$, and $|y-z| < 8r_k$. Applying the Minkowski inequality we get

$$\begin{aligned}
J_{11} &\le \int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\Big[\iint_{\substack{|y-x|<t\\ |y-z|<t\\ y\in 4B_k}}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}\Big|^2\frac{1}{t^{n+2\rho+1}}dydt\Big]^{\frac{1}{2}}dzdx\\
&\le \int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\Big[\int_{|y-z|<8r_k}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\Big(\int_{|x-x_k|-4r_k}^{\infty}\frac{dt}{t^{n+2\rho+1}}\Big)dy\Big]^{\frac{1}{2}}dzdx\\
&\le C\sum_k\int_{Q_k}|b(z)|\Big(\int_{|y-z|<8r_k}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}dy\Big)^{\frac{1}{2}}dz\int_{(E^*)^c}\frac{1}{|x-x_k|^{(n+2\rho)/2}}dx\\
&\le C\sum_k\int_{Q_k}|b(z)|dz\\
&\le C\|b\|_1. \qquad (3.4)
\end{aligned}$$

As for J_{12}, we have

$$J_{12} = \int_{(E^*)^c}\sum_k\Big[\Big(\iint_{\substack{|y-x|<t\\ t\le|y-x_k|+2r_k\\ y\in(4B_k)^c}} + \iint_{\substack{|y-x|<t\\ t>|y-x_k|+2r_k\\ y\in(4B_k)^c}}\Big)\Big|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}b_k(z)dz\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big]^{\frac{1}{2}}dx.$$

Let

$$\begin{aligned}
J_{12}^1 &= \int_{(E^*)^c}\sum_k\Big[\iint_{\substack{|y-x|<t\\ t\le|y-x_k|+2r_k\\ y\in(4B_k)^c}}\Big|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}b_k(z)dz\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big]^{\frac{1}{2}}dx,\\
J_{12}^2 &= \int_{(E^*)^c}\sum_k\Big[\iint_{\substack{|y-x|<t\\ t>|y-x_k|+2r_k\\ y\in(4B_k)^c}}\Big|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}b_k(z)dz\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big]^{\frac{1}{2}}dx.
\end{aligned}$$

Then $J_{12} \le J_{12}^1 + J_{12}^2$. By $z \in Q_k$, $x \in (E^*)^c$ and $y \in (4B_k)^c$, it is easy to see that $|y-z| \sim |y-x_k|$ and $|x-x_k| \le |x-y| + |y-x_k| \le t + |y-x_k| \le 2|y-x_k| + 2r_k < 3|y-x_k|$.

Applying the Minkowski inequality again, we get

$$\begin{aligned}
J_{12}^1 &\le \int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\Big[\iint_{\substack{|y-x|<t\\ t\le|y-x_k|+2r_k\\ |y-z|<t\\ y\in(4B_k)^c}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Big]^{\frac{1}{2}}dzdx\\
&\le \int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\Big[\int_{y\in(4B_k)^c}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\Big(\int_{|y-z|}^{|y-x_k|+2r_k}\frac{dt}{t^{n+2\rho+1}}\Big)dy\Big]^{\frac{1}{2}}dzdx\\
&\le C\int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\Big[\int_{y\in(4B_k)^c}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{r_k}{|y-z|^{n+2\rho+1}}dy\Big]^{\frac{1}{2}}dzdx
\end{aligned}$$

$$
\begin{aligned}
&\le C\int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\Big[\int_{y\in(4B_k)^c}\frac{|\Omega(y-z)|^2}{|y-z|^{n+\frac{1}{2}}}\frac{r_k}{|y-x_k|^{2n+\frac{1}{2}}}dy\Big]^{\frac{1}{2}}dzdx\\
&\le C\sum_k\int_{Q_k}|b(z)|r_k^{\frac{1}{2}}\Big(\int_{y\in(4B_k)^c}\frac{|\Omega(y-z)|^2}{|y-z|^{n+\frac{1}{2}}}dy\Big)^{\frac{1}{2}}dz\int_{(E^*)^c}\frac{1}{|x-x_k|^{n+\frac{1}{4}}}dx\\
&= C\sum_k\int_{Q_k}|b(z)|dz\\
&\le C\|b\|_1. \qquad (3.5)
\end{aligned}
$$

Now, we give the estimate of J_{12}^2. Note that $Q_k\subset B_k\subset\{z:|y-z|<t\}$ since $y\in(4B_k)^c$ and $t>|y-x_k|+2r_k$. In addition, $|x-x_k|<|x-y|+|y-x_k|<3t$. Then by the cancelation property of $b(x)$ on Q_k, we have

$$
\begin{aligned}
J_{12}^2 &= \int_{(E^*)^c}\sum_k\Big[\iint_{\substack{|y-x|<t\\ t>|y-x_k|+2r_k\\ y\in(4B_k)^c}}\Big|\int_{|y-z|<t}\Big(\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\Big)b_k(z)dz\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big]^{\frac{1}{2}}dx\\
&\le \int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\Big[\int_{y\in(4B_k)^c}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\Big|^2\\
&\quad\times\Big(\int_{\substack{|y-x|<t\\ t>|y-x_k|+2r_k\\ |y-z|<t}}\frac{1}{t^{n+2\rho+1}}dt\Big)dy\Big]^{\frac{1}{2}}dzdx\\
&\le \int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\Big[\int_{y\in(4B_k)^c}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\Big|^2\\
&\quad\times\Big(\int_{|y-x_k|+2r_k}^{\infty}\frac{(\log\frac{t}{r_k})^{2+2\epsilon}}{t^{n+2\rho+1}(\log\frac{t}{r_k})^{2+2\epsilon}}dt\Big)dy\Big]^{\frac{1}{2}}dzdx\\
&\le \sum_k\int_{(E^*)^c}\int_{Q_k}|b(z)|\Big[\int_{y\in(4B_k)^c}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\Big|^2\\
&\quad\times\Big(\int_{|y-x_k|+2r_k}^{\infty}\frac{(\log\frac{t}{r_k})^{2+2\epsilon}}{t^{2\rho-n+1}(\frac{|x-x_k|}{3})^{2n}(\log\frac{|x-x_k|}{3r_k})^{2+2\epsilon}}dt\Big)dy\Big]^{\frac{1}{2}}dzdx,
\end{aligned}
$$

where $0<\epsilon<\min\{1/2,(\lambda-2)n/2,\rho-n/2,\alpha-\frac{3}{2}\}$. By Lemma 2.2 and Lemma 2.3, we have

$$
\begin{aligned}
J_{12}^2 &\le C\sum_k\int_{(E^*)^c}\int_{Q_k}|b(z)|\Big[\int_{y\in(4B_k)^c}\Big(\frac{1+|\Omega(y-x_k)|}{|y-x_k|^{n-\rho}(\log\frac{|y-x_k|}{r_k})^{\alpha}}\Big)^2\\
&\quad\times\Big(\frac{[\log(\frac{|y-x_k|}{r_k}+2)]^{2+2\epsilon}}{(|y-x_k|+2r_k)^{2\rho-n}}\frac{1}{(\frac{|x-x_k|}{3})^{2n}(\log\frac{|x-x_k|}{3r_k})^{2+2\epsilon}}\Big)dy\Big]^{\frac{1}{2}}dzdx\\
&\le C\sum_k\int_{Q_k}|b(z)|dz\Big[\int_{y\in(4B_k)^c}\frac{(1+|\Omega(y-x_k)|)^2}{|y-x_k|^n(\log\frac{|y-x_k|}{r_k})^{2\alpha}}[\log(\frac{|y-x_k|}{r_k}+2)]^{2+2\epsilon}dy\Big]^{\frac{1}{2}}\\
&\quad\times\int_{(E^*)^c}\frac{1}{(\frac{|x-x_k|}{3})^n(\log\frac{|x-x_k|}{3r_k})^{1+\epsilon}}dx.
\end{aligned}
$$

Denote that

$$
\begin{aligned}
I &= \Big[\int_{y\in(4B_k)^c}\frac{(1+|\Omega(y-x_k)|)^2}{|y-x_k|^n(\log\frac{|y-x_k|}{r_k})^{2\alpha}}[\log(\frac{|y-x_k|}{r_k}+2)]^{2+2\epsilon}dy\Big]^{\frac12}\\
&= \Big[\int_{S^{n-1}}(1+|\Omega(y')|)^2d\sigma(y')\Big]^{\frac12}\Big[\int_{4r_k}^{\infty}\frac{\Big(\log(\frac{r}{r_k}+2)\Big)^{2+2\epsilon}}{r(\log\frac{r}{r_k})^{2\alpha}}dr\Big]^{\frac12}\\
&\le C\Big(\int_{4r_k}^{\infty}\frac{(\log(\frac{r}{r_k}+2))^{2+2\epsilon}}{r(\log\frac{r}{r_k})^{2\alpha}}dr\Big)^{\frac12}\\
&= C\Big(\int_{4}^{\infty}\frac{(\log(t+2))^{2+2\epsilon}}{t(\log t)^{2\alpha}}dt\Big)^{\frac12}\\
&\le C\Big(\int_{4}^{\infty}\frac{(\log(2t))^{2+2\epsilon}}{t(\log t)^{2\alpha}}dt\Big)^{\frac12}\\
&= C\Big(\int_{4}^{\infty}\frac{\Big((\frac{\log 2}{\log t}+1)\log t\Big)^{2+2\epsilon}}{t(\log t)^{2\alpha}}dt\Big)^{\frac12}\\
&\le C\Big(\int_{4}^{\infty}\frac{1}{(\log t)^{2\alpha-2-2\epsilon}}d(\log t)\Big)^{\frac12}\\
&= C.
\end{aligned}
$$

Thus

$$
J_{12}^2\le C\sum_k\int_{Q_k}|b(z)|dz\int_{(E^*)^c}\frac{1}{(\frac{|x-x_k|}{3})^n(\log\frac{|x-x_k|}{3r_k})^{1+\epsilon}}dx\le C\|b\|_1. \tag{3.6}
$$

By (3.4)–(3.6), we obtain

$$
J_1\le C\|b\|_1. \tag{3.7}
$$

As for J_2, note that $\frac{1}{t}<\frac{1}{|y-z|}$, so $\frac{1}{t^{2\rho-n-\epsilon}}<\frac{1}{|y-z|^{2\rho-n-\epsilon}}$. For $y\in 4B_k$, $x\in(E^*)^c$ and $z\in Q_k$, we have $|y-x|>|x-x_k|-|y-x_k|>|x-x_k|/2$, $|y-z|<8r_k$ and $|x-y|\sim|x-x_k|$. By the Minkowski inequality and $\epsilon<(\lambda-2)n/2$, we have

$$
\begin{aligned}
J_2 &\le \int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\Big[\iiint_{\substack{|y-x|\ge t\\ |y-z|<t\\ |y-z|<8r_k\\ y\in 4B_k}}(\frac{t}{t+|x-y|})^{2n+2\epsilon}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Big]^{\frac12}dzdx\\
&\le \int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\Big[\int_{|y-z|<8r_k}\frac{|\Omega(y-z)|^2}{(|x-x_k|/2)^{2n+2\epsilon}|y-z|^{2n-2\rho}}\\
&\quad\times\Big(\int_0^{|y-x|}\frac{t^{2n+2\epsilon}}{t^{2n+\epsilon+1}|y-z|^{2\rho-n-\epsilon}}dt\Big)dy\Big]^{\frac12}dzdx\\
&\le \int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\Big[\int_{|y-z|<8r_k}\frac{|\Omega(y-z)|^2|y-x|^{\epsilon}}{(|x-x_k|/2)^{2n+2\epsilon}|y-z|^{n-\epsilon}}dy\Big]^{\frac12}dzdx\\
&\le C\int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\Big[\int_{|y-z|<8r_k}\frac{|\Omega(y-z)|^2}{|x-x_k|^{2n+\epsilon}|y-z|^{n-\epsilon}}dy\Big]^{\frac12}dzdx
\end{aligned}
$$

$$\begin{aligned}
&\le C\sum_k\int_{Q_k}|b(z)|\Big[\int_{|y-z|<8r_k}\frac{|\Omega(y-z)|^2}{|y-z|^{n-\epsilon}}dy\Big]^{\frac12}dz\int_{(E^*)^c}\frac{1}{|x-x_k|^{n+\frac{\epsilon}{2}}}dx\\
&\le C\sum_k\int_{Q_k}|b(z)|dz\\
&\le C\|b\|_1.
\end{aligned}$$

Now let us estimate J_3. Denote

$$\begin{aligned}
J_{31} &= \int_{(E^*)^c}\sum_k\Big[\iint_{\substack{|y-x|\ge t\\ y\in(4B_k)^c\\ t\le|y-x_k|+C_\epsilon r_k}}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\Big|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}b_k(z)dz\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big]^{\frac12}dx,\\
J_{32} &= \int_{(E^*)^c}\sum_k\Big[\iint_{\substack{|y-x|\ge t\\ y\in(4B_k)^c\\ t>|y-x_k|+C_\epsilon r_k}}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\Big|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}b_k(z)dz\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big]^{\frac12}dx,
\end{aligned}$$

where $C_\epsilon=e^{(2+2\epsilon)/\epsilon}$, then $J_3\le J_{31}+J_{32}$. By $y\in(4B_k)^c$ and $z\in Q_k$, we have $|y-z|\sim|y-x_k|$, $|y-x_k|\le|y-z|+|z-x_k|\le t+2r_k$. Moreover, for $\theta>0$, we have the following inequality

$$\int_{|y-x_k|-2r_k}^{|y-x_k|+C_\epsilon r_k}\frac{1}{t^{\theta+1}}dt\le\frac{Cr_k}{|y-x_k|^{\theta+1}}. \tag{3.8}$$

Applying the Minkowski inequality

$$\begin{aligned}
J_{31} &\le \int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\Big[\iint_{\substack{|y-x|\ge t\\ y\in(4B_k)^c\\ t\le|y-x_k|+C_\epsilon r_k\\ |y-z|<t}}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Big]^{\frac12}dzdx\\
&= \int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\Big[\Big(\iint_{\substack{|y-x|\ge t\\ y\in(4B_k)^c\\ t\le|y-x_k|+C_\epsilon r_k\\ |y-z|<t\\ |x-x_k|\le2|y-x_k|}}+\iint_{\substack{|y-x|\ge t\\ y\in(4B_k)^c\\ t\le|y-x_k|+C_\epsilon r_k\\ |y-z|<t\\ |x-x_k|>2|y-x_k|}}\Big)\\
&\quad\times\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Big]^{\frac12}dzdx.
\end{aligned}$$

Let

$$J_{31}^1=\int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\Big[\iint_{\substack{|y-x|\ge t\\ y\in(4B_k)^c\\ t\le|y-x_k|+C_\epsilon r_k\\ |y-z|<t\\ |x-x_k|\le2|y-x_k|}}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Big]^{\frac12}dzdx,$$

$$J_{31}^2=\int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\Big[\iint_{\substack{|y-x|\ge t\\ y\in(4B_k)^c\\ t\le|y-x_k|+C_\epsilon r_k\\ |y-z|<t\\ |x-x_k|>2|y-x_k|}}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Big]^{\frac12}dzdx.$$

Then $J_{31} \leq J_{31}^1 + J_{31}^2$. By (3.8) we get

$$\begin{aligned}
J_{31}^1 &\leq \int_{(E^*)^c} \sum_k \int_{Q_k} |b(z)| \Big[\int_{\substack{y\in(4B_k)^c \\ |x-x_k|\leq 2|y-x_k|}} \Big(\int_{|y-x_k|-2r_k}^{|y-x_k|+C_\epsilon r_k} \frac{dt}{t^{n+2\rho+1}} \Big) \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} dy \Big]^{\frac12} dzdx \\
&\leq C \int_{(E^*)^c} \sum_k \int_{Q_k} |b(z)| \Big[\int_{\substack{y\in(4B_k)^c \\ |x-x_k|\leq 2|y-x_k|}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{r_k}{|y-x_k|^{n+2\rho+1}} dy \Big]^{\frac12} dzdx \\
&\leq C \int_{(E^*)^c} \sum_k \int_{Q_k} |b(z)| \Big[\int_{y\in(4B_k)^c} \frac{|\Omega(y-z)|^2}{|y-z|^{n+\frac12}} \frac{r_k}{|x-x_k|^{2n+\frac12}} dy \Big]^{\frac12} dzdx \\
&= C \sum_k r_k^{\frac12} \int_{Q_k} |b(z)| \Big[\int_{y\in(4B_k)^c} \frac{|\Omega(y-z)|^2}{|y-z|^{n+\frac12}} dy \Big]^{\frac12} dz \int_{(E^*)^c} \frac{1}{|x-x_k|^{n+\frac14}} dx \\
&\leq C \sum_k \int_{Q_k} |b(z)| dz \\
&\leq C\|b\|_1. \qquad (3.9)
\end{aligned}$$

Now we consider J_{31}^2. Note that $0 < \epsilon < \min\{1/2, (\lambda-2)n/2, \rho - n/2, \alpha - \frac32\}$, $|y-x| > |x-x_k| - |y-x_k| \geq |x-x_k|/2$, and $|y-z| \sim |y-x_k|$. By (3.8) we have

$$\begin{aligned}
J_{31}^2 &\leq \int_{(E^*)^c} \sum_k \int_{Q_k} |b(z)| \Big[\int_{\substack{y\in(4B_k)^c \\ |x-x_k|> 2|y-x_k|}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \Big(\int_{|y-x_k|-2r_k}^{|y-x_k|+C_\epsilon r_k} \big(\frac{t}{t+|x-y|}\big)^{2n+2\epsilon} \\
&\quad \times \frac{dt}{t^{n+2\rho+1}} \Big) dy \Big]^{\frac12} dzdx \\
&\leq \int_{(E^*)^c} \sum_k \int_{Q_k} |b(z)| \Big[\int_{\substack{y\in(4B_k)^c \\ |x-x_k|> 2|y-x_k|}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \Big(\int_{|y-x_k|-2r_k}^{|y-x_k|+C_\epsilon r_k} \frac{t^{n+2\epsilon-2\rho-1}}{|x-y|^{2n+2\epsilon}} \\
&\quad \times dt \Big) dy \Big]^{\frac12} dzdx \\
&\leq C \int_{(E^*)^c} \sum_k \int_{Q_k} |b(z)| \Big[\int_{\substack{y\in(4B_k)^c \\ |x-x_k|> 2|y-x_k|}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{r_k}{|x-x_k|^{2n+2\epsilon}} \\
&\quad \times \frac{1}{|y-x_k|^{2\rho-n-2\epsilon+1}} dy \Big]^{\frac12} dzdx \\
&\leq C \int_{(E^*)^c} \sum_k \int_{Q_k} |b(z)| \Big[\int_{\substack{y\in(4B_k)^c \\ |x-x_k|> 2|y-x_k|}} \frac{|\Omega(y-z)|^2}{|y-z|^{n-2\epsilon+1}} \frac{r_k}{|x-x_k|^{2n+2\epsilon}} dy \Big]^{1/2} dzdx \\
&\leq C \sum_k \int_{Q_k} |b(z)| \Big(\int_{y\in(4B_k)^c} \frac{|\Omega(y-z)|^2}{|y-z|^{n-2\epsilon+1}} dy \Big)^{1/2} dz \int_{(E^*)^c} \frac{r_k^{\frac12}}{|x-x_k|^{n+\epsilon}} dx \\
&\leq C\|b\|_1. \qquad (3.10)
\end{aligned}$$

Finally, let us estimate J_{32}. By $y \in (4B_k)^c$ and $t > |y-x_k| + C_\epsilon r_k$, we have $Q_k \subset B_k \subset \{z : |y-z| < t\}$. On the other hand, it is easy to see that

$$t + |x-y| \geq t + |x-x_k| - |y-x_k| \geq |y-x_k| + C_\epsilon r_k + |x-x_k| - |y-x_k| = |x-x_k| + C_\epsilon r_k.$$

Hence by the cancelation property of b on Q_k and applying the Minkowski inequality, we get

$$\begin{aligned}
J_{32} &\leq \int_{(E^*)^c}\sum_k\int_{Q_k}|b(z)|\Big[\iint_{\substack{y\in(4B_k)^c\\|y-x_k|+C_\epsilon r_k<t\\|y-z|<t\\t\leq|y-x|}}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\Big|^2\\
&\quad\times\frac{dydt}{t^{n+2\rho+1}}\Big]^{\frac12}dzdx\\
&= \sum_k\int_{(E^*)^c}\int_{Q_k}|b(z)|\Big[\iint_{\substack{y\in(4B_k)^c\\|y-x_k|+C_\epsilon r_k<t\\|y-z|<t\\t\leq|y-x|}}\frac{t^{\lambda n}(\log\frac{t+|x-y|}{r_k})^{2+2\epsilon}}{(t+|x-y|)^{2n}(\log\frac{t+|x-y|}{r_k})^{2+2\epsilon}}\\
&\quad\times\frac{1}{(t+|x-y|)^{\lambda n-2n}}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big]^{\frac12}dzdx\\
&\leq C\sum_k\int_{(E^*)^c}\int_{Q_k}\frac{|b(z)|}{(|x-x_k|+C_\epsilon r_k)^n(\log\frac{|x-x_k|+C_\epsilon r_k}{r_k})^{1+\epsilon}}\Big[\iint_{\substack{y\in(4B_k)^c\\|y-x_k|+C_\epsilon r_k<t\\|y-z|<t\\t\leq|y-x|}}\\
&\quad\frac{t^{\lambda n}[\log\frac{t+|x-y|}{r_k}]^{2+2\epsilon}}{(t+|x-y|)^{\lambda n-2n}}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big]^{\frac12}dzdx\\
&\leq C\sum_k\int_{(E^*)^c}\int_{Q_k}\frac{|b(z)|}{|x-x_k|^n(\log\frac{|x-x_k|+C_\epsilon r_k}{r_k})^{1+\epsilon}}\Big[\int_{\substack{y\in(4B_k)^c\\|y-x|\geq|y-x_k|+C_\epsilon r_k}}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}\\
&\quad-\frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\Big|^2\Big(\int_{|y-x_k|+C_\epsilon r_k}^{|y-x|}\frac{t^{\lambda n}(\log\frac{t+|x-y|}{r_k})^{2+2\epsilon}}{(t+|x-y|)^{\lambda n-2n}t^{n+2\rho+1}}dt\Big)dy\Big]^{1/2}dzdx.
\end{aligned}$$

Notice that, the function $g(s)=\frac{(\log s)^{2+2\epsilon}}{s^\epsilon}$ is decreasing when $s>e^{(2+2\epsilon)/\epsilon}$ and

$$\frac{t+|x-y|}{r_k}\geq\frac{|y-x_k|+C_\epsilon r_k+|x-y|}{r_k}\geq\frac{|y-x_k|+C_\epsilon r_k}{r_k}>C_\epsilon=e^{(2+2\epsilon)/\epsilon}.$$

Then

$$\frac{(\log\frac{t+|x-y|}{r_k})^{2+2\epsilon}}{(\frac{t+|x-y|}{r_k})^\epsilon}=g\Big(\frac{t+|x-y|}{r_k}\Big)\leq g\Big(\frac{|y-x_k|+C_\epsilon r_k}{r_k}\Big)=\frac{(\log\frac{|y-x_k|+C_\epsilon r_k}{r_k})^{2+2\epsilon}}{(\frac{|y-x_k|+C_\epsilon r_k}{r_k})^\epsilon},$$

that is

$$\frac{(\log\frac{t+|x-y|}{r_k})^{2+2\epsilon}}{(t+|x-y|)^\epsilon}\leq\frac{(\log\frac{|y-x_k|+C_\epsilon r_k}{r_k})^{2+2\epsilon}}{(|y-x_k|+C_\epsilon r_k)^\epsilon}.$$

Since $t\leq|x-y|$, then $\frac{1}{t+|x-y|}\leq\frac{1}{2t}$. Together this with the above inequality we get

$$\begin{aligned}
&\int_{|y-x_k|+C_\epsilon r_k}^{|y-x|}\frac{(\log\frac{t+|x-y|}{r_k})^{2+2\epsilon}}{(t+|x-y|)^{\lambda n-2n}t^{n+2\rho+1-\lambda n}}dt\\
&=\int_{|y-x_k|+C_\epsilon r_k}^{|y-x|}\frac{(\log\frac{t+|x-y|}{r_k})^{2+2\epsilon}}{(t+|x-y|)^\epsilon}\frac{1}{(t+|x-y|)^{\lambda n-2n-\epsilon}t^{n+2\rho+1-\lambda n}}dt
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{|y-x_k|+C_\epsilon r_k}^{\infty} \frac{(\log \frac{|y-x_k|+C_\epsilon r_k}{r_k})^{2+2\epsilon}}{(|y-x_k|+C_\epsilon r_k)^{\epsilon}} \frac{1}{t^{2\rho-n+1-\epsilon}}dt \\
&= C\frac{(\log \frac{|y-x_k|+C_\epsilon r_k}{r_k})^{2+2\epsilon}}{(|y-x_k|+C_\epsilon r_k)^{\epsilon}} \frac{1}{(|y-x_k|+C_\epsilon r_k)^{2\rho-n-\epsilon}} \\
&= C\frac{(\log \frac{|y-x_k|+C_\epsilon r_k}{r_k})^{2+2\epsilon}}{(|y-x_k|+C_\epsilon r_k)^{2\rho-n}}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
J_{32} \quad \leq \quad & C\sum_k \int_{(E^*)^c} \int_{Q_k} \frac{|b(z)|}{|x-x_k|^n (\log \frac{|x-x_k|+C_\epsilon r_k}{r_k})^{1+\epsilon}} \Big[\int_{\substack{y\in(4B_k)^c \\ |y-x|\geq|y-x_k|+C_\epsilon r_k}} \Big| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \\
& -\frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\Big|^2 \frac{(\log \frac{|y-x_k|+C_\epsilon r_k}{r_k})^{2+2\epsilon}}{(|y-x_k|+C_\epsilon r_k)^{2\rho-n}} dy\Big]^{\frac{1}{2}} dzdx.
\end{aligned}$$

Applying Lemma 2.2, we have

$$\begin{aligned}
J_{32} \quad \leq \quad & C\sum_k \int_{(E^*)^c} \int_{Q_k} \frac{|b(z)|}{|x-x_k|^n (\log \frac{|x-x_k|+C_\epsilon r_k}{r_k})^{1+\epsilon}} \Big[\int_{y\in(4B_k)^c} \Big(\frac{1+|\Omega(y-x_k)|}{|y-x_k|^{n-\rho} (\log \frac{|y-x_k|}{r_k})^{\alpha}} \Big)^2 \\
& \times \frac{(\log \frac{|y-x_k|+C_\epsilon r_k}{r_k})^{2+2\epsilon}}{(|y-x_k|+C_\epsilon r_k)^{2\rho-n}} dy\Big]^{\frac{1}{2}} dzdx \\
\leq \quad & C\sum_k \int_{Q_k} |b(z)|dz \Big[\int_{y\in(4B_k)^c} \frac{(1+|\Omega(y-x_k)|)^2 (\log \frac{|y-x_k|+C_\epsilon r_k}{r_k})^{2+2\epsilon}}{|y-x_k|^n (\log \frac{|y-x_k|}{r_k})^{2\alpha}} dy\Big]^{\frac{1}{2}} \\
& \times \int_{(E^*)^c} \frac{1}{|x-x_k|^n (\log \frac{|x-x_k|+C_\epsilon r_k}{r_k})^{1+\epsilon}} dx.
\end{aligned}$$

Let

$$\begin{aligned}
I' \quad = \quad & \Big(\int_{y\in(4B_k)^c} \frac{(1+|\Omega(y-x_k)|)^2 (\log \frac{|y-x_k|+C_\epsilon r_k}{r_k})^{2+2\epsilon}}{|y-x_k|^n (\log \frac{|y-x_k|}{r_k})^{2\alpha}} dy \Big)^{\frac{1}{2}} \\
= \quad & \Big(\int_{S^{n-1}} (1+|\Omega(y')|)^2 d\sigma(y') \Big)^{1/2} \Big(\int_{4r_k}^{\infty} \frac{(\log(\frac{r}{r_k}+C_\epsilon))^{2+2\epsilon}}{r(\log \frac{r}{r_k})^{2\alpha}} dr \Big)^{\frac{1}{2}} \\
\leq \quad & \Big[\Big(\int_{S^{n-1}} d\sigma(y') \Big)^{1/2} + \Big(\int_{S^{n-1}} |\Omega(y')|^2 d\sigma(y') \Big)^{\frac{1}{2}} \Big] \Big[\int_4^{\infty} \frac{(\log(t+C_\epsilon))^{2+2\epsilon}}{t(\log t)^{2\alpha}} dt \Big]^{\frac{1}{2}} \\
\leq \quad & C\Big[\int_4^{\infty} \frac{(\log(C't))^{2+2\epsilon}}{t(\log t)^{2\alpha}} dt \Big]^{\frac{1}{2}} (where \quad C' = 1+\frac{C_\epsilon}{4} > 1) \\
= \quad & C\Big[\int_4^{\infty} \frac{(\frac{\log C'}{\log t}+1)^{2+2\epsilon} (\log t)^{2+2\epsilon}}{t(\log t)^{2\alpha}} dt \Big]^{\frac{1}{2}} \\
\leq \quad & C\Big[\int_4^{\infty} \frac{1}{(\log t)^{2\alpha-2-2\epsilon}} d(\log t) \Big]^{\frac{1}{2}} \\
= \quad & C.
\end{aligned}$$

Thus

$$J_{32} \leq C \sum_k \int_{Q_k} |b(z)| dz \leq C \|b\|_1. \tag{3.11}$$

By (3.9) – (3.11) we obtain $J_3 \leq C\|b\|_1$. Then we finish the proof of Theorem 3.1. □

Applying the Marcinkiewicz interpolation theorem between the $(L^1, L^{1,\infty})$-boundedness in Theorem 3.1 and the (L^2, L^2)-boundedness in Theorem A, we obtain immediately the $L^p(\mathbb{R}^n)$ boundedeness of the operator $\mu_\lambda^{*,\rho}$ for $1 < p < 2$. Combining the result of Theorem A again we have the $L^p(\mathbb{R}^n)$ boundedeness of the operator $\mu_\lambda^{*,\rho}$ for all $1 < p < \infty$.

Theorem 3.2. *Suppose that Ω satisfies the same conditions as in Theorem 3.1, then for $\rho > n/2$, $\lambda > 2$ and $1 < p < \infty$, there is*

$$\|\mu_\lambda^{*,\rho}(f)(x)\|_p \leq C\|f\|_p.$$

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