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Berezin Transform on the L^2 Space with Respect to the Invariant Measure

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Abstract

In this paper we study about some of the ergodicity properties of the Berezin transform B on the L^2 space of the disk with respect to the invariant measure.

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1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $dA(z)$ be the area measure on $\mathbb{D}$ normalized so that the area of the disk is 1. Let $L^2(\mathbb{D}, dA)$ be the Hilbert space of Lebesgue measurable functions f on $\mathbb{D}$ with

$$\|f\|_2 = \left[\int_{\mathbb{D}} |f(z)|^2 dA(z)\right]^{\frac{1}{2}} < \infty.$$

The inner product is defined as

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z)$$

for $f, g \in L^2(\mathbb{D}, dA)$. The Bergman space $L^2_a(\mathbb{D})$ is the set of those functions in $L^2(\mathbb{D}, dA)$ that are analytic on $\mathbb{D}$. The Bergman space $L^2_a(\mathbb{D})$ is a closed subspace of $L^2(\mathbb{D}, dA)$, and so there is an orthogonal projection P from $L^2(\mathbb{D}, dA)$ onto $L^2_a(\mathbb{D})$. Let $K(z, \bar{w})$ be the function on $\mathbb{D}\times\mathbb{D}$ defined by $K(z, \bar{w}) = \overline{K_z(w)} = \frac{1}{(1-z\bar{w})^2}$. The function $K(z, \bar{w})$ is called the Bergman kernel of $\mathbb{D}$ or the reproducing kernel of $L^2_a(\mathbb{D})$ because the formula

$$f(z) = \int_{\mathbb{D}} f(w)K(z, \bar{w})dA(w)$$

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reproduces each f in $L_a^2(\mathbb{D})$. For any $n \geq 0, n \in \mathbb{Z}$, let $e_n(z) = \sqrt{n+1}z^n$. The sequence $\{e_n\}$ forms an orthonormal basis for $L_a^2(\mathbb{D})$ and $K(z,\bar{w}) = \sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)} = \frac{1}{(1-z\bar{w})^2}$. Let $k_a(z) = \frac{K(z,\bar{a})}{\sqrt{K(a,\bar{a})}} = \frac{1-|a|^2}{(1-\bar{a}z)^2}$. These functions k_a are called the normalized reproducing kernels of $L_a^2(\mathbb{D})$; it is clear that they are unit vectors in $L_a^2(\mathbb{D})$. For any $a \in \mathbb{D}$, let ϕ_a be the analytic mapping on $\mathbb{D}$ defined by $\phi_a(z) = \frac{a-z}{1-\bar{a}z}, z \in \mathbb{D}$. An easy calculation shows [9] that the derivative of ϕ_a at z is equal to $-k_a(z)$. It follows that the real Jacobian determinant of ϕ_a at z is

$$J_{\phi_a}(z) = |k_a(z)|^2 = \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}.$$

Let Aut$(\mathbb{D})$ be the Lie group of all automorphisms (biholomorphic mappings) of $\mathbb{D}$. Let $L^\infty(\mathbb{D}, dA)$ be the Banach space of all essentially bounded measurable functions f on $\mathbb{D}$ with

$$\|f\|_\infty = \text{ ess sup } \{|f(z)| : z \in \mathbb{D}\}.$$

Let $H^\infty(\mathbb{D})$ be the space of bounded analytic functions on $\mathbb{D}$. By a harmonic function we mean a complex valued function on $\mathbb{D}$ whose Laplacian is identically 0.

If $f \in L^1(\mathbb{D}, dA)$, the Berezin transform of f is, by definition,

$$(Bf)(w) = \widetilde{f}(w) = \langle fk_w, k_w \rangle = \int_{\mathbb{D}} \frac{(1-|w|^2)^2}{|1-\bar{w}z|^4} f(z) dA(z), w \in \mathbb{D},$$

where k_w is the normalized reproducing kernel at $w \in \mathbb{D}$ given by $k_w(z) = \frac{1-|w|^2}{(1-\bar{w}z)^2}$. Notice that $k_w \in L^\infty(\mathbb{D})$ for all $w \in \mathbb{D}$, so the definition makes sense. On $\mathbb{D}$, the only measure left invariant by all Mobius transformations $z \mapsto e^{i\theta}\frac{w-z}{1-\bar{w}z} := e^{i\theta}\phi_w(z), w \in \mathbb{D}, \theta \in \mathbb{R}$ is the pseudo-hyperbolic measure $d\eta(z) = \frac{dA(z)}{(1-|z|^2)^2}$. The invariance may be verified by direct computation. It turns out [4] that the Berezin transform behaves well with respect to the invariant measures. The mapping $B : f \to \widetilde{f}$ is a contractive linear operator on each of the spaces $L^p(\mathbb{D}, d\eta(z)), 1 \leq p \leq \infty$ and $L^1(\mathbb{D}, d\eta) \subset L^1(\mathbb{D}, dA)$. Let $\mathcal{L}(L^2(\mathbb{D}, d\eta))$ be the set of all bounded linear operators from $L^2(\mathbb{D}, d\eta)$ into itself.

The Berezin transform plays an important role in the theory of Toeplitz operators on the Bergman space and the Fock space. In this paper we discuss about some of the ergodicity properties of the Berezin transform defined on the L^2 space of the disk with respect to the invariant measure. In section 2 we investigate the boundedness of the Berezin transform on various L^p spaces. We observe that there is no nonzero fixed points of the Berezin transform B on $L^2(\mathbb{D}, d\eta)$ and B is a positive contraction with norm equal to $\frac{\pi}{4}$. In section 3 we studied about some ergodicity properties of the Berezin transform B and characterize those functions $g \in L^2(\mathbb{D}, d\eta)$ that are in range of B.

2 Berezin transform

In this section we show that the Berezin transform is a contractive linear operator on $L^p(\mathbb{D}, d\eta(z))$ where $d\eta(z) = K(z,z)dA(z)$ and $1 \leq p \leq \infty$. We also show that the Berezin transform $B \in \mathcal{L}(L^2(\mathbb{D}, d\eta))$ is a positive contraction and spectrum of B is the interval $[0, \frac{\pi}{4}]$.

Theorem 2.1. *The Berezin transform B is a contractive linear operator on each of the spaces $L^p(\mathbb{D}, d\eta(z)), 1 \leq p \leq \infty$. Further, B is a self-adjoint operator from $L^2(\mathbb{D}, d\eta)$ into itself.*

Proof. Notice that $L^1(\mathbb{D}, d\eta) \subset L^1(\mathbb{D}, dA)$. Since the Berezin transform is defined on the space $L^1(\mathbb{D}, dA)$ hence B is defined on $L^1(\mathbb{D}, d\eta)$. Further $|(Bf)(w)| = \left|\int_{\mathbb{D}} f(z)|k_w(z)|^2 dA(z)\right| \leq B(|f|)(w)$. Thus

$$\begin{aligned}
\int_{\mathbb{D}} |(Bf)(w)|K(w,w)dA(w) &\leq \int_{\mathbb{D}} \left(\int_{\mathbb{D}} |f(z)||k_w(z)|^2 dA(z)\right) K(w,w)dA(w) \\
&= \int_{\mathbb{D}} |f(z)| \int_{\mathbb{D}} |K(z,w)|^2 dA(w)dA(z) \\
&= \int_{\mathbb{D}} |f(z)|K(z,z)dA(z).
\end{aligned}$$

The change of the order of integration being justified by the positivity of the integrand. It thus follows that B is a contraction on $L^1(\mathbb{D}, d\eta)$. The same is true for $L^\infty(\mathbb{D})$ as

$$|\widetilde{f}(w)| = |\langle fk_w, k_w\rangle| \leq \|fk_w\|_2\|k_w\|_2 \leq \|f\|_\infty\|k_w\|_2^2 = \|f\|_\infty.$$

Hence the result follows from the Marcinkiewicz interpolation theorem. We now verify that B is a self-adjoint operator on $L^2(\mathbb{D}, d\eta)$. For $f \in L^2(\mathbb{D}, d\eta)$,

$$\begin{aligned}
\langle Bf, f\rangle_{L^2(\mathbb{D},d\eta)} &= \int_{\mathbb{D}} (Bf)(z)\overline{f(z)}K(z,z)dA(z) \\
&= \int_{\mathbb{D}} \left(\int_{\mathbb{D}} (f \circ \phi_z)(w)dA(w)\right) \overline{f(z)}K(z,z)dA(z) \\
&= \int_{\mathbb{D}} \left(\int_{\mathbb{D}} f(w)|k_z(w)|^2 dA(w)\right) \overline{f(z)}K(z,z)dA(z) \\
&= \int_{\mathbb{D}} \int_{\mathbb{D}} f(w)|K(z,w)|^2 dA(w)\overline{f(z)}dA(z) \\
&= \int_{\mathbb{D}} f(w)K(w,w)dA(w) \int_{\mathbb{D}} \overline{f(z)} \frac{|K(z,w)|^2}{K(w,w)} dA(z) \\
&= \int_{\mathbb{D}} f(w) \overline{\left(\int_{\mathbb{D}} f(z)|k_w(z)|^2 dA(z)\right)} d\eta(w) \\
&= \langle f, Bf\rangle_{L^2(\mathbb{D},d\eta)}.
\end{aligned}$$

□

Lemma 2.2. *A function $f \in L^1(\mathbb{D}, dA)$ is harmonic if and only if $Bf = f$.*

Proof. If $f \in L^1(\mathbb{D}, dA)$ is harmonic, then so is $f \circ \phi_a$ for any $a \in \mathbb{D}$; by the mean value property,

$$(Bf)(z) = \int_{\mathbb{D}} f(\phi_z(w))dA(w) = (f \circ \phi_z)(0) = f(z).$$

Conversely, if $Bf = f$ then f is harmonic. This follows from [1]. □

From Lemma 2.2 it is not difficult to see that if $f \in L^1(\mathbb{D}, d\eta)$ then $Bf = f$ if and only if f is harmonic. This is so as $L^1(\mathbb{D}, d\eta) \subset L^1(\mathbb{D}, dA)$. We shall now show that if $f \in L^2(\mathbb{D}, d\eta)$ is harmonic then $f \equiv 0$.

Lemma 2.3. *If $f \in L^2(\mathbb{D}, d\eta)$ is harmonic then $f \equiv 0$.*

Proof. Let

$$M(r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt.$$

This is a nonnegative and nondecreasing function of r. Further,

$$\|f\|^2_{L^2(\mathbb{D}, d\eta)} = \int_0^1 M(r) \frac{2r}{(1-r^2)^2} dr < \infty.$$

So $M(r)$ must tend to zero as $r \to 1$. Thus $M(r) \equiv 0$, whence $f \equiv 0$. □

From Lemma 2.3 it follows that there is no nonzero fixed point of B in $L^2(\mathbb{D}, d\eta)$. We shall now discuss about Fourier-Helgason transform [5], [3] on the disk. It maps a function $f(z)$ on the disk into a function $\widehat{f}(t,b)$ of $t \in \mathbb{R}$ and b on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. In fact

$$\widehat{f}(t,b) = \int_{\mathbb{D}} f(x) e_{t,b}(x) d\eta(x)$$

where $e_{t,b}(x) = \left(\frac{1-|x|^2}{|b-x|^2}\right)^{\frac{1}{2}+it}$, $x \in \mathbb{D}, t \in \mathbb{R}$ and $b \in \mathbb{T}$. On L^2 with respect to the invariant measure, the Berezin transform is a Fourier multiplier with respect to the Fourier-Helgason transform; the multiplier function being $(t^2 + \frac{1}{4})\frac{\pi}{\cosh(\pi t)}$, $t \in \mathbb{R}$. That is, for the Berezin transform B one has $\widehat{(Bf)}(t,b) = m(t)\widehat{f}(t,b)$ where $m(t) = (t^2 + \frac{1}{4})\frac{\pi}{\cosh(\pi t)}$. For more details see [5] and [2].

Lemma 2.4. *The following is true for the Berezin transform B as an operator on $L^2(\mathbb{D}, d\eta)$:*

(i) *B is positive.*

(ii) *B^n converges to 0 in SOT.*

(iii) *B^n converges to 0 in norm.*

(iv) *The operator $I - B$ not only has closed range but even is invertible.*

(v) *The spectrum of B is the interval $[0, \frac{\pi}{4}]$ and $\|B\| = \frac{\pi}{4}$.*

(vi) *-1 is not an eigenvalue of B.*

(vii) $\frac{1}{n}\sum_{k=0}^{n-1} B^k \to 0$ *in the strong operator topology.*

Proof. Observe that the function $m(t)$ has a maximum at $t = 0$ with value $\frac{\pi}{4}$. By spectral theorem, B has thus norm $\frac{\pi}{4}$, which is strictly less than 1. Using the Fourier-Helgason transform, one has (by Plancherel theorem, which also holds [5] for this transform) $\langle Bf, f\rangle = \langle\widehat{(Bf)}, \widehat{f}\rangle = \int_{\mathbb{R}}\int_{\mathbb{T}} m(t)|\widehat{f}(t,b)|^2 dtdb \geq 0$ since the multiplier function $m(t) = (t^2 + \frac{1}{4})\frac{\pi}{\cosh(\pi t)}$ is positive. Thus the operator B is positive. This also gives the spectral decomposition of B. Let $E(\lambda)$ be the resolution of identity for the self-adjoint operator B. Then $\|B^n f\|^2 = \int_{[0,\frac{\pi}{4}]} |\lambda^n|^2 d\langle E(\lambda)f, f\rangle$. According to the Lebesgue monotone convergence theorem, this tends to $\|(I - E(1-))f\|^2 = \|P_{\ker(B-I)}f\|^2$. Furthermore, $\ker(I - B) = \{0\}$ since 1 is not in the spectrum of B by Lemma 2.3, so $\|B^n f\|$ tends to zero. In fact, even $\|B^n\|$ tends to zero as $\|B\| < 1$. Thus the operator $I - B$ not only has closed range but even is invertible. Finally, the spectrum of B is the closure of the essential range of the multiplier function above, which is the interval $[0, \pi/4]$. So -1 is not even in the spectrum. Since $\overline{\text{range}(I - B)} \oplus \ker(I - B) = L^2(\mathbb{D}, d\eta) \oplus \{0\} = L^2(\mathbb{D}, d\eta)$, and B is a contraction, it follows from [8] that $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} B^k h = P_{\ker(I-B)} = 0$ for all $h \in L^2(\mathbb{D}, d\eta)$. □

Lemma 2.5. *Let B be the Berezin transform defined on* $L^2(\mathbb{D}, d\eta)$. *Then*

(i) $\ker(I - B) = \ker((I - B)^2)$.

(ii) $\overline{\textit{range}((I - B)^2)} = L^2(\mathbb{D}, d\eta)$.

(iii) *range*$((I - B)^2)$ *is closed and range*$(I - B) =$ *range*$((I - B)^2) = L^2(\mathbb{D}, d\eta)$.

Proof. (i) Let $f \in \ker((I - B)^2)$. Then the element $g = (I - B)f$ is in $\ker(I - B) \cap \text{range}(I - B) = \{0\} \cap L^2(\mathbb{D}, d\eta) = \{0\}$. Hence $f \in \ker(I - B)$. Thus $\ker((I - B)^2) \subseteq \ker(I - B)$. It is not difficult to see that $\ker(I - B) \subseteq \ker((I - B)^2)$. (ii) Since $(I - B)$ is self adjoint, we have $(\text{range}(I - B))^\perp = \ker(I - B) = \ker((I - B)^2) = (\text{range}((I - B)^2))^\perp$ and note that $\text{range}(I - B)$ is closed. Hence by Hahn Banach theorem, $L^2(\mathbb{D}, d\eta) = \text{range}(I - B) = \overline{\text{range}((I - B)^2)}$. (iii)We shall show that $\text{range}(I - B)$ is closed if and only if $\text{range}((I - B)^2)$ is closed. Suppose $I - B$ has closed range. Then from [7] it follows that $\text{range}(I - B)(I - B^*) = \text{range}((I - B)^2)$ is closed and $\text{range}(I - B) = \text{range}((I - B)(I - B^*)) = \text{range}((I - B)^2)$. Conversely, if $(I - B)(I - B^*)$ has closed range then $L^2(\mathbb{D}, d\eta) = \text{range}((I - B)(I - B^*)) \oplus \ker((I - B)(I - B^*)) = \text{range}((I - B)(I - B^*)) \oplus \ker(I - B)^* \subset \text{range}(I - B) \oplus \ker(I - B)^* \subset L^2(\mathbb{D}, d\eta)$ which also implies that $I - B$ has closed range. □

3 Ergodicity properties of the Berezin transform

In this section, we prove an ergodicity properties of the Berezin transform for sums that are weighted with binomial coefficients. We shall also show that $\sum_{k=1}^{\infty} c_{n,k}B^k$ converges to 0 in the strong operator topology in $\mathcal{L}(L_a^2(\mathbb{D}))$ for any uniformly regular matrix $(c_{n,k})$.

Definition 3.1. A matrix $(c_{n,k}), k, n = 1, 2, \cdots$ of real numbers is called uniformly regular if

$$\sup_n \sum_{k=1}^{\infty} |c_{n,k}| \le C < \infty;$$

$$\lim_{n\to\infty} \sup_k |c_{n,k}| = 0;$$

$$\lim_{n\to\infty} \sum_k c_{n,k} = 1.$$

Theorem 3.2. *Let B be the Berezin transform defined on $L^2(\mathbb{D}, d\eta)$ and for $g \in L^2(\mathbb{D}, d\eta)$, $K_n g = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}B^k g$. Then*

(i) *K_n converges to 0 in the strong operator topology.*

(ii) *$\sum_{k=1}^{\infty} c_{n,k}B^k$ converges to 0 in the strong operator topology in $\mathcal{L}(L^2(\mathbb{D}, d\eta))$ for any uniformly regular matrix $(c_{n,k})$.*

Proof. (i) From Lemma 2.4 it follows that the operator B is positive, hence self-adjoint and range$(I-B)$ is closed. Further, if $g \in \ker(I-B) = \{0\}$, then $K_n g = g = 0$ for all n and the map K_n is a linear contraction for all n. We shall show that if $g \in (\ker(I-B))^{\perp} = \text{range}(I-B) = L^2(\mathbb{D}, d\eta)$, then $\|K_n g\|$ converges to 0 as $n \to \infty$. In order to show that $\|K_n g\|$ converges to 0 for $g \in (\ker(I-B))^{\perp} = \text{range}(I-B) = L^2(\mathbb{D}, d\eta)$, let $g = Bf - f$ for some $f \in L^2(\mathbb{D}, d\eta)$.

Then

$$\begin{aligned} K_n g &= \tfrac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left(B^{k+1}f - B^k f\right) \\ &= -\tfrac{1}{2^n}f + \tfrac{1}{2^n}\sum_{k=1}^{n}\left(\binom{n}{k-1} - \binom{n}{k}\right)B^k f + \frac{1}{2^n}B^{n+1}f. \end{aligned}$$

Since

$$\left|\binom{n}{k-1} - \binom{n}{k}\right| = \frac{|2k-n-1|}{k}\binom{n}{k-1},$$

we get $\|K_n g\| \le \frac{1}{2^n}\left(2 + \sum_{k=1}^{n}\frac{|2k-n-1|}{k}\binom{n}{k-1}\right)\|g\|$. It will suffice to show that

$$\lim_{n\to\infty}\frac{1}{2^n}\sum_{k=1}^{n}\frac{|2k-n+1|}{k}\binom{n}{k} = 0. \tag{1}$$

Let $0 < \alpha < \frac{1}{2}$. Let $\lfloor x \rfloor$ denote the largest integer not greater than x. Using Stirling's formula, it is easy to prove that for some constant $C > 0$, we always have

$$\begin{aligned} \sum_{k=n-\lfloor \alpha n-1 \rfloor}^{n}\binom{n}{k} &= \sum_{k=0}^{\lfloor \alpha n-1 \rfloor}\binom{n}{k} \\ &\le Cn\left(\tfrac{1}{\alpha^{\alpha}(1-\alpha)^{(1-\alpha)}}\right)^n. \end{aligned}$$

Since $\alpha^{\alpha}(1-\alpha)^{(1-\alpha)} > \frac{1}{2}$, we have

$$\limsup_{n\to\infty} \frac{1}{2^n}\sum_{k=1}^{n} \frac{|2k-n+1|}{k}\binom{n}{k} = \limsup_{n\to\infty} \frac{1}{2^n}\sum_{k=\lfloor \alpha n\rfloor}^{n-\lfloor \alpha n\rfloor} \frac{|2k-n+1|}{k}\binom{n}{k}. \tag{2}$$

However, for $\alpha n - 1 \le k \le n - \alpha n + 1$, we have $\frac{|2k-n+1|}{k} \le \frac{(1-2\alpha)n+3}{\alpha n-1}$, which together with (2) yields

$$\limsup_{n\to\infty} \frac{1}{2^n}\sum_{k=1}^{n} \frac{|2k-n+1|}{k}\binom{n}{k} \le \frac{1-2\alpha}{\alpha}.$$

Taking α arbitrarily close to $\frac{1}{2}$ gives (1).

Now we shall prove (ii). Since $B^n \to 0$ in norm hence for $f \in L^2(\mathbb{D}, d\eta)$, $\langle B^n f, f\rangle \to 0$ as $n \to \infty$. We shall show that $\sum_k c_{n,k} B^k$ converges to 0 in the strong operator topology in $\mathcal{L}(L^2(\mathbb{D}, d\eta))$ for any uniformly regular matrix $(c_{n,k})$. Notice that for $f \in L^2(\mathbb{D}, d\eta)$,

$$\begin{aligned} \|\sum_k c_{n,k} B^k f\|^2 &\le \sum_k \sum_j c_{n,k} c_{n,j} \langle B^k f, B^j f\rangle \\ &\le \textstyle\sum_k \sum_j |c_{n,k} c_{n,j} \langle B^k f, B^j f\rangle|. \end{aligned} \tag{3}$$

Let us fix $0 < \epsilon < 1$. Since for $f \in L^2(\mathbb{D}, d\eta)$, $\|B^n f\| \to 0$, we can find $L > 0$ such that for $i > L$ and $j \ge 0$,

$$\|B^i f\|^2 - \|B^{i+j} f\|^2 \le \epsilon^2$$

and $|\langle B^i f, f\rangle| \le \epsilon$. Then

$$\begin{aligned} |\langle B^i f, f\rangle - \langle B^{i+j} f, B^j f\rangle| &= |\langle B^i f, f\rangle - \langle B^j B^{i+j} f, f\rangle| \\ &\le \|B^i f - B^j B^{i+j} f\| \|f\| \\ &= \left(\|B^i f - B^j B^{i+j} f\|^2\right)^{\frac{1}{2}} \|f\| \\ &= \left(\|B^i f\|^2 - 2\|B^{i+j} f\|^2 + \|B^j B^{i+j} f\|^2\right)^{\frac{1}{2}} \|f\| \\ &\le \left(\|B^i f\|^2 - \|B^{i+j} f\|^2\right)^{\frac{1}{2}} \|f\| \\ &\le \epsilon \|f\| \end{aligned}$$

and therefore

$$|\langle B^{i+j} f, B^j f\rangle| \le \epsilon(1 + \|f\|)$$

for all $i > L$ and $j \ge 0$, i.e., for $|k - j| \ge i$ the inequality

$$|\langle B^k f, B^j f\rangle| \le \epsilon(1 + \|f\|)$$

is valid. We shall fix $\delta > 0$, and let N be such a natural number that $\sup_k |c_{n,k}| < \delta$ for $n \ge N$. Then the expression (3) for $n \ge N$ could be estimated as follows:

$$\begin{aligned}
\sum_k \sum_j |c_{n,k}c_{n,j}\langle B^k f, B^j f\rangle| &= \sum_{|k-j|\le i} |c_{n,k}c_{n,j}\langle B^k f, B^j f\rangle| \\
&+ \sum_{|k-j|>i} |c_{n,k}c_{n,j}\langle B^k f, B^j f\rangle| \\
&\le \sum_k |c_{n,k}|\delta\|f\|^2(2i+1) \\
&+ \sum_k \sum_j |c_{n,k}c_{n,j}|\epsilon(1+\|f\|) \\
&\le C\delta\|f\|^2(2i+1) + C^2\epsilon(1+\|f\|).
\end{aligned}$$

Since ϵ, δ are arbitrary, it follows that $\sum_k c_{n,k}B^k$ converges to 0 in the strong operator topology in $\mathcal{L}(L^2(\mathbb{D}, d\eta))$ for any uniformly regular matrix $(c_{n,k})$. □

Theorem 3.3. *Let B be the Berezin transform defined on $L^2(\mathbb{D}, d\eta)$ and let $K_n = \frac{1}{2^n}\sum_{j=0}^n \binom{n}{j} B^j$. Then $K_n \in \mathcal{L}(L^2(\mathbb{D}, d\eta))$ and $K_n \to 0$ in norm.*

Proof. The norm of the operator $B \in \mathcal{L}(L^2(\mathbb{D}, d\eta))$ is equal to $\frac{\pi}{4}$ which is less than 1. Hence B is power bounded. Now $K_n = \frac{1}{2^n}\sum_{j=0}^n \binom{n}{j} B^j = \left(\frac{I+B}{2}\right)^n = S^n$ where $S = \frac{I+B}{2}$. By a result of Katznelson-Tzafriri [6], we have $\sigma(S) \cap \{z \in \mathbb{C} : |z| = 1\} \subset \{1\}$. Observe also that the operator $I - B$ is invertible and Range$(I - B) = L^2(\mathbb{D}, d\eta)$. Thus $1 \notin \sigma(B)$. Hence $1 \notin \sigma(S)$ as $I - S = I - (\frac{I+B}{2}) = \frac{I-B}{2}$. Thus $\sigma(S) \cap \{z \in \mathbb{C} : |z| = 1\} = \emptyset$. Hence $\|S^n\| = \|K_n\| \to 0$. □

Corollary 3.4. *Let $U \in \mathcal{L}(L^2(\mathbb{D}, d\eta))$ be unitary and B be the Berezin transform defined on $L^2(\mathbb{D}, d\eta)$. Then*

$$1 - \frac{\pi}{4} \le \|U - B\| \le 1 + \frac{\pi}{4}.$$

Proof. Let $f \in L^2(\mathbb{D}, d\eta)$ be such that $\|f\| = 1$. Then

$$\|(U - B)f\|^2 = \langle (I + B^2 - U^*B - BU)f, f\rangle \ge 1 + \|Bf\|^2 - 2\|Bf\| = (1 - \|Bf\|)^2.$$

But since B is positive,

$$\inf_{\|f\|=1} \|Bf\| = \inf_{\|f\|=1} \langle Bf, f\rangle$$

and by Lemma 2.4

$$\sup_{\|f\|=1} \|Bf\| = \sup_{\|f\|=1} \langle Bf, f\rangle.$$

Hence

$$\begin{aligned}
\|(U - B)\| &\ge \sup_{\|f\|=1} |1 - \|Bf\|| \\
&= \sup_{\|f\|=1} |1 - \langle Bf, f\rangle| \\
&= \sup_{\|f\|=1} |\langle (I - B)f, f\rangle| \\
&= \|I - B\| \ge \|I\| - \|B\| = 1 - \tfrac{\pi}{4}.
\end{aligned}$$

This proves the left inequality. Again by Lemma 2.4

$$\begin{aligned}
\|(U-B)\| &= \sup_{\|f\|=1}\|Uf-Bf\| \\
&\le \sup_{\|f\|=1}(1+\|Bf\|) \\
&= \sup_{\|f\|=1}\langle (I+B)f,f\rangle \\
&= \|I+B\| \le \|I\|+\|B\| = 1+\tfrac{\pi}{4}
\end{aligned}$$

and the result follows. □

Since $B^n \to 0$ in norm hence it is not difficult to check that $g \in \text{range}(I-B)$ if and only if $\sum_{k=0}^{\infty} B^k g$ is convergent. The following lemma is a special case of this result.

Lemma 3.5. *Let B be the Berezin transform defined on $L^2(\mathbb{D}, d\eta)$. Then Range$B$ is the set of all $g \in L^2(\mathbb{D}, d\eta)$ for which the series $\sum_{k=0}^{\infty}(I-B)^k g$ converges with respect to the norm of $L^2(\mathbb{D}, d\eta)$. In this case if $f = \sum_{k=0}^{\infty}(I-B)^k g$ then $f \in (\ker B)^{\perp}$ and $Bf = g$.*

Proof. Suppose Q is the orthogonal projection of $L^2(\mathbb{D}, d\eta)$ onto $(\ker B)^{\perp}$ and E_λ is the spectral resolution of B, with $E_{0^-} = 0, E_1$ the identity on $L^2(\mathbb{D}, d\eta)$, and E_λ right-continuous at each λ. Suppose $n \ge 0$ and $h \in L^2(\mathbb{D}, d\eta)$. Then

$$h - \sum_{k=0}^{n}(I-B)^k Bh = (I-B)^{n+1}h. \tag{4}$$

Thus, with $f = Qh$,

$$\begin{aligned}
\left\| f - \sum_{k=0}^{n}(I-B)^k Bf \right\|^2 &= \|(I-B)^{n+1}f\|^2 \\
&= \int_{0^-}^{1}(1-\lambda)^{2(n+1)}d\langle E_\lambda f, f\rangle.
\end{aligned}$$

Since Q commutes with B and, hence, with $E_0, E_0 f$ is in $\ker B \cap (\ker B)^{\perp} = \{0\}$. Hence, the measure of $\{0\}$ with respect to the measure $\langle E(.)f, f\rangle$ is 0. In view of this and the uniform boundedness and convergence to 0 on $(0,1]$ of $(1-\lambda)^{2(n+1)}$; the integral $\int_{0^-}^{1}(1-\lambda)^{2(n+1)}d\langle E_\lambda f, f\rangle \to 0$ as $n \to \infty$. Hence, $\sum_{k=0}^{\infty}(I-B)^k Bh$ has limit f, the unique pre-image of Bh in $(\ker B)^{\perp}$; in other words, the vector of least norm mapped by B onto Bh. Conversely, if $\sum_{k=0}^{\infty}(I-B)^k g = e$ then $(I-B)e = e - g$. Thus $g = Be$. □

Theorem 3.6. *Let B be the Berezin transform on $L^2(\mathbb{D}, d\eta)$. Then $g \in$ range$B$ if and only if $\sum_{k=0}^{\infty}\|(I-B^2)^{\frac{k}{2}}g\|^2 < \infty$. Further, the series $\sum_{k=0}^{\infty}(I-B^2)^k Bg$ converges and if $\sum_{k=0}^{\infty}(I-B^2)^k Bg = e$ then $g = Be$.*

Proof. From Lemma 3.5 it follows that (by taking the inner product of both sides of (4) by f)

$$\begin{aligned} ||f||^2 - \sum_{k=0}^{n} ||(I-B^2)^{\frac{k}{2}} Bf||^2 &= \langle (I-B^2)^{n+1} f, f\rangle \\ &= \int_{0^-}^{1} (1-\lambda^2)^{(n+1)} d\langle E_\lambda f, f\rangle, \end{aligned}$$

the integral tending to 0 by an argument similar to that in Lemma 3.5. Conversely, suppose that $\sum_{k=0}^{\infty} ||(I-B^2)^{\frac{k}{2}} g||^2 < \infty$. Since, with $Qg = g_1$ and $(I-Q)g = g_2$,

$$\begin{aligned} \sum_{k=0}^{n} ||(I-B^2)^{\frac{k}{2}} g||^2 &= \sum_{k=0}^{n} \{\langle (I-B^2)^k g_1, g_1\rangle + \langle (I-B^2)^k g_2, g_2\rangle\} \\ &\geq \sum_{k=0}^{n} ||g_2||^2. \end{aligned}$$

Hence $g \in (\ker B)^{\perp} = \overline{\mathrm{range} B}$ as B is positive. Since $\left(\sum_{k=n}^{m} (1-\lambda^2)^k \lambda\right)^2 \leq \sum_{k=n}^{m} (1-\lambda^2)^k$ on $[0,1]$, the spectral theorem gives

$$\begin{aligned} \left\| \sum_{k=n}^{m} (I-B^2)^k Bg \right\|^2 &\leq \sum_{k=n}^{m} \langle (I-B^2)^k g, g\rangle \\ &= \sum_{k=n}^{m} ||(I-B^2)^{\frac{k}{2}} g||^2. \end{aligned}$$

Therefore, $\sum_{k=0}^{\infty} (I-B^2)^k Bg = e$ exists in $L^2(\mathbb{D}, d\eta)$. Then $(I-B^2)e = e - Bg$ so that $g - Be$ is in $\ker B \cap (\ker B)^{\perp}$ and $g = Be$. Hence $g \in \mathrm{range} B$. $\square$

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