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CONVERGENCE TO ATTRACTORS UNDER PERTURBATIONS

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Abstract

We show that if for any initial point there exists a trajectory of a nonexpansive set-valued mapping attracted by a given set, then this property is stable under small perturbations of the mapping.

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1 Introduction and statement of the main result

The study of the convergence of iterations of mappings of contractive type has been an important topic in Nonlinear Functional Analysis since Banach's seminal paper [1] on the existence of a unique fixed point for a strict contraction. It is well known that Banach's fixed point theorem also yields convergence of iterates to the unique fixed point. During the last fifty years or so, many developments have taken place in this area. Interesting results have also been obtained regarding set-valued mappings, where the situation is more difficult and less understood. See, for example, [2, 6, 9-13] and the references cited therein. As already mentioned above, one of the methods used for proving the classical Banach theorem is to show the convergence of Picard iterations, which holds for any initial point. In the case of set-valued mappings, we do not have convergence of all trajectories of the dynamical system induced by the given mapping. Convergent trajectories have to be constructed in a

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special way. For example, in [6], if at the moment $t = 0, 1, \ldots$ we have reached a point x_t, then we choose an element of $T(x_t)$ (here T is the mapping) such that x_{t+1} approximates the best approximation of x_t from $T(x_t)$. Since our mapping acts on a general complete metric space we cannot, in general, choose x_{t+1} as the best approximation of x_t by elements of $T(x_t)$. Instead, we choose x_{t+1} to approximate the best approximation up to a positive number ϵ_t, such that the sequence $\{\epsilon_t\}_{t=0}^{\infty}$ is summable. This method allowed Nadler [6] to obtain the existence of a fixed point of a strictly contractive set-valued mapping and the authors of [2] to obtain more general results. In view of this state of affairs, it is important to study convergence of the iterates of both single- and set-valued mappings in the presence of errors.

In particular, it is natural to ask if convergence of the iterates of nonexpansive mappings will be preserved in the presence of computational errors. In [3] we provide affirmative answers to this question. Related results can be found, for example, in [4, 5, 7, 8]. More precisely, in [3] we show that if all exact iterates of a given nonexpansive mapping converge (to fixed points), then this convergence continues to hold for inexact orbits with summable errors. In [8] we continued to study the influence of computational errors on the convergence of iterates of nonexpansive mappings in both Banach and metric spaces. We show there that if all the orbits of a nonexpansive self-mapping of a metric space X converge to some closed subset F of X, then all inexact orbits with summable errors also converge to F. On the other hand, we also construct examples which show that the convergence of inexact orbits no longer holds when the errors are not summable.

In this paper we study the existence of convergent iterations in the presence of computational errors for a nonexpansive set-valued mapping. We show that if for any initial point there exists a trajectory of a nonexpansive set-valued mapping attracted by a given set, then this property is stable under small perturbations of the mapping. More precisely, we prove two assertions. In the first assertion we show that for a given positive number δ, if the perturbations are small enough, then for any initial state there exists a trajectory which is attracted by a δ-neighborhood of the attractor. In the second assertion we show that under the same assumptions for any initial state there exists a trajectory with a subsequence which is attracted by the attractor.

Let (X,ρ) be a metric space. For each $x \in X$ and each nonempty set $A \subset X$, put

$$\rho(x,A) = \inf\{\rho(x,y) : \ y \in A\}.$$

For each pair of nonempty sets $A, B \subset X$, set

$$H(A,B) = \max\{\sup_{x\in A}\rho(x,B),\ \sup_{y\in B}\rho(y,A)\}.$$

Let $T : X \to 2^X \setminus \{\emptyset\}$ satisfy

$$H(T(x),T(y)) \le \rho(x,y) \text{ for all } x,y \in X. \tag{1.1}$$

Theorem 1.1. *Assume that F is a nonempty subset of X and that for each $x \in X$, there is a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that $x_0 = x$ and $x_{i+1} \in T(x_i)$ for all integers $i \ge 0$, and*

$$\lim_{i\to\infty}\rho(x_i,F) = 0.$$

Assume further that

$$\{\epsilon_i\}_{i=0}^{\infty} \subset (0,\infty), \ \sum_{i=0}^{\infty} \epsilon_i < \infty. \tag{1.2}$$

For each integer $i \geq 0$*, let* $T_i : X \to 2^X \setminus \{\emptyset\}$ *satisfy*

$$H(T_i(x), T(x)) \leq \epsilon_i, \ x \in X. \tag{1.3}$$

Then the following two assertions hold.

1. Let $\delta > 0$*. For each* $x \in X$*, there exists a sequence* $\{x_i\}_{i=0}^{\infty}$ *such that* $x_0 = x$*, for each integer* $i \geq 0$*,*

$$x_{i+1} \in T_i(x_i),$$

and

$$\rho(x_i, F) \leq \delta \text{ for all sufficiently large integers } i \geq 0.$$

2. For each $x \in X$*, there exists a sequence* $\{x_i\}_{i=0}^{\infty}$ *such that* $x_0 = x$ *and* $\liminf_{i\to\infty} \rho(x_i, F) = 0$.

2 Proof of Theorem 1.1

Lemma 2.1. *Let* $q \geq 0$ *be an integer. Let the sequence* $\{x_i\}_{i=q}^{\infty} \subset X$ *satisfy*

$$x_{i+1} \in T(x_i) \tag{2.1}$$

for each integer $i \geq q$*.*

Then there is a sequence $\{y_i\}_{i=q}^{\infty} \subset X$ *such that*

$$y_q = x_q, \ y_{i+1} \in T_i(y_i) \text{ for all integers } i \geq q, \tag{2.2}$$

and for all integers $j \geq q+1$*,*

$$\rho(y_j, x_j) \leq \sum_{i=q}^{j-1} 2\epsilon_i. \tag{2.3}$$

Proof. We construct the sequence $\{y_i\}_{i=q}^{\infty} \subset X$ by induction. Set

$$y_q = x_q. \tag{2.4}$$

By (2.1) and (1.3),

$$\rho(x_{q+1}, T_q(x_q)) \leq H(T(x_q), T_q(x_q)) \leq \epsilon_q$$

and there is

$$y_{q+1} \in T_q(y_q) \tag{2.5}$$

such that

$$\rho(x_{q+1}, y_{q+1}) \leq 2\epsilon_q. \tag{2.6}$$

By (2.6), (2.3) holds with $j = q+1$.

Assume that $s \geq q+1$ is an integer and that the elements of $\{y_i\}_{i=q}^{s} \subset X$ satisfy

$$y_{i+1} \in T_i(y_i),\ i = q, \ldots, s-1,$$

$y_q = x_q$, and that (2.3) holds for $j = q+1, \ldots, s$. (Clearly this assumption holds for $s = q+1$ (see (2.4)-(2.6)). By (1.3), (2.1), (1.1) and (2.3) (with $j = s$),

$$\rho(x_{s+1}, T_s(y_s)) \leq \rho(x_{s+1}, T(y_s)) + H(T(y_s), T_s(y_s))$$

$$\leq \rho(x_{s+1}, T(y_s)) + \epsilon_s \leq H(T(x_s), T(y_s)) + \epsilon_s \leq \rho(x_s, y_s) + \epsilon_s \leq \sum_{i=q}^{s-1} 2\epsilon_i + \epsilon_s.$$

By the above relations there is

$$y_{s+1} \in T_s(y_s)$$

such that

$$\rho(x_{s+1}, y_{s+1}) < \sum_{i=q}^{s} 2\epsilon_i.$$

Clearly, the assumption we made concerning s is now seen to hold for $s+1$ too. Therefore the sequence $\{y_i\}_{i=q}^{\infty}$ has indeed been constructed by induction. Lemma 2.1 is proved. □

Proof of Assertion 1. Let $x \in X$. By (1.2) there is a natural number k_0 such that

$$\sum_{i=k_0}^{\infty} \epsilon_i < \delta/8. \tag{2.7}$$

There is a sequence $\{x_i\}_{i=0}^{k_0} \subset X$ such that

$$x_0 = x,\ x_{i+1} \in T_i(x_i),\ i = 0, \ldots, k_0 - 1. \tag{2.8}$$

By the assumptions of the theorem, there is a sequence $\{z_i\}_{i=k_0}^{\infty} \subset X$ such that

$$z_{k_0} = x_{k_0},$$

$$z_{i+1} \in T_i(z_i),\ i = k_0, k_0+1, \ldots,$$

and

$$\lim_{i\to\infty} \rho(z_i, F) = 0. \tag{2.9}$$

By Lemma 2.1, (2.7) and (2.9), there exists a sequence $\{x_i\}_{i=k_0}^{\infty} \subset X$ such that

$$x_{i+1} \in T_i(x_i) \text{ for all integers } i \geq k_0, \tag{2.10}$$

and for all integers $j \geq k_0 + 1$,

$$\rho(z_j, x_j) \leq \sum_{i=k_0}^{j-1} 2\epsilon_i < \delta/4. \tag{2.11}$$

By (2.9) and (2.11), there is a natural number $k_1 > k_0$ such that for all integers $j \geq k_1$,

$$\rho(z_j, F) \leq \delta/4.$$

It follows from this inequality that

$$\rho(x_j, F) \leq \rho(x_j, z_j) + \rho(z_j, F) < \delta/4 + \delta/4 < \delta$$

for all integers $j \geq k_1$. Assertion 1 is proved. □

Proof of Assertion 2. Let $x \in X$. Set

$$S_0 = 0,\ x_0 = x. \tag{2.12}$$

Assume that $q \geq 0$ is an integer and that we have already defined a strictly increasing sequence of nonnegative integers S_i, $i = 0, \ldots, q$, and a sequence $\{x_i\}_{i=0}^{S_q} \subset X$ such that (2.12) holds,

$$x_{i+1} \in T_i(x_i) \tag{2.13}$$

$$\text{for all integers } i \text{ satisfying } 0 \leq i < S_q,$$

and for all integers j satisfying $1 \leq j \leq q$,

$$\rho(x_{S_j}, F) \leq 1/j. \tag{2.14}$$

(Note that for $q = 0$ this assumption holds). By (1.2) there is a natural number $T_1 > S_q + 4$ such that

$$\sum_{i=T_1}^{\infty} \epsilon_i < (4(q+1))^{-1}. \tag{2.15}$$

There is a sequence $\{x_i\}_{i=S_q}^{T_1} \subset X$ such that

$$x_{i+1} \in T_i(x_i),\ i = S_q, \ldots, T_1 - 1. \tag{2.16}$$

By the assumption of the theorem, there is a sequence $\{z_i\}_{i=T_1}^{\infty} \subset X$ such that

$$z_{T_1} = x_{T_1},$$

$$z_{i+1} \in T(z_i) \text{ for all integers } i \geq T_1$$

and

$$\lim_{i\to\infty} \rho(z_i, F) = 0. \tag{2.17}$$

By Lemma 2.1, (2.17) and (2.15), there is a sequence $\{y_i\}_{i=T_1}^{\infty} \subset X$ such that

$$y_{T_1} = z_{T_1},$$

$$y_{i+1} \in T_i(y_i) \text{ for all integers } i \geq T_1, \tag{2.18}$$

and for all integers $j > T_1$,

$$\rho(y_j, z_j) \leq \sum_{i=T_1}^{j-1} 2\epsilon_i < 2\sum_{i=T_1}^{\infty} \epsilon_i < (2(q+1))^{-1}. \tag{2.19}$$

By (2.17) there is an integer $S_{q+1} > T_1 + 4$ such that

$$\rho(z_{S_{q+1}}, F) < (4(q+1))^{-1}. \tag{2.20}$$

By (2.19),

$$\rho(y_{S_{q+1}}, F) \leq \rho(y_{S_{q+1}}, z_{S_{q+1}}) + \rho(z_{S_{q+1}}, F) \leq (2(q+1))^{-1} + (4(q+1))^{-1}. \tag{2.21}$$

For integers $i = T_1 + 1, \ldots, S_{q+1}$, set

$$x_i = y_i. \tag{2.22}$$

By (2.22), (2.13), (2.16), (2.18) and (2.17),

$$x_{i+1} \in T_i(x_i),\ i = 0, \ldots, S_{q+1} - 1.$$

By (2.21) and (2.22),

$$\rho(x_{S_{q+1}}, F) \leq (q+1)^{-1}.$$

Thus the assumption we made regarding q also holds for $q+1$. Therefore we have constructed by induction a strictly increasing sequence of nonnegative integers $\{S_q\}_{q=0}^{\infty}$ and a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that

$$x_{i+1} \in T_i(x_i) \text{ for all integers } i \geq 0$$

and

$$\rho(x_{S_q}, F) \leq q^{-1} \text{ for all integes } q \geq 1.$$

This completes the proof of Assertion 2 and of Theorem 1.1 itself.

□

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