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When Relativistic Mass Meets Hyperbolic Geometry

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Abstract

It is admitted in the literature on special relativity that, being velocity dependent, relativistic mass is a wild notion in the sense that it does not conform with the Minkowskian four-vector formalism. The resulting lack of clear consensus on the basic role of relativistic mass in special relativity has some influence in diminishing its use in modern books. Fortunately, relativistic mechanics is regulated by the hyperbolic geometry of Bolyai and Lobachevsky just as classical mechanics is regulated by Euclidean geometry. Guided by analogies that Euclidean geometry and classical mechanics share with hyperbolic geometry and relativistic mechanics, the objective of this article is to tame the relativistic mass by placing it under the umbrella of the Minkowskian formalism, and to present interesting consequences.

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1 Introduction

Steve Smale's predilection to geometric mechanics is well-known. An overview of Smale's involvement with geometric mechanics, without entering into technical details, is presented by Marsden in [14]. Accordingly, this article on the hyperbolic geometric interpretation of the relativistic mass is dedicated to the 80th Anniversary of Steve Smale.

It is admitted in the literature on special relativity that the notion of the relativistic mass does not conform with the Minkowskian four-vector formalism of special relativity. The failure to recognize that relativistic mass meshes well with the Minkowskian formalism, and to advance the role of relativistic mass in hyperbolic geometry created a void that resulted in the trend to reject the special relativistic concept of relativistic mass.

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Employing the hyperbolic geometric viewpoint of special relativity in [30, 32, 38, 42], the objective of this article is to tame the relativistic mass by recognizing the role it plays in hyperbolic geometry and, hence, placing it under the umbrella of Minkowski's formalism. Straightforward consequences then follow. Our main result is based on Theorem 5.2 in hyperbolic geometry and relativistic mechanics, which is illustrated by its Newtonian counterpart, Theorem 5.3, in Euclidean geometry and classical mechanics.

Invariant mass in special relativity is the rest mass of an object, which is the Newtonian mass of the object [50] as measured by an observer moving along with the object. In contrast, the relativistic mass of a moving object with invariant mass m and velocity $\mathbf{v}$ relative to an inertial rest frame is observer's dependent, given by $m\gamma_{\mathbf{v}}$, relative to the rest frame, where $\gamma_{\mathbf{v}}$ is the Lorentz factor of special relativity.

The idea of relativistic mass dates back to Lorentz's 1904 paper *"Electromagnetic Phenomena in a System Moving With Any Velocity Less Than That of Light"* that introduced the "longitudinal" and "transverse" electromagnetic masses of the electron. A single velocity dependent mass, then, resulted in the relativistic mass, which was introduced by Lewis in 1908. But, the term "relativistic mass" appeared later. "Relativistic mass" came into common usage in the relativity physics literature of the 1920s by books written by Pauli, Eddington, and Born.

The use of the relativistic mass notion in special relativity was promoted by several prominent physicists including Lewis, Tolman, Born, and Fock. However, it is presently falling into disuse in the modern literature on special relativity, pointing to the lack of clear consensus on its basic role. Accordingly, the overall trend in the literature on relativity physics is one of moving away from relativistic mass [17].

On 21 September 1908, Herman Minkowski (1864 – 1909) presented a lecture on "space and time" at the 80th Assembly of the German Natural Scientists and Physicians in Cologne, where he argued famously that certain circumstances require to discard the view of physical space as a Euclidean three-space in favor of a four-dimensional geometry characterized by the invariance of a certain quadratic form [49]. This day is generally considered to be the birthday of the four-dimensional geometric version of special relativity theory [10], resulting in Minkowski's reformulation of Einstein's theory [31]. Today, more than 100 years later, the relativistic mass notion still, seemingly, does not conform with the Minkowskian four-vector formalism of special relativity.

Accordingly, R.W. Brehme emphasizes the advantage of teaching relativity with four-vectors along with the presentation of what he describes as "the bane of the relativistic mass" [2]. Indeed, in strong support of Brehme's opinion that the relativistic mass does not mesh up with the Minkowskian four-vector formalism approach to special relativity, C.G. Adler admits in [1] that the "relativistic mass is a concept in turmoil":

> "Any one who has tried to teach special relativity using the four-vector spacetime approach knows that relativistic mass and four-vectors make for an ill-conceived marriage. . . . The solution is for physics teachers to understand that relativistic mass is a concept in turmoil. If they choose to use it in their course, they should caution the students to this effect."
>
> C.G. Adler, 1987

Yet, W. Rindler expresses in [21] a strong support of the relativistic mass. In fact, Lev B. Okun makes in [18] the case that the concept of relativistic mass is no longer even pedagogically useful. However, T.R. Sandin has argued otherwise [23]. See also [32, pp. 358-359].

The aim of this article is to capture the correct relativistic mass of a particle system and, hence, to demonstrate that the relativistic mass is additive and that, furthermore, it meshes extraordinarily well with the Minkowskian four-vector formalism of special relativity. To achieve the goal, we uncover the link between the relativistic mass and the four-vector formalism. It turns out that the link is the relativistic mass and its associated invariant mass of a system of particles, expressed quantitatively in terms of the invariant masses of the constituent particles and their internal velocities relative to each other. Paradoxically, in the presence of internal velocities, the invariant mass of a system of particles is greater than the sum of the invariant masses of its constituent particles, as noted in [38] and [40]. This paradoxical result is well-known in the literature, where it is obtained from the additivity of energy and momentum and their conservation [16, pp. 185-186]. The resulting *relativistically invariant mass paradox* [40] that stems from the non-additivity of invariant mass is quantitatively attributed in this article to the presence of internal velocities in particle systems.

In order to uncover the invariant mass of a system of particles and its relationship with the four-vector formalism of special relativity, we present the Lorentz transformation boost in terms of its underlying Einstein velocity addition. Accordingly, Sec. 2 is devoted to the presentation of Einstein velocity addition. The latter is associated with Thomas gyration [36, 34], leading to its group-like structure known as a *gyrocommutative gyrogroup*, discovered in 1988 [27], presented in [30, 39], and studied in [30, 32, 38, 42]. In Sec. 3 we present the Lorentz transformation boost and its application to four-momenta in terms of Einstein velocity addition.

The formalism developed in Secs. 2 – 3 enables the invariant and relativistic mass of a system of particles, along with the role they play in the four-vector formalism of special relativity, to be expressed quantitatively, in Secs. 4 – 5, in terms of internal relative velocities. It turns out that owing to the presence of internal relative velocities between constituent particles, the invariant mass of a system exceeds the sum of the invariant masses of its constituent particles. This paradoxical result is correctly explained in the literature in terms of energy and momentum conservation considerations. This paradoxical result is further clarified in the main result of this article.

The main result of this article is Theorem 5.2 in Sec. 5. It proves that a four-vector equation, (5.3), for the unknowns m_0 (the relativistically invariant mass of a particle system) and $\mathbf{v}_0$ (the center of momentum (CM) frame velocity of the particle system relative to a given inertial rest frame) has a unique solution, which is presented along with its hyperbolic geometric properties. It is the identity, (5.3), of Theorem 5.2 that places the Einstein relativistic mass $m_0\gamma_{\mathbf{v}_0}$ in the context of the Minkowskian four-vector formalism of special relativity.

Owing to its novelty, elegance, and compatibility with the Minkowskian four-vector formalism approach to special relativity, our main result is surprising. Hence, in order to demystify the main result in Theorem 5.2, we present its classical counterpart, Theorem 5.3, which is trivial and well-known, and with which Theorem 5.2 shares remarkable analogies.

The main result is enhanced in Sec. 6 by the observation that our relativistic mass is additive.

Finally, in Sec. 7 we demonstrate the application of Theorem 5.2 about the invariant mass of a system in the interpretation of observations in astrophysics, and in Sec. 8 we demonstrate its application in the interpretation of observations in particle physics. Theorem 5.2 is, thus, remarkable in that (i) it places the relativistic mass under the umbrella of the Minkowskian four-vector formalism of special relativity; and (ii) it unifies the interpretation of observations in astrophysics, on the scale of the cosmos, and particle physics, on the subatomic scale.

2 Einstein Velocity Addition

Relativistic mechanics and its Einstein addition law of Einsteinian, relativistically admissible velocities is woven into the fabric of hyperbolic geometry just as classical mechanics and its addition law of Newtonian velocities, which is the ordinary vector addition, is woven into the fabric of Euclidean geometry [30, 32, 38, 42, 44, 45].

Let c be any positive constant, let $(\mathbb{R}^n, +, \cdot)$ be the Euclidean n-space, and let

$$\mathbb{R}^n_c = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < c\} \tag{2.1}$$

be the c-ball of all relativistically admissible velocities of material particles. It is the open ball of radius c, centered at the origin of $\mathbb{R}^n$, consisting of all vectors $\mathbf{v}$ in $\mathbb{R}^n$ with magnitude $\|\mathbf{v}\|$ smaller than c.

Einstein velocity addition in the c-ball $\mathbb{R}^n_c$ of all relativistically admissible velocities is given by the equation [30], [24, Eq. 2.9.2],[15, p. 55],[9],

$$\mathbf{u}\oplus\mathbf{v} = \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\left\{\mathbf{u}+\frac{1}{\gamma_{\mathbf{u}}}\mathbf{v}+\frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\} \tag{2.2}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$, where $\gamma_{\mathbf{u}}$ is the gamma factor

$$\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}}} \tag{2.3}$$

in the c-ball $\mathbb{R}^n_c$. Here $\mathbf{u}\cdot\mathbf{v}$ and $\|\mathbf{v}\|$ represent the inner product and the norm in the ball, which the ball $\mathbb{R}^n_c$ inherits from its space $\mathbb{R}^n$.

In physical applications, $\mathbb{R}^n = \mathbb{R}^3$ is the Euclidean 3-space, which is the space of all classical, Newtonian velocities, and $\mathbb{R}^n_c = \mathbb{R}^3_c \subset \mathbb{R}^3$ is the c-ball of $\mathbb{R}^3$ of all relativistically admissible, Einsteinian velocities. Furthermore, the constant c represents in physical applications the vacuum speed of light. For applications in hyperbolic geometry, however, n is any positive integer.

Einstein addition (2.2) of relativistically admissible velocities was introduced by Einstein in his 1905 paper [6] [7, p. 141] that founded the special theory of relativity, where the magnitudes of the two sides of Einstein addition (2.2) are presented. One has to remember here that the Euclidean 3-vector algebra was not so widely known in 1905 and, consequently, was not used by Einstein. Einstein calculated in [6] the behavior of the velocity components parallel and orthogonal to the relative velocity between inertial systems, which is as close as one can get without vectors to the vectorial version (2.2).

We naturally use the abbreviation $\mathbf{u}\ominus\mathbf{v} = \mathbf{u}\oplus(-\mathbf{v})$ for Einstein subtraction, so that, for instance, $\mathbf{v}\ominus\mathbf{v} = \mathbf{0}$, $\ominus\mathbf{v} = \mathbf{0}\ominus\mathbf{v} = -\mathbf{v}$ and, in particular,

$$\ominus(\mathbf{u}\oplus\mathbf{v}) = \ominus\mathbf{u}\ominus\mathbf{v} \tag{2.4}$$

and

$$\ominus\mathbf{u}\oplus(\mathbf{u}\oplus\mathbf{v}) = \mathbf{v} \tag{2.5}$$

for all $\mathbf{u}, \mathbf{v}$ in the ball $\mathbb{R}^n_c$, in full analogy with vector addition and subtraction. Identity (2.4) is known as the *automorphic inverse property*, and Identity (2.5) is known as the *left cancellation law* of Einstein addition [38]. We may note that Einstein addition does not obey the naive right counterpart of the left cancellation law (2.5) since, in general,

$$(\mathbf{u}\oplus\mathbf{v})\ominus\mathbf{v} \neq \mathbf{u} \tag{2.6}$$

However, this seemingly lack of a *right cancellation law* of Einstein addition is repaired in [32, Table 2.1, p. 33].

In the Newtonian limit of large c, $c \to \infty$, the ball $\mathbb{R}^n_c$ expands to the whole of its space $\mathbb{R}^n$, as we see from (2.1), and Einstein addition $\oplus$ in $\mathbb{R}^n_c$ reduces to the ordinary vector addition + in $\mathbb{R}^n$, as we see from (2.2) and (2.3).

Einstein addition and the gamma factor are related by the *gamma identity*,

$$\gamma_{\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) \tag{2.7}$$

which can be written, equivalently, as

$$\gamma_{\ominus\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) \tag{2.8}$$

for all $\mathbf{u}, \mathbf{v}\in\mathbb{R}^n_c$. Here, (2.8) is obtained from (2.7) by replacing $\mathbf{u}$ by $\ominus\mathbf{u} = -\mathbf{u}$ in (2.7).

An important identity that follows immediately from (2.3) is

$$\frac{\mathbf{v}^2}{c^2} = \frac{\gamma_{\mathbf{v}}^2 - 1}{\gamma_{\mathbf{v}}^2} \tag{2.9}$$

and, similarly, an important identity that follows immediately from (2.8) is

$$\frac{\mathbf{u}\cdot\mathbf{v}}{c^2} = 1 - \frac{\gamma_{\ominus\mathbf{u}\oplus\mathbf{v}}}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}} \tag{2.10}$$

It is the gamma identity (2.7) that signaled the emergence of hyperbolic geometry in special relativity when it was first studied by Sommerfeld [25] and Varičak [46, 47] in terms of *rapidities*, a term coined by Robb [22]. In fact, the gamma identity plays a role in hyperbolic geometry, analogous to the law of cosines in Euclidean geometry. Historically, it formed the first link between special relativity and the hyperbolic geometry of Bolyai and Lobachevsky, recently leading to the novel trigonometry in hyperbolic geometry that became known as *gyrotrigonometry*, developed in [38, Ch. 12], [42, Ch. 4] and [29, 37].

Einstein addition is known since 1905. Its resulting gamma identities (2.7) – (2.8) were known to Sommerfeld and Varičak a few years later. In this article we need one more result about Einstein addition, which we place in (2.12) below. Since Einstein addition is neither commutative nor associative, we have, in general,

$$\ominus(\mathbf{w}\oplus\mathbf{u})\oplus(\mathbf{w}\oplus\mathbf{v}) \neq \ominus\mathbf{u}\oplus\mathbf{v} \tag{2.11}$$

However, we do have the related identity

$$\|\ominus(\mathbf{w}\oplus\mathbf{u})\oplus(\mathbf{w}\oplus\mathbf{v})\| = \|\ominus\mathbf{u}\oplus\mathbf{v}\| \tag{2.12}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n_c$. The proof of (2.12) follows from the gyrocommutative gyrogroup structure of Einstein addition, and is found in [30, Theorem 2.40], [32, Theorem 3.13], [38, Theorem 3.12, p. 57], [42, Eqs. (4.19),(4.24)]. Interested readers may prove Identity (2.12) straightforwardly, by using a computer software for symbolic manipulation, like Mathematica or Maple.

A trip beyond Einstein velocity addition law leads us to Thomas gyration, which is the mathematical abstraction of the relativistic effect known as Thomas precession [30, Sec. 3, pp. 6-8][32, Sec. 10.5].

Einstein addition is noncommutative. While

$$\|\mathbf{u}\oplus\mathbf{v}\| = \|\mathbf{v}\oplus\mathbf{u}\|\,, \tag{2.13}$$

we have, in general,

$$\mathbf{u}\oplus\mathbf{v} \neq \mathbf{v}\oplus\mathbf{u} \tag{2.14}$$

$\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$. Moreover, Einstein addition is also nonassociative. In general

$$(\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w} \neq \mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w}) \tag{2.15}$$

$\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n_c$.

It seems that following the breakdown of commutativity and associativity in Einstein addition some mathematical regularity has been lost in the transition from Newton's velocity vector addition in $\mathbb{R}^n$ to Einstein's velocity addition (2.2) in $\mathbb{R}^n_c$. This is, however, not the case since Thomas gyration comes to the rescue [30, 32, 38, 42, 48, 20]. Accordingly, owing to the presence of Thomas gyration, the Einstein groupoid $(\mathbb{R}^n_c, \oplus)$ has a grouplike structure [28] that we naturally call the *Einstein gyrogroup* [30].

3 The Lorentz Boost and the Four-Velocity

A Lorentz transformation is a linear transformation of spacetime coordinates that fixes the spacetime origin. A Lorentz boost, $L(\mathbf{v})$, is a Lorentz transformation without rotation, parametrized by a velocity parameter $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3_c$.

Being linear, the Lorentz boost has a matrix representation $L_m(\mathbf{v})$, which turns out to be [15],

$$L_m(\mathbf{v}) =$$
$$\begin{pmatrix} \gamma_{\mathbf{v}} & c^{-2}\gamma_{\mathbf{v}}v_1 & c^{-2}\gamma_{\mathbf{v}}v_2 & c^{-2}\gamma_{\mathbf{v}}v_3 \\ \gamma_{\mathbf{v}}v_1 & 1+c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_1^2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_1v_2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_1v_3 \\ \gamma_{\mathbf{v}}v_2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_1v_2 & 1+c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_2^2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_2v_3 \\ \gamma_{\mathbf{v}}v_3 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_1v_3 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_2v_3 & 1+c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_3^2 \end{pmatrix} \tag{3.1}$$

Employing the matrix representation (3.1) of the Lorentz transformation boost, the Lorentz boost application to spacetime coordinates takes the form

$$L(\mathbf{v})\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = L_m(\mathbf{v})\begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} =: \begin{pmatrix} t' \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} t' \\ \mathbf{x}' \end{pmatrix} \tag{3.2}$$

where $\mathbf{v} = (v_1, v_2, v_3)^t \in \mathbb{R}^3_c$, $\mathbf{x} = (x_1, x_2, x_3)^t \in \mathbb{R}^3$, $\mathbf{x}' = (x'_1, x'_2, x'_3)^t \in \mathbb{R}^3$, and $t, t' \in \mathbb{R}$, where exponent t denotes transposition.

In our approach to geometry and mechanics, analogies with classical results form the right tool. Hence, we emphasize that in the Newtonian limit of large vacuum speed of light c, $c \to \infty$, the Lorentz boost $L(\mathbf{v})$, (3.1) – (3.2), reduces to the Galilei boost $G(\mathbf{v})$, $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$,

$$\begin{aligned} G(\mathbf{v})\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} &= \lim_{c\to\infty} L(\mathbf{v})\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & 1 & 0 & 0 \\ v_2 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= \begin{pmatrix} t \\ x_1 + v_1 t \\ x_2 + v_2 t \\ x_3 + v_3 t \end{pmatrix} = \begin{pmatrix} t \\ \mathbf{x} + \mathbf{v}t \end{pmatrix} \end{aligned} \tag{3.3}$$

where $\mathbf{x} = (x_1, x_2, x_3)^t \in \mathbb{R}^3$ and $t \in \mathbb{R}$.

As we see from (3.2) – (3.3), our spacetime coordinates are $(t, \mathbf{x})^t$ and, as a result, the Lorentz boost matrix representation $L_m(\mathbf{v})$ in (3.1) is *non-symmetric* for $c \neq 1$. In contrast, some authors present spacetime coordinates as $(ct, \mathbf{x})^t$, resulting in a *symmetric* Lorentz boost matrix representation found, for instance, in [11, Eq. (11.98), pp. 541].

Since in our approach to special relativity analogies with classical results form the right tool, the representation of spacetime coordinates as $(t,\mathbf{x})^t$ is more advantageous than its representation as $(ct,\mathbf{x})^t$. Indeed, unlike the latter, the former allows one to recover the Galilei boost from the Lorentz boost by taking the Newtonian limit of large speed of light c, as shown in (3.3).

As a result of adopting $(t,\mathbf{x})^t$ rather than $(ct,\mathbf{x})^t$ as our four-vector that represents four-position, our four-velocity is given by $(\gamma_{\mathbf{v}},\gamma_{\mathbf{v}}\mathbf{v})$ rather than $(\gamma_{\mathbf{v}}c,\gamma_{\mathbf{v}}\mathbf{v})$, $\mathbf{v}\in\mathbb{R}_c^3$. Similarly, our four-momentum is given by

$$\begin{pmatrix} p_0 \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \frac{E}{c^2} \\ \mathbf{p} \end{pmatrix} = m\begin{pmatrix} \gamma_{\mathbf{v}} \\ \gamma_{\mathbf{v}}\mathbf{v} \end{pmatrix} \tag{3.4}$$

rather than the standard four-momentum, which is given by $(p_0,\mathbf{p})^t = (E/c,\mathbf{p})^t = (m\gamma_{\mathbf{v}}c, m\gamma_{\mathbf{v}}\mathbf{v})^t$, as found in most relativity physics books. According to (3.4) the relativistically invariant mass (that is, rest mass) m of a particle is the ratio of the particle's four-momentum $(p_0,\mathbf{p})^t$ to its four-velocity $(\gamma_{\mathbf{v}},\gamma_{\mathbf{v}}\mathbf{v})^t$.

For the sake of simplicity, and without loss of generality, some authors normalize the vacuum speed of light to $c = 1$ as, for instance, in [8]. We, however, prefer to leave c as a free positive parameter, enabling related modern results to be reduced to classical ones under the limit of large c, $c \to \infty$ as, for instance, in the transition from a Lorentz boost into a corresponding Galilei boost in (3.1) – (3.3).

The Lorentz boost (3.1) – (3.2) can be written vectorially in the form

$$L(\mathbf{u})\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \gamma_{\mathbf{u}}(t + \frac{1}{c^2}\mathbf{u}\cdot\mathbf{x}) \\ \gamma_{\mathbf{u}}\mathbf{u}t + \mathbf{x} + \frac{1}{c^2}\frac{\gamma_{\mathbf{v}}^2}{1+\gamma_{\mathbf{v}}}(\mathbf{u}\cdot\mathbf{x})\mathbf{u} \end{pmatrix} \tag{3.5}$$

Being written in a vector form, the Lorentz boost in (3.5) survives unimpaired in higher dimensions. Rewriting (3.5) in higher dimensional spaces, with $\mathbf{x} = \mathbf{v}t$, $\mathbf{u},\mathbf{v}\in\mathbb{R}_c^n \subset \mathbb{R}^n$, we have

$$\begin{aligned} L(\mathbf{u})\begin{pmatrix} t \\ \mathbf{v}t \end{pmatrix} &= \begin{pmatrix} \gamma_{\mathbf{u}}(t + \frac{1}{c^2}\mathbf{u}\cdot\mathbf{v}t) \\ \gamma_{\mathbf{u}}\mathbf{u}t + \mathbf{v}t + \frac{1}{c^2}\frac{\gamma_{\mathbf{v}}^2}{1+\gamma_{\mathbf{v}}}(\mathbf{u}\cdot\mathbf{v}t)\mathbf{u} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\gamma_{\mathbf{u}\oplus\mathbf{v}}}{\gamma_{\mathbf{v}}}t \\ \frac{\gamma_{\mathbf{u}\oplus\mathbf{v}}}{\gamma_{\mathbf{v}}}(\mathbf{u}\oplus\mathbf{v})t \end{pmatrix} \end{aligned} \tag{3.6}$$

Equation (3.6) reveals explicitly the way Einstein velocity addition underlies the Lorentz boost. The second equation in (3.6) follows from the first by (2.7) and (2.2).

The special case of $t = \gamma_{\mathbf{v}}$ in (3.6) proves useful, giving rise to the elegant identity,

$$L(\mathbf{u})\begin{pmatrix} \gamma_{\mathbf{v}} \\ \gamma_{\mathbf{v}}\mathbf{v} \end{pmatrix} = \begin{pmatrix} \gamma_{\mathbf{u}\oplus\mathbf{v}} \\ \gamma_{\mathbf{u}\oplus\mathbf{v}}(\mathbf{u}\oplus\mathbf{v}) \end{pmatrix} \tag{3.7}$$

of the Lorentz boost of four-velocities, $\mathbf{u},\mathbf{v}\in\mathbb{R}_c^n$.

The four-vector $m(\gamma_{\mathbf{v}},\gamma_{\mathbf{v}}\mathbf{v})^t$ is the four-momentum of a particle with invariant mass m and velocity $\mathbf{v}$ relative to a given inertial rest frame $\Sigma_{\mathbf{0}}$. Let $\Sigma_{\ominus\mathbf{u}}$ be an inertial frame that moves with velocity $\ominus\mathbf{u} = -\mathbf{u}$ relative to the rest frame $\Sigma_{\mathbf{0}}$, $\mathbf{u},\mathbf{v}\in\mathbb{R}^n_c$. Then, the particle with velocity $\mathbf{v}$ relative to $\Sigma_{\mathbf{0}}$ has velocity $\mathbf{u}\oplus\mathbf{v}$ relative to the frame $\Sigma_{\ominus\mathbf{u}}$. In full agreement and, owing to the linearity of the Lorentz boost, it follows from (3.7) that the four-momentum of the particle relative to the frame $\Sigma_{\ominus\mathbf{u}}$ is

$$L(\mathbf{u})m\begin{pmatrix}\gamma_{\mathbf{v}}\\ \gamma_{\mathbf{v}}\mathbf{v}\end{pmatrix} = mL(\mathbf{u})\begin{pmatrix}\gamma_{\mathbf{v}}\\ \gamma_{\mathbf{v}}\mathbf{v}\end{pmatrix} = m\begin{pmatrix}\gamma_{\mathbf{u}\oplus\mathbf{v}}\\ \gamma_{\mathbf{u}\oplus\mathbf{v}}(\mathbf{u}\oplus\mathbf{v})\end{pmatrix} \tag{3.8}$$

Interestingly, while the Lorentz boost expressed in terms of coordinate time t is linear, as we see in this section, the Lorentz boost expressed in terms of proper time τ is nonlinear, as shown in [33, 35] and in [38, Secs. 11.14-11.15].

It follows from the linearity of the Lorentz boost in (3.7) that

$$\begin{aligned} L(\mathbf{w})\sum_{k=1}^{N} m_k\begin{pmatrix}\gamma_{\mathbf{v}_k}\\ \gamma_{\mathbf{v}_k}\mathbf{v}_k\end{pmatrix} &= \sum_{k=1}^{N} m_k L(\mathbf{w})\begin{pmatrix}\gamma_{\mathbf{v}_k}\\ \gamma_{\mathbf{v}_k}\mathbf{v}_k\end{pmatrix} \\ &= \sum_{k=1}^{N} m_k\begin{pmatrix}\gamma_{\mathbf{w}\oplus\mathbf{v}_k}\\ \gamma_{\mathbf{w}\oplus\mathbf{v}_k}(\mathbf{w}\oplus\mathbf{v}_k)\end{pmatrix} \\ &= \begin{pmatrix}\sum_{k=1}^{N} m_k\gamma_{\mathbf{w}\oplus\mathbf{v}_k}\\ \sum_{k=1}^{N} m_k\gamma_{\mathbf{w}\oplus\mathbf{v}_k}(\mathbf{w}\oplus\mathbf{v}_k)\end{pmatrix} \end{aligned} \tag{3.9}$$

The chain of equations (3.9) reveals the interplay of Einstein addition, $\oplus$, in $\mathbb{R}^n_c$ and vector addition, $+$, in $\mathbb{R}^n$ that appears implicitly in the sigma-notation for scalar and vector addition.

The (Minkowski) norm of a four-vector is Lorentz transformation invariant. The norm of the four-position $(t,\mathbf{x})^t$ is

$$\left\|\begin{pmatrix}t\\ \mathbf{x}\end{pmatrix}\right\| = \sqrt{t^2-\frac{\|\mathbf{x}\|^2}{c^2}} \tag{3.10}$$

and, accordingly, the norm of the four-velocity is

$$\left\|\begin{pmatrix}\gamma_{\mathbf{v}}\\ \gamma_{\mathbf{v}}\mathbf{v}\end{pmatrix}\right\| = \gamma_{\mathbf{v}}\left\|\begin{pmatrix}1\\ \mathbf{v}\end{pmatrix}\right\| = \gamma_{\mathbf{v}}\sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}} = 1 \tag{3.11}$$

4 The Invariant Mass of a System of Particles

In obtaining the result in (3.8) we exploit the linearity of the Lorentz boost. We will now further exploit that linearity, demonstrated in (3.9), to obtain the relativistically invariant mass of a system of particles. Being observer's invariant, we refer the Newtonian, rest mass, m, to as the (relativistically) invariant mass, as opposed to the common relativistic mass, $m\gamma_{\mathbf{v}}$, which is observer's dependent.

Let

$$S = S(m_k, \mathbf{v}_k, \Sigma_{\mathbf{0}}, k = 1, \ldots, N) \tag{4.1}$$

be an isolated system of N noninteracting material particles the k-th particle of which has invariant mass $m_k > 0$ and velocity $\mathbf{v}_k \in \mathbb{R}^n_c$ relative to an inertial frame $\Sigma_\mathbf{0}$, $k = 1, \ldots, N$.

Classically, the Newtonian mass m_{newton} of the system S equals the sum of the Newtonian masses of its constituent particles, that is

$$m_{newton} = \sum_{k=1}^{N} m_k \tag{4.2}$$

and it forms the total mass of the system. Relativistically, however, this need not be the case since dark matter may emerge, as we will see in Theorem 5.2 of Sec. 5.

Accordingly, we wish to determine the relativistically invariant mass m_0 of the system S, and the velocity $\mathbf{v}_0$ relative to $\Sigma_\mathbf{0}$ of a fictitious inertial frame, called the center of momentum (CM) frame, relative to which the three-momentum of S vanishes.

Assuming that the four-momentum is additive, the sum of the four-momenta of the N particles of the system S gives the four-momentum $(m_0\gamma_{\mathbf{v}_0}, m_0\gamma_{\mathbf{v}_0}\mathbf{v}_0)^t$ of S. Accordingly,

$$\sum_{k=1}^{N} m_k \begin{pmatrix} \gamma_{\mathbf{v}_k} \\ \gamma_{\mathbf{v}_k}\mathbf{v}_k \end{pmatrix} = m_0 \begin{pmatrix} \gamma_{\mathbf{v}_0} \\ \gamma_{\mathbf{v}_0}\mathbf{v}_0 \end{pmatrix} \tag{4.3}$$

where

(i) the invariant masses $m_k > 0$ and the velocities $\mathbf{v}_k$, $k = 1, ..., N$, relative to $\Sigma_\mathbf{0}$ of the constituent particles of S are given, while

(ii) the invariant mass m_0 of S and the velocity $\mathbf{v}_0$ of the CM frame of S relative to $\Sigma_\mathbf{0}$ are to be determined uniquely by the *Resultant Relativistically Invariant Mass Theorem*, which is Theorem 5.2 in Sec. 5.

If $m_0 > 0$ and $\mathbf{v}_0 \in \mathbb{R}^n_c$ that satisfy (4.3) exist then, as anticipated, the three-momentum of the system S relative to its CM frame vanishes since, by (3.8) and (4.3), the four-momentum of S relative to its CM frame is given by

$$\begin{aligned} L(\ominus\mathbf{v}_0) \sum_{k=1}^{N} m_k \begin{pmatrix} \gamma_{\mathbf{v}_k} \\ \gamma_{\mathbf{v}_k}\mathbf{v}_k \end{pmatrix} &= L(\ominus\mathbf{v}_0) m_0 \begin{pmatrix} \gamma_{\mathbf{v}_0} \\ \gamma_{\mathbf{v}_0}\mathbf{v}_0 \end{pmatrix} \\ &= m_0 \begin{pmatrix} \gamma_{\ominus\mathbf{v}_0\oplus\mathbf{v}_0} \\ \gamma_{\ominus\mathbf{v}_0\oplus\mathbf{v}_0}(\ominus\mathbf{v}_0\oplus\mathbf{v}_0) \end{pmatrix} \\ &= m_0 \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} \end{aligned} \tag{4.4}$$

5 The Resultant Relativistically Invariant Mass Theorem

We need the following Lemma 5.1 for the proof of our main result, which is the *resultant relativistically invariant mass Theorem* 5.2.

Lemma 5.1. *Let N be any positive integer, and let* $m_k > 0$ *and* $\mathbf{v}_k \in \mathbb{R}_c^n$*,* $k = 1, \ldots, N$*, be N positive numbers and N points of an Einstein gyrogroup* $\mathbb{R}_c^n = (\mathbb{R}_c^n, \oplus)$*. Then*

$$
\begin{aligned}
&(\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \frac{\mathbf{v}_k}{c})^2 \\
&= (\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k})^2 - \left\{ (\sum_{k=1}^{N} m_k)^2 + 2 \sum_{\substack{j,k=1 \\ j<k}}^{h} m_j m_k (\gamma_{\ominus \mathbf{v}_j \oplus \mathbf{v}_k} - 1) \right\}
\end{aligned}
\tag{5.1}
$$

Proof. The proof is given by the following chain of equations, which are numbered for subsequent explanation.

$$
\begin{aligned}
(\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \frac{\mathbf{v}_k}{c})^2 &\overbrace{=\!=}^{(1)} \sum_{k=1}^{N} m_k^2 \gamma_{\mathbf{v}_k}^2 \frac{\mathbf{v}_k^2}{c^2} + 2 \sum_{\substack{j,k=1 \\ j<k}}^{h} m_j m_k \gamma_{\mathbf{v}_j} \gamma_{\mathbf{v}_k} \frac{\mathbf{v}_j \cdot \mathbf{v}_k}{c^2} \\
&\overbrace{=\!=}^{(2)} \sum_{k=1}^{N} m_k^2 (\gamma_{\mathbf{v}_k}^2 - 1) + 2 \sum_{\substack{j,k=1 \\ j<k}}^{h} m_j m_k (\gamma_{\mathbf{v}_j} \gamma_{\mathbf{v}_k} - \gamma_{\ominus \mathbf{v}_j \oplus \mathbf{v}_k}) \\
&\overbrace{=\!=}^{(3)} \sum_{k=1}^{N} m_k^2 \gamma_{\mathbf{v}_k}^2 - \sum_{k=1}^{N} m_k^2 + 2 \sum_{\substack{j,k=1 \\ j<k}}^{h} m_j m_k \gamma_{\mathbf{v}_j} \gamma_{\mathbf{v}_k} - 2 \sum_{\substack{j,k=1 \\ j<k}}^{h} m_j m_k \gamma_{\ominus \mathbf{v}_j \oplus \mathbf{v}_k} \\
&\overbrace{=\!=}^{(4)} (\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k})^2 - \left\{ \sum_{k=1}^{N} m_k^2 + 2 \sum_{\substack{j,k=1 \\ j<k}}^{h} m_j m_k \gamma_{\ominus \mathbf{v}_j \oplus \mathbf{v}_k} \right\} \\
&\overbrace{=\!=}^{(5)} (\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k})^2 - \left\{ (\sum_{k=1}^{N} m_k)^2 + 2 \sum_{\substack{j,k=1 \\ j<k}}^{h} m_j m_k (\gamma_{\ominus \mathbf{v}_j \oplus \mathbf{v}_k} - 1) \right\}
\end{aligned}
\tag{5.2}
$$

The assumption $\mathbf{v}_k \in \mathbb{R}_c^n$ implies, by (2.3), that all gamma factors in (5.1) – (5.2) are real and greater than 1. Derivation of the numbered equalities in (5.2) follows:

(1) This equation is obtained by an expansion of the square of a sum of vectors in $\mathbb{R}^n$.

(2) Follows from (1) by (2.9) – (2.10).

(3) Follows from (2) by an obvious expansion.

(4) Follows from (3) by an expansion of the square of a sum of real numbers.

(5) Follows from (4) by an expansion of another square of a sum of real numbers.

□

Lemma 5.1 proves useful in the proof of the following Resultant Relativistically Invariant Mass Theorem:

Theorem 5.2. *Let* $(\mathbb{R}_c^n, \oplus)$ *be an Einstein gyrogroup, and let* $\mathbf{v}_k \in \mathbb{R}_c^n$ *and* $m_k > 0$*,* $k = 1, 2, \dots, N$*, be N elements of* $\mathbb{R}_c^n$ *and N positive constants. Then, there exist unique* $\mathbf{v}_0 \in \mathbb{R}_c^n$ *and* $m_0 > 0$ *such that*

$$\sum_{k=1}^{N} m_k \begin{pmatrix} \gamma_{\mathbf{v}_k} \\ \gamma_{\mathbf{v}_k} \mathbf{v}_k \end{pmatrix} = m_0 \begin{pmatrix} \gamma_{\mathbf{v}_0} \\ \gamma_{\mathbf{v}_0} \mathbf{v}_0 \end{pmatrix} \tag{5.3}$$

Furthermore, $\mathbf{v}_0$ *and* m_0 *satisfy the following three identities for all* $\mathbf{w} \in \mathbb{R}_c^n$ *(including, in particular, the interesting special case of* $\mathbf{w} = \mathbf{0}$*):*

$$\mathbf{w} \oplus \mathbf{v}_0 = \frac{\sum_{k=1}^{N} m_k \gamma_{\mathbf{w} \oplus \mathbf{v}_k} (\mathbf{w} \oplus \mathbf{v}_k)}{\sum_{k=1}^{N} m_k \gamma_{\mathbf{w} \oplus \mathbf{v}_k}} \tag{5.4}$$

$$\gamma_{\mathbf{w} \oplus \mathbf{v}_0} = \frac{\sum_{k=1}^{N} m_k \gamma_{\mathbf{w} \oplus \mathbf{v}_k}}{m_0} \tag{5.5}$$

$$\gamma_{\mathbf{w} \oplus \mathbf{v}_0} (\mathbf{w} \oplus \mathbf{v}_0) = \frac{\sum_{k=1}^{N} m_k \gamma_{\mathbf{w} \oplus \mathbf{v}_k} (\mathbf{w} \oplus \mathbf{v}_0)}{m_0} \tag{5.6}$$

where

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k \right)^2 + 2 \sum_{\substack{j,k=1 \\ j<k}}^{N} m_j m_k (\gamma_{\ominus(\mathbf{w} \oplus \mathbf{v}_j) \oplus (\mathbf{w} \oplus \mathbf{v}_k)} - 1)} \tag{5.7}$$

Proof. Let us consider the following four equations, which are specialized from (5.4) – (5.7) with $\mathbf{w} = \mathbf{0}$:

$$\mathbf{v}_0 = \frac{\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \mathbf{v}_k}{\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k}} \tag{5.8}$$

$$\gamma_{\mathbf{v}_0} = \frac{\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k}}{m_0} \tag{5.9}$$

$$\gamma_{\mathbf{v}_0} \mathbf{v}_0 = \frac{\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \mathbf{v}_0}{m_0} \tag{5.10}$$

and

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k \right)^2 + 2 \sum_{\substack{j,k=1 \\ j<k}}^{N} m_j m_k (\gamma_{\ominus \mathbf{v}_j \oplus \mathbf{v}_k} - 1)} \tag{5.11}$$

The proof consists of two parts. In the first part of the proof we show that if equation (5.3) for the unknowns $\mathbf{v}_0 \in \mathbb{R}^n$ and $m_0 \in \mathbb{R}$ possesses a solution, then the solution must be given uniquely by $\mathbf{v}_0$ of (5.8) and m_0 of (5.11), with $\mathbf{v}_0 \in \mathbb{R}_c^n$ and $m_0 > 0$, satisfying (5.9) – (5.10).

In the second part of the proof we show that $\mathbf{v}_0$ of (5.8) and m_0 of (5.11), indeed, form a solution of (5.3) for the unknowns $\mathbf{v}_0 \in \mathbb{R}^n_c$ and $m_0 > 0$, and that the solution satisfies (5.4) – (5.7).

Part I: If $m_0 \in \mathbb{R}$ and $\mathbf{v}_0 \in \mathbb{R}^n$ that satisfy (5.3) exist, then the norms of the two sides of (5.3) are equal while, by (3.11), the norm of the right-hand side of (5.3) is m_0. Hence, the norm of the left-hand side of (5.3) equals m_0 as well, obtaining the following chain of equations, which are numbered for subsequent explanation:

$$
\begin{aligned}
m_0^2 &\overbrace{=\!=}^{(1)} \left\| \sum_{k=1}^{N} m_k \begin{pmatrix} \gamma_{\mathbf{v}_k} \\ \gamma_{\mathbf{v}_k}\mathbf{v}_k \end{pmatrix} \right\|^2 \\
&\overbrace{=\!=}^{(2)} \left\| \begin{pmatrix} \sum_{k=1}^{N} m_k\gamma_{\mathbf{v}_k} \\ \sum_{k=1}^{N} m_k\gamma_{\mathbf{v}_k}\mathbf{v}_k \end{pmatrix} \right\|^2 \\
&\overbrace{=\!=}^{(3)} (\sum_{k=1}^{N} m_k\gamma_{\mathbf{v}_k})^2 - (\sum_{k=1}^{N} m_k\gamma_{\mathbf{v}_k}\frac{\mathbf{v}_k}{c})^2 \\
&\overbrace{=\!=}^{(4)} (\sum_{k=1}^{N} m_k)^2 + 2\sum_{\substack{j,k=1\\ j<k}}^{h} m_j m_k \gamma_{\ominus\mathbf{v}_j\oplus\mathbf{v}_k} - 1
\end{aligned}
\tag{5.12}
$$

Derivation of the numbered equalities in (5.12) follows:

(1) This equation follows from the result that the norm of the left-hand side of (5.3) equals the norm of the right-hand side of (5.3), the latter being m_0 by (3.11).

(2) Follows from (1) by "four-vector" addition of $(n+1)$-vectors (where $n = 3$ in physical applications).

(3) Follows from (2) by (3.10).

(4) Follows from (3) by Identity (5.1) of Lemma 5.1.

It follows from the upper entry of (5.3) that

$$m_0 > 0 \tag{5.13}$$

We thus obtained in (5.12) the desired equation, (5.11), for m_0.

Hence, if m_0 and $\mathbf{v}_0$ that satisfy (5.3) exist, m_0 is positive and must be given by (5.11).

By assumption, $\mathbf{v}_0$ satisfies (5.3). Equation (5.3) is equivalent to two equations, formed by the upper entry and by the lower entry of (5.3). Dividing the lower entry of (5.3) by its upper entry, noting that $m_0 \neq 0$ by (5.13), we obtain (5.8). Owing to the convexity of $\mathbb{R}^n_c$, (5.8) implies $\mathbf{v}_0 \in \mathbb{R}^n_c$.

Similarly, dividing the upper entry of (5.3) by $m_0 > 0$ we obtain (5.9), and dividing the lower entry of (5.3) by $m_0 > 0$ we obtain (5.10).

Hence, if m_0 and $\mathbf{v}_0$ that satisfy (5.3) exist, then $m_0 > 0$, $\mathbf{v}_0 \in \mathbb{R}^n_c$, and they must be given by (5.11) and (5.8), and satisfy (5.9) – (5.10).

Part II: In part I we have shown that if (5.3) possesses a solution for the unknowns $\mathbf{v}_0 \in \mathbb{R}^n$ and $m_0 \in \mathbb{R}$, then $\mathbf{v}_0 \in \mathbb{R}^n_c$ is given uniquely by (5.8) and $m_0 > 0$ is given uniquely by (5.11), satisfying (5.9) – (5.10). We will now show that, indeed, $\mathbf{v}_0 \in \mathbb{R}^n_c$, given by (5.8), and $m_0 > 0$, given by (5.11), form a solution of (5.3) that satisfies (5.4) – (5.7). Accordingly, in this second part of the proof we assume that $\mathbf{v}_0$ and $m_0 > 0$ are given by (5.8) and (5.11).

It follows from Identity (5.1) of Lemma 5.1, along with m_0 of (5.11) that

$$(\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \frac{\mathbf{v}_k}{c})^2 = (\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k})^2 - m_0^2 \tag{5.14}$$

Hence, by (5.8) and (5.14), we have the following chain of equations, which are numbered for subsequent explanation:

$$\begin{aligned} \frac{\mathbf{v}_0^2}{c^2} &\overbrace{=\!=}^{(1)} \frac{(\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \frac{\mathbf{v}_k}{c})^2}{(\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k})^2} \\ &\overbrace{=\!=}^{(2)} \frac{(\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k})^2 - m_0^2}{(\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k})^2} \\ &=\!= 1 - \frac{m_0^2}{(\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k})^2} \end{aligned} \tag{5.15}$$

Derivation of the numbered equalities in (5.15) follows:

(1) This equation is given by assumption, (5.8).

(2) Follows from (1) by (5.14).

It follows from (5.15) that

$$\gamma_{\mathbf{v}_0} = \frac{1}{\sqrt{1 - \frac{\mathbf{v}_0^2}{c^2}}} = \frac{\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k}}{m_0} \tag{5.16}$$

thus verifying (5.9).

Following (5.9) and (5.8) we have

$$\gamma_{\mathbf{v}_0} \mathbf{v}_0 = \frac{\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \mathbf{v}_k}{m_0} \tag{5.17}$$

thus verifying (5.10).

Finally, (5.16) implies that m_0 and $\mathbf{v}_0$ satisfy the upper entry of (5.3) and, similarly, (5.17) implies that m_0 and $\mathbf{v}_0$ satisfy the lower entry of (5.3). Hence, the pair consisting of m_0 and $\mathbf{v}_0$ forms a solution of (5.3). We have thus shown that $\mathbf{v}_0$ and m_0 given by (5.8) and (5.11) form a solution of (5.3), and that this solution satisfies (5.9) – (5.10).

To complete the proof it remains to show that the pair $(m_0, \mathbf{v}_0)$ satisfies (5.4) – (5.7) as well.

Let us first show that m_0 of (5.11) satisfies (5.7). Indeed, following (2.3) and (2.12) we have

$$\gamma_{\ominus(\mathbf{w}\oplus\mathbf{v}_j)\oplus(\mathbf{w}\oplus\mathbf{v}_k)} = \gamma_{\ominus\mathbf{v}_j\oplus\mathbf{v}_k} \tag{5.18}$$

implying that the right-hand sides of (5.7) and (5.11) are equal, so that m_0 is independent of $\mathbf{w}$, as desired. As such, m_0 is given by each of (5.7) and (5.11).

We have thus shown that the unique solution of (5.3) is formed by $\mathbf{v}_0\in\mathbb{R}^n_c$ and $m_0 > 0$ that are given by (5.8) and (5.11), and that the solution satisfies (5.9) – (5.10). It, therefore, remains to show that the solution satisfies (5.4) – (5.6) as well.

Applying the Lorentz boost $L(\mathbf{w})$ to each side of (5.3), we have the equivalent equation

$$L(\mathbf{w})\sum_{k=1}^{N} m_k \begin{pmatrix} \gamma_{\mathbf{v}_k} \\ \gamma_{\mathbf{v}_k}\mathbf{v}_k \end{pmatrix} = L(\mathbf{w}) m_0 \begin{pmatrix} \gamma_{\mathbf{v}_0} \\ \gamma_{\mathbf{v}_0}\mathbf{v}_0 \end{pmatrix} \tag{5.19}$$

Following the linearity of the Lorentz boost, illustrated in (3.8) and (3.9), (5.19) can be written as the equation

$$\sum_{k=1}^{N} m_k \begin{pmatrix} \gamma_{\mathbf{w}\oplus\mathbf{v}_k} \\ \gamma_{\mathbf{w}\oplus\mathbf{v}_k}(\mathbf{w}\oplus\mathbf{v}_k) \end{pmatrix} = m_0 \begin{pmatrix} \gamma_{\mathbf{w}\oplus\mathbf{v}_0} \\ \gamma_{\mathbf{w}\oplus\mathbf{v}_0}(\mathbf{w}\oplus\mathbf{v}_0) \end{pmatrix} \tag{5.20}$$

Equation (5.20) is identical with (5.3) in which $\mathbf{v}_k\in\mathbb{R}^n_c$ is replaced by $\mathbf{w}\oplus\mathbf{v}_k\in\mathbb{R}^n_c$, $k = 0, 1, \ldots, N$.

But, the unique solution of (5.3) is the pair $(m_0 > 0, \mathbf{v}_0\in\mathbb{R}^n_c)$ that satisfies (5.8) – (5.11). Hence, the unique solution of (5.20) is the pair $(m_0 > 0, \mathbf{w}\oplus\mathbf{v}_0\in\mathbb{R}^n_c)$ that satisfies (5.4) – (5.7). Hence, the unique solution $(m_0 > 0, \mathbf{v}_0\in\mathbb{R}^n_c)$ of (5.3) satisfies not only (5.8) – (5.11) but, more generally, (5.4) – (5.7), and the proof is complete. □

We have thus established in Theorem 5.2 the following four results concerning an isolated system S, (4.1),

$$S = S(m_k, \mathbf{v}_k, \Sigma_{\mathbf{0}}, k = 1, \ldots, N) \tag{5.21}$$

of N noninteracting material particles the k-th particle of which has invariant mass $m_k > 0$ and velocity $\mathbf{v}_k\in\mathbb{R}^n_c$ relative to an inertial frame $\Sigma_{\mathbf{0}}$, $k = 1, \ldots, N$:

(1) The relativistically invariant (or, rest) mass m_0 of the system S is given by

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2\sum_{\substack{j,k=1\\ j<k}}^{N} m_j m_k (\gamma_{\ominus\mathbf{v}_j\oplus\mathbf{v}_k} - 1)} \tag{5.22}$$

according to (5.7) with $\mathbf{w} = \mathbf{0}$.

(2) The relativistic mass of the system S is

$$m_0\gamma_{\mathbf{v}_0} \tag{5.23}$$

relative to the rest frame $\Sigma_{\mathbf{0}}$, where $\mathbf{v}_0$ is the velocity of the CM frame of S relative to $\Sigma_{\mathbf{0}}$, given by

$$\mathbf{v}_0 = \frac{\sum_{k=1}^{N} m_k\gamma_{\mathbf{v}_k}\mathbf{v}_k}{\sum_{k=1}^{N} m_k\gamma_{\mathbf{v}_k}} \tag{5.24}$$

according to (5.4) with $\mathbf{w} = \mathbf{0}$.

(3) Like energy and momentum, the relativistic mass is additive, that is, in particular for the system S relative to the rest frame $\Sigma_{\mathbf{0}}$, by (5.5) with $\mathbf{w} = \mathbf{0}$,

$$m_0\gamma_{\mathbf{v}_0} = \sum_{k=1}^{N} m_k\gamma_{\mathbf{v}_k} \tag{5.25}$$

(4) The relativistic mass $m_0\gamma_{\mathbf{v}_0}$ of a system meshes extraordinarily well with the Minkowskian four-vector formalism of special relativity. In particular, for the system S relative to the rest frame $\Sigma_{\mathbf{0}}$, we have, by (5.3),

$$\sum_{k=1}^{N} \begin{pmatrix} m_k\gamma_{\mathbf{v}_k} \\ m_k\gamma_{\mathbf{v}_k}\mathbf{v}_k \end{pmatrix} = \begin{pmatrix} m_0\gamma_{\mathbf{v}_0} \\ m_0\gamma_{\mathbf{v}_0}\mathbf{v}_0 \end{pmatrix} \tag{5.26}$$

where m_0 and $\mathbf{v}_0$ are given by (5.22) and (5.24).

Thus, the relativistically invariant mass m_0 of a particle system S in (5.22) gives rise to its associated relativistic mass $m_0\gamma_{\mathbf{v}_0}$ relative to the rest frame $\Sigma_{\mathbf{0}}$. The latter, in turn, brings in (5.26) the concept of the relativistic mass into conformity with the Minkowskian four-vector formalism of special relativity. Moreover, we will see in Secs. 7 and 8 that the relativistically invariant mass m_0 of a particle system S provides a natural interpretation of observations in astrophysics and in particle physics.

To appreciate the power and elegance of Theorem 5.2 in relativistic mechanics in terms of novel analogies that it shares with familiar results in classical mechanics, we present below the classical counterpart, Theorem 5.3, of Theorem 5.2. The latter is obtained from the former by approaching the Newtonian limit when c tends to infinity. The resulting Theorem 5.3 is immediate, and its importance in classical mechanics is well-known.

Theorem 5.3. *Let $(\mathbb{R}^n, +)$ be a group of Newtonian velocities, and let $\mathbf{v}_k \in \mathbb{R}^n$ and $m_k > 0$, $k = 1, 2, \ldots, N$, be N elements of $\mathbb{R}^n$ and N positive constants. There exist unique $\mathbf{v}_0 \in \mathbb{R}^n$ and $m_0 > 0$ such that*

$$\sum_{k=1}^{N} m_k \begin{pmatrix} 1 \\ \mathbf{v}_k \end{pmatrix} = m_0 \begin{pmatrix} 1 \\ \mathbf{v}_0 \end{pmatrix}. \tag{5.27}$$

Furthermore, $\mathbf{v}_0$ and m_0 satisfy the following identities for all $\mathbf{w} \in \mathbb{R}^n$ (including, in particular, the interesting special case of $\mathbf{w} = \mathbf{0}$):

$$\mathbf{w} + \mathbf{v}_0 = \frac{\sum_{k=1}^{N} m_k(\mathbf{w} + \mathbf{v}_k)}{\sum_{k=1}^{N} m_k} \tag{5.28}$$

and

$$m_0 = \sum_{k=1}^{N} m_k \tag{5.29}$$

Proof. While the proof of Theorem 5.3 is trivial, our point is to present a proof that emphasizes how Theorem 5.3 is derived from Theorem 5.2. Indeed, in the limit as $c \to \infty$, the results of Theorem 5.2 tend to corresponding results of Theorem 5.3, noting that in this limit gamma factors tend to 1. Accordingly, Theorem 5.3 is a special case of Theorem 5.2 corresponding to $c = \infty$. □

Identity (5.28) of Theorem 5.3 is immediate. Yet, it is geometrically important. The geometric importance of the validity of (5.28) for all $\mathbf{w}\in\mathbb{R}^n$ lies on its implication that the velocity $\mathbf{v}_0$ of the CM frame of a particle system relative to a given inertial rest frame in classical mechanics is independent of the choice of the origin of the classical velocity space with its underlying standard model of Euclidean geometry.

Unlike Identity (5.28) of Theorem 5.3, which is immediate, its counterpart in Theorem 5.2, Identity (5.4), is not immediate. Yet, in full analogy with Theorem 5.3, the validity of Identity (5.4) in Theorem 5.2 for all $\mathbf{w}\in\mathbb{R}^n_c$ is geometrically important. This geometric importance of Identity (5.4) lies on its implication that the velocity $\mathbf{v}_0$ of the CM frame of a particle system relative to a given inertial rest frame in relativistic mechanics is independent of the choice of the origin of the relativistic velocity space with its underlying Beltrami-Klein ball model of hyperbolic geometry. A study of special relativity in terms of its underlying hyperbolic geometry is presented in [30, 32, 38, 41, 42, 44, 45].

Finally, it is the identity, (5.3), of Theorem 5.2 that places the Einstein relativistic mass $m_0\gamma_{\mathbf{v}_0}$ in the context of the Minkowskian four-vector formalism of special relativity.

6 The Relativistic Mass is Additive

Suppose that the system S, (5.21), is made up of M subsystems each itself a system of particles. Let $m_{0,p}$ be the relativistically invariant mass and $\mathbf{v}_{0,p}$ the CM frame velocity of the pth subsystem, $p = 1,\dots,M$, so that the relativistic mass of the pth subsystem is $m_{0,p}\gamma_{\mathbf{v}_{0,p}}$.

Then, the relativistic mass $m_0\gamma_{\mathbf{v}_0}$ of the system S, given by (5.22) – (5.24), is additive, that is, it possesses the relativistic mass additivity property

$$m_0\gamma_{\mathbf{v}_0} = \sum_{p=1}^{M} m_{0,p}\gamma_{\mathbf{v}_{0,p}} \tag{6.1}$$

For simplicity, we prove the relativistic mass additivity property (6.1) for the case of $M = 2$ subsystems, the proof for any $M > 2$ being similar.

Let us, therefore, view the system S of N particles, (5.21) – (5.24), with $N \geq 3$, as a system of the two subsystems S_1 and S_2,

$$\begin{aligned} S_1 &= S_1(m_k, \mathbf{v}_k, \Sigma_{\mathbf{0}}, k = 1,\dots,N_1) \\ S_2 &= S_2(m_k, \mathbf{v}_k, \Sigma_{\mathbf{0}}, k = N_1+1,\dots,N) \end{aligned} \tag{6.2}$$

for any fixed N_1, $1 < N_1 < N$.

Then, the relativistically invariant masses $m_{0,1}$ and $m_{0,2}$ of the subsystems S_1 and S_2

and their CM frame velocities, $\mathbf{v}_{0,1}$ and $\mathbf{v}_{0,2}$ relative to $\Sigma_{\mathbf{0}}$, respectively, are

$$
\begin{aligned}
m_{0,1} &= \sqrt{\left(\sum_{k=1}^{N_1} m_k\right)^2 + 2\sum_{\substack{j,k=1\\ j<k}}^{N_1} m_j m_k (\gamma_{\ominus\mathbf{v}_j\oplus\mathbf{v}_k} - 1)} \\
m_{0,2} &= \sqrt{\left(\sum_{k=N_1+1}^{N} m_k\right)^2 + 2\sum_{\substack{j,k=N_1+1\\ j<k}}^{N} m_j m_k (\gamma_{\ominus\mathbf{v}_j\oplus\mathbf{v}_k} - 1)}
\end{aligned}
\tag{6.3}
$$

and

$$
\begin{aligned}
\mathbf{v}_{0,1} &= \frac{\sum_{k=1}^{N_1} m_k \gamma_{\mathbf{v}_k} \mathbf{v}_k}{\sum_{k=1}^{N_1} m_k \gamma_{\mathbf{v}_k}} \\
\mathbf{v}_{0,2} &= \frac{\sum_{k=N_1+1}^{N} m_k \gamma_{\mathbf{v}_k} \mathbf{v}_k}{\sum_{k=N_1+1}^{N} m_k \gamma_{\mathbf{v}_k}}
\end{aligned}
\tag{6.4}
$$

possessing the relativistic mass additivity property

$$
m_{0,1}\gamma_{\mathbf{v}_{0,1}} + m_{0,2}\gamma_{\mathbf{v}_{0,2}} = m_0\gamma_{\mathbf{v}_0}
\tag{6.5}
$$

The proof of the additivity property (6.5) follows from (5.25) immediately. Indeed, by applying the identity in (5.25) to each of the particle systems S_1, S_2 and S, we have the chain of equations

$$
\begin{aligned}
m_{0,1}\gamma_{\mathbf{v}_{0,1}} + m_{0,2}\gamma_{\mathbf{v}_{0,2}} &= \sum_{k=1}^{N_1} m_k\gamma_{\mathbf{v}_k} + \sum_{k=N_1+1}^{N} m_k\gamma_{\mathbf{v}_k} \\
&= \sum_{k=1}^{N} m_k\gamma_{\mathbf{v}_k} \\
&= m_0\gamma_{\mathbf{v}_0}
\end{aligned}
\tag{6.6}
$$

The relativistically invariant mass m_0 of a system of particles S, (5.21) – (5.24), leads to its associated relativistic mass, (5.23). Falling under the umbrella of the Minkowskian four-vector formalism, (5.3), and owing to its additivity property, (6.1), the relativistic mass is no longer the ugly duckling of Einstein's special theory of relativity. Rather, it is the swan of the theory, thus "putting to rest mass misconceptions" in support of Rindler's opinion about the usefulness of the relativistic mass concept [21].

7 The Relativistically Invariant Mass of a System in Astrophysics

The resultant relativistically invariant mass m_0, (5.7),

$$
m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2\sum_{\substack{j,k=1\\ j<k}}^{N} m_j m_k (\gamma_{\ominus\mathbf{v}_j\oplus\mathbf{v}_k} - 1)}
\tag{7.1}
$$

of a particle system $S = S(m_k, \mathbf{v}_k, \Sigma_\mathbf{0}, N)$ comprises of two distinct kinds of relativistically invariant mass that represent the Newtonian contribution and the relativistic contribution. The two distinct kinds of mass are:

(*i*) The Newtonian mass m_{newton},

$$m_{newton} := \sum_{k=1}^{N} m_k \tag{7.2}$$

which is the sum of the invariant, rest masses of the particles that constitute the system S, as in (4.2).

(*ii*) The dark mass m_{dark},

$$m_{dark} := \sqrt{2 \sum_{\substack{j,k=1 \\ j<k}}^{N} m_j m_k (\gamma_{\ominus \mathbf{v}_j \oplus \mathbf{v}_k} - 1)} \tag{7.3}$$

The dark mass of a particle system S, given by (7.3), depends on the *velocity dispersion* of S, that is, on the spread of internal velocities $\mathbf{v}_{jk} = \ominus\mathbf{v}_j \oplus \mathbf{v}_k$, $1 \le j < k \le N$, of the constituent particles of S relative to each other. In other words, the dark mass in (7.3) measures the extent to which the system S deviates away from rigidity. Gravitationally, dark mass behaves just like ordinary mass, as postulated in cosmology [4, p. 37]. However, it is undetectable by all means other than gravity since it is fictitious, or virtual, in the sense that it is generated solely by relative motion between constituent objects of the system.

We thus see from (7.3) that the dark mass of a system may be viewed as a measure of mass that results solely form the velocity dispersion of the system. In astrophysics, the velocity dispersion of stars or galaxies in a cluster is estimated by measuring the radial velocities of selected constituents. Once the velocity distribution is known, the cluster's mass is calculated by using the *virial theorem* [3].

Dark matter was introduced into cosmology as an *ad hoc* postulate, hypothesized to provide observed missing gravitational force [5]. In contrast, dark mass emerges here as a consequence of the covariance of Einstein's special theory of relativity, and it stems from relative motion between constituent objects of a system. All relative velocities between the constituent particles of a *rigid* system vanish, so that if the system S is rigid, then $\ominus\mathbf{v}_j \oplus \mathbf{v}_k = \mathbf{0}$, $j,k = 1,\ldots,N$. This, in turn, implies by (7.3) that the dark mass of a rigid system vanishes.

The mass m_{newton} and the dark mass m_{dark} of a system S are relativistically invariant, and are composed according to the Pythagorean formula

$$m_0 = \sqrt{m_{newton}^2 + m_{dark}^2} \tag{7.4}$$

giving rise to the invariant resultant rest mass m_0 of the system S, as we see from (9.8) – (7.3).

It should be remarked that our dark matter is predicted by special relativity considerations. Hence, it need not account for the whole dark matter observed by astrophysicists in the cosmos, because there could be contributions from general relativity and, perhaps, other unknown sources.

8 The Relativistically Invariant Mass of a System in Particle Physics

Following the four-momentum in (3.4) we have

$$\left\| \begin{pmatrix} p_0 \\ \mathbf{p} \end{pmatrix} \right\| = \sqrt{\frac{E^2}{c^4} - \frac{\|\mathbf{p}\|^2}{c^2}} = m \tag{8.1}$$

where, by (3.4),

$$\begin{aligned} E &= m\gamma_{\mathbf{v}} c^2 \\ \mathbf{p} &= m\gamma_{\mathbf{v}} \mathbf{v} \end{aligned} \tag{8.2}$$

Assuming that both energy, E, and three-momentum, $\mathbf{p}$, are additive is equivalent to assuming that the four-momentum is additive. The latter assumption, in turn, led us to identity (4.3) that we, now, write as

$$\begin{pmatrix} \frac{E}{c^2} \\ \mathbf{p} \end{pmatrix} = \sum_{k=1}^{N} \begin{pmatrix} \frac{E_k}{c^2} \\ \mathbf{p}_k \end{pmatrix} = \sum_{k=1}^{N} m_k \begin{pmatrix} \gamma_{\mathbf{v}_k} \\ \gamma_{\mathbf{v}_k} \mathbf{v}_k \end{pmatrix} = m_0 \begin{pmatrix} \gamma_{\mathbf{v}_0} \\ \gamma_{\mathbf{v}_0} \mathbf{v}_0 \end{pmatrix} \tag{8.3}$$

In (8.3),

$$\begin{aligned} E_k &= m_k \gamma_{\mathbf{v}_k} c^2 \\ \mathbf{p}_k &= m_k \gamma_{\mathbf{v}_k} \mathbf{v}_k \end{aligned} \tag{8.4}$$

are the energy and momentum of the k-th particle of the system S, $k = 1, ..., N$, and accordingly,

$$\begin{aligned} E &= \sum_{k=1}^{N} E_k \\ \mathbf{p} &= \sum_{k=1}^{N} \mathbf{p}_k \end{aligned} \tag{8.5}$$

are the energy and momentum of the system S.

Furthermore, as in (4.3), $\mathbf{v}_0$ is the velocity of the CM frame of S relative to the rest frame $\Sigma_{\mathbf{0}}$, and m_0 is the resultant invariant mass of S.

Noting (3.11), the norms of the two extreme sides of (8.3) give the equation

$$m_0 = \sqrt{\frac{E^2}{c^4} - \frac{\|\mathbf{p}\|^2}{c^2}} \tag{8.6}$$

where E and $\mathbf{p}$ are given by (8.5). Identity (8.6) demonstrates, by the relativistic four-vector formalism, that the resultant mass m_0 of a particle system S in (5.22) is relativistically invariant, being the norm of a four-vector.

Identity (8.6), written equivalently as

$$E^2 = m_0^2 c^4 + \|\mathbf{p}\|^2 c^2 \tag{8.7}$$

is known in particle physics as the *energy-momentum relation*. For a particle in its inertial rest frame, the relation (8.7) reduces to Einstein's famous formula

$$E = m_0 c^2 \tag{8.8}$$

The energy-momentum relation (8.6) is used in particle physics to calculate the relativistically invariant mass m_0 of a system of particles in terms of the total energy E and momentum $\mathbf{p}$ of the system. However,

(*i*) the equality between m_0 in (8.6) and m_0 in (5.22);

(*ii*) the compatibility of m_0 in (5.22) with the four-vector formalism of special relativity, as seen in (8.3); and

(*iii*) the decomposition (7.4) of m_0 into Newtonian mass and dark mass,

have gone unnoticed.

As an illustrative example, let us consider two particles with rest (or, Newtonian) masses m_1 and m_2, and velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ relative to an inertial rest frame $\Sigma_{\mathbf{0}}$, respectively. If these particles were to collide and stick, the rest mass m_0 and the velocity $\mathbf{v}_0$ relative to $\Sigma_{\mathbf{0}}$ of the resulting composite particle would satisfy the four-momentum conservation law (4.3), that is

$$m_0 \begin{pmatrix} \gamma_{\mathbf{v}_0} \\ \gamma_{\mathbf{v}_0}\mathbf{v}_0 \end{pmatrix} = m_1 \begin{pmatrix} \gamma_{\mathbf{v}_1} \\ \gamma_{\mathbf{v}_1}\mathbf{v}_1 \end{pmatrix} + m_2 \begin{pmatrix} \gamma_{\mathbf{v}_2} \\ \gamma_{\mathbf{v}_2}\mathbf{v}_2 \end{pmatrix} \tag{8.9}$$

Hence, by (5.22) and (7.4),

$$\begin{aligned} m_0 &= \sqrt{(m_1+m_2)^2 + 2m_1 m_2(\gamma_{\ominus\mathbf{v}_1\oplus\mathbf{v}_2} - 1)} \\ &= \sqrt{m_{newton}^2 + m_{dark}^2} \end{aligned} \tag{8.10}$$

where

$$\begin{aligned} m_{newton} &= m_1 + m_2 \\ m_{dark} &= 2m_1 m_2(\gamma_{\ominus\mathbf{v}_1\oplus\mathbf{v}_2} - 1) > 0 \end{aligned} \tag{8.11}$$

and, by (5.24),

$$\mathbf{v}_0 = \frac{m_1\gamma_{\mathbf{v}_1}\mathbf{v}_1 + m_2\gamma_{\mathbf{v}_2}\mathbf{v}_2}{m_1\gamma_{\mathbf{v}_1} + m_2\gamma_{\mathbf{v}_2}} \tag{8.12}$$

Hence, the relativistic mass of the composite particle is $m_0\gamma_{\mathbf{v}_0}$, where m_0 is given by (8.10), and $\mathbf{v}_0$ is given by (8.12).

It is clear from (8.10) –(8.11) that the Newtonian mass, m_{newton}, is conserved during the collision. It is only the total invariant mass, m_0, which is increased following the collision owing to the emergence of the dark mass m_{dark}.

Examples of particles that collide and stick, as described in (8.9) – (8.12), are observed in experimental searches for new particles in high-energy particle colliders.

We thus see that the concept of the relativistic mass fits well under the umbrella of the four-vector formalism of special relativity, and that the resulting dark mass emerges naturally not only in the interpretation of observations in astrophysics, demonstrated in Sec. 7, but also in the interpretation of observations in particle physics, demonstrated in this section.

Following M.H.L. Pryce [19], and L.R. Lehner and O.M. Moreschi [12], and several other authors, it is commonly believed that a satisfactory relativistic center of mass definition does not exist. However, by employing the relativistic mass $m_0\gamma_{\mathbf{v}_0}$ of a system S, we present the relativistic center of mass of an isolated, *disintegrated* system in [38, Sec. 11.18]. Applications of the relativistic mass $m_0\gamma_{\mathbf{v}_0}$ of a system S in hyperbolic geometry are presented in [26], [43] and [44, 45].

9 Conclusion

Seeking a way to place the relativistic mass $m_0\gamma_{\mathbf{v}_0}$ of a particle system S under the umbrella of the Minkowskian four-vector formalism of special relativity, we have uncovered the novel, relativistically invariant, or rest, mass m_0 of a particle system, presented in (9.8) below. Furthermore, following the discovery of m_0 in (9.8), we have uncovered remarkable analogies that Newtonian and Einsteinian mechanics share.

To see the analogies, let us consider the following well known classical results, (9.2) – (9.4) below, which are involved in the calculation of the Newtonian resultant mass m_0 and the classical center of momentum (CM) of a Newtonian system of particles, and to which we will subsequently present our Einsteinian analogs that have been discovered in Theorem 5.2. Let

$$S = S(m_k, \mathbf{v}_k, \Sigma_{\mathbf{0}}, k = 1, \ldots, N), \qquad \mathbf{v}_k \in \mathbb{R}^n \tag{9.1}$$

be an isolated Newtonian system of N noninteracting material particles the k-th particle of which has mass m_k and Newtonian uniform velocity $\mathbf{v}_k$ relative to an inertial frame $\Sigma_{\mathbf{0}}$, $k = 1, \ldots, N$. Furthermore, let m_0 be the resultant mass of S, considered as the mass of a virtual particle located at the center of momentum (CM) of S, and let $\mathbf{v}_0$ be the Newtonian velocity relative to $\Sigma_{\mathbf{0}}$ of the Newtonian CM frame of S. Then we have the following well-known identities:

$$1 = \frac{1}{m_0}\sum_{k=1}^{N} m_k \tag{9.2}$$

and

$$\begin{aligned} \mathbf{v}_0 &= \frac{1}{m_0}\sum_{k=1}^{N} m_k \mathbf{v}_k \\ \mathbf{w} + \mathbf{v}_0 &= \frac{1}{m_0}\sum_{k=1}^{N} m_k(\mathbf{w} + \mathbf{v}_k) \end{aligned} \tag{9.3}$$

where the binary operation + is the common vector addition in $\mathbb{R}^n$, and where

$$m_0 = \sum_{k=1}^{N} m_k \tag{9.4}$$

$\mathbf{v}, \mathbf{w}_k \in \mathbb{R}^3$, $m_k > 0$, $k = 0, 1, \ldots, N$.

In full analogy with (9.1), let

$$S = S(m_k, \mathbf{v}_k, \Sigma_{\mathbf{0}}, k = 1, \ldots, N), \qquad \mathbf{v}_k \in \mathbb{R}_c^n \tag{9.5}$$

be an isolated Einsteinian system of N noninteracting material particles the k-th particle of which has invariant mass m_k and Einsteinian uniform velocity $\mathbf{v}_k$ relative to an inertial frame $\Sigma_{\mathbf{0}}$, $k = 1, \ldots, N$. Furthermore, let m_0 be the resultant mass of S, considered as the mass of a virtual particle located at the center of mass of S (calculated in (5.12)), and let $\mathbf{v}_0$ be the Einsteinian velocity relative to $\Sigma_{\mathbf{0}}$ of the Einsteinian center of momentum (CM) frame of the Einsteinian system S. Then, as shown in Theorem 5.2, the relativistic analogs of the Newtonian expressions in (9.2) – (9.4) are, respectively, the following Einsteinian expressions in (9.6) – (9.8),

$$\begin{aligned} \gamma_{\mathbf{v}_0} &= \frac{1}{m_0} \sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \\ \gamma_{\mathbf{u}\oplus\mathbf{v}_0} &= \frac{1}{m_0} \sum_{k=1}^{N} m_k \gamma_{\mathbf{u}\oplus\mathbf{v}_k} \end{aligned} \tag{9.6}$$

and

$$\begin{aligned} \gamma_{\mathbf{v}_0} \mathbf{v}_0 &= \frac{1}{m_0} \sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \mathbf{v}_k \\ \gamma_{\mathbf{w}\oplus\mathbf{v}_0} (\mathbf{w}\oplus\mathbf{v}_0) &= \frac{1}{m_0} \sum_{k=1}^{N} m_k \gamma_{\mathbf{w}\oplus\mathbf{v}_k} (\mathbf{w}\oplus\mathbf{v}_k) \end{aligned} \tag{9.7}$$

where the binary operation $\oplus$ is the Einstein velocity addition in $\mathbb{R}_c^n$, given by (2.2), and where

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2 \sum_{\substack{j,k=1 \\ j<k}}^{N} m_j m_k (\gamma_{\ominus\mathbf{v}_j\oplus\mathbf{v}_k} - 1)} \tag{9.8}$$

$\mathbf{w}, \mathbf{v}_k \in \mathbb{R}_c^3$, $m_k > 0$, $k = 0, 1, \ldots, N$. Here m_0 is the relativistic invariant mass of the Einsteinian system S, supposed concentrated at the relativistic center of mass of S, and $\mathbf{v}_0$ is the Einsteinian velocity relative to $\Sigma_{\mathbf{0}}$ of the Einsteinian CM frame of the Einsteinian system S.

To conform with the Minkowskian four-vector formalism of special relativity, both m_0 and $\mathbf{v}_0$ are determined in Theorem 5.2 as the unique solution of the Minkowskian four-vector equation (5.3).

We finally wrote (9.8) as, (7.4),

$$m_0 = \sqrt{m_{newton}^2 + m_{dark}^2} \tag{9.9}$$

viewing the relativistically invariant, or rest, mass m_0 of the system S as a Pythagorean composition of the Newtonian rest mass, m_{newton} and the dark mass, m_{dark} of S. The mass m_{dark} is *dark* in the sense that it is the mass of virtual matter that does not collide and does not emit radiation. Following observations in cosmology, one may postulate that our dark mass reveals its presence only gravitationally. we have shown qualitatively that (9.9) explains observations in both astrophysics and particle physics.

We should remark that the presence of our dark mass is dictated by the hyperbolic geometry that underlies special relativity. Hence, it need not account for the whole mass of dark matter observed by astrophysicists in the cosmos, because there could be contributions from general relativity and, perhaps, other unknown sources.

It is well known that Newtonian resultant masses m_0 in Theorem 5.3 of particle systems play a role in the introduction of barycentric coordinates into Euclidean geometry, where they are employed, for instance, for the determination of various triangle centers, as demonstrated in [44, 45].

Surprisingly, in full analogy with Newtonian resultant masses, Einsteinian relativistic resultant masses $m_0\gamma_{\mathbf{v}_0}$ in Theorem 5.2 of particle systems play a role in the introduction of barycentric coordinates into hyperbolic geometry, where they are employed, for instance, for the determination of various hyperbolic triangle centers, as demonstrated in [44, 45].

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