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ON THE STATIONARY OSEEN EQUATIONS IN $\mathbb{R}^3$

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Abstract

The stationary Oseen equations are studied in $\mathbb{R}^3$ in its general form, that is, with a non-constant divergenceless function on the convective term. We prove existence, uniqueness and regularity results in weighted Sobolev spaces. From this new approach, we also state existence, uniqueness and regularity results for the generalized Oseen model which describes the rotating flows. The proofs are based on Laplace, Stokes and Oseen theories.

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1 Introduction

We study the linearized Navier-Stokes equations at the steady state, that is, the Oseen equations

$$-\Delta \boldsymbol{u} + (\boldsymbol{a}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \boldsymbol{f}, \quad \operatorname{div}\boldsymbol{u} = g \quad \text{in } \mathbb{R}^3. \tag{1.1}$$

Here $\boldsymbol{u}$ and π are unknown functions denoting the fluid velocity vector and the pressure function, respectively. The data are the external forces $\boldsymbol{f}$ acting on the fluid, a function g and a divergenceless function $\boldsymbol{a}$.

The presence of a non-constant function $\boldsymbol{a}$, which is motivated by the rotating flows, does not allow the use of the potential theory. Indeed, we have no fundamental solution at all. Under the assumptions that the datum $\boldsymbol{f}$ belongs to some weighted Sobolev spaces and

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the datum $\boldsymbol{a}$ belongs to some Lebesgue spaces, we will prove the existence of solution to the Oseen model (1.1). Namely, we state the existence, uniqueness and regularity of solutions when

(case i) $\boldsymbol{a} \in \mathbf{L}^3(\mathbb{R}^3)$ with $\operatorname{div} \boldsymbol{a} = 0$,

(case ii) $\boldsymbol{a} \in \mathbf{L}^3_{\text{loc}}(\mathbb{R}^3)$ with $\operatorname{div} \boldsymbol{a} = 0$, satisfying

$$\exists k > 0,\ \exists R_0 > 0: \quad \boldsymbol{a}(\boldsymbol{x}) = k\mathbf{e}_1, \quad |\boldsymbol{x}| \geq R_0.$$

Note that in the case ii, $\boldsymbol{a} \notin \mathbf{L}^3(\mathbb{R}^3)$. Thus it is not included nor in the Oseen equations in the general form (1.1) nor in the simpler form of the Oseen equations

$$-\Delta \boldsymbol{u} + k\frac{\partial \boldsymbol{u}}{\partial x_1} + \nabla \pi = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = g \quad \text{ in } \mathbb{R}^3. \tag{1.2}$$

We refer that this classical steady Oseen model is studied by several authors [AR1, F, Fi, KS]. Also the study of the linear Oseen problem with $\boldsymbol{f} = \operatorname{div}\mathbf{F}$ as a particular datum can be found in [SY].

We also study the Oseen model in the form, for a constant $k > 0$, as in [FHM, KNP]

$$-\Delta \boldsymbol{u} + k\frac{\partial \boldsymbol{u}}{\partial x_1} + (\boldsymbol{a}\cdot\nabla)\boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = g \quad \text{ in } \mathbb{R}^3. \tag{1.3}$$

The model (1.1) could be seen as a particular case ($k = 0$) of the model (1.3), however the additional term $k\frac{\partial \boldsymbol{u}}{\partial x_1}$ is not a difficulty but it helps the existence results. We refer to [FHM, KNP] and the references therein, where the study of rotating fluids is based on the decomposition of singular kernel in Fourier space and on the Littlewood-Paley theory.

Here we restricted our study to $\mathbb{R}^3$ but by identical arguments it can be done to $\mathbb{R}^n$ for any $n \in \mathbb{N}$, $n \geq 2$, provided that $\boldsymbol{a} \in \mathbf{L}^n(\mathbb{R}^n)$.

The outline of the work is as follows. Next section we introduce the functional framework in the scope of the weighted Sobolev spaces. We prove that the Oseen problem (1.1) case i has a unique weak solution in a Hilbert space, a unique generalized solution under the L^p-theory, strong solutions and very weak solutions in Sections 3, 4, 5 and 6, respectively. Section 7 is devoted to the proof of existence results of the weak, generalized and strong solutions for the generalized Oseen problem (1.3). Finally in Section 8, we solve the Oseen problem (1.1) case ii.

2 Preliminar results in weighted Sobolev spaces

Let us introduce the weighted Sobolev spaces with the logarithmic factor [AGG],

$$\begin{aligned} W^{m,p}_\alpha(\mathbb{R}^3) \quad = \quad & \{u \in \mathcal{D}'(\mathbb{R}^3);\ \forall \boldsymbol{\lambda} \in \mathbb{N}^3, \\ & 0 \leq |\boldsymbol{\lambda}| \leq \varkappa,\ \rho^{\alpha-m+|\boldsymbol{\lambda}|}\ln^{-1}(1+\rho^2)D^{\boldsymbol{\lambda}}u \in \mathbf{L}^p(\mathbb{R}^3); \\ & \varkappa + 1 \leq |\boldsymbol{\lambda}| \leq m,\ \rho^{\alpha-m+|\boldsymbol{\lambda}|}D^{\boldsymbol{\lambda}}u \in \mathbf{L}^p(\mathbb{R}^3)\} \end{aligned}$$

for any positive integer m, real numbers $p > 1$, $\alpha \in \mathbb{R}$ and

$$\varkappa = \begin{cases} m - 3/p - \alpha & \text{if } 3/p + \alpha \in \{1, \cdots, m\} \\ -1 & \text{otherwise}, \end{cases}$$

and for $m = 0$, we set

$$W_\alpha^{0,p}(\mathbb{R}^3) = \{u \in \mathcal{D}'(\mathbb{R}^3);\ \rho^\alpha u \in L^p(\mathbb{R}^3)\}.$$

Here $\rho = \sqrt{1+|\boldsymbol{x}|^2}$ denotes the weight function where $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $|\cdot|$ is the usual euclidean norm in $\mathbb{R}^3$. Throughout this paper, the bold type characters denote vector distributions or vector Sobolev spaces.

These spaces obey the following embedding

$$W_\alpha^{m,p}(\mathbb{R}^3) \hookrightarrow W_{\alpha-1}^{m-1,p}(\mathbb{R}^3) \tag{2.1}$$

if and only if $m > 0$ and $3/p+\alpha \neq 1$. Note that $W_0^{1,p}(\mathbb{R}^3)$ is a reflexive Banach space endowed with the norm

$$\|u\|_{W_0^{1,p}(\mathbb{R}^3)} = \begin{cases} \left(\left\|\frac{\rho^{-1}}{\ln(1+\rho^2)}u\right\|^p_{L^p(\mathbb{R}^3)} + \|\nabla u\|^p_{\mathbf{L}^p(\mathbb{R}^3)}\right)^{1/p} & \text{if } p = 3 \\ (\|\rho^{-1}u\|^p_{L^p(\mathbb{R}^3)} + \|\nabla u\|^p_{\mathbf{L}^p(\mathbb{R}^3)})^{1/p} & \text{otherwise.} \end{cases}$$

For a detailed study of these spaces we refer to [AGG, K]. However, let us recall some properties that we will need in this paper. It is known that the space $\mathcal{D}(\mathbb{R}^3)$ is dense in $W_\alpha^{m,p}(\mathbb{R}^3)$. So that its dual space, denoted by $W_{-\alpha}^{-m,p'}(\mathbb{R}^3)$, where $p' = p/(p-1)$ is the conjugate exponent of p, is a space of distributions. The embedding (2.1) also holds for $m \leq 0$ and $3/p+\alpha \neq 3$. Remark that $W_0^{-1,p}(\mathbb{R}^3)$ has the following characterization, for all $p > 1$,

$$W_0^{-1,p}(\mathbb{R}^3) = \{f \in \mathcal{D}'(\mathbb{R}^3);\ f = f_0 + \mathrm{div}\boldsymbol{f},\ f_0 \in E, \boldsymbol{f} \in \mathbf{L}^p(\mathbb{R}^3)\},$$

where E is one of the spaces, $W_1^{0,p}(\mathbb{R}^3)$ if $p \neq 3/2$, or

$$W_{1,1}^{0,p}(\mathbb{R}^3) = \{u \in L^p(\mathbb{R}^3);\ \rho\ln(1+\rho)u \in L^p(\mathbb{R}^3)\}, \quad \text{if } p = 3/2.$$

For $p > 3/2$ we can take $f_0 = 0$ since the Hardy inequality holds

$$\forall u \in W_0^{1,q}(\mathbb{R}^3)\,, \quad \|u\|_{W_0^{1,q}(\mathbb{R}^3)} \leq C\|\nabla u\|_{\mathbf{L}^q(\mathbb{R}^3)} \quad \text{if } q < 3. \tag{2.2}$$

Moreover, for all $p > 1$, $\mathcal{D}(\mathbb{R}^3)$ is also dense in $W_0^{-1,p}(\mathbb{R}^3)$.

For $1 < p < 3$, we have the Sobolev embedding $W_0^{1,p}(\mathbb{R}^3) \hookrightarrow L^{p*}(\mathbb{R}^3)$, with $\frac{1}{p*} = \frac{1}{p} - \frac{1}{3}$. Consequently, by duality we have, for $p' > 3/2$, $L^q(\mathbb{R}^3) \hookrightarrow W_0^{-1,p'}(\mathbb{R}^3)$, with $\frac{1}{q} = \frac{1}{p'} + \frac{1}{3}$.

It is known that $W_0^{1,3}(\mathbb{R}^3) \hookrightarrow BMO(\mathbb{R}^3)$, where $BMO(\mathbb{R}^3)$ is the space of locally integrable functions v such that

$$\sup_Q \frac{1}{|Q|}\int_Q |v(\boldsymbol{x}) - v_Q| d\boldsymbol{x} < \infty,$$

where Q is an arbitrary cube and $v_Q = \frac{1}{|Q|}\int_Q v(\boldsymbol{x})d\boldsymbol{x}$ is the average of v on Q.

For $p \geq 3$, the space $W_0^{1,p}(\mathbb{R}^3)$ contains the constants and the weighted Poincaré inequality holds

$$\exists C = C(p) > 0 : \quad \forall u \in W_0^{1,p}(\mathbb{R}^3), \quad \|u\|_{W_0^{1,p}(\mathbb{R}^3)/\mathbb{R}} \leq C\|\nabla u\|_{\mathbf{L}^p(\mathbb{R}^3)}. \tag{2.3}$$

For $3/p \notin \{1, \cdots, m\}$, the spaces $W_0^{m,p}(\mathbb{R}^3)$ and $W_1^{m+1,p}(\mathbb{R}^3)$ contain the polynomial functions $\mathbb{P}_{[m-3/p]}$ of degree lesser or equal than $[m-3/p]$, where $[s]$ stands for the integer part of $s \in \mathbb{R}_0^+$.

Hence further C will denote a generic positive constant that may vary from line to line.

Let us recall the following results about the behaviour at large distances of some functions. We begin by considering the case $1 < p < 3$ (see [AR] or [G, pp. 60]).

Lemma 2.1. *Assume* $1 < p < 3$ *and* $u \in \mathcal{D}'(\mathbb{R}^3)$ *such that* $\nabla u \in \mathbf{L}^p(\mathbb{R}^3)$. *Then there exists a unique constant* u_∞ *defined by*

$$u_\infty = \frac{1}{4\pi} \lim_{|\boldsymbol{x}|\to\infty} \int_{S_2} u(\sigma|\boldsymbol{x}|) d\sigma, \tag{2.4}$$

such that $u - u_\infty \in W_0^{1,p}(\mathbb{R}^3)$ *and where* S_2 *denotes the unit sphere of* $\mathbb{R}^3$. *Moreover, we have the following properties:*

$$u - u_\infty \in L^{3p/(3-p)}(\mathbb{R}^3), \tag{2.5}$$

with the estimate

$$\|u - u_\infty\|_{L^{3p/(3-p)}(\mathbb{R}^3)} \leq C \|\nabla u\|_{\mathbf{L}^p(\mathbb{R}^3)}, \tag{2.6}$$

$$\lim_{|\boldsymbol{x}|\to\infty} \int_{S_2} |u(\sigma|\boldsymbol{x}|) - u_\infty| d\sigma = \lim_{|\boldsymbol{x}|\to\infty} \int_{S_2} |u(\sigma|\boldsymbol{x}|) - u_\infty|^p d\sigma = 0 \tag{2.7}$$

and

$$\int_{S_2} |u(r\sigma) - u_\infty|^p d\sigma \leq C\, r^{p-3} \int_{\{\boldsymbol{x}\in\mathbb{R}^3,\, |\boldsymbol{x}|>r\}} |\nabla u|^p d\boldsymbol{x}, \tag{2.8}$$

with $r > 0$.

Definition 2.2. A function u will be said to tend weakly to a constant u_∞ at infinity if

$$\lim_{|\boldsymbol{x}|\to\infty} \int_{S_2} |u(\sigma|\boldsymbol{x}|) - u_\infty| d\sigma = 0.$$

Remark 2.3. Lemma 2.1 implies that, if $1 < p < 3$ and $u \in \mathcal{D}'(\mathbb{R}^3)$ such that $\nabla u \in \mathbf{L}^p(\mathbb{R}^3)$, then the previous definition is equivalent to

$$u - u_\infty \in W_0^{1,p}(\mathbb{R}^3).$$

We now give a result for the case $p > 3$ which can be obtained from the Sobolev inequalities.

Lemma 2.4. *Let* r *and* p *be two reals such that* $1 < r < \infty$ *and* $p > 3$. *Let* $u \in L^r(\mathbb{R}^3)$ *and* $\nabla u \in \mathbf{L}^p(\mathbb{R}^3)$. *Then* $u \in C(\mathbb{R}^3)$ *and*

$$\lim_{|\boldsymbol{x}|\to\infty} u(\boldsymbol{x}) = 0 \ \textit{pointwise}\ . \tag{2.9}$$

Defining

$$X_\alpha^{1,p}(\mathbb{R}^3) = \{v \in W_\alpha^{1,p}(\mathbb{R}^3);\ \tfrac{\partial v}{\partial x_1} \in W_\alpha^{-1,p}(\mathbb{R}^3)\};$$
$$X_\alpha^{2,p}(\mathbb{R}^3) = \{v \in W_\alpha^{2,p}(\mathbb{R}^3);\ \tfrac{\partial v}{\partial x_1} \in W_\alpha^{0,p}(\mathbb{R}^3)\},$$

where α will be taken equal to 0 or 1, we recall that $\mathcal{D}(\mathbb{R}^3)$ is dense in $X_\alpha^{1,p}(\mathbb{R}^3)$ and $X_\alpha^{2,p}(\mathbb{R}^3)$ (see [AR1]). Note that the operator $\partial_i : X_\alpha^{2,p}(\mathbb{R}^3) \to X_\alpha^{1,p}(\mathbb{R}^3)$ is continuous.

Finally let us state the following auxiliary results.

Proposition 2.5. *For $u \in X_0^{1,p}(\mathbb{R}^3)$, we have*

i) if $1 < p < 3$, then $u \in L^{4p/(4-p)}(\mathbb{R}^3) \cap L^{3p/(3-p)}(\mathbb{R}^3)$ and the following estimate holds

$$\|u\|_{L^{4p/(4-p)}(\mathbb{R}^3)} + \|u\|_{L^{3p/(3-p)}(\mathbb{R}^3)} \leq C\|u\|_{X_0^{1,p}(\mathbb{R}^3)}. \tag{2.10}$$

ii) if $p = 3$, then there exists a unique constant λ such that $u + \lambda \in L^{12}(\mathbb{R}^3) \cap BMO$. In particular, $u + \lambda \in L^r(\mathbb{R}^3)$ for any $r \geq 12$, and the following estimate holds

$$\|u + \lambda\|_{L^r(\mathbb{R}^3)} \leq C\|u\|_{X_0^{1,p}(\mathbb{R}^3)}. \tag{2.11}$$

iii) if $3 < p < 4$, then there exists a unique constant λ such that $u + \lambda \in L^{4p/(4-p)}(\mathbb{R}^3) \cap C(\mathbb{R}^3)$ and the following estimate holds

$$\|u + \lambda\|_{L^{4p/(4-p)}(\mathbb{R}^3)} \leq C\|u\|_{X_0^{1,p}(\mathbb{R}^3)}. \tag{2.12}$$

Proof. i) This proof can be found in [AN].

ii) For $u \in X_0^{1,3}(\mathbb{R}^3)$, we have $-\Delta u + \frac{\partial u}{\partial x_1} \in W_0^{-1,3}(\mathbb{R}^3)$. From scalar Oseen theory [AR, Theorem 4.4], there exists a unique $v \in X_0^{1,3}(\mathbb{R}^3) \cap L^{12}(\mathbb{R}^3)$ verifying

$$\Delta v + \frac{\partial v}{\partial x_1} = \Delta u + \frac{\partial u}{\partial x_1} \quad \text{in } \mathbb{R}^3. \tag{2.13}$$

By uniqueness argument, we show that $\nabla u = \nabla v$, that means, there exists a unique constant λ such that $u + \lambda = v$. As $v \in BMO$, we obtain $u + \lambda \in L^{12}(\mathbb{R}^3) \cap BMO$, and consequently $u + \lambda \in L^r(\mathbb{R}^3)$ for any $r \geq 12$, verifying (2.11).

iii) For $u \in X_0^{1,p}(\mathbb{R}^3)$, similarly to the case ii, there exist a unique $v \in X_0^{1,p}(\mathbb{R}^3) \cap L^{4p/(4-p)}(\mathbb{R}^3)$ verifying (2.13) and a unique constant λ such that $u + \lambda = v$. Then we get $u + \lambda \in L^{4p/(4-p)}(\mathbb{R}^3)$ verifying (2.12). Applying Lemma 2.4 we conclude that $u + \lambda \in C(\mathbb{R}^3)$. □

Proposition 2.6. *For $u \in X_0^{2,p}(\mathbb{R}^3)$, we have*

i) if $1 < p < 3/2$, then $u \in L^{2p/(2-p)}(\mathbb{R}^3) \cap L^{3p/(3-2p)}(\mathbb{R}^3)$;

ii) if $3/2 \leq p < 2$, then there exists a unique constant λ such that $u + \lambda \in L^q(\mathbb{R}^3)$ for all $q \geq 2p/(2-p)$;

iii) if $1 < p < 3$, then $\nabla u \in \mathbf{L}^{4p/(4-p)}(\mathbb{R}^3) \cap \mathbf{L}^{3p/(3-p)}(\mathbb{R}^3)$;

iv) if $p = 3$, then there exists a unique $\lambda \in \mathbb{P}_1$, independent on x_1, such that $\nabla(u + \lambda) \in \mathbf{L}^{12}(\mathbb{R}^3) \cap \mathbf{BMO}(\mathbb{R}^3)$. In particular, $\nabla(u + \lambda) \in \mathbf{L}^r(\mathbb{R}^3)$ for any $r \geq 12$;

v) if $3 < p < 4$, then there exists a unique $\lambda \in \mathbb{P}_1$, independent on x_1, such that $\nabla(u + \lambda) \in \mathbf{L}^{4p/(4-p)}(\mathbb{R}^3) \cap \mathbf{L}^\infty(\mathbb{R}^3)$.

Proof. The proof of i)-ii) can be found in [AN], while iii)-v) easily follow from Proposition 2.5. □

3 Weak solutions in $\mathbf{W}_0^{1,2}(\mathbb{R}^3)$

Let us introduce the spaces

$$\begin{aligned}
\mathcal{V} &= \{\boldsymbol{v} \in \mathcal{D}(\mathbb{R}^3); \quad \operatorname{div} \boldsymbol{v} = 0 \quad \text{in } \mathbb{R}^3\};\\
\mathbf{H}_p &= \{\boldsymbol{v} \in \mathbf{L}^p(\mathbb{R}^3); \quad \operatorname{div} \boldsymbol{v} = 0 \quad \text{in } \mathbb{R}^3\};\\
\mathbf{V}_p &= \{\boldsymbol{v} \in \mathbf{W}_0^{1,p}(\mathbb{R}^3); \quad \operatorname{div} \boldsymbol{v} = 0 \quad \text{in } \mathbb{R}^3\},
\end{aligned}$$

and we suppose in this section that $\boldsymbol{a} \in \mathbf{H}_3$.

Definition 3.1. We say that $\boldsymbol{u} \in \mathbf{V}_2$ is a weak solution to the problem (1.1), with $g = 0$, if it satisfies

$$\forall \boldsymbol{v} \in \mathbf{V}_2, \quad \int_{\mathbb{R}^3} \nabla \boldsymbol{u} : \nabla \boldsymbol{v}\, d\boldsymbol{x} + \int_{\mathbb{R}^3} \nabla \boldsymbol{u} : (\boldsymbol{a} \otimes \boldsymbol{v}) d\boldsymbol{x} = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \tag{3.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality bracket $\mathbf{W}_0^{-1,2}(\mathbb{R}^3) \times \mathbf{W}_0^{1,2}(\mathbb{R}^3)$.

Remark 3.2. The Sobolev embedding theorem yields $W_0^{1,p}(\mathbb{R}^3) \hookrightarrow L^{p*}(\mathbb{R}^3)$ if $p < 3$. If $\boldsymbol{a} \in \mathbf{L}^3(\mathbb{R}^3)$, then for $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{W}_0^{1,2}(\mathbb{R}^3)$, we have $\boldsymbol{a} \otimes \boldsymbol{v} \in \mathbf{L}^2(\mathbb{R}^3)$ and the second integral in (3.1) is meanful. Moreover, thanks to the density of $\mathcal{V}$ in $\mathbf{V}_2$ (see [AA]), we have the property

$$\forall \boldsymbol{v} \in \mathbf{V}_2, \quad \int_{\mathbb{R}^3} \nabla \boldsymbol{v} : (\boldsymbol{a} \otimes \boldsymbol{v}) d\boldsymbol{x} = 0. \tag{3.2}$$

Theorem 3.3. *Given* $\boldsymbol{f} \in \mathbf{W}_0^{-1,2}(\mathbb{R}^3)$, $\boldsymbol{a} \in \mathbf{H}_3$ *and* $g = 0$, *there exists a unique weak solution* $\boldsymbol{u} \in \mathbf{V}_2$ *of the problem (3.1) and moreover* $\boldsymbol{u}$ *satisfies the estimate*

$$\|\boldsymbol{u}\|_{\mathbf{W}_0^{1,2}(\mathbb{R}^3)} \leq C \|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,2}(\mathbb{R}^3)} \tag{3.3}$$

and the energy equality

$$\int_{\mathbb{R}^3} |\nabla \boldsymbol{u}|^2 d\boldsymbol{x} = \langle \boldsymbol{f}, \boldsymbol{u} \rangle_{\mathbf{W}_0^{-1,2}(\mathbb{R}^3) \times \mathbf{W}_0^{1,2}(\mathbb{R}^3)}.$$

Besides, there exists a unique function $\pi \in L^2(\mathbb{R}^3)$ *such that* $(\boldsymbol{u}, \pi)$ *solves the problem (1.1) in the sense of distributions, and the following estimate holds*

$$\|\pi\|_{L^2(\mathbb{R}^3)} \leq C(1 + \|\boldsymbol{a}\|_{\mathbf{L}^3(\mathbb{R}^3)}) \|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,2}(\mathbb{R}^3)}. \tag{3.4}$$

Proof. Let $\{R_m\}_{m \in \mathbb{N}}$ be an increasing sequence of positive reals such that $\lim_{m \to \infty} R_m = \infty$ and the sequence of approximate problems defined in the open balls centered at the origin $B_m = B_{R_m}(0) \subset \mathbb{R}^3$:

Find $\boldsymbol{u}_m \in \mathbf{J}_m$ such that, for all $\boldsymbol{v} \in \mathbf{J}_m$,

$$\int_{B_m} \nabla \boldsymbol{u}_m : \nabla \boldsymbol{v}\, d\boldsymbol{x} + \int_{B_m} \nabla \boldsymbol{u}_m : (\boldsymbol{a} \otimes \boldsymbol{v}) d\boldsymbol{x} = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\mathbf{H}^{-1}(B_m) \times \mathbf{H}_0^1(B_m)}, \tag{3.5}$$

where

$$\mathbf{J}_m = \{\boldsymbol{v} \in \mathbf{H}_0^1(B_m);\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } B_m\}.$$

Note that if $\boldsymbol{f} \in \mathbf{W}_0^{-1,2}(\mathbb{R}^3)$, then its restriction to B_m belongs to $\mathbf{H}^{-1}(B_m)$ and it verifies

$$\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(B_m)} \leq \|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,2}(\mathbb{R}^3)}. \tag{3.6}$$

The existence of a unique solution, $\boldsymbol{u}_m \in \mathbf{J}_m$, of (3.5) is a consequence of the Lax-Milgram Lemma and the property

$$\forall \boldsymbol{v} \in \mathbf{J}_m, \quad \int_{B_m} \nabla \boldsymbol{v} : (\boldsymbol{a} \otimes \boldsymbol{v}) d\boldsymbol{x} = 0.$$

Let us take $\boldsymbol{v} = \boldsymbol{u}_m$ as a test function in (3.5). Thus the energy equality holds

$$\int_{B_m} |\nabla \boldsymbol{u}_m|^2 d\boldsymbol{x} = \langle \boldsymbol{f}, \boldsymbol{u}_m \rangle = -\int_{B_m} \mathbf{F} \cdot \nabla \boldsymbol{u}_m d\boldsymbol{x},$$

observing that $\boldsymbol{f} = \operatorname{div} \mathbf{F}$ with

$$\mathbf{F} \in \mathbf{L}^2(\mathbb{R}^3): \quad \|\mathbf{F}\|_{\mathbf{L}^2(\mathbb{R}^3)} \leq C \|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,2}(\mathbb{R}^3)}.$$

Consequently the estimate holds

$$\|\nabla \boldsymbol{u}_m\|^2_{\mathbf{L}^2(B_m)} \leq \|\mathbf{F}\|_{\mathbf{L}^2(B_m)} \|\nabla \boldsymbol{u}_m\|_{\mathbf{L}^2(B_m)} \leq C \|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,2}(\mathbb{R}^3)} \|\nabla \boldsymbol{u}_m\|_{\mathbf{L}^2(B_m)}.$$

Denoting by $\tilde{\boldsymbol{u}}_m$, the extended function by zero in $\mathbb{R}^3 \setminus B_m$, we have $\tilde{\boldsymbol{u}}_m \in \mathbf{W}_0^{1,2}(\mathbb{R}^3)$ and

$$\|\nabla \tilde{\boldsymbol{u}}_m\|_{\mathbf{L}^2(\mathbb{R}^3)} \leq C \|\boldsymbol{f}\|_{\mathbf{W}^{-1,2}(\mathbb{R}^3)}. \tag{3.7}$$

From the estimates (3.7) and (2.2) then, extracting subsequences if necessary, we get $\tilde{\boldsymbol{u}}_m \rightharpoonup \boldsymbol{u}$ in $\mathbf{W}_0^{1,2}(\mathbb{R}^3)$ and

$$\|\boldsymbol{u}\|_{\mathbf{W}_0^{1,2}(\mathbb{R}^3)} \leq C \|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,2}(\mathbb{R}^3)}.$$

In order to prove that $\boldsymbol{u}$ is a weak solution to (3.1), let $\boldsymbol{v} \in \mathcal{D}(\mathbb{R}^3)$ such that $\operatorname{div} \boldsymbol{v} = 0$. Choosing $n \in \mathbb{N}$ such that $\operatorname{supp} \boldsymbol{v} \subset B_n$, then for any $m \geq n$ it follows

$$\forall \boldsymbol{v} \in \boldsymbol{\mathcal{V}}, \quad \int_{\mathbb{R}^3} \nabla \tilde{\boldsymbol{u}}_m : \nabla \boldsymbol{v} \, d\boldsymbol{x} + \int_{\mathbb{R}^3} \nabla \tilde{\boldsymbol{u}}_m : (\boldsymbol{a} \otimes \boldsymbol{v}) d\boldsymbol{x} = \langle \boldsymbol{f}, \boldsymbol{v} \rangle. \tag{3.8}$$

So we can pass to the limit in (3.8) and therefore $\boldsymbol{u}$ satisfies (3.1), for any $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}$. Thanks to the density of $\boldsymbol{\mathcal{V}}$ in $\mathbf{V}_2$ (cf. [AA]), it is clear that (3.1) holds for any $\boldsymbol{v} \in \mathbf{V}_2$. Hence, the estimate (3.3) arises.

The solution $\boldsymbol{u}$ is unique using the standard argument of taking two solutions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ to the linear weak formulation (3.1). Indeed defining $\boldsymbol{w} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, by (3.2) it verifies

$$\int_{\mathbb{R}^3} |\nabla \boldsymbol{w}|^2 d\boldsymbol{x} = 0,$$

then $\boldsymbol{w} = \mathbf{0}$ and $\boldsymbol{u}_1 = \boldsymbol{u}_2$.

Since $\boldsymbol{a} \otimes \boldsymbol{u} \in \mathbf{L}^2(\mathbb{R}^3)$, so that $\boldsymbol{f} + \Delta \boldsymbol{u} - \operatorname{div}(\boldsymbol{a} \otimes \boldsymbol{u}) \in \mathbf{W}_0^{-1,2}(\mathbb{R}^3)$. Hence, we have

$$\forall \boldsymbol{v} \in V_2, \quad \langle \boldsymbol{f} + \Delta \boldsymbol{u} - \operatorname{div}(\boldsymbol{a} \otimes \boldsymbol{u}), \boldsymbol{v} \rangle = 0.$$

Therefore, De Rham Theorem in weighted spaces (cf. [AA]) guarantees the existence of a unique pressure $\pi \in L^2(\mathbb{R}^3)$ such that $(\boldsymbol{u}, \pi)$ satisfies the Oseen problem (1.1) in the sense of distributions. Moreover, we get

$$\|\nabla \pi\|_{\mathbf{W}_0^{-1,2}(\mathbb{R}^3)} \leq \|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,2}(\mathbb{R}^3)} + C\Big(\|\boldsymbol{u}\|_{\mathbf{W}_0^{1,2}(\mathbb{R}^3)} + \|\boldsymbol{a} \otimes \boldsymbol{u}\|_{\mathbf{L}^2(\mathbb{R}^3)}\Big).$$

Using the estimate (3.3), it follows

$$\|\nabla \pi\|_{\mathbf{W}_0^{-1,2}(\mathbb{R}^3)} \leq C\Big(\|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,2}(\mathbb{R}^3)} + \|\boldsymbol{a} \otimes \boldsymbol{u}\|_{\mathbf{L}^2(\mathbb{R}^3)}\Big).$$

Applying Hölder inequality, Sobolev embedding and using again the estimate (3.3) we obtain

$$\|\nabla\pi\|_{\mathbf{W}_0^{-1,2}(\mathbb{R}^3)} \le C(1+\|\boldsymbol{a}\|_{\mathbf{L}^3(\mathbb{R}^3)})\|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,2}(\mathbb{R}^3)}.$$

Finally, using the fact that the operator gradient is an isomorphism from $L^2(\mathbb{R}^3)$ to $\mathbf{W}_0^{-1,2}(\mathbb{R}^3) \perp \mathbf{V}_2$ (see [AA]) we conclude (3.4). The previous notation means the subspace of $\mathbf{W}_0^{-1,2}(\mathbb{R}^3)$ orthogonal to $\mathbf{V}_2$, *i.e.*,

$$\begin{aligned} \mathbf{W}_0^{-1,2}(\mathbb{R}^3) \perp \mathbf{V}_2 &= \{\boldsymbol{f} \in \mathbf{W}_0^{-1,2}(\mathbb{R}^3);\ \forall \boldsymbol{v} \in \mathbf{V}_2,\ \langle \boldsymbol{f}, \boldsymbol{v}\rangle = 0\} \\ &= \left(\mathbf{W}_0^{-1,2}(\mathbb{R}^3)/\mathbf{V}_2\right)'. \end{aligned}$$

□

Theorem 3.4. *Given* $\boldsymbol{f} \in \mathbf{W}_0^{-1,2}(\mathbb{R}^3)$, $\boldsymbol{a} \in \mathbf{H}_3$ *and* $g \in L^2(\mathbb{R}^3)$, *there exists a unique solution* $(\boldsymbol{u},\pi) \in \mathbf{W}_0^{1,2}(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ *of the problem (1.1) such that*

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathbf{W}_0^{1,2}(\mathbb{R}^3)} &\le C\Big(\|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,2}(\mathbb{R}^3)} + \|g\|_{L^2(\mathbb{R}^3)}(1+\|\boldsymbol{a}\|_{\mathbf{L}^3(\mathbb{R}^3)})\Big) := M && (3.9) \\ \|\pi\|_{L^2(\mathbb{R}^3)} &\le M(1+\|\boldsymbol{a}\|_{\mathbf{L}^3(\mathbb{R}^3)}), && (3.10) \end{aligned}$$

and satisfying the energy equality

$$\int_{\mathbb{R}^3} |\nabla \boldsymbol{u}|^2 d\boldsymbol{x} = \langle \boldsymbol{f}, \boldsymbol{u}\rangle_{\mathbf{W}_0^{-1,2}(\mathbb{R}^3)\times\mathbf{W}_0^{1,2}(\mathbb{R}^3)} + \int_{\mathbb{R}^3} \pi g\, d\boldsymbol{x}\ . \qquad (3.11)$$

Proof. Given $g \in L^2(\mathbb{R}^3)$, since the operator div is an isomorphism from $\mathbf{W}_0^{1,2}(\mathbb{R}^3)/\mathbf{V}_2$ to $L^2(\mathbb{R}^3)$ (see [AA, Proposition 2.2]) there exists $\boldsymbol{w} \in \mathbf{W}_0^{1,2}(\mathbb{R}^3)$ such that $\operatorname{div}\boldsymbol{w} = g$ with the estimate

$$\|\boldsymbol{w}\|_{\mathbf{W}_0^{1,2}(\mathbb{R}^3)} \le C\|g\|_{L^2(\mathbb{R}^3)}.$$

Since $\boldsymbol{f} - \operatorname{div}(\boldsymbol{a}\otimes\boldsymbol{w}) \in \mathbf{W}_0^{-1,2}(\mathbb{R}^3)$, then we can solve the Oseen problem:

$$-\Delta\boldsymbol{v} + (\boldsymbol{a}\cdot\nabla)\boldsymbol{v} + \nabla\pi = \boldsymbol{f} + \Delta\boldsymbol{w} - \operatorname{div}(\boldsymbol{a}\otimes\boldsymbol{w}) \quad \text{and} \quad \operatorname{div}\boldsymbol{v} = 0 \quad \text{in } \mathbb{R}^3.$$

Indeed, the existence and uniqueness of $\boldsymbol{v} \in \mathbf{V}_2$ and $\pi \in L^2(\mathbb{R}^3)$ follow from Theorem 3.3. Moreover, we have

$$\begin{aligned} \|\boldsymbol{v}\|_{\mathbf{W}_0^{1,2}(\mathbb{R}^3)} &\le C(\|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,2}(\mathbb{R}^3)} + \|\Delta\boldsymbol{w}\|_{\mathbf{W}_0^{-1,2}(\mathbb{R}^3)} + \|\boldsymbol{a}\|_{\mathbf{L}^3(\mathbb{R}^3)}\|\boldsymbol{w}\|_{\mathbf{L}^6(\mathbb{R}^3)}) \\ \|\pi\|_{L^2(\mathbb{R}^3)} &\le C(\|\boldsymbol{f}+\Delta\boldsymbol{w}\|_{\mathbf{W}_0^{-1,2}(\mathbb{R}^3)} + \|\boldsymbol{a}\otimes\boldsymbol{w}\|_{\mathbf{L}^2(\mathbb{R}^3)})\left(1+\|\boldsymbol{a}\|_{\mathbf{L}^3(\mathbb{R}^3)}\right). \end{aligned}$$

Choosing $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w} \in \mathbf{W}_0^{1,2}(\mathbb{R}^3)$ then $(\boldsymbol{u},\pi) \in \mathbf{W}_0^{1,2}(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ is the required solution and it satisfies (3.9) and (3.10). □

4 Generalized solutions in $\mathbf{W}_0^{1,p}(\mathbb{R}^3)$

Let us state the following generalized result using the known Stokes theory in $\mathbb{R}^3$.

Lemma 4.1. *For $1 < p < \infty$, let $\boldsymbol{f} \in \mathbf{W}_0^{-1,2}(\mathbb{R}^3) \cap \mathbf{W}_0^{-1,p}(\mathbb{R}^3)$ satisfy the compatibility condition: for any $i = 1,2,3$*

$$\langle f_i, 1\rangle_{W_0^{-1,p}(\mathbb{R}^3)\times W_0^{1,p'}(\mathbb{R}^3)} = 0 \quad \text{if } p \leq 3/2, \tag{4.1}$$

$g \in L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ and $\boldsymbol{a} \in \mathbf{H}_3$. If additionally $\boldsymbol{a} \in \mathbf{L}^{6p/(6-p)}(\mathbb{R}^3)$ for $1 < p < 6$ or $\boldsymbol{a} \in \mathbf{L}^\infty(\mathbb{R}^3)$ for $p \geq 6$, then the solution $(\boldsymbol{u}, \pi) \in \mathbf{W}_0^{1,2}(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ given by Theorem 3.4 is such that $\boldsymbol{u} \in \mathbf{W}_0^{1,p}(\mathbb{R}^3)$ and $\pi \in L^p(\mathbb{R}^3)$.

Proof. Considering that $\boldsymbol{f} \in \mathbf{W}_0^{-1,2}(\mathbb{R}^3)$, $g \in L^2(\mathbb{R}^3)$ and $\boldsymbol{a} \in \mathbf{H}_3$, Theorem 3.4 yields the existence and uniqueness of $(\boldsymbol{u}, \pi)$ in the space $\mathbf{W}_0^{1,2}(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ verifying

$$-\Delta \boldsymbol{u} + \operatorname{div}(\boldsymbol{a} \otimes \boldsymbol{u}) + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = g \quad \text{in } \mathbb{R}^3, \tag{4.2}$$

taking into account that $\operatorname{div} \boldsymbol{a} = 0$. From the Sobolev embedding (cf. Remark 3.2), note that $\boldsymbol{u} \in \mathbf{L}^6(\mathbb{R}^3)$.

Case 1: $1 < p < 6$. We have $\boldsymbol{a} \otimes \boldsymbol{u} \in \mathbf{L}^p(\mathbb{R}^3)$ considering that $\boldsymbol{a} \in \mathbf{L}^{6p/(6-p)}(\mathbb{R}^3)$. Then $\boldsymbol{f} - \operatorname{div}(\boldsymbol{a} \otimes \boldsymbol{u}) \in \mathbf{W}_0^{-1,p}(\mathbb{R}^3)$. Moreover, we have, for all $i = 1,2,3$,

$$\langle f_i - \operatorname{div}(\boldsymbol{a} u_i), 1\rangle_{W_0^{-1,p}(\mathbb{R}^3)\times W_0^{1,p'}(\mathbb{R}^3)} = 0, \quad \text{if } p \leq 3/2.$$

Thus the Stokes theory [AA, Proposition 3.3] guarantees the existence of a solution $(\boldsymbol{v}, \eta) \in \mathbf{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$, where η is unique and $\boldsymbol{v}$ is unique, up to a constant of $\mathbb{R}^3$ if $p \geq 3$, of the Stokes problem

$$-\Delta \boldsymbol{v} + \nabla \eta = \boldsymbol{f} - \operatorname{div}(\boldsymbol{a} \otimes \boldsymbol{u}), \quad \operatorname{div} \boldsymbol{v} = g \quad \text{in } \mathbb{R}^3. \tag{4.3}$$

Thus the uniqueness argument implies first that the harmonic function $\eta - \pi$ belonging to $\mathbf{L}^p(\mathbb{R}^3) + \mathbf{L}^2(\mathbb{R}^3)$ is necessarily equal to zero and with similar argument, we obtain also $\nabla \boldsymbol{u} = \nabla \boldsymbol{v} \in \mathbf{L}^p(\mathbb{R}^3) \cap \mathbf{L}^2(\mathbb{R}^3)$. Note that $\boldsymbol{u} = \boldsymbol{v}$ if $p < 3$ and $\boldsymbol{u} = \boldsymbol{v} + \mathbf{k} \in \mathbf{W}_0^{1,p}(\mathbb{R}^3)$ with $\mathbf{k} \in \mathbb{R}^3$, if $p \geq 3$.

Case 2: $p \geq 6$. For all $2 \leq q \leq 6$, we have $\boldsymbol{f} \in \mathbf{W}_0^{-1,q}(\mathbb{R}^3)$ and $\boldsymbol{a} \otimes \boldsymbol{u} \in \mathbf{L}^2(\mathbb{R}^3) \cap \mathbf{L}^6(\mathbb{R}^3)$. Then we can apply case 1 to obtain $\boldsymbol{u} \in \mathbf{W}_0^{1,q}(\mathbb{R}^3)$ for all $2 \leq q < 6$. In particular, we get $\boldsymbol{u} \in \mathbf{L}^6(\mathbb{R}^3) \cap \mathbf{L}^\infty(\mathbb{R}^3)$. Then $\boldsymbol{a} \otimes \boldsymbol{u} \in \mathbf{L}^2(\mathbb{R}^3) \cap \mathbf{L}^\infty(\mathbb{R}^3) \hookrightarrow \mathbf{L}^p(\mathbb{R}^3)$ and arguing as in the case 1 we conclude the proof of Lemma 4.1. □

Lemma 4.2. *Let $1 < p < \infty$ and $\boldsymbol{a} \in \mathbf{H}_p$. Then, there exist $\boldsymbol{\psi} \in \mathbf{V}_p$ such that*

$$\boldsymbol{a} = \nabla \times \boldsymbol{\psi}, \tag{4.4}$$

and a sequence $\{\boldsymbol{a}_m\}_{m\in\mathbb{N}} \subset \boldsymbol{\mathcal{V}}$ such that

$$\boldsymbol{a}_m \to \boldsymbol{a} \quad \text{in } \mathbf{L}^p(\mathbb{R}^3). \tag{4.5}$$

Proof. For $\boldsymbol{a} \in \mathbf{H}_p$, we get $\nabla \times \boldsymbol{a} \in \mathbf{W}_0^{-1,p}(\mathbb{R}^3) \perp \mathbb{P}_{[1-3/p']}$. From Laplace theory (see [AGG]), there exists $\boldsymbol{\psi} \in \mathbf{W}_0^{1,p}(\mathbb{R}^3)$ such that $-\Delta\boldsymbol{\psi} = \nabla \times \boldsymbol{a}$ in $\mathbb{R}^3$ with

$$\|\nabla\boldsymbol{\psi}\|_{\mathbf{L}^p(\mathbb{R}^3)} \leq C\|\boldsymbol{a}\|_{\mathbf{L}^p(\mathbb{R}^3)}.$$

Then, we have $\operatorname{div}\boldsymbol{\psi} \in \mathbf{L}^p(\mathbb{R}^3)$ such that $\Delta(\operatorname{div}\boldsymbol{\psi}) = 0$ in $\mathbb{R}^3$. Thus it results that $\operatorname{div}\boldsymbol{\psi} = 0$ and consequently $\boldsymbol{\psi} \in \mathbf{V}_p$. Since $\operatorname{div}\boldsymbol{a} = 0$, we deduce that

$$\Delta(\nabla \times \boldsymbol{\psi}) = -\nabla \times (\nabla \times \boldsymbol{a}) = \Delta\boldsymbol{a}.$$

Taking $\boldsymbol{w} = \nabla \times \boldsymbol{\psi} - \boldsymbol{a} \in \mathbf{L}^p(\mathbb{R}^3)$ we obtain $\Delta\boldsymbol{w} = 0$ and we conclude $\boldsymbol{w} = \mathbf{0}$ and (4.4) holds.

Since $\mathcal{D}(\mathbb{R}^3)$ is dense in $W_0^{1,p}(\mathbb{R}^3)$, we have

$$\boldsymbol{\psi} = \lim_{m\to+\infty} \boldsymbol{\psi}_m \quad \text{in } \mathbf{W}_0^{1,p}(\mathbb{R}^3), \quad \boldsymbol{\psi}_m \in \mathcal{D}(\mathbb{R}^3).$$

Taking $\boldsymbol{a}_m = \nabla \times \boldsymbol{\psi}_m \in \boldsymbol{\mathcal{V}}$, we obtain clearly (4.5). □

The following results deal with the existence to the problem (1.1) under the data into Banach spaces with $p-$exponent.

Theorem 4.3. *For $1 < p < 3$, let $\boldsymbol{f} \in \mathbf{W}_0^{-1,p}(\mathbb{R}^3)$ satisfy (4.1), $g \in L^p(\mathbb{R}^3)$ and $\boldsymbol{a} \in \mathbf{H}_3$. If there exists a positive constant K, only dependent on p, such that*

$$\|\boldsymbol{a}\|_{\mathbf{L}^3(\mathbb{R}^3)} < K, \tag{4.6}$$

then the Oseen problem (1.1) has a unique solution $(\boldsymbol{u}, \pi) \in \mathbf{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ such that

$$\|\boldsymbol{u}\|_{\mathbf{W}_0^{1,p}(\mathbb{R}^3)} + \|\pi\|_{L^p(\mathbb{R}^3)} \leq C_K(\|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)} + \|g\|_{L^p(\mathbb{R}^3)}). \tag{4.7}$$

Proof. By (4.1) and [AGG], we have $\boldsymbol{f} = \operatorname{div} \mathbf{F}$ with $\mathbf{F} \in L^p(\mathbb{R}^3)^{3\times3}$. Then, there exist sequences $\{\mathbf{F}_m\} \subset \mathcal{D}(\mathbb{R}^3)^{3\times3}$ and $\{g_m\} \subset \mathcal{D}(\mathbb{R}^3)$ such that

$$\boldsymbol{f}_m = \operatorname{div}\mathbf{F}_m \to \boldsymbol{f} \text{ in } \mathbf{W}_0^{-1,p}(\mathbb{R}^3) \quad \text{and} \quad g_m \to g \text{ in } L^p(\mathbb{R}^3),$$

with $\boldsymbol{f}_m$ satisfying the condition (4.1). From Lemma 4.2, we can take a sequence $\{\boldsymbol{a}_m\} \subset \boldsymbol{\mathcal{V}}$ convergent to $\boldsymbol{a}$ in $\mathbf{L}^3(\mathbb{R}^3)$. Thanks to Lemma 4.1, there exists a unique solution

$$\boldsymbol{u}_m \in \mathbf{W}_0^{1,2}(\mathbb{R}^3) \cap \mathbf{W}_0^{1,p}(\mathbb{R}^3), \quad \pi_m \in L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$$

satisfying

$$-\Delta\boldsymbol{u}_m + \nabla\pi_m = \boldsymbol{f}_m - \operatorname{div}(\boldsymbol{a}_m \otimes \boldsymbol{u}_m), \quad \operatorname{div}\boldsymbol{u}_m = g_m \text{ in } \mathbb{R}^3. \tag{4.8}$$

From the Stokes theory [AA, Theorem 3.3] and applying Hölder and Sobolev inequalities, we obtain

$$\begin{aligned}
&\|\boldsymbol{u}_m\|_{\mathbf{W}_0^{1,p}(\mathbb{R}^3)} + \|\pi_m\|_{L^p(\mathbb{R}^3)} \leq \\
&\leq c(\|\boldsymbol{f}_m\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)} + \|\boldsymbol{a}_m \otimes \boldsymbol{u}_m\|_{\mathbf{L}^p(\mathbb{R}^3)} + \|g_m\|_{L^p(\mathbb{R}^3)}) \\
&\leq c_1(\|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)} + \|\boldsymbol{a}\|_{\mathbf{L}^3(\mathbb{R}^3)}\|\boldsymbol{u}_m\|_{\mathbf{L}^{p*}(\mathbb{R}^3)} + \|g\|_{L^p(\mathbb{R}^3)}) \\
&\leq c_1(\|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)} + c_2\|\boldsymbol{a}\|_{\mathbf{L}^3(\mathbb{R}^3)}\|\boldsymbol{u}_m\|_{\mathbf{W}_0^{1,p}(\mathbb{R}^3)} + \|g\|_{L^p(\mathbb{R}^3)}),
\end{aligned}$$

where $c_1,\ c_2 > 0$ are the Stokes and Sobolev constants, respectively. By the assumption (4.6), it follows

$$\begin{aligned}(1 - c_1 c_2 K)\|\boldsymbol{u}_m\|_{\mathbf{W}_0^{1,p}(\mathbb{R}^3)} &\le c_1(\|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)} + \|g\|_{L^p(\mathbb{R}^3)});\\ \|\pi_m\|_{L^p(\mathbb{R}^3)} &\le c_1(\|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)} + \|g\|_{L^p(\mathbb{R}^3)}).\end{aligned}$$

Taking $0 < K < 1/(c_1c_2)$, we can extract subsequences of $\boldsymbol{u}_m$ and π_m, still denoted by $\boldsymbol{u}_m$ and π_m, such that

$$\boldsymbol{u}_m \rightharpoonup \boldsymbol{u} \text{ in } \mathbf{W}_0^{1,p}(\mathbb{R}^3) \quad \text{and} \quad \pi_m \rightharpoonup \pi \text{ in } L^p(\mathbb{R}^3),$$

where $(\boldsymbol{u},\pi) \in \mathbf{W}_0^{1,p}(\mathbb{R}^3)\times L^p(\mathbb{R}^3)$ verifies (1.1) and the estimate (4.7) holds. So the proof of Theorem 4.3 is finished. □

Remark 4.4. For all $p < 3$ and $\boldsymbol{u} \in \mathbf{W}_0^{1,p}(\mathbb{R}^3)$, as $\boldsymbol{a} \in \mathbf{L}^3(\mathbb{R}^3)$ then we have $\boldsymbol{a}\otimes\boldsymbol{u} \in \mathbf{L}^p(\mathbb{R}^3)$ and $\operatorname{div}(\boldsymbol{a}\otimes\boldsymbol{u}) \in \mathbf{W}_0^{-1,p}(\mathbb{R}^3)$.

Finally the following existence result can be stated via a dual argument.

Theorem 4.5. *For $p \ge 3$, let $\boldsymbol{f} \in \mathbf{W}_0^{-1,p}(\mathbb{R}^3)$, $g \in L^p(\mathbb{R}^3)$ and $\boldsymbol{a} \in \mathbf{H}_3$ satisfy (4.6). Then the Oseen problem (1.1) has a solution $(\boldsymbol{u},\pi) \in \mathbf{W}_0^{1,p}(\mathbb{R}^3)\times L^p(\mathbb{R}^3)$, where $\boldsymbol{u}$ is unique up to a constant vector, such that*

$$\|\boldsymbol{u}\|_{\mathbf{W}_0^{1,p}(\mathbb{R}^3)/\mathbb{P}_{[1-3/p]}} + \|\pi\|_{L^p(\mathbb{R}^3)} \le C_K(\|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)} + \|g\|_{L^p(\mathbb{R}^3)}). \tag{4.9}$$

Proof. We will follow the duality argument already used in [AG]. On one hand, Green formula yields, for all $\boldsymbol{v} \in \mathbf{W}_0^{1,p'}(\mathbb{R}^3)$ and $(\boldsymbol{u},\pi) \in \mathbf{W}_0^{1,p}(\mathbb{R}^3)\times L^p(\mathbb{R}^3)$

$$\begin{aligned}&\langle -\Delta\boldsymbol{u} + (\boldsymbol{a}\cdot\nabla)\boldsymbol{u} + \nabla\pi, \boldsymbol{v}\rangle_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)\times\mathbf{W}_0^{1,p'}(\mathbb{R}^3)} =\\ &= \langle \boldsymbol{u}, -\Delta\boldsymbol{v} - \operatorname{div}(\boldsymbol{a}\otimes\boldsymbol{v})\rangle_{\mathbf{W}_0^{1,p}(\mathbb{R}^3)\times\mathbf{W}_0^{-1,p'}(\mathbb{R}^3)} - \langle\pi, \operatorname{div}\boldsymbol{v}\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)},\end{aligned}$$

taking into account that $\boldsymbol{a} \in \mathbf{H}_3$ and $\boldsymbol{v} \in \mathbf{W}_0^{1,p'}(\mathbb{R}^3) \hookrightarrow \mathbf{L}^{3p/(2p-3)}(\mathbb{R}^3)$ imply that $\boldsymbol{a}\otimes\boldsymbol{v} \in \mathbf{L}^{p'}(\mathbb{R}^3)$ and consequently $\operatorname{div}(\boldsymbol{a}\otimes\boldsymbol{v}) \in \mathbf{W}_0^{-1,p'}(\mathbb{R}^3)$.

On the other hand, for all $\eta \in L^{p'}(\Omega)$,

$$\langle \boldsymbol{u}, \nabla\eta\rangle_{\mathbf{W}_0^{1,p}(\mathbb{R}^3)\times\mathbf{W}_0^{-1,p'}(\mathbb{R}^3)} = -\langle\operatorname{div}\boldsymbol{u}, \eta\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)}.$$

Then the Oseen problem (1.1) has the following equivalent variational formulation:
Find $(\boldsymbol{u},\pi) \in \mathbf{W}_0^{1,p}(\mathbb{R}^3)\times L^p(\mathbb{R}^3)$ such that for all $\boldsymbol{v} \in \mathbf{W}_0^{1,p'}(\mathbb{R}^3), \eta \in L^{p'}(\mathbb{R}^3)$,

$$\begin{aligned}&\langle \boldsymbol{u}, -\Delta\boldsymbol{v} - \operatorname{div}(\boldsymbol{a}\otimes\boldsymbol{v}) + \nabla\eta\rangle_{\mathbf{W}_0^{1,p}(\mathbb{R}^3)\times\mathbf{W}_0^{-1,p'}(\mathbb{R}^3)} - \langle\pi, \operatorname{div}\boldsymbol{v}\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)} =\\ &= \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)\times\mathbf{W}_0^{1,p'}(\mathbb{R}^3)} - \langle g, \eta\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)}.\end{aligned} \tag{4.10}$$

According to Theorem 4.3 applied with $p' \le 3/2$, for each $(\boldsymbol{f}', g') \in \mathbf{W}_0^{-1,p'}(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)$ satisfying

$$\langle f_i', 1\rangle_{W_0^{-1,p'}(\mathbb{R}^3)\times W_0^{1,p}(\mathbb{R}^3)} = 0,$$

there exists a unique solution $(\boldsymbol{v},\eta) \in \mathbf{W}_0^{1,p'}(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)$ such that

$$-\Delta \boldsymbol{v} - (\boldsymbol{a}\cdot\nabla)\boldsymbol{v} + \nabla\eta = \boldsymbol{f'}, \quad \operatorname{div}\boldsymbol{v} = g' \quad \text{in } \mathbb{R}^3,$$

with the estimate

$$\|\boldsymbol{v}\|_{\mathbf{W}_0^{1,p'}(\mathbb{R}^3)} + \|\eta\|_{L^{p'}(\mathbb{R}^3)} \le C_K(\|\boldsymbol{f'}\|_{\mathbf{W}_0^{-1,p'}(\mathbb{R}^3)} + \|g'\|_{L^{p'}(\mathbb{R}^3)}).$$

Observe that the mapping

$$T : (\boldsymbol{f'}, g') \mapsto \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)\times\mathbf{W}_0^{1,p'}(\mathbb{R}^3)} - \langle g,\eta\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)},$$

is linear and continuous with

$$\begin{aligned}|T(\boldsymbol{f'},g')| &\le \|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)}\|\boldsymbol{v}\|_{\mathbf{W}_0^{1,p'}(\mathbb{R}^3)} + \|g\|_{L^p(\mathbb{R}^3)}\|\eta\|_{L^{p'}(\mathbb{R}^3)}\\ &\le C_K\Big(\|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)} + \|g\|_{L^p(\mathbb{R}^3)}\Big)\Big(\|\boldsymbol{f'}\|_{\mathbf{W}_0^{-1,p'}(\mathbb{R}^3)} + \|g'\|_{L^{p'}(\mathbb{R}^3)}\Big).\end{aligned}$$

Note that $\boldsymbol{f'}$ belongs to $\mathbf{W}_0^{-1,p'}(\mathbb{R}^3)$ and $\boldsymbol{f'} \perp \mathbb{R}^3$. Thus the Riesz representation Theorem guarantees the existence of a unique $(\boldsymbol{u},\pi) \in (\mathbf{W}_0^{1,p}(\mathbb{R}^3)/\mathbb{P}_{[1-3/p]})\times L^p(\mathbb{R}^3)$ such that

$$T(\boldsymbol{f'},g') = \langle \boldsymbol{u},\boldsymbol{f'}\rangle_{\mathbf{W}_0^{1,p}(\mathbb{R}^3)\times\mathbf{W}_0^{-1,p'}(\mathbb{R}^3)} - \langle \pi, g'\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)},$$

with

$$\|\boldsymbol{u}\|_{\mathbf{W}_0^{1,p}(\mathbb{R}^3)} + \|\pi\|_{L^p(\mathbb{R}^3)} \le C_K(\|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)} + \|g\|_{L^p(\mathbb{R}^3)}).$$

By definition of T, it follows

$$\begin{aligned}&\langle \boldsymbol{f},\boldsymbol{v}\rangle_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)\times\mathbf{W}_0^{1,p'}(\mathbb{R}^3)} - \langle g,\eta\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)}\\ &= \langle \boldsymbol{u},\boldsymbol{f'}\rangle_{\mathbf{W}_0^{1,p}(\mathbb{R}^3)\times\mathbf{W}_0^{-1,p'}(\mathbb{R}^3)} - \langle \pi, g'\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)}\end{aligned}$$

which is the variational formulation (4.10). □

Remark 4.6. Supposing that the data $\boldsymbol{f} \in \mathbf{W}_0^{-1,p}(\mathbb{R}^3)\cap\mathbf{W}_0^{-1,q}(\mathbb{R}^3)$ satisfies

$$\langle f_i, 1\rangle_{W_0^{-1,p}(\mathbb{R}^3)\times W_0^{1,p'}(\mathbb{R}^3)} = \langle f_i, 1\rangle_{W_0^{-1,q}(\mathbb{R}^3)\times W_0^{1,q'}(\mathbb{R}^3)} = 0 \quad \text{if } p,q \le 3/2,$$

$g \in L^p(\mathbb{R}^3)\cap L^q(\mathbb{R}^3)$ and $\boldsymbol{a} \in \mathbf{H}_3$ satisfies (4.6), from Theorems 4.3 and 4.5 there exists a solution $(\boldsymbol{u},\pi) \in (\mathbf{W}_0^{1,p}(\mathbb{R}^3)\cap\mathbf{W}_0^{1,q}(\mathbb{R}^3))\times(L^p(\mathbb{R}^3)\cap L^q(\mathbb{R}^3))$ to the Oseen problem (1.1), for any $1 < p,q < \infty$.

5 Strong solutions in $\mathbf{W}_0^{2,p}(\mathbb{R}^3)$ and in $\mathbf{W}_1^{2,p}(\mathbb{R}^3)$

We begin by proving the existence of a unique strong solution in $\mathbf{W}_0^{2,p}(\mathbb{R}^3)\times W_0^{1,p}(\mathbb{R}^3)$ for $1 < p < 3$ in the following sense.

Theorem 5.1. *For* $1 < p < 3$, *let* $\boldsymbol{f} \in \mathbf{L}^p(\mathbb{R}^3)$, $g \in W_0^{1,p}(\mathbb{R}^3)$ *and* $\boldsymbol{a} \in \mathbf{H}_3$ *satisfy (4.6). Then the Oseen problem (1.1) has a unique solution* $(\boldsymbol{u},\pi) \in (\mathbf{W}_0^{2,p}(\mathbb{R}^3)/\mathbb{P}_{[2-3/p]}) \times W_0^{1,p}(\mathbb{R}^3)$ *such that*

$$\|\boldsymbol{u}\|_{\mathbf{W}_0^{2,p}(\mathbb{R}^3)/\mathbb{P}_{[2-3/p]}} + \|\pi\|_{W_0^{1,p}(\mathbb{R}^3)} \le C(1+C_K)(\|\boldsymbol{f}\|_{\mathbf{L}^p(\mathbb{R}^3)} + \|g\|_{W_0^{1,p}(\mathbb{R}^3)}). \tag{5.1}$$

Proof. For all $1 < p < 3$, Sobolev embedding holds

$$\mathbf{W}_0^{1,3p'/(3+p')}(\mathbb{R}^3) \hookrightarrow \mathbf{L}^{p'}(\mathbb{R}^3)$$

and by duality we obtain

$$\mathbf{L}^p(\mathbb{R}^3) \hookrightarrow \mathbf{W}_0^{-1,3p/(3-p)}(\mathbb{R}^3). \tag{5.2}$$

Since $g \in W_0^{1,p}(\mathbb{R}^3) \hookrightarrow L^{3p/(3-p)}(\mathbb{R}^3)$ and $\boldsymbol{f} \in \mathbf{W}_0^{-1,3p/(3-p)}(\mathbb{R}^3)$, Theorems 4.3 and 4.5 guarantee the existence of a solution

$$(\boldsymbol{u},\pi) \in \mathbf{W}_0^{1,3p/(3-p)}(\mathbb{R}^3) \times L^{3p/(3-p)}(\mathbb{R}^3)$$

to the Oseen problem (1.1) with the estimate

$$\|\boldsymbol{u}\|_{\mathbf{W}_0^{1,3p/(3-p)}(\mathbb{R}^3)/\mathbb{P}_{[2-3/p]}} + \|\pi\|_{L^{3p/(3-p)}(\mathbb{R}^3)} \le C_K(\|\boldsymbol{f}\|_{\mathbf{L}^p(\mathbb{R}^3)} + \|g\|_{W_0^{1,p}(\mathbb{R}^3)}).$$

Note that the compatibility condition (4.1) is not required because we have $3p/(3-p) > 3/2$. Besides, we have $(\boldsymbol{a}\cdot\nabla)\boldsymbol{u} \in \mathbf{L}^p(\mathbb{R}^3)$. We can apply the Stokes regularity theory (cf. [AA, Theorem 3.8]) to deduce the existence of $(\boldsymbol{v},\eta) \in \mathbf{W}_0^{2,p}(\mathbb{R}^3) \times W_0^{1,p}(\mathbb{R}^3)$ verifying (4.3), unique up to an element of $\mathbb{P}_{[2-3/p]} \times \{0\}$.

Moreover, the estimate holds

$$\inf_{\boldsymbol{\lambda}\in\mathbb{P}_{[2-3/p]}} \|\boldsymbol{v}+\boldsymbol{\lambda}\|_{\mathbf{W}_0^{2,p}(\mathbb{R}^3)} + \|\eta\|_{W_0^{1,p}(\mathbb{R}^3)} \le$$
$$\le C(\|\boldsymbol{f}\|_{\mathbf{L}^p(\mathbb{R}^3)} + \|\boldsymbol{a}\|_{\mathbf{L}^3(\mathbb{R}^3)}\|\nabla\boldsymbol{u}\|_{\mathbf{L}^{3p/(3-p)}(\mathbb{R}^3)} + \|g\|_{W_0^{1,p}(\mathbb{R}^3)}),$$

with C denoting a constant only dependent on p.

Let $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}$ and $\theta = \eta - \pi$, then

$$-\Delta\boldsymbol{w} + \nabla\theta = 0 \quad \text{and} \quad \operatorname{div}\boldsymbol{w} = 0 \quad \text{in } \mathbb{R}^3,$$

with $\theta \in L^{3p/(3-p)}(\mathbb{R}^3)$ and $\boldsymbol{w} \in \mathbf{W}_0^{1,3p/(3-p)}(\mathbb{R}^3)$. As $\Delta\theta = 0$ in $\mathbb{R}^3$, then $\theta = 0$ and $\boldsymbol{w} \in \mathbb{P}_{[2-3/p]} \subset \mathbf{W}_0^{2,p}(\mathbb{R}^3)$. Consequently, $\boldsymbol{u} \in \mathbf{W}_0^{2,p}(\mathbb{R}^3)$ and $\pi \in W_0^{1,p}(\mathbb{R}^3)$ and we obtain the estimate (5.1). □

Remark 5.2. Observe that Theorem 5.1 does not include the case $p \ge 3$, under its assumptions, that is, $\boldsymbol{f} \in \mathbf{L}^p(\mathbb{R}^3), g \in W_0^{1,p}(\mathbb{R}^3)$ and $\boldsymbol{a} \in \mathbf{H}_3$. Indeed if it would be possible to find $\boldsymbol{u} \in \mathbf{W}_0^{2,p}(\mathbb{R}^3)$ and $\pi \in W_0^{1,p}(\mathbb{R}^3)$ such that

$$(\boldsymbol{a}\cdot\nabla)\boldsymbol{u} = \Delta\boldsymbol{u} - \nabla\pi + \boldsymbol{f} \in \mathbf{L}^p(\mathbb{R}^3),$$

it would happen a contradiction, since $\boldsymbol{a} \in \mathbf{L}^3(\mathbb{R}^3)$ and $\nabla\boldsymbol{u} \notin \mathbf{L}^{3p/(3-p)}(\mathbb{R}^3)$.

In order to present a strong solution for $p \ge 3$, we state the following results.

Theorem 5.3. *For $p \geq 3$, let $\boldsymbol{f} \in \mathbf{L}^p(\mathbb{R}^3)$, $g \in W_0^{1,p}(\mathbb{R}^3)$ and $\boldsymbol{a} \in \mathbf{H}_3$ satisfy (4.6). If we additionally assume $\boldsymbol{f} \in \mathbf{L}^q(\mathbb{R}^3)$, $g \in W_0^{1,q}(\mathbb{R}^3)$ and $\boldsymbol{a} \in \mathbf{L}^{3pq/(q(3+p)-3p)}(\mathbb{R}^3)$ for some $3p/(3+p) \leq q < 3$, then the solution $(\boldsymbol{u},\pi) \in \mathbf{W}_0^{2,q}(\mathbb{R}^3) \times W_0^{1,q}(\mathbb{R}^3)$ given by Theorem 5.1 belongs also to $\mathbf{W}_0^{2,p}(\mathbb{R}^3) \times W_0^{1,p}(\mathbb{R}^3)$ and it satisfies*

$$\|\boldsymbol{u}\|_{\mathbf{W}_0^{2,p}(\mathbb{R}^3)/\mathbb{P}_{[2-3/p]}} + \|\pi\|_{W_0^{1,p}(\mathbb{R}^3)} \leq C(1+C_K)(\|\boldsymbol{f}\|_{\mathbf{L}^p(\mathbb{R}^3)} + \|g\|_{W_0^{1,p}(\mathbb{R}^3)}). \tag{5.3}$$

Proof. Since $\boldsymbol{f} \in \mathbf{L}^q(\mathbb{R}^3)$ and $g \in W_0^{1,q}(\mathbb{R}^3)$, for $3/2 \leq q < 3$, we can apply Theorem 5.1. Then there exists a unique solution $(\boldsymbol{u},\pi) \in (\mathbf{W}_0^{2,q}(\mathbb{R}^3)/\mathbb{R}^3) \times W_0^{1,q}(\mathbb{R}^3)$ satisfying the Oseen problem (1.1). Thus it results $(\boldsymbol{a}\cdot\nabla)\boldsymbol{u} \in \mathbf{L}^p(\mathbb{R}^3)$ since $\nabla\boldsymbol{u} \in \mathbf{L}^{3q/(3-q)}(\mathbb{R}^3)$ and $\boldsymbol{a} \in \mathbf{L}^{3pq/(q(3+p)-3p)}(\mathbb{R}^3)$ for some $3p/(3+p) \leq q < 3$. Next proceeding as in the proof of Theorem 5.1 we can conclude that $(\boldsymbol{u},\pi) \in (\mathbf{W}_0^{2,p}(\mathbb{R}^3)/\mathbb{P}_{[2-3/p]}) \times W_0^{1,p}(\mathbb{R}^3)$ verifies (5.3). □

Note that in Theorem 5.3, $\boldsymbol{a} \in \mathbf{L}^\infty(\mathbb{R}^3)$ if $q = 3p/(3+p)$, and if q is close to 3 then $3pq/(q(3+p)-3p)$ is close to p.

Finally, we take $\boldsymbol{f}$ in weighted $\mathbf{L}^p(\mathbb{R}^3)$, more precisely $\boldsymbol{f} \in \mathbf{W}_1^{0,p}(\mathbb{R}^3)$, and the data g in the corresponding weighted Sobolev space $W_1^{1,p}(\mathbb{R}^3)$.

Theorem 5.4. *Let $p \neq 3/2$, $\boldsymbol{f} \in \mathbf{W}_1^{0,p}(\mathbb{R}^3)$ satisfy (4.1), $g \in W_1^{1,p}(\mathbb{R}^3)$ and $\boldsymbol{a} \in \mathbf{H}_3 \cap \mathbf{L}^\infty(\mathbb{R}^3)$ satisfy (4.6) and*

$$\exists L > 0, \quad |\boldsymbol{a}(\boldsymbol{x})| \leq \frac{L}{|\boldsymbol{x}|}, \quad a.e.\ \boldsymbol{x} \in \mathbb{R}^3. \tag{5.4}$$

Then the Oseen problem (1.1) has a unique solution $(\boldsymbol{u},\pi) \in \mathbf{W}_1^{2,p}(\mathbb{R}^3)/\mathbb{P}_{[1-3/p]} \times W_1^{1,p}(\mathbb{R}^3)$ satisfying

$$\|\boldsymbol{u}\|_{\mathbf{W}_1^{2,p}(\mathbb{R}^3)/\mathbb{P}_{[1-3/p]}} + \|\pi\|_{W_1^{1,p}(\mathbb{R}^3)} \leq C(1+LC_K)(\|\boldsymbol{f}\|_{\mathbf{W}_1^{0,p}(\mathbb{R}^3)} + \|g\|_{W_1^{1,p}(\mathbb{R}^3)}). \tag{5.5}$$

Proof. Since $W_1^{1,p}(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ and for $p \neq 3/2$ we have $\mathbf{W}_1^{0,p}(\mathbb{R}^3) \hookrightarrow \mathbf{W}_0^{-1,p}(\mathbb{R}^3)$, thanks to Theorems 4.3 and 4.5, there exists a solution $(\boldsymbol{u},\pi) \in \mathbf{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ to the Oseen problem (1.1) satisfying (4.9).

Considering the assumption (5.4), we get $(\boldsymbol{a}\cdot\nabla)\boldsymbol{u} \in \mathbf{W}_1^{0,p}(\mathbb{R}^3)$. Moreover, we have, for all $i = 1,2,3$,

$$\langle(\boldsymbol{a}\cdot\nabla)u_i + f_i, 1\rangle_{W_1^{0,p}(\mathbb{R}^3)\times W_{-1}^{0,p'}(\mathbb{R}^3)} = 0, \quad \text{if } p \leq 3/2.$$

Thus the conditions of Stokes regularity result are fulfilled (cf. [AA, Theorem 3.1]) and we can conclude the existence of $(\boldsymbol{v},\eta) \in \mathbf{W}_1^{2,p}(\mathbb{R}^3) \times W_1^{1,p}(\mathbb{R}^3)$, unique up to an element of $\mathbb{P}_{[1-3/p]} \times \{0\}$ and satisfying

$$\inf_{\lambda\in\mathbb{P}_{[1-3/p]}} \|\boldsymbol{v}+\boldsymbol{\lambda}\|_{\mathbf{W}_1^{2,p}(\mathbb{R}^3)} + \|\eta\|_{W_1^{1,p}(\mathbb{R}^3)} \leq$$
$$\leq C(\|\boldsymbol{f}\|_{\mathbf{W}_1^{0,p}(\mathbb{R}^3)} + L\|\nabla\boldsymbol{u}\|_{\mathbf{L}^p(\mathbb{R}^3)} + \|g\|_{W_1^{1,p}(\mathbb{R}^3)}).$$

Then the proof's conclusion of Theorem 5.4 is identical to the one of Theorem 5.1. □

Remark 5.5. Observe that $W_1^{0,p}(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$, $W_1^{1,p}(\mathbb{R}^3) \hookrightarrow W_0^{1,p}(\mathbb{R}^3)$ and if $p \geq 3$ then $W_1^{0,p}(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ for all $q \in [3p/(3+p), 3[$. The assumptions on $\boldsymbol{a}$ stated in Theorem 5.4 are also stronger of those in Theorems 5.1 and 5.3.

Remark 5.6. i) It is known that $L^3(\mathbb{R}^3) \hookrightarrow L^{3,\infty}(\mathbb{R}^3)$ and $1/|\boldsymbol{x}| \in L^{3,\infty}(\mathbb{R}^3)$, where $L^{3,\infty}(\mathbb{R}^3)$ is the space of measurable functions v defined on $\mathbb{R}^3$ satisfying

$$\exists C > 0, \quad \forall t > 0, \quad t^3 \text{meas}\{\boldsymbol{x} \in \mathbb{R}^3;\ |v(\boldsymbol{x})| > t\} \leq C.$$

ii) When $p = 3/2$, the previous existence result holds provided we suppose that $\boldsymbol{f} \in \mathbf{W}_1^{0,3/2}(\mathbb{R}^3) \cap \mathbf{W}_0^{-1,3/2}(\mathbb{R}^3)$ (see (2.1)).

6 Very weak solutions in $\mathbf{L}^p(\mathbb{R}^3)$

In this section we show how very weak solutions to the problem (1.1) can easily been obtained from the existence of strong solutions as in Section 5 via a dual argument. We begin by precising the meaning of very weak variational formulation.

Lemma 6.1. *For $p > 3/2$, the problem of finding a pair $(\boldsymbol{u}, \pi) \in \mathbf{L}^p(\mathbb{R}^3) \times W_0^{-1,p}(\mathbb{R}^3)$ verifying (1.1), with $\boldsymbol{f} \in \mathbf{W}_0^{-2,p}(\mathbb{R}^3)$ and $g \in W_0^{-1,p}(\mathbb{R}^3)$, has the following variational formulation*

$$\begin{aligned}\int_{\mathbb{R}^3} \boldsymbol{u} \cdot (-\Delta \boldsymbol{v} - div(\boldsymbol{a} \otimes \boldsymbol{v}) + \nabla \eta) d\boldsymbol{x} - \langle \pi, div\, \boldsymbol{v} \rangle_{W_0^{-1,p}(\mathbb{R}^3) \times W_0^{1,p'}(\mathbb{R}^3)} = \\ = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\mathbf{W}_0^{-2,p}(\mathbb{R}^3) \times \mathbf{W}_0^{2,p'}(\mathbb{R}^3)} - \langle g, \eta \rangle_{W_0^{-1,p}(\mathbb{R}^3) \times W_0^{1,p'}(\mathbb{R}^3)}\end{aligned} \tag{6.1}$$

for all $\boldsymbol{v} \in \mathbf{W}_0^{2,p'}(\mathbb{R}^3)$ and $\eta \in W_0^{1,p'}(\mathbb{R}^3)$.

Proof. By a density argument of $\mathcal{D}(\mathbb{R}^3)$ into respectively $W_0^{2,p'}(\mathbb{R}^3)$ and $W_0^{1,p'}(\mathbb{R}^3)$, we have the following equalities

$$\begin{aligned}\langle -\Delta \boldsymbol{u}, \boldsymbol{v} \rangle_{\mathbf{W}_0^{-2,p}(\mathbb{R}^3) \times \mathbf{W}_0^{2,p'}(\mathbb{R}^3)} &= \langle \boldsymbol{u}, -\Delta \boldsymbol{v} \rangle_{\mathbf{L}^p(\mathbb{R}^3) \times \mathbf{L}^{p'}(\mathbb{R}^3)} \\ \langle \nabla \pi, \boldsymbol{v} \rangle_{\mathbf{W}_0^{-2,p}(\mathbb{R}^3) \times \mathbf{W}_0^{2,p'}(\mathbb{R}^3)} &= -\langle \pi, \text{div} \boldsymbol{v} \rangle_{W_0^{-1,p}(\mathbb{R}^3) \times W_0^{1,p'}(\mathbb{R}^3)} \\ \langle \text{div} \boldsymbol{u}, \eta \rangle_{W_0^{-1,p}(\mathbb{R}^3) \times W_0^{1,p'}(\mathbb{R}^3)} &= -\langle \boldsymbol{u}, \nabla \eta \rangle_{\mathbf{L}^p(\mathbb{R}^3) \times \mathbf{L}^{p'}(\mathbb{R}^3)}.\end{aligned}$$

Since $\boldsymbol{a} \in \mathbf{H}_3$ and $p > 3/2$, on one hand $\boldsymbol{a} \otimes \boldsymbol{u} \in \mathbf{L}^{3p/(3+p)}(\mathbb{R}^3) \hookrightarrow \mathbf{W}_0^{-1,p}(\mathbb{R}^3)$ and $(\boldsymbol{a} \cdot \nabla)\boldsymbol{u} = \text{div}(\boldsymbol{a} \otimes \boldsymbol{u}) \in \mathbf{W}_0^{-1,3p/(3+p)}(\mathbb{R}^3) \hookrightarrow \mathbf{W}_0^{-2,p}(\mathbb{R}^3)$. On the other hand, for any $\boldsymbol{v} \in \mathbf{W}_0^{2,p'}(\mathbb{R}^3)$ we have $\nabla \boldsymbol{v} \in \mathbf{W}_0^{1,p'}(\mathbb{R}^3) \hookrightarrow \mathbf{L}^{3p/(2p-3)}(\mathbb{R}^3)$ and $\text{div}(\boldsymbol{a} \otimes \boldsymbol{v}) = (\boldsymbol{a} \cdot \nabla)\boldsymbol{v} \in \mathbf{L}^{p'}(\mathbb{R}^3)$. Then we obtain

$$\begin{aligned}\langle \text{div}(\boldsymbol{a} \otimes \boldsymbol{u}), \boldsymbol{v} \rangle_{\mathbf{W}_0^{-2,p}(\mathbb{R}^3) \times \mathbf{W}_0^{2,p'}(\mathbb{R}^3)} &= -\langle \boldsymbol{a} \otimes \boldsymbol{u}, \nabla \boldsymbol{v} \rangle_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3) \times \mathbf{W}_0^{1,p'}(\mathbb{R}^3)} \\ &= \langle \boldsymbol{u}, -\text{div}(\boldsymbol{a} \otimes \boldsymbol{v}) \rangle_{\mathbf{L}^p(\mathbb{R}^3) \times \mathbf{L}^{p'}(\mathbb{R}^3)}.\end{aligned}$$

Thus Lemma 6.1 holds. □

Definition 6.2. We say that $(\boldsymbol{u}, \pi) \in \mathbf{L}^p(\mathbb{R}^3) \times W_0^{-1,p}(\mathbb{R}^3)$ is a very weak solution to the problem (1.1) if it satisfies (6.1).

Theorem 6.3. *For $p > 3/2$, let $\boldsymbol{f} \in \mathbf{W}_0^{-2,p}(\mathbb{R}^3)$ satisfy the compatibility condition: for any $i = 1, 2, 3$*

$$\langle f_i, 1 \rangle_{W_0^{-2,p}(\mathbb{R}^3) \times W_0^{2,p'}(\mathbb{R}^3)} = 0 \quad \textit{if } p \leq 3, \tag{6.2}$$

$g \in W_0^{-1,p}(\mathbb{R}^3)$ and $\boldsymbol{a} \in \mathbf{H}_3$ satisfy (4.6). Then the Oseen problem (1.1) has a unique very weak solution $(\boldsymbol{u},\pi) \in \mathbf{L}^p(\mathbb{R}^3) \times W_0^{-1,p}(\mathbb{R}^3)$ such that

$$\|\boldsymbol{u}\|_{\mathbf{L}^p(\mathbb{R}^3)} + \|\pi\|_{W_0^{-1,p}(\mathbb{R}^3)} \le C(1+C_K)(\|\boldsymbol{f}\|_{\mathbf{W}_0^{-2,p}(\mathbb{R}^3)} + \|g\|_{W_0^{-1,p}(\mathbb{R}^3)}). \tag{6.3}$$

Proof. In accordance to Theorem 5.1 for $p' < 3, \boldsymbol{f}' \in \mathbf{L}^{p'}(\mathbb{R}^3)$ and $g' \in W_0^{1,p'}(\mathbb{R}^3)$, there exists a unique solution $(\boldsymbol{v},\eta) \in (\mathbf{W}_0^{2,p'}(\mathbb{R}^3)/\mathbb{P}_{[2-3/p']}) \times W_0^{1,p'}(\mathbb{R}^3)$ to the problem

$$-\Delta \boldsymbol{v} - (\boldsymbol{a}\cdot\nabla)\boldsymbol{v} + \nabla\eta = \boldsymbol{f}', \quad \operatorname{div}\boldsymbol{v} = g' \quad \text{in } \mathbb{R}^3,$$

satisfying the estimate

$$\|\boldsymbol{v}\|_{\mathbf{W}_0^{2,p'}(\mathbb{R}^3)/\mathbb{P}_{[2-3/p']}} + \|\eta\|_{W_0^{1,p'}(\mathbb{R}^3)} \le C(1+C_K)(\|\boldsymbol{f}'\|_{\mathbf{L}^{p'}(\mathbb{R}^3)} + \|g'\|_{W_0^{1,p'}(\mathbb{R}^3)}).$$

Define the mapping

$$T : (\boldsymbol{f}', g') \mapsto \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{\mathbf{W}_0^{-2,p}(\mathbb{R}^3)\times\mathbf{W}_0^{2,p'}(\mathbb{R}^3)} - \langle g, \eta\rangle_{W_0^{-1,p}(\mathbb{R}^3)\times W_0^{1,p'}(\mathbb{R}^3)}.$$

Then T is linear and using (6.2) T is continuous with

$$\begin{aligned}|T(\boldsymbol{f}', g')| &\le \|\boldsymbol{f}\|_{\mathbf{W}_0^{-2,p}(\mathbb{R}^3)}\|\boldsymbol{v}\|_{\mathbf{W}_0^{2,p'}(\mathbb{R}^3)/\mathbb{P}_{[2-3/p']}} + \|g\|_{W_0^{-1,p}(\mathbb{R}^3)}\|\eta\|_{W_0^{1,p'}(\mathbb{R}^3)} \\ &\le C(1+C_K)\Big(\|\boldsymbol{f}\|_{\mathbf{W}_0^{-2,p}(\mathbb{R}^3)} + \|g\|_{W_0^{-1,p}(\mathbb{R}^3)}\Big)\Big(\|\boldsymbol{f}'\|_{\mathbf{L}^{p'}(\mathbb{R}^3)} + \|g'\|_{W_0^{1,p'}(\mathbb{R}^3)}\Big).\end{aligned}$$

Thanks to the Riesz representation Theorem, there exists a unique $(\boldsymbol{u},\pi) \in \mathbf{L}^p(\mathbb{R}^3) \times W_0^{-1,p}(\mathbb{R}^3)$ such that

$$T(\boldsymbol{f}', g') = \langle \boldsymbol{u}, \boldsymbol{f}'\rangle_{\mathbf{L}^p(\mathbb{R}^3)\times\mathbf{L}^{p'}(\mathbb{R}^3)} - \langle \pi, g'\rangle_{W_0^{-1,p}(\mathbb{R}^3)\times W_0^{1,p'}(\mathbb{R}^3)},$$

and the estimate (6.3) is satisfied. By definition of T, it follows the very weak variational formulation (6.1). □

Remark 6.4. Using Theorem 5.4 with $p' \neq 3/2$ and similar dual argument we can conclude that for $p > 3/2$ such that $p \neq 3, \boldsymbol{f} \in \mathbf{W}_{-1}^{-2,p}(\mathbb{R}^3)$, $g \in W_{-1}^{-1,p}(\mathbb{R}^3)$ and $\boldsymbol{a} \in \mathbf{H}_3$ satisfying (4.6) and (5.4), there exists a very weak solution $(\boldsymbol{u},\pi) \in \mathbf{W}_{-1}^{0,p}(\mathbb{R}^3) \times W_{-1}^{-1,p}(\mathbb{R}^3)$ to the problem (1.1) in the following sense

$$\begin{aligned}\langle \boldsymbol{u}, -\Delta\boldsymbol{v} - \operatorname{div}(\boldsymbol{a}\otimes\boldsymbol{v}) + \nabla\eta\rangle_{\mathbf{W}_{-1}^{0,p}(\mathbb{R}^3)\times\mathbf{W}_1^{0,p'}(\mathbb{R}^3)} - \langle \pi, \operatorname{div}\boldsymbol{v}\rangle_{W_{-1}^{-1,p}(\mathbb{R}^3)\times W_1^{1,p'}(\mathbb{R}^3)} = \\ = \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{\mathbf{W}_{-1}^{-2,p}(\mathbb{R}^3)\times\mathbf{W}_1^{2,p'}(\mathbb{R}^3)} - \langle g, \eta\rangle_{W_{-1}^{-1,p}(\mathbb{R}^3)\times W_1^{1,p'}(\mathbb{R}^3)}\end{aligned}$$

for all $\boldsymbol{v} \in \mathbf{W}_1^{2,p'}(\mathbb{R}^3)$ and $\eta \in W_1^{1,p'}(\mathbb{R}^3)$. Indeed the above weak formulation has meanful for $p > 3/2$, since $\boldsymbol{a} \in \mathbf{H}_3$ and $\nabla\boldsymbol{v} \in \mathbf{W}_1^{1,p'}(\mathbb{R}^3) \hookrightarrow \mathbf{W}_1^{0,3p/(2p-3)}(\mathbb{R}^3)$ we have then $\operatorname{div}(\boldsymbol{a}\otimes\boldsymbol{v}) = (\boldsymbol{a}\cdot\nabla)\boldsymbol{v} \in \mathbf{W}_1^{0,p'}(\mathbb{R}^3)$.

7 The generalized Oseen problem (1.3)

In this section, we study the generalized Oseen problem (1.3):

$$-\Delta \boldsymbol{u} + k\frac{\partial \boldsymbol{u}}{\partial x_1} + (\boldsymbol{a}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \boldsymbol{f}, \quad \operatorname{div}\boldsymbol{u} = g \quad \text{in } \mathbb{R}^3.$$

As in sections 3, 4 and 5 where we applied the Stokes theory, here we can apply the theory concerning the Oseen system (1.2).

We introduce the following Banach space

$$Z_p(\mathbb{R}^3) = \{v \in L^p(\mathbb{R}^3);\ \frac{\partial v}{\partial x_1} \in W_0^{-2,p}(\mathbb{R}^3)\}.$$

Next we prove some useful results.

Lemma 7.1. *For every* $1 < p < \infty$ *and* $g \in Z_p(\mathbb{R}^3)$ *verifying*

$$\forall\lambda \in \mathbb{P}_{[2-3/p']}, \quad \langle \frac{\partial g}{\partial x_1}, \lambda\rangle_{W_0^{-2,p}(\mathbb{R}^3)\times W_0^{2,p'}(\mathbb{R}^3)} = 0, \tag{7.1}$$

there exists $\boldsymbol{w} \in \mathbf{X}_0^{1,p}(\mathbb{R}^3)$ *such that* $\operatorname{div}\boldsymbol{w} = g$ *in* $\mathbb{R}^3$ *and satisfies the estimate*

$$\|\boldsymbol{w}\|_{\mathbf{X}_0^{1,p}(\mathbb{R}^3)} \leq C\|g\|_{Z_p(\mathbb{R}^3)}. \tag{7.2}$$

Moreover $\boldsymbol{w} \in \mathbf{L}^{4p/(4-p)}(\mathbb{R}^3)\cap\mathbf{L}^{3p/(3-p)}(\mathbb{R}^3)$ *if* $1 < p < 3$, $\boldsymbol{w} \in \mathbf{L}^{12}(\mathbb{R}^3)\cap\mathbf{BMO}$ *if* $p = 3$, *and* $\boldsymbol{w} \in \mathbf{L}^{4p/(4-p)}(\mathbb{R}^3)\cap\mathbf{L}^{\infty}(\mathbb{R}^3)$ *if* $3 < p < 4$ *with estimates corresponding to (2.10), (2.11) and (2.12).*

Proof. Let $g \in Z_p(\mathbb{R}^3)$. From $g \in L^p(\mathbb{R}^3)$ there exists $v \in W_0^{2,p}(\mathbb{R}^3)$ verifying

$$\Delta v = g \text{ in } \mathbb{R}^3,$$

where v is unique up to a polynomial function of $\mathbb{P}_{[2-3/p]}$ (see [AGG]). We can choose v such that there exists a constant $C_1 > 0$ satisfying

$$\|\nabla v\|_{\mathbf{W}_0^{1,p}(\mathbb{R}^3)} \leq C_1\|g\|_{L^p(\mathbb{R}^3)}. \tag{7.3}$$

From $\frac{\partial g}{\partial x_1} \in W_0^{-2,p}(\mathbb{R}^3) \perp \mathbb{P}_{[2-3/p']}$ there exists a unique $z \in L^p(\mathbb{R}^3)$ verifying

$$\Delta z = \frac{\partial g}{\partial x_1} \text{ in } \mathbb{R}^3$$

and it is such that the following estimate holds

$$\|\nabla z\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)} \leq C_2\|z\|_{L^p(\mathbb{R}^3)} \leq C_3\|\frac{\partial g}{\partial x_1}\|_{W_0^{-2,p}(\mathbb{R}^3)}. \tag{7.4}$$

Then $\frac{\partial v}{\partial x_1} - z \in W_0^{1,p}(\mathbb{R}^3) + L^p(\mathbb{R}^3)$ is harmonic and $\frac{\partial v}{\partial x_1} - z \in \mathbb{P}_{[1-3/p]}$, that means, $\frac{\partial v}{\partial x_1} = z$ if $p < 3$ and $\frac{\partial v}{\partial x_1} - z = \text{constant}$ if $p \geq 3$.

Next let us take $\boldsymbol{w} = \nabla v$. Hence $\boldsymbol{w} \in \mathbf{W}_0^{1,p}(\mathbb{R}^3)$, $\operatorname{div} \boldsymbol{w} = g$ and $\frac{\partial \boldsymbol{w}}{\partial x_1} = \nabla(\frac{\partial v}{\partial x_1}) = \nabla z \in \mathbf{W}_0^{-1,p}(\mathbb{R}^3)$.

Using (7.3) and (7.4) it follows

$$\|\boldsymbol{w}\|_{\mathbf{X}_0^{1,p}(\mathbb{R}^3)} = \|\boldsymbol{w}\|_{\mathbf{W}_0^{1,p}(\mathbb{R}^3)} + \|\frac{\partial \boldsymbol{w}}{\partial x_1}\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)} \leq \max\{C_1, C_3\}\|g\|_{Z_p(\mathbb{R}^3)}.$$

Consequently we get (7.2). Applying Proposition 2.5 we conclude the proof of Lemma 7.1. □

Remark 7.2. The condition (7.1) is equivalent to

$$\langle \frac{\partial g}{\partial x_1}, x_1\rangle_{W_0^{-2,p}(\mathbb{R}^3)\times W_0^{2,p'}(\mathbb{R}^3)} = 0, \quad \text{if } p' \geq 3.$$

Indeed for all $\lambda \in \mathbb{P}_{[2-3/p']}$ such that $\frac{\partial \lambda}{\partial x_1} = 0$, we have

$$\langle \frac{\partial g}{\partial x_1}, \lambda\rangle_{W_0^{-2,p}(\mathbb{R}^3)\times W_0^{2,p'}(\mathbb{R}^3)} = -\langle g, \frac{\partial \lambda}{\partial x_1}\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)} = 0,$$

taking into account the density of $\mathcal{D}(\mathbb{R}^3)$ in $X_0^{2,p'}(\mathbb{R}^3)$.

Lemma 7.3. *Let $g \in Z_p(\mathbb{R}^3)$ verifying the compatibility condition (7.1). Then, there exists a sequence $g_m \in \mathcal{D}(\mathbb{R}^3)$ satisfying (7.1) and such that $g_m \to g$ in $Z_p(\mathbb{R}^3)$.*

Proof. Let $g \in Z_p(\mathbb{R}^3)$. Applying Lemma 7.1 there exists $\boldsymbol{w} \in \mathbf{X}_0^{1,p}(\mathbb{R}^3)$ verifying $\operatorname{div} \boldsymbol{w} = g$ and (7.2). Since $\mathcal{D}(\mathbb{R}^3)$ is dense into $\mathbf{X}_0^{1,p}(\mathbb{R}^3)$, there exists a sequence $\{\boldsymbol{w}_m\}_{m\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^3)$ such that $\boldsymbol{w}_m \to \boldsymbol{w}$ in $\mathbf{X}_0^{1,p}(\mathbb{R}^3)$. Taking $g_m = \operatorname{div} \boldsymbol{w}_m \in \mathcal{D}(\mathbb{R}^3)$ it follows $g_m \to g$ in $Z_p(\mathbb{R}^3)$. Observe that g_m satisfies (7.1). □

The first existence result to the problem (1.3) concerns the Hilbertian case $p = 2$.

Proposition 7.4. *Assume that $\boldsymbol{f} \in \mathbf{W}_0^{-1,2}(\mathbb{R}^3)$, $g \in Z_2(\mathbb{R}^3)$ and $\boldsymbol{a} \in \mathbf{H}_3$. Then the problem (1.3) has a unique solution $(\boldsymbol{u}, \pi) \in (\mathbf{X}_0^{1,2}(\mathbb{R}^3) \cap \mathbf{L}^4(\mathbb{R}^3)) \times L^2(\mathbb{R}^3)$. Moreover, the following estimate holds*

$$\|\boldsymbol{u}\|_{\mathbf{X}_0^{1,2}(\mathbb{R}^3)} + \|\boldsymbol{u}\|_{\mathbf{L}^4(\mathbb{R}^3)} + \|\pi\|_{L^2(\mathbb{R}^3)} \leq C\Big(\|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,2}(\mathbb{R}^3)} + (1 + \|\boldsymbol{a}\|_{\mathbf{L}^3(\mathbb{R}^3)})\|g\|_{Z_2(\mathbb{R}^3)}\Big). \tag{7.5}$$

Proof. Let $g \in Z_2(\mathbb{R}^3)$. Applying Lemma 7.1 and Remark 7.2, there exists $\boldsymbol{w} \in \mathbf{X}_0^{1,2}(\mathbb{R}^3)$ satisfying $\operatorname{div} \boldsymbol{w} = g$ and (7.2) with $p = 2$. Next let us take the unique solution $(\boldsymbol{v}, \pi) \in (\mathbf{X}_0^{1,2}(\mathbb{R}^3) \cap \mathbf{L}^4(\mathbb{R}^3)) \times L^2(\mathbb{R}^3)$ to the Oseen problem (cf. [AN, Lemma 4.1])

$$-\Delta \boldsymbol{v} + k\frac{\partial \boldsymbol{v}}{\partial x_1} + (\boldsymbol{a}\cdot\nabla)\boldsymbol{v} + \nabla\pi = \boldsymbol{f} + \Delta\boldsymbol{w} - k\frac{\partial \boldsymbol{w}}{\partial x_1} - (\boldsymbol{a}\cdot\nabla)\boldsymbol{w}, \quad \operatorname{div} \boldsymbol{v} = 0 \text{ in } \mathbb{R}^3,$$

satisfying the estimate

$$\|\boldsymbol{v}\|_{\mathbf{X}_0^{1,2}(\mathbb{R}^3)} + \|\pi\|_{L^2(\mathbb{R}^3)} \leq C(\|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,2}(\mathbb{R}^3)} + \|\boldsymbol{w}\|_{\mathbf{X}_0^{1,2}(\mathbb{R}^3)} + \|\boldsymbol{a}\|_{\mathbf{L}^3(\mathbb{R}^3)}\|\boldsymbol{w}\|_{\mathbf{W}_0^{1,2}(\mathbb{R}^3)}).$$

Letting $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$, we obtain $\boldsymbol{u} \in \mathbf{X}_0^{1,2}(\mathbb{R}^3)$ that satisfies (1.3) and (7.5). □

Proposition 7.5. *Assume that* $1 < p < \infty$, $\boldsymbol{f} \in \mathbf{W}_0^{-1,2}(\mathbb{R}^3) \cap \mathbf{W}_0^{-1,p}(\mathbb{R}^3)$, $g \in Z_2(\mathbb{R}^3) \cap Z_p(\mathbb{R}^3)$ *satisfy the compatibility conditions (4.1) and (7.1), respectively, and* $\boldsymbol{a} \in \mathbf{H}_3$. *If* $\boldsymbol{a} \in \mathbf{L}^{4p/(4-p)}(\mathbb{R}^3)$ *for* $1 < p < 4$ *and* $\boldsymbol{a} \in \mathbf{L}^p(\mathbb{R}^3)$ *for* $p \geq 4$, *then the pair* $(\boldsymbol{u}, \pi) \in \mathbf{X}_0^{1,2}(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ *given by Proposition 7.4 belongs also to* $\mathbf{X}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$. *Moreover*

$$\boldsymbol{u} \in \mathbf{L}^{4p/(4-p)}(\mathbb{R}^3) \cap \mathbf{L}^6(\mathbb{R}^3) \quad \textit{if } p < 2; \tag{7.6}$$

$$\boldsymbol{u} \in \mathbf{L}^4(\mathbb{R}^3) \cap \mathbf{L}^{3p/(3-p)}(\mathbb{R}^3) \quad \textit{if } 2 \leq p < 3; \tag{7.7}$$

$$\boldsymbol{u} \in \mathbf{L}^r(\mathbb{R}^3), \quad \forall r \geq 4, \quad \textit{if } p = 3; \tag{7.8}$$

$$\boldsymbol{u} \in \mathbf{L}^4(\mathbb{R}^3) \cap \mathbf{L}^\infty(\mathbb{R}^3), \quad \textit{if } p > 3. \tag{7.9}$$

Proof. Under the assumptions on $\boldsymbol{f}, g$ and $\boldsymbol{a}$, Proposition 7.4 yields the existence of a unique $(\boldsymbol{u}, \pi) \in \mathbf{X}_0^{1,2}(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ verifying

$$-\Delta \boldsymbol{u} + k \frac{\partial \boldsymbol{u}}{\partial x_1} + \nabla \pi = \boldsymbol{f} - \operatorname{div}(\boldsymbol{a} \otimes \boldsymbol{u}) \quad \text{in } \mathbb{R}^3.$$

From Proposition 2.5, we have $\boldsymbol{u} \in \mathbf{L}^4(\mathbb{R}^3) \cap \mathbf{L}^6(\mathbb{R}^3)$.

Case 1: $1 < p < 4$. Considering that $\boldsymbol{a} \in \mathbf{L}^{4p/(4-p)}(\mathbb{R}^3)$, we get $\boldsymbol{a} \otimes \boldsymbol{u} \in \mathbf{L}^p(\mathbb{R}^3)$ and then $\operatorname{div}(\boldsymbol{a} \otimes \boldsymbol{u}) \in \mathbf{W}_0^{-1,p}(\mathbb{R}^3) \perp \mathbb{P}_{[1-3/p']}$. From the Oseen theory [AR1, Theorem 2.2] there exists a unique solution $(\boldsymbol{v}, \eta) \in \mathbf{X}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ to the problem

$$-\Delta \boldsymbol{v} + k \frac{\partial \boldsymbol{v}}{\partial x_1} + \nabla \eta = \boldsymbol{f} - \operatorname{div}(\boldsymbol{a} \otimes \boldsymbol{u}) \in \mathbf{W}_0^{-1,p}(\mathbb{R}^3), \quad \operatorname{div} \boldsymbol{v} = g \text{ in } \mathbb{R}^3. \tag{7.10}$$

Moreover $\boldsymbol{v} \in \mathbf{L}^{4p/(4-p)}(\mathbb{R}^3)$. Then a uniqueness argument implies that $\pi = \eta$ and next $\boldsymbol{u} = \boldsymbol{v}$. Thanks to Proposition 2.5 it follows (7.6)-(7.9).

Case 2: $p \geq 4$. As $\boldsymbol{f} \in \mathbf{W}_0^{-1,2}(\mathbb{R}^3) \cap \mathbf{W}_0^{-1,q}(\mathbb{R}^3)$ and $g \in Z_2(\mathbb{R}^3) \cap Z_q(\mathbb{R}^3)$, for any $2 \leq q < 4$, thanks to the case 1 we get $\boldsymbol{u} \in \mathbf{X}_0^{1,2}(\mathbb{R}^3) \cap \mathbf{X}_0^{1,q}(\mathbb{R}^3)$ and $\pi \in L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ for any $2 \leq q < 4$. In particular, choosing $3 < q < 4$ we obtain $\boldsymbol{u} \in \mathbf{L}^\infty(\mathbb{R}^3)$ and then $\nabla \cdot (\boldsymbol{a} \otimes \boldsymbol{u}) \in \mathbf{W}_0^{-1,p}(\mathbb{R}^3)$ and also (7.9) holds. From the Oseen theory [AR1, Theorem 2.2] there exists a solution $(\boldsymbol{v}, \eta) \in \mathbf{X}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ to the problem (7.10), where η is unique and $\boldsymbol{v}$ is unique up to an element of $\mathbb{R}^3$. By a similar argument to the case 1, we show that $(\nabla \boldsymbol{u}, \pi) = (\nabla \boldsymbol{v}, \eta)$ and $(\boldsymbol{u}, \pi) \in \mathbf{X}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ verifying (7.9). □

Now we are able to prove the existence result in the non-Hilbertian case $p \neq 2$.

Theorem 7.6. *Assume that* $1 < p < \infty$, $\boldsymbol{f} \in \mathbf{W}_0^{-1,p}(\mathbb{R}^3)$, $g \in Z_p(\mathbb{R}^3)$ *and* $\boldsymbol{a} \in \mathbf{H}_3$ *satisfy (4.1), (7.1) and (4.6), respectively. Then the problem (1.3) has a solution* $(\boldsymbol{u}, \pi) \in \mathbf{X}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$, *where* π *is unique and* $\boldsymbol{u}$ *is unique if* $1 < p < 4$ *and up to an element of* $\mathbb{R}^3$ *otherwise. Moreover, we have the estimate*

$$\|\boldsymbol{u}\|_{\mathbf{W}_0^{1,p}(\mathbb{R}^3)/\mathbb{P}_{[1-4/p]}} + \|\frac{\partial \boldsymbol{u}}{\partial x_1}\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)} + \|\pi\|_{L^p(\mathbb{R}^3)} \leq C_K(\|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)} + \|g\|_{Z_p(\mathbb{R}^3)})$$

and also that $\boldsymbol{u} \in \mathbf{L}^{4p/(4-p)}(\mathbb{R}^3) \cap \mathbf{L}^{3p/(3-p)}(\mathbb{R}^3)$ *if* $1 < p < 3$, $\boldsymbol{u} \in \mathbf{L}^r(\mathbb{R}^3)$ *for all* $r \geq 12$ *if* $p = 3$, *and* $\boldsymbol{u} \in \mathbf{L}^{4p/(4-p)}(\mathbb{R}^3) \cap \mathbf{L}^\infty(\mathbb{R}^3)$ *if* $3 < p < 4$.

Proof. By (4.1), we have $\boldsymbol{f} = \operatorname{div}\mathbf{F}$ with $\mathbf{F} \in L^p(\mathbb{R}^3)^{3\times 3}$. Thus, there exists a sequence $\{\mathbf{F}_m\} \subset \mathcal{D}(\mathbb{R}^3)^{3\times 3}$ such that $\mathbf{F}_m \to \mathbf{F}$ in $\mathbf{L}^p(\mathbb{R}^3)$. Set $\boldsymbol{f}_m = \operatorname{div}\mathbf{F}_m$. Then $\boldsymbol{f}_m$ satisfying the condition (4.1) and $\boldsymbol{f}_m$ converges to $\boldsymbol{f}$ in $\mathbf{W}_0^{-1,p}(\mathbb{R}^3)$. Using the density properties of $\mathcal{D}(\mathbb{R}^3)$ into $Z_p(\mathbb{R}^3)$ (cf. Lemma 7.3) and of $\boldsymbol{\mathcal{V}}$ into $\mathbf{H}_3$ (cf. Lemma 4.2), there exist $\{g_m\} \subset \mathcal{D}(\mathbb{R}^3)$ and $\{\boldsymbol{a}_m\} \subset \boldsymbol{\mathcal{V}}$ such that

$$g_m \to g \text{ in } Z_p(\mathbb{R}^3) \quad \text{and} \quad \boldsymbol{a}_m \to \boldsymbol{a} \text{ in } \mathbf{H}_3.$$

Applying Proposition 7.5, there exists $(\boldsymbol{u}_m,\pi_m) \in \mathbf{X}_0^{1,p}(\mathbb{R}^3)\times L^p(\mathbb{R}^3)$ solution of the problem

$$-\Delta \boldsymbol{u}_m + k\frac{\partial \boldsymbol{u}_m}{\partial x_1} + \operatorname{div}(\boldsymbol{a}_m \otimes \boldsymbol{u}_m) + \nabla \pi_m = \boldsymbol{f}_m, \quad \operatorname{div}\boldsymbol{u}_m = g_m \text{ in } \mathbb{R}^3,$$

where $\boldsymbol{f}_m$, g_m and $\boldsymbol{a}_m$ converge to $\boldsymbol{f}, g$ and $\boldsymbol{a}$ in $\mathbf{W}_0^{-1,p}(\mathbb{R}^3)$, $Z_p(\mathbb{R}^3)$ and $\mathbf{L}^3(\mathbb{R}^3)$, respectively. Now applying the Oseen theory [AR1, Theorem 2.2], there exists a unique solution $(\boldsymbol{v}_m,\eta_m) \in \mathbf{X}_0^{1,p}(\mathbb{R}^3)\times L^p(\mathbb{R}^3)$ satisfying

$$-\Delta \boldsymbol{v}_m + k\frac{\partial \boldsymbol{v}_m}{\partial x_1} + \nabla \eta_m = \boldsymbol{f}_m - \operatorname{div}(\boldsymbol{a}_m \otimes \boldsymbol{u}_m), \quad \operatorname{div}\boldsymbol{v}_m = g_m \text{ in } \mathbb{R}^3.$$

Moreover, the estimate holds

$$\|\boldsymbol{v}_m\|_{\mathbf{W}_0^{1,p}(\mathbb{R}^3)/\mathbb{P}_{[1-4/p]}} + \|\frac{\partial \boldsymbol{v}_m}{\partial x_1}\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)} + \|\eta_m\|_{L^p(\mathbb{R}^3)} \le$$
$$\le C(\|\boldsymbol{f}_m - \operatorname{div}(\boldsymbol{a}_m\otimes\boldsymbol{u}_m)\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)} + \|g_m\|_{L^p(\mathbb{R}^3)} + \|\frac{\partial g_m}{\partial x_1}\|_{W_0^{-2,p}(\mathbb{R}^3)}).$$

By the uniqueness argument, it is clear that $(\nabla \boldsymbol{v}_m,\eta_m) = (\nabla \boldsymbol{u}_m,\pi_m)$. In order to obtain an estimate independent on m we split into two cases.

Case 1: $p < 3$. Observing that $\boldsymbol{u}_m \in \mathbf{W}_0^{1,p}(\mathbb{R}^3) \hookrightarrow \mathbf{L}^{p*}(\mathbb{R}^3)$, we have

$$\|\operatorname{div}(\boldsymbol{a}_m\otimes\boldsymbol{u}_m)\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)} \le \|\boldsymbol{a}_m\otimes\boldsymbol{u}_m\|_{\mathbf{L}^p(\mathbb{R}^3)} \le C\|\boldsymbol{a}_m\|_{\mathbf{L}^3(\mathbb{R}^3)}\|\boldsymbol{u}_m\|_{\mathbf{W}_0^{1,p}(\mathbb{R}^3)}.$$

Case 2: $p \ge 3$. Observing that $(\boldsymbol{a}_m\cdot\nabla)\boldsymbol{u}_m \in \mathbf{L}^{3p/(3+p)}(\mathbb{R}^3) \hookrightarrow \mathbf{W}_0^{-1,p}(\mathbb{R}^3)$, we have

$$\|(\boldsymbol{a}_m\cdot\nabla)\boldsymbol{u}_m\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)} \le C\|(\boldsymbol{a}_m\cdot\nabla)\boldsymbol{u}_m\|_{\mathbf{L}^{3p/(3+p)}(\mathbb{R}^3)} \le C\|\boldsymbol{a}_m\|_{\mathbf{L}^3(\mathbb{R}^3)}\|\boldsymbol{u}_m\|_{\mathbf{W}_0^{1,p}(\mathbb{R}^3)}.$$

Hence in both cases it follows

$$\|\boldsymbol{u}_m\|_{\mathbf{W}_0^{1,p}(\mathbb{R}^3)/\mathbb{P}_{[1-4/p]}} + \|\frac{\partial \boldsymbol{u}_m}{\partial x_1}\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)} + \|\pi_m\|_{L^p(\mathbb{R}^3)} \le$$
$$\le C(\|\boldsymbol{f}_m\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)} + \|\boldsymbol{a}_m\|_{\mathbf{L}^3(\mathbb{R}^3)}\|\boldsymbol{u}_m\|_{\mathbf{W}_0^{1,p}(\mathbb{R}^3)} + \|g_m\|_{Z_p(\mathbb{R}^3)})$$
$$\le C(\|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)} + \|\boldsymbol{a}\|_{\mathbf{L}^3(\mathbb{R}^3)}\|\boldsymbol{u}_m\|_{\mathbf{W}_0^{1,p}(\mathbb{R}^3)} + \|g\|_{Z_p(\mathbb{R}^3)}).$$

Then, proceeding as in the proof of Theorem 4.3 and also using Proposition 2.5, we conclude Theorem 7.6. □

Remark 7.7. If $3 < p < 4$, Lemma 2.4 yields that the solution $\boldsymbol{u}$ given at Theorem 7.6 is a continuous function satisfying (2.9). If $p < 3$, the unique solution to (1.1) in accordance to Theorem 7.6 tends weakly to zero at infinity (cf. Lemma 2.1 and Definition 2.2).

Next, let us state the existence of strong solutions to (1.3).

Theorem 7.8. *Assume that* $1 < p < 3$, $\boldsymbol{f} \in \mathbf{L}^p(\mathbb{R}^3)$, $g \in X_0^{1,p}(\mathbb{R}^3)$ *and* $\boldsymbol{a} \in \mathbf{H}_3$ *such that (4.6) is satisfied. Then the problem (1.3) has a unique solution* $(\boldsymbol{u},\pi) \in \mathbf{X}_0^{2,p}(\mathbb{R}^3) \times W_0^{1,p}(\mathbb{R}^3)$ *satisfying*

$$\nabla \boldsymbol{u} \in \mathbf{L}^{4p/(4-p)}(\mathbb{R}^3) \cap \mathbf{L}^{3p/(3-p)}(\mathbb{R}^3); \tag{7.11}$$

$$\boldsymbol{u} \in \mathbf{L}^{3p/(3-2p)}(\mathbb{R}^3) \cap \mathbf{L}^{2p/(2-p)}(\mathbb{R}^3) \quad \textit{if } p < 3/2; \tag{7.12}$$

$$\boldsymbol{u} \in \mathbf{L}^{q}(\mathbb{R}^3), \quad \forall q \geq 2p/(2-p) \quad \textit{if } 3/2 \leq p < 2. \tag{7.13}$$

Proof. As $1 < p < 3$, note that $\mathbf{L}^p(\mathbb{R}^3) \hookrightarrow \mathbf{W}_0^{-1,3p/(3-p)}(\mathbb{R}^3)$, see (5.2). Under the assumption on g we get $g \in W_0^{1,p}(\mathbb{R}^3) \hookrightarrow L^{3p/(3-p)}(\mathbb{R}^3)$ and the embedding $W_0^{2,3p/(4p-3)}(\mathbb{R}^3) \hookrightarrow W_0^{1,p'}(\mathbb{R}^3)$ implies that $\frac{\partial g}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^3) \hookrightarrow W_0^{-2,3p/(3-p)}(\mathbb{R}^3)$. Moreover for $1 < p < 3$ we get $3/2 < p* = 3p/(3-p) < \infty$. Thus Theorem 7.6 guarantees the existence of a solution $(\boldsymbol{u},\pi) \in \mathbf{X}_0^{1,p*}(\mathbb{R}^3) \times L^{p*}(\mathbb{R}^3)$ to the problem (1.3). Then $(\boldsymbol{a}\cdot\nabla)\boldsymbol{u} \in \mathbf{L}^p(\mathbb{R}^3)$ and we can apply the Oseen regularity theory [AR1, Theorem 2.6] to conclude that $(\boldsymbol{u},\pi) \in \mathbf{X}_0^{2,p}(\mathbb{R}^3) \times W_0^{1,p}(\mathbb{R}^3)$. Moreover, by Proposition 2.6 we have (7.11)-(7.13). □

Theorem 7.9. *For* $p \geq 3$, *let* $\boldsymbol{f} \in \mathbf{L}^p(\mathbb{R}^3)$, $g \in W_0^{1,p}(\mathbb{R}^3)$ *and* $\boldsymbol{a} \in \mathbf{H}_3$ *satisfy (4.6). If we additionally assume* $\boldsymbol{f} \in \mathbf{L}^q(\mathbb{R}^3)$, $g \in W_0^{1,q}(\mathbb{R}^3)$ *and* $\boldsymbol{a} \in \mathbf{L}^{3pq/(q(3+p)-3p)}(\mathbb{R}^3)$ *for some* $3p/(3+p) \leq q < 3$, *then the solution* $(\boldsymbol{u},\pi) \in \mathbf{X}_0^{2,q}(\mathbb{R}^3) \times W_0^{1,q}(\mathbb{R}^3)$ *given by Theorem 7.8 belongs also to* $\mathbf{X}_0^{2,p}(\mathbb{R}^3) \times W_0^{1,p}(\mathbb{R}^3)$ *and it satisfies*

$$\nabla u \in \mathbf{L}^{4q/(4-q)}(\mathbb{R}^3) \cap \mathbf{L}^{3q/(3-q)}(\mathbb{R}^3). \tag{7.14}$$

Proof. Since $\boldsymbol{f} \in \mathbf{L}^q(\mathbb{R}^3)$ and $g \in W_0^{1,q}(\mathbb{R}^3)$, with $3/2 \leq q < 3$, we can apply Theorem 7.8. Then there exists a unique solution $(\boldsymbol{u},\pi) \in \mathbf{X}_0^{2,q}(\mathbb{R}^3) \times W_0^{1,q}(\mathbb{R}^3)$ satisfying the generalized Oseen problem (1.3). As in the proof of Theorem 5.3 it results $(\boldsymbol{a}\cdot\nabla)\boldsymbol{u} \in \mathbf{L}^p(\mathbb{R}^3)$. Analogously to the proof of Theorem 7.8, applying the Oseen regularity theory [AR1, Theorem 2.6] we can conclude that $(\boldsymbol{u},\pi) \in \mathbf{X}_0^{2,p}(\mathbb{R}^3) \times W_0^{1,p}(\mathbb{R}^3)$, and it verifies (7.14). □

In order to prove the existence of stronger solutions of the generalized Oseen problem (1.3) under smoother data, let us state the following result.

Lemma 7.10. *i) For every* $p \neq 3/2$ *and* $g \in X_1^{1,p}(\mathbb{R}^3)$ *verifying*

$$\forall \lambda \in \mathbb{P}_{[2-3/p']}, \quad \langle \frac{\partial g}{\partial x_1}, \lambda \rangle_{W_1^{-1,p}(\mathbb{R}^3)\times W_{-1}^{1,p'}(\mathbb{R}^3)} = 0, \tag{7.15}$$

there exists $\boldsymbol{w} \in \mathbf{X}_1^{2,p}(\mathbb{R}^3)$ *such that div* $\boldsymbol{w} = g$ *in* $\mathbb{R}^3$ *and satisfies the estimate*

$$\|\boldsymbol{w}\|_{\mathbf{X}_1^{2,p}(\mathbb{R}^3)} \leq C\|g\|_{X_1^{1,p}(\mathbb{R}^3)}.$$

ii) If moreover $g \in Z_q(\mathbb{R}^3)$ *verifying (7.1), with* p *changed in* q, *then we can choose* $\boldsymbol{w} \in \mathbf{X}_1^{2,p}(\mathbb{R}^3) \cap \mathbf{X}_0^{1,q}(\mathbb{R}^3)$ *with the corresponding estimate.*

Proof. i) On one hand (cf. [AGG]), from $g \in W_1^{1,p}(\mathbb{R}^3)$ there exists $v \in W_1^{3,p}(\mathbb{R}^3)$ verifying

$$\Delta v = g \text{ in } \mathbb{R}^3, \quad \text{with} \quad \|\nabla v\|_{\mathbf{W}_1^{2,p}(\mathbb{R}^3)} \le C\|g\|_{W_1^{1,p}(\mathbb{R}^3)}.$$

On the other hand (cf. [AGG]), from $\frac{\partial g}{\partial x_1} \in W_1^{-1,p}(\mathbb{R}^3) \perp \mathbb{P}_{[2-3/p']}$ there exists a unique $z \in W_1^{1,p}(\mathbb{R}^3)$ verifying

$$\Delta z = \frac{\partial g}{\partial x_1} \text{ in } \mathbb{R}^3, \quad \text{with} \quad \|\nabla z\|_{\mathbf{W}_1^{0,p}(\mathbb{R}^3)} \le C\|\frac{\partial g}{\partial x_1}\|_{W_1^{-1,p}(\mathbb{R}^3)}.$$

Then $\frac{\partial v}{\partial x_1} - z \in W_1^{2,p}(\mathbb{R}^3) + W_1^{1,p}(\mathbb{R}^3)$ is harmonic and $\frac{\partial v}{\partial x_1} - z \in \mathbb{P}_{[1-3/p]}$, that means, $\frac{\partial v}{\partial x_1} = z$ if $p < 3$ and $\frac{\partial v}{\partial x_1} - z =$ constant if $p \ge 3$.

Next let us take $\boldsymbol{w} = \nabla v$. Hence $\boldsymbol{w} \in \mathbf{W}_1^{2,p}(\mathbb{R}^3)$, $\operatorname{div} \boldsymbol{w} = g$ and $\frac{\partial \boldsymbol{w}}{\partial x_1} = \nabla(\frac{\partial v}{\partial x_1}) = \nabla z \in \mathbf{W}_1^{0,p}(\mathbb{R}^3)$. Thus we can proceed as in the proof of Lemma 7.1 to conclude i).

ii) Using a similar argument as in i), we can choose $\boldsymbol{w}$ also belonging to $\mathbf{X}_0^{1,q}(\mathbb{R}^3)$. □

In accordance with Oseen theory, we can prove the following regularity result.

Theorem 7.11. *Let $p > 3/2$, $q = 3p/(3+p)$, $\boldsymbol{f} \in \mathbf{W}_1^{0,p}(\mathbb{R}^3) \cap \mathbf{W}_0^{-1,q}(\mathbb{R}^3)$ satisfy, for all $i = 1,2,3$,*

$$\langle f_i, 1\rangle_{W_0^{-1,q}(\mathbb{R}^3)\times W_0^{1,q'}(\mathbb{R}^3)} = 0, \quad \text{if } 3/2 < p \le 3, \tag{7.16}$$

$g \in X_1^{1,p}(\mathbb{R}^3) \cap Z_q(\mathbb{R}^3)$ satisfy (7.15) and

$$\langle \frac{\partial g}{\partial x_1}, x_1\rangle_{W_0^{-2,q}(\mathbb{R}^3)\times W_0^{2,q'}(\mathbb{R}^3)} = 0, \quad \text{if } q \le 3/2, \tag{7.17}$$

and $\boldsymbol{a} \in \mathbf{H}_3$ satisfy (4.6) and (5.4). Then the Oseen problem (1.3) has a unique solution $(\boldsymbol{u},\pi)$ satisfying

$$\boldsymbol{u} \in \mathbf{X}_1^{2,p}(\mathbb{R}^3) \cap \mathbf{X}_0^{1,p}(\mathbb{R}^3) \cap \mathbf{X}_0^{1,q}(\mathbb{R}^3); \qquad \pi \in W_1^{1,p}(\mathbb{R}^3) \cap L^q(\mathbb{R}^3). \tag{7.18}$$

Proof. In order to apply Theorem 7.6, we consider the embedding $\mathbf{W}_1^{0,p}(\mathbb{R}^3) \hookrightarrow \mathbf{W}_0^{-1,p}(\mathbb{R}^3)$, for $p \ne 3/2$. Then there exists a solution $(\boldsymbol{u},\pi) \in \mathbf{X}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ to the Oseen problem (1.3). Thus the assumption (5.4) implies that $(\boldsymbol{a}\cdot\nabla)\boldsymbol{u} \in \mathbf{W}_1^{0,p}(\mathbb{R}^3)$.

We can also apply Theorem 7.6 for the existence of a solution $(\boldsymbol{z},\eta) \in \mathbf{X}_0^{1,q}(\mathbb{R}^3) \times L^q(\mathbb{R}^3)$ to the Oseen problem (1.3). Since $q = 3p/(3+p) < 3$, we get $\boldsymbol{z} = \boldsymbol{u}$ and $\eta = \pi$. Thus we have $(\boldsymbol{a}\cdot\nabla)\boldsymbol{u} \in \mathbf{L}^{3q/(3+q)}(\mathbb{R}^3) \hookrightarrow \mathbf{W}_0^{-1,q}(\mathbb{R}^3)$.

Since $g \in X_1^{1,p}(\mathbb{R}^3) \cap Z_q(\mathbb{R}^3)$ satisfies (7.15) and (7.17), considering Lemma 7.10 there exists $\boldsymbol{w} \in \mathbf{X}_1^{2,p}(\mathbb{R}^3) \cap \mathbf{X}_0^{1,q}(\mathbb{R}^3)$ such that $\operatorname{div} \boldsymbol{w} = g$ in $\mathbb{R}^3$. Set now $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{w}$, then $\boldsymbol{v} \in \mathbf{X}_0^{1,p}(\mathbb{R}^3) \cap \mathbf{X}_0^{1,q}(\mathbb{R}^3)$ and

$$-\Delta \boldsymbol{v} + k\frac{\partial \boldsymbol{v}}{\partial x_1} + \nabla\pi = \boldsymbol{f} - (\boldsymbol{a}\cdot\nabla)\boldsymbol{u} + \Delta\boldsymbol{w} - k\frac{\partial \boldsymbol{w}}{\partial x_1}, \quad \operatorname{div}\boldsymbol{v} = 0 \text{ in } \mathbb{R}^3. \tag{7.19}$$

Since the function $\mathbf{F} := \boldsymbol{f} - (\boldsymbol{a}\cdot\nabla)\boldsymbol{u} + \Delta\boldsymbol{w} - k\frac{\partial \boldsymbol{w}}{\partial x_1} \in W_1^{0,p}(\mathbb{R}^3) \cap W_0^{-1,q}(\mathbb{R}^3)$ satisfies the compatibility conditions, for $3/2 < p \le 3$,

$$\langle F_i, 1\rangle_{W_1^{0,p}(\mathbb{R}^3)\times W_{-1}^{0,p'}(\mathbb{R}^3)} = \langle F_i, 1\rangle_{W_0^{-1,q}(\mathbb{R}^3)\times W_0^{1,q'}(\mathbb{R}^3)} = 0,$$

then applying the Oseen regularity theory [AR1, Theorem 2.12] we deduce that $\boldsymbol{v} \in \mathbf{X}_1^{2,p}(\mathbb{R}^3)$ and $\pi \in W_1^{1,p}(\mathbb{R}^3)$. Then we conclude that π and $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$ satisfy (7.18). □

Remark 7.12. The solution $\boldsymbol{u}$ found in Theorem 7.11 also belongs to $\mathbf{L}^p(\mathbb{R}^3) \cap \mathbf{L}^{12p/(12+p)}(\mathbb{R}^3)$. Additionally, if $p < 4$ then $\boldsymbol{u} \in \mathbf{L}^{4p/(4-p)}(\mathbb{R}^3)$, and if $p < 3$ then $\boldsymbol{u} \in \mathbf{L}^{3p/(3-p)}(\mathbb{R}^3)$.

8 The Oseen problem (1.1) with $\boldsymbol{a}$ not in $\mathbf{L}^3(\mathbb{R}^3)$

In this section we study the Oseen model (1.1) case ii). For the sake of simplicity, let us set $k = 1$, which means that $\boldsymbol{a} \in \mathbf{L}^3_{loc}(\mathbb{R}^3)$ is such that $\operatorname{div}\boldsymbol{a} = 0$ and

$$\exists R_0 > 0: \quad \boldsymbol{a}(\boldsymbol{x}) = \mathbf{e}_1, \quad |\boldsymbol{x}| \ge R_0, \tag{8.1}$$

are satisfied.

Theorem 8.1. *Let $\boldsymbol{a} \in \mathbf{L}^3_{loc}(\mathbb{R}^3)$ satisfy $\operatorname{div}\boldsymbol{a} = 0$ and (8.1). For $\boldsymbol{f} \in \mathbf{W}_0^{-1,2}(\mathbb{R}^3)$ and $g \in Z_2(\mathbb{R}^3)$, there exists a unique pair $(\boldsymbol{u}, \pi) \in (\mathbf{X}_0^{1,2}(\mathbb{R}^3) \cap \mathbf{L}^4(\mathbb{R}^3)) \times L^2(\mathbb{R}^3)$ solving (1.1) such that*

$$\|\boldsymbol{u}\|_{\mathbf{W}_0^{1,2}(\mathbb{R}^3)} + \|\frac{\partial \boldsymbol{u}}{\partial x_1}\|_{\mathbf{W}_0^{-1,2}(\mathbb{R}^3)} + \|\pi\|_{L^2(\mathbb{R}^3)} \le C\Big(\|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,2}(\mathbb{R}^3)} + (1 + \|\boldsymbol{a} - \mathbf{e}_1\|_{\mathbf{L}^3(\mathbb{R}^3)})\|g\|_{Z_2(\mathbb{R}^3)}\Big).$$

Moreover such solution verifies the energy equality (3.11).

Proof. First note that $\operatorname{div}(\boldsymbol{a} - \mathbf{e}_1) = 0$ in $\mathbb{R}^3$ and $\operatorname{supp}(\boldsymbol{a} - \mathbf{e}_1)$ is a compact set into the ball B_{R_0}. Then we obtain $\boldsymbol{a} - \mathbf{e}_1 \in \mathbf{H}_3$. We can apply Proposition 7.4 with $k = 1$ to the problem

$$-\Delta\boldsymbol{u} + \frac{\partial \boldsymbol{u}}{\partial x_1} + ((\boldsymbol{a} - \mathbf{e}_1)\cdot\nabla)\boldsymbol{u} + \nabla\pi = \boldsymbol{f}, \quad \operatorname{div}\boldsymbol{u} = g \quad \text{in } \mathbb{R}^3,$$

concluding the existence and uniqueness of the required solution to (1.1). Moreover such solution verifies the energy equality (3.11), taking into account that

$$\langle \frac{\partial \boldsymbol{u}}{\partial x_1}, \boldsymbol{u}\rangle_{\mathbf{W}_0^{-1,2}(\mathbb{R}^3)\times\mathbf{W}_0^{1,2}(\mathbb{R}^3)} = 0 \quad \text{and} \quad \int_{\mathbb{R}^3} \nabla\boldsymbol{u} : (\boldsymbol{a} - \mathbf{e}_1)\otimes\boldsymbol{u}\, d\boldsymbol{x} = 0.$$

□

Remark 8.2. The condition $\boldsymbol{a} \in \mathbf{L}^3(\mathbb{R}^3)$ is then not necessary to the existence of a solution $(\boldsymbol{u}, \pi) \in \mathbf{X}_0^{1,2}(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ of the problem (1.1).

Theorem 8.3. *For $1 < p < \infty$, let $\boldsymbol{f} \in \mathbf{W}_0^{-1,p}(\mathbb{R}^3)$ and $g \in Z_p(\mathbb{R}^3)$ verify (4.1) and (7.1), respectively. Let $\boldsymbol{a}$ be as in Theorem 8.1 such that*

$$\|\boldsymbol{a} - \mathbf{e}_1\|_{\mathbf{L}^3(B_{R_0})} \le K, \tag{8.2}$$

where K is the constant introduced in (4.6). Then the Oseen problem (1.1) has a solution $(\boldsymbol{u},\pi)\in \mathbf{X}_0^{1,p}(\mathbb{R}^3)\times L^p(\mathbb{R}^3)$ *such that*

$$\|\boldsymbol{u}\|_{\mathbf{W}_0^{1,p}(\mathbb{R}^3)/\mathbb{P}_{[1-4/p]}}+\|\frac{\partial \boldsymbol{u}}{\partial x_1}\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)}+\|\pi\|_{L^p(\mathbb{R}^3)}\leq C_K(\|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^3)}+\|g\|_{Z_p(\mathbb{R}^3)})$$

and also that $\boldsymbol{u}\in \mathbf{L}^{4p/(4-p)}(\mathbb{R}^3)\cap \mathbf{L}^{3p/(3-p)}(\mathbb{R}^3)$ *if* $1<p<3$, $\boldsymbol{u}\in \mathbf{L}^r(\mathbb{R}^3)$ *for all* $r\geq 12$ *if* $p=3$, *and* $\boldsymbol{u}\in \mathbf{L}^{4p/(4-p)}(\mathbb{R}^3)\cap \mathbf{L}^{\infty}(\mathbb{R}^3)$ *if* $3<p<4$.

Proof. We proceed as in the proof of Theorem 8.1, taking

$$\mathbf{c}=\boldsymbol{a}-\mathbf{e}_1\in \mathbf{H}_3.$$

Next we observe that $\|\mathbf{c}\|_{\mathbf{L}^3(\mathbb{R}^3)}\leq K$. Then according to Theorem 7.6 there exists a solution $(\boldsymbol{u},\pi)\in \mathbf{X}_0^{1,p}(\mathbb{R}^3)\times L^p(\mathbb{R}^3)$ to (1.1) which concludes the desired existence result. □

Remark 8.4. Similarly to Theorem 7.8, for $1<p<3$, if $\boldsymbol{f}\in \mathbf{L}^p(\mathbb{R}^3)$, $g\in X_0^{1,p}(\mathbb{R}^3)$ and $\boldsymbol{a}$ as given at Theorem 8.3, then the Oseen problem (1.1) has a unique solution $(\boldsymbol{u},\pi)\in \mathbf{X}_0^{2,p}(\mathbb{R}^3)\times W_0^{1,p}(\mathbb{R}^3)$ as in Theorem 7.8. Analogously to Theorem 7.9.

Remark 8.5. Analogously to Theorem 7.11, for $p>3/2$, $\boldsymbol{f}$ and g under the conditions of Theorem 7.11 and $\boldsymbol{a}$ as given at Theorem 8.3 and satisfy

$$\exists L>0: \quad \sqrt{(a_1(\boldsymbol{x})-1)^2+a_2^2(\boldsymbol{x})+a_3^2(\boldsymbol{x})}\leq \frac{L}{|\boldsymbol{x}|}, \quad \text{a.e. } \boldsymbol{x}\in\mathbb{R}^3,$$

then the Oseen problem (1.1) has a unique solution $(\boldsymbol{u},\pi)\in (\mathbf{X}_1^{2,p}(\mathbb{R}^3)/\mathbb{P}_{[1-4/p]})\times W_1^{1,p}(\mathbb{R}^3)$ as in Theorem 7.11.

References

[AA] F. Alliot and C. Amrouche, The Stokes problem in $\mathbb{R}^n$: an approach in weighted Sobolev spaces. *Math. Meth. Mod. Appl. Sci.* **9-5** (1999), pp 723-754.

[AG] C. Amrouche and V. Girault, Decomposition of vector spaces and application to the Stokes problem in arbitrary dimensions. *Czechoslovak Math. J.* **44-119** (1994), pp 109-140.

[AGG] C. Amrouche, V. Girault and J. Giroire, Weighted Sobolev spaces for Laplace's equation in $\mathbb{R}^n$. *J. Math. Pures Appl.* **73-9** (1994), pp 579-606.

[AN] C. Amrouche and H.H. Nguyen, The stationary three-dimensional Navier-Stokes Equations with a non-zero constant velocity at infinity. *Math. Meth. Appl. Sci.* **31-18** (2008), pp 2147-2171.

[AR] C. Amrouche and U. Razafison, Weighted Sobolev spaces for a scalar model of the stationary Oseen equations in $\mathbb{R}^3$. *J. Math. Fluid Mech.* **9** (2007), pp 181-210.

[AR1] C. Amrouche and U. Razafison, The stationary Oseen equations in $\mathbb{R}^3$. An approach in weighted Sobolev spaces. *J. Math. Fluid Mech.* **9** (2007), pp 211-225.

[F] R. Farwig, A variational approach in weighted Sobolev spaces to the operator $-\Delta + \partial/\partial x_1$ in exterior domains of $\mathbb{R}^3$. *Math. Z.* **210-3** (1992), pp 409-447.

[F1] R. Farwig, The stationary Navier-Stokes equations in a 3D-exterior domain. *Recent topics on mathematical theory of viscous incompressible fluid*, Tsukuba, (1996). Lecture Notes Numer. Appl. Anal., **16** (1998), pp 53-115.

[FS] R. Farwig, H. Sohr, Weighted estimates for the Oseen equations and the Navier-Stokes equations in exterior domains. *Proc. 3rd International Conference on the Navier-Stokes Equations: Theory and Numerical Methods*, Oberwolfach, Germany, June 511, , J. G. Heywood (ed.), World Sci. Publishings, River Edge, NJ, (1994) 1998. Ser. Adv. Math. Appl. Sci., **47** (1998), pp 11-30.

[FHM] R. Farwig, T. Hishida and O. Müller, L^q-theory of a singular "winding" integral operator arising from fluid dynamics. *Pacific J. Math.* **215** (2004), pp 297-312.

[Fi] R. Finn, On the exterior stationary problem for the Navier-Stokes equations, and associated perturbation problems. *Arch. Ration. Mech. Anal.* **19** (1965), pp 363-406.

[G] G. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations. Linearized steady problems*. Springer Tracts in Natural Philosophy **38**, New York, (1994).

[KS] T. Kobayashi and Y. Shibata, On the Oseen equation in the three-dimensional exterior domains. *Math. Ann.* **310-1** (1998), pp 1-45.

[KNP] S. Kračmar, S. Nečasová and P. Penel, On the weak solutions to the Oseen-type problem arising from flow around a rotating rigid body in the whole space. *WSEAS Trans. on Math.* **5-3** (2006), pp 243-251.

[K] A. Kufner, *Weighted Sobolev spaces*. Wiley, Chichester, (1985).

[SY] Y. Shibata and M. Yamazaki, Uniform estimates in the velocity at infinity for stationary solutions to the Navier-Stokes exterior problem. *Japanese J. Math. (New Series)* **31** (2005), pp 225-279.