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A REMARK ON SIMPLE SCATTERING THEORY

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Abstract

Scattering theory between the fractional power $H_0 = \kappa^{-1}(-\Delta)^{\kappa/2}$ ($\kappa \geq 1$) of negative Laplacian and the Hamiltonian $H = H_0 + V$ perturbed by short- and long-range potentials considered in [14] is revisited and a new proof of the existence and asymptotic completeness of wave operators is given with utilizing the smooth operator technique.

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1 Introduction

We consider a free Hamiltonian defined in a Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$

$$H_0 = \kappa^{-1}(-\Delta)^{\kappa/2},$$

and a perturbation

$$H = H_0 + V \tag{1.1}$$

of H_0 by a simple two-body potential V. Here $\kappa \geq 1$,

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2},$$

and V is decomposed into a sum $V = V_S(x) + V_L(x)$ of two real-valued measurable functions $V_S(x)$ and $V_L(x)$ on $\mathbb{R}^n$ which satisfy the following conditions.

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Assumption 1.1. There exists a constant $0 < \delta < 1$ such that

$$\|\langle x\rangle^{1+\delta} V_S(x)(1+H_0)^{-1}\| < \infty, \tag{1.2}$$

where $\|\cdot\|$ denotes the operator norm and $\langle x\rangle = \sqrt{1+|x|^2}$.

Assumption 1.2. Let $\delta \in (0,1)$ be the same constant as in Assumption 1.1. For all multi-indices α there exists a constant $C_\alpha > 0$ such that for all $x \in \mathbb{R}^n$

$$|\partial_x^\alpha V_L(x)| \le C_\alpha \langle x\rangle^{-|\alpha|-\delta}, \tag{1.3}$$

where $\partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_n)^{\alpha_n}$ for a mult-index $\alpha = (\alpha_1, \dots, \alpha_n)$.

Under these assumptions, V is relatively bounded with respect to H_0 with H_0-bound < 1. Thus H defines a selfadjoint operator with domain $\mathcal{D}(H) = \mathcal{D}(H_0) = H^\kappa(\mathbb{R}^n)$, the Sobolev space of order κ. Therefore the solution of the Schrödinger equation

$$\frac{1}{i}\frac{\partial u}{\partial t}(t) + Hu(t) = 0, \quad u(0) = f(\in \mathcal{D}(H))$$

is given by a unitary group e^{-itH} $(t \in \mathbb{R})$ as follows.

$$u(t) = e^{-itH} f.$$

Similarly for the free Hamiltonian H_0, the solution $u_0(t)$ of the corresponding Schrödinger equation with initial condition $u_0(0) = g$ is given by

$$u_0(t) = e^{-itH_0} g.$$

We remark that the operator H_0 is written as a pseudodifferential operator with symbol $H_0(\xi) = \kappa^{-1}|\xi|^\kappa$ $(\kappa \ge 1)$. Namely for $f \in \mathcal{D}(H_0)$

$$\begin{aligned} H_0 f(x) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} \kappa^{-1} |\xi|^\kappa \hat{f}(\xi) d\xi \\ &= (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i(x-y)\xi} \kappa^{-1} |\xi|^\kappa f(y) dy d\xi \end{aligned}$$

with $\hat{f} = \mathcal{F} f$ denoting the Fourier transform of f. We will use a convention $\widehat{d\xi} = (2\pi)^{-n} d\xi$. Then this expression is written as follows.

$$H_0 f(x) = \iint_{\mathbb{R}^{2n}} e^{i(x-y)\xi} \kappa^{-1} |\xi|^\kappa f(y) dy \widehat{d\xi}.$$

The problem of simple scattering theory is whether the both of the limits

$$W_1^\pm g = \lim_{t\to\pm\infty} e^{itH} J e^{-itH_0} g \quad (g \in \mathcal{H}) \tag{1.4}$$

$$W_2^\pm f = \lim_{t\to\pm\infty} e^{itH_0} J^{-1} e^{-itH} f \quad (f \in \mathcal{H}_c) \tag{1.5}$$

exist, if one constructs the identification operator J suitably. This problem is called the problem of the existence and asymptotic completeness of wave operators. Here we used the notation $\mathcal{H}_c$ to denote the spectrally continuous subspace of H. The asymptotic completeness means the existence of the limits $W_2^\pm g$ ($g \in \mathcal{H}_c$), and this is equivalent to the equality

$$\mathcal{R}(W_1^\pm) = \mathcal{H}_c. \tag{1.6}$$

From the definition above of $W_1^\pm$, we see that if the wave operators $W_1^\pm$ exist, the following holds.

$$e^{-isH}W_1^\pm = W_1^\pm e^{-isH_0} \quad (\forall s \in \mathbb{R}).$$

Taking Laplace transform of both sides, we easily see that for any Borel set B

$$E_H(B)W_1^\pm = W_1^\pm E_0(B),$$

where $E_H(B)$ and $E_0(B)$ are the spectral measures for H and H_0 respectively. From this follows that

$$\mathcal{R}(W_1^\pm) \subset \mathcal{H}_c.$$

Therefore to prove the asymptotic completeness it suffices to prove the converse inclusion

$$\mathcal{H}_c \subset \mathcal{R}(W_1^\pm). \tag{1.7}$$

The case $-$ is treated similarly to the case $+$, so that we consider the $+$ case only in the following.

Assume for a moment that the existence of the wave operator W_1^+ has been proved, and suppose that for a given $f \in \mathcal{H}_c$, there is a sequence $t_k \to \infty$ (as $k \to \infty$) such that the limit

$$W_2^+ f = \lim_{k\to\infty} e^{it_kH_0}J^{-1}e^{-it_kH}f \tag{1.8}$$

exists. Then we have

$$\begin{aligned} f &= \lim_{k\to\infty} e^{itH_k}Je^{-it_kH_0}e^{it_kH_0}J^{-1}e^{-it_kH}f \\ &= W_1^+W_2^+f \\ &\in \mathcal{R}(W_1^+), \end{aligned}$$

and the proof of (1.7) is complete.

The existence of wave operator

$$W_1^\pm g = \lim_{t\to\pm\infty} e^{itH}Je^{-itH_0}g \quad (g \in \mathcal{H})$$

is shown similarly to that of the existence of the limit (1.8). Thus the concern of scattering theory is to see how the existence of the limit (1.8) is shown, and several proofs are known for both short-range and long-range perturbations with respect to $H_0 = -\Delta$. In our previous paper [14] we extended the results of scattering theory to the Hamiltonian $H_0 = \kappa^{-1}(-\Delta)^{\kappa/2}$ ($\kappa \geq 1$) including the relativistic Hamiltonian $H_0 = \sqrt{-\Delta}$ with vanishing mass. The present paper is a continuation of [14] and gives a new simpler proof of the existence and asymptotic completeness of the wave operators associated to $H_0 = \kappa^{-1}(-\Delta)^{\kappa/2}$ ($\kappa \geq 1$).

While the scattering theory for the case $\kappa = 2$ is as mentioned fairly well investigated, it seems that only short-range perturbations have been dealt with (e.g., [5], [20], [22], [25]) concerning the relativistic Hamiltonians. The immediate motivation of our work started in [14] was to find a proof of the asymptotic completeness in the case of long-range perturbations with respect to the relativistic Hamiltonian $H_0 = \sqrt{-\Delta}$. For the relativistic Hamiltonian with positive mass including general pseudodifferential operators, there are preliminary works by Weder [23], [24] which investigated the spectral properties of those operators. Some inverse problems have also been investigated by [3], [4], [6] for the case of relativistic Hamiltonians as well as in the case of Dirac equations.

For illustrating the purpose of giving a new proof in spite of the well-established state of the present scattering theory, we will review some history of scattering theory for the perturbations of the Hamiltonian $H_0 = -\Delta$. The proof of the asymptotic completeness for this case had been treated by stationary method in the early age of the scattering theory. (E.g., [1], [9], [10], [15], [16], [19] for the short-range case. For other earlier results on trace class perturbations, etc., see e.g., [8] and references therein.) Around almost the same time Lax-Phillips [17] developed an abstract time-dependent scattering theory as well as gave concrete applications of the abstract theory to the acoustic wave equations. A little bit later Enss [2] gave a time-dependent method for treating the Schrödinger scattering theory. The similarity between the Lax-Phillips theory and Enss method was later noticed in [13]. In both approaches what is essential is the micro-local decomposition of the identity as defined in section 2 of [14]. Then it is shown that the incoming part vanishes as time goes to $+\infty$ by Ruelle [18] argument. The remaining outgoing part is treated by analyzing the propagation properties of the free unperturbed evolution. The proof has been simplified considerably in the Lax-Phillips-Enss method compared to the former proof by the stationary approach. There had been known however another time-dependent method developed by T. Kato [7] (1966) called smooth operator technique. A sufficient condition for the existence and the asymptotic completeness to hold is given in Theorem 3.9 of [7] in a time-dependent form[1]. This condition was later extended and utilized by Sigal-Soffer [21] in proving the asymptotic completeness of channel wave operators for N-body scattering problem with short-range pair potentials. Their improvement is found in Lemma 3.4 of [21]. The point of their argument is as follows with some simplification for the sake of illustration restricting the case to the short-range simple two-body perturbations. Suppose that a bounded operator F defined on $\mathcal{H}$ satisfies as a sesquibilinear form

$$i[H,F] = i(HF - FH) = F_1^2 + M_1(t) \text{ on } \mathcal{H}_c \times \mathcal{H}_c \tag{1.9}$$

for some selfadjoint operator F_1 and bounded operator $M_1(t)$ continuous in t with respect to operator norm such that $\|M_1(t)\| \in L^1(\mathbb{R})$. Then one has for $\tau > \sigma$ and $f \in \mathcal{H}_c$

$$\begin{aligned}
|(e^{i\tau H}Fe^{-i\tau H}f - e^{i\sigma H}Fe^{-i\sigma H}f, f)| &= \left| \int_\sigma^\tau \frac{d}{dt}(e^{itH}Fe^{-itH}f, f)dt \right| \\
&= \left| \int_\sigma^\tau (e^{itH}i[H,F]e^{-itH}f, f)dt \right| \\
&\geq \left| \int_\sigma^\tau (e^{itH}F_1^2e^{-itH}f, f)dt \right| - \int_\sigma^\tau \|M_1(t)\|dt\|f\|^2.
\end{aligned}$$

[1] A rather complete exposition of the method of smooth operator is found in a recent book [26] in Chapters 0, 1, and 2. Especially an explanation of the technique relevant to the present paper is given in section 5 of Chapter 0.

As the left hand side is bounded by a constant times $\|f\|^2$ uniformly with respect to t, one has the convergence of the following integral for $f \in \mathcal{H}_c$

$$\int_\sigma^\tau \|F_1 e^{-itH} f\|^2 dt \le M_1^2 \|f\|^2 \tag{1.10}$$

for some constant $M_1 > 0$ independent of $\tau > \sigma$. If one can make a similar assumption with respect to the unperturbed operator H_0 and has

$$i[H_0, F] = F_0^2 + M_0(t) \text{ on } \mathcal{H} \times \mathcal{H} \tag{1.11}$$

for a selfadjoint operator F_0 and some bounded norm continuous operator $M_0(t)$ such that $\|M_0(t)\| \in L^1(\mathbb{R})$, one then obtains an estimate similar to the above for $g \in \mathcal{H}$

$$\int_\sigma^\tau \|F_0 e^{-itH_0} g\|^2 dt \le M_0^2 \|g\|^2 \tag{1.12}$$

for some constant $M_0 > 0$ independent of $\tau > \sigma$. Assume now that one has a factorization

$$i(HF - FH_0) = F_1^* F_0 + M(t)^* \tag{1.13}$$

with $\|M(t)\| \in L^1(\mathbb{R})$. Then one gets for $f \in \mathcal{H}_c$ and $g \in \mathcal{H}$

$$\begin{aligned}&(e^{i\tau H_0} F^* e^{-i\tau H} f - e^{i\sigma H_0} F^* e^{-i\sigma H} f, g)\\ &= \int_\sigma^\tau (F_1 e^{-itH} f, F_0 e^{-itH_0} g) dt + \int_\sigma^\tau (M(t) f, g) dt.\end{aligned} \tag{1.14}$$

Applying (1.10) and (1.12) to the right hand side one obtains

$$\begin{aligned}&|(e^{i\tau H_0} F^* e^{-i\tau H} f - e^{i\sigma H_0} F^* e^{-i\sigma H} f, g)|\\ &\le M_0 \left(\int_\sigma^\tau \|F_1 e^{-itH} f\|^2 dt \right)^{\frac{1}{2}} \|g\| + \int_\sigma^\tau \|M(t)\| dt \|f\| \|g\|,\end{aligned} \tag{1.15}$$

which proves the existence of the limit

$$\lim_{t \to \infty} e^{itH_0} F^* e^{-itH} f \tag{1.16}$$

for $f \in \mathcal{H}_c$. If one can show the existence of a sequence $t_k \to \infty$ $(k \to \infty)$ for each $f \in \mathcal{H}_c$ such that

$$\|e^{-it_k H} f - F^* e^{-it_k H} f\| \to 0 \quad (\text{as } k \to \infty),$$

one has the existence of the limit

$$\lim_{k \to \infty} e^{it_k H_0} e^{-it_k H} f \tag{1.17}$$

for $f \in \mathcal{H}_c$ and the proof of asymptotic completeness is complete.

We will do this in a more refined manner to include the long-rang potentials so that one needs to modify the definition of wave operators and introduce time-independent modifier J following [14]. In the next section 2 we will prepare the known fact about scattering state, i.e. about the vector in $\mathcal{H}$ which belongs to the continuous spectral subspace $\mathcal{H}_c$ of $\mathcal{H}$. In section 3 we will state the definition of time-independent modifier or identification operator J following [14]. In the final section 4 we will give a refinement of the above argument adapted to the long-range case with introducing a time-dependent factor $J^* P^\varepsilon(t)$ instead of the factor F^* above and will conclude the description of a new proof.

2 Scattering State

We denote by $E_H(B)$ the spectral measure for H, and use the notation $\mathcal{H}_c(a,b) = E_H([a,b])\mathcal{H} \subset \mathcal{H}_c$. Under Assumptions 1.1 and 1.2, it is known that the closed linear hull of the set $\bigcup_{0<a<b<\infty} \mathcal{H}_c(a,b)$ equals $\mathcal{H}_c$. Also it is known that the following holds.

Lemma 2.1. *For any $f \in \mathcal{H}_c(a,b) = E_H([a,b])\mathcal{H}$ $(0 < a < b < \infty)$ with $\langle x\rangle^2 f \in \mathcal{H} = L^2(R^n)$, there exists a sequence $t_k \to \pm\infty$ as $k \to \pm\infty$ such that for any $\phi \in C_0^\infty(\mathbb{R})$ and $R > 0$*

$$\|\chi_{\{x\in\mathbb{R}^n \| x|<R\}} e^{-it_kH} f\| \to 0, \tag{2.1}$$

$$\|(\phi(H) - \phi(H_0))e^{-it_kH} f\| \to 0, \tag{2.2}$$

$$\left\|\left(\frac{x}{t_k} - |D_x|^{\kappa-2}D_x\right)e^{-it_kH} f\right\| \to 0 \tag{2.3}$$

as $k \to \pm\infty$, where $D = D_x = -i\partial_x$ and χ_B denotes the characteristic function of a set B.

Proof of the lemma is found in section 5 of [14], and we omit it here. We remark that when $H = H_0$ Lemma 2.1 holds with the sequence $t_k \to \infty$ replaced by $t \to \infty$ for any $f \in \mathcal{H} = L^2(\mathbb{R}^n)$ with $\langle x\rangle^2 f \in \mathcal{H}$.

Let a function $\rho(\lambda) \in C^\infty(\mathbb{R})$ satisfy the following.

$$0 \le \rho(\lambda) \le 1,$$
$$\rho(\lambda) = \begin{cases} 1 & (\lambda \le -1) \\ 0 & (\lambda \ge 0) \end{cases}$$
$$\rho'(\lambda) \le 0,$$
$$\rho(\lambda)^{\frac{1}{2}}, \quad |\rho'(\lambda)|^{\frac{1}{2}} \in C^\infty(\mathbb{R}).$$

Define for $\lambda \in \mathbb{R}$, R, $\varepsilon > 0$, $\theta > 0$

$$\phi_\varepsilon(\lambda < R) = \rho((\lambda - R)/\varepsilon),$$
$$\phi_\varepsilon(\lambda > R) = 1 - \phi_\varepsilon(\lambda < R),$$
$$\phi(\lambda < \theta) = \phi_\theta(\lambda < 2\theta)$$

and choose $\chi(\lambda) \in C^\infty(\mathbb{R})$ with $0 \le \chi(\lambda) \le 1$ such that

$$\chi(\lambda) = \begin{cases} 1 & (\lambda \in [a,b]), \\ 0 & (\lambda \le a/2 \text{ or } \ge 2b). \end{cases}$$

We now set

$$p^\varepsilon(x/t,\xi) = \phi(|x/t - \nabla_\xi H_0(\xi)|^2 < \varepsilon)\chi(H_0(\xi))^2. \tag{2.4}$$

We let $P^\varepsilon(t)$ be the pseudodifferential operator with symbol $p^\varepsilon(x/t,\xi)$.

For $f \in \mathcal{H}_c(a,b)$ satisfying $\langle x\rangle^2 f \in \mathcal{H}$, there is a sequence $t_k \to \infty$ $(k \to \infty)$ which satisfies the conditions of Lemma 2.1. In particular from the relation $\phi(0 < \varepsilon) = 1$, $\nabla_\xi H_0(\xi) = |\xi|^{\kappa-2}\xi$, (2.2), and (2.3) of Lemma 2.1, we have $\lim_{k\to\infty}(e^{-it_kH} f - \chi(H_0)e^{-it_kH} f) = 0$ and

$$\|e^{-it_kH} f - P^\varepsilon(t_k)e^{-it_kH} f\| \to 0 \quad (k \to \infty) \tag{2.5}$$

for $f = E_H([a,b])f \in \mathcal{H}_c(a,b)$ with $\langle x\rangle^2 f \in \mathcal{H}$. As J has a bounded inverse J^{-1} as we will remark at the end of section 3, the operators $e^{itH_0}J^{-1}P^\varepsilon(t)e^{-itH}$ form a uniformly bounded family with respect to $t \in \mathbb{R}$. Therefore to prove the existence of the limit (1.8) for $f \in \mathcal{H}_c$, it suffices to show the existence of the following limit

$$\lim_{k\to\infty} e^{it_kH_0}J^{-1}P^\varepsilon(t_k)e^{-it_kH}f \tag{2.6}$$

for $f \in \mathcal{H}_c(a,b)$ with $\langle x\rangle^2 f \in \mathcal{H}$.

3 Identification operator J

The identification operator J in (1.4) and (1.5) is a bounded operator from $\mathcal{H} = L^2(\mathbb{R}^n)$ into itself and is defined as follows as in section 4 of [14].

$$\begin{aligned} Jf(x) &= (2\pi)^{-n}\iint e^{i(\varphi(x,\xi)-y\xi)}f(y)dyd\xi \\ &= (2\pi)^{-n/2}\int e^{i\varphi(x,\xi)}\hat{f}(\xi)d\xi. \end{aligned} \tag{3.1}$$

Here the phase function $\varphi(x,\xi)$ is constructed as a solution of an eikonal equation for the Hamiltonian (1.1) and satisfies the following theorem (Theorem 4.4 of [14]).

Theorem 3.1. *Let $d_2 > d_1 > 0$ and $-1 < \sigma_- < \sigma_+ < 1$ be fixed. Then there is $R = R_{d_1,d_2,\sigma_\pm} > 1$ and a real-valued C^∞ function $\varphi(x,\xi)$ of $(x,\xi) \in \mathbb{R}^{2n}$ such that the following holds:*

i) For $d_2 \ge |\xi| \ge d_1$, $|x| \ge R$ and $\cos(x,\xi) \ge \sigma_+$ or $\cos(x,\xi) \le \sigma_-$

$$\kappa^{-1}|\nabla_x\varphi(x,\xi)|^\kappa + V_L(x) = \kappa^{-1}|\xi|^\kappa. \tag{3.2}$$

ii) For any multi-indices α,β there is a constant $C_{\alpha\beta} > 0$ such that for $d_2 \ge |\xi| \ge d_1$ and $x \in \mathbb{R}^n$

$$|\partial_x^\alpha\partial_\xi^\beta(\varphi(x,\xi) - x\cdot\xi)| \le C_{\alpha\beta}\langle x\rangle^{1-\delta-|\alpha|}\langle\xi\rangle^{1-\kappa}. \tag{3.3}$$

In particular for $|\alpha| \ne 0$, we have for $\delta_0, \delta_1 \ge 0$ with $\delta_0 + \delta_1 = \delta$

$$|\partial_x^\alpha\partial_\xi^\beta(\varphi(x,\xi) - x\cdot\xi)| \le C_{\alpha\beta}R^{-\delta_0}\langle x\rangle^{1-\delta_1-|\alpha|}\langle\xi\rangle^{1-\kappa}. \tag{3.4}$$

iii) Set for $f \in \mathcal{S}$

$$Tf(x) = (HJ - JH_0)f(x). \tag{3.5}$$

Then we have

$$Tf(x) = \iint e^{i(\varphi(x,\xi)-y\xi)}\{a(x,\xi) + V_S(x)\}f(y)dy\widehat{d\xi}. \tag{3.6}$$

Here

$$a(x,\xi) = \kappa^{-1}|\nabla_x\varphi(x,\xi)|^\kappa + V_L(x) - \kappa^{-1}|\xi|^\kappa + r(x,\xi), \tag{3.7}$$

where

$$r(x,\xi) = -i \iint e^{i(x-y)\eta} \nabla_y \cdot \left(\int_0^1 |\widetilde{\nabla}_x \varphi(x,\xi,y) + \theta\eta|^{\kappa-2} (\widetilde{\nabla}_x \varphi(x,\xi,y) + \theta\eta) d\theta \right) dy d\widehat{\eta}, \tag{3.8}$$

and

$$\widetilde{\nabla}_x \varphi(x,\xi,y) = \int_0^1 \nabla_x \varphi(y + \theta(x-y),\xi) d\theta.$$

The symbol $a(x,\xi)$ *satisfies for* $d_2 \geq |\xi| \geq d_1$, $|x| \geq R$ *and any* α,β

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq \begin{cases} C_{\alpha\beta} \langle x \rangle^{-1-\delta-|\alpha|} \langle \xi \rangle^{1-\kappa}, & \cos(x,\xi) \in [-1,\sigma_-] \cup [\sigma_+,1], \\ C_{\alpha\beta} \langle x \rangle^{-\delta-|\alpha|}, & \cos(x,\xi) \in [\sigma_-,\sigma_+]. \end{cases} \tag{3.9}$$

We remark that the factor $\langle \xi \rangle^{1-\kappa}$ in the bounds above is effective just in each region $d_1 \leq |\xi| \leq d_2$ and the constant $C_{\alpha\beta}$ depends on d_1 and d_2.

As stated above, J is defined for $f \in \mathcal{S}$

$$Jf(x) = \iint e^{i(\varphi(x,\xi)-y\xi)} f(y) dy d\widehat{\xi}.$$

Since the regions $d_2 \geq |\xi| \geq d_1$ of definition for the phase function $\varphi(x,\xi)$ are enlarged if we wait enough until late or early time t near $+\infty$ or $-\infty$, they in total cover the whole region $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. Thus J is regarded to have been defined on the whole Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$. When it is thought to be constructed in such a way, this J is known (Theorem 3.3 in [11]) to have a bounded inverse J^{-1}. Thus we can define $W_1(t)$ and $W_2(t)$ as follows:

$$W_1(t) = e^{itH} J e^{-itH_0}, \quad W_2(t) = e^{itH_0} J^{-1} e^{-itH}.$$

From ii) of Theorem 3.1 and the factor $\chi(H_0)^2$ in $P^\varepsilon(t)$, we have that the operator $(J^{-1} - J^*)P^\varepsilon(t)$ is compact. From (2.1) of Lemma 2.1, we have $\text{w-lim}_{k\to\pm\infty} e^{-it_k H} f = 0$. Thus to prove the existence of the limit (2.6) it suffices to prove the existence of the limit.

$$\lim_{k\to\infty} e^{it_k H_0} J^* P^\varepsilon(t_k) e^{-it_k H} f \quad (f \in \mathcal{H}_c(a,b) \text{ with } \langle x \rangle^2 f \in \mathcal{H}). \tag{3.10}$$

4 A proof of the asymptotic completeness

To prove the asymptotic completeness we have seen that it is sufficient to prove the existence of the limit (3.10) for $f \in \mathcal{H}_c(a,b)$ with $\langle x \rangle^2 f \in \mathcal{H}$ where $0 < a < b < \infty$ are fixed. We will prove a little bit strongly that the limit

$$\lim_{t\to\infty} e^{itH_0} J^* P^\varepsilon(t) e^{-itH} f \tag{4.1}$$

exists, owing to the introduction of the factor $P^\varepsilon(t)$ in (2.6).

As we will fix $\varepsilon > 0$ sufficiently small below, we will write $P^\varepsilon(t)$ just as $P(t)$. The necessary and sufficient condition for the limit (4.1) to exist is that when $\tau > \sigma \to \infty$

$$\|e^{i\tau H_0} J^* P(\tau) e^{-i\tau H} f - e^{i\sigma H_0} J^* P(\sigma) e^{-i\sigma H} f\| \to 0. \tag{4.2}$$

The norm is equal to the following by the fundamental theorem of calculus.

$$\left\|\int_\sigma^\tau \frac{d}{dt}(e^{itH_0}J^*P(t)e^{-itH}f)dt\right\|. \tag{4.3}$$

The integrand is equal to

$$\frac{d}{dt}(e^{itH_0}J^*P(t)e^{-itH}f) = e^{itH_0}\{-iT^*P(t) + iJ^*(V_S P(t) - P(t)V_S) \\ +iJ^*[V_L, P(t)] + J^*(i[H_0, P(t)] + \partial_t P(t))\}e^{-itH}f. \tag{4.4}$$

The term $-iT^*P(t) + iJ^*(V_S P(t) - P(t)V_S) + iJ^*[V_L, P(t)]$ on the right hand side is a compact operator and decays in the order $O(t^{-1-\delta})$ with respect to t as $t \to \infty$ by Theorem 3.1, iii) and the assumptions on V_S and V_L, and hence is integrable with respect to $t \geq 1$.

The remaining term $i[H_0, P(t)] + \partial_t P(t)$ is treated by the following lemma[2].

Lemma 4.1. Let $P^\varepsilon(t)$ $(t \geq 1)$ be the pseudodifferential operator with symbol $p^\varepsilon(x/t, \xi)$ defined by (2.4). Then there are operator valued functions $S(t)$ and $R(t)$ $(t \geq 1)$ continuous in uniform operator topology such that the following holds.

$$i[H_0, P^\varepsilon(t)] + \partial_t P^\varepsilon(t) = \frac{1}{t}S(t) + R(t). \tag{4.5}$$

Here $S(t)$ is a nonnegative selfadjoint operator and $R(t)$ satisfies the following estimate for some constant $C > 0$ independent of $t \geq 1$.

$$S(t) \geq 0, \quad \|R(t)\| \leq Ct^{-2}. \tag{4.6}$$

Proof It suffices to show the lemma for $\varepsilon = 1$. I.e. we assume that the symbol $p(x/t, \xi)$ of $P^\varepsilon(t)$ is of the following form.

$$p(x/t, \xi) = \phi(|x/t - \nabla_\xi H_0(\xi)|^2 < 1)\chi(H_0(\xi))^2 =: p_t(x, \xi). \tag{4.7}$$

We note that the symbol $p_t(x, \xi)$ satisfies for any multi-index α, β

$$\sup_{x,\xi \in \mathbb{R}^n} |\partial_x^\alpha \partial_\xi^\beta p_t(x, \xi)| \leq C_{\alpha\beta} t^{-|\alpha|}, \tag{4.8}$$

where constant $C_{\alpha\beta} > 0$ is independent of $t \geq 1$. By a direct computation we have the following.

$$(i[H_0, p_t(X, D)] + \partial_t p_t(X, D))f(x) \\ = (2\pi)^{-n/2} \int e^{ix\cdot\xi} \left\{\nabla_\xi H_0(\xi) \cdot \nabla_x p_t(x, \xi) + \partial_t p_t(x, \xi) + r_t(x, \xi)\right\} \hat{f}(\xi) d\xi, \tag{4.9}$$

where $r_t(x, \xi)$ satisfies

$$|\partial_x^\alpha \partial_\xi^\beta r_t(x, \xi)| \leq C_{\alpha\beta} t^{-2-|\alpha|}. \tag{4.10}$$

[2]The following lemma is an extension of Lemma 4.2 in [12].

In fact we compute for $f \in \mathcal{S}$ as follows.

$$\begin{aligned} i[H_0, p_t(X,D)]f(x) &= i \iint e^{i(x-y)\xi} H_0(\xi) \iint e^{i(y-z)\eta} p_t(y,\eta) f(z) dz d\widehat{\eta} dy d\widehat{\xi} \\ &\quad - i \iint e^{i(x-y)\xi} p_t(x,\xi) H_0(\xi) f(y) dy d\widehat{\xi} \\ &= i \iint e^{i(x-z)\xi} H_0(\xi) \left[\iint e^{i(y-z)(\eta-\xi)} p_t(y,\eta) dy d\widehat{\eta} - p_t(x,\xi) \right] f(z) dz d\widehat{\xi}. \end{aligned}$$

Noting that

$$\begin{aligned} \iint e^{i(y-z)(\eta-\xi)} p_t(y,\eta) dy d\widehat{\eta} &= \iint e^{iy\eta} p_t(z+y, \xi+\eta) dy d\widehat{\eta} \\ &= \iint e^{iy\eta} p_t(z+y,\xi) dy d\widehat{\eta} + \iint e^{iy\eta} \eta \cdot \int_0^1 \nabla_\xi p_\gamma(z+y, \xi+\theta\eta) d\theta dy d\widehat{\eta} \\ &= p_t(z,\xi) + \sum_{j=1}^n \iint e^{iy\eta} \eta_j \int_0^1 \partial_{\xi_j} p_t(z+y, \xi+\theta\eta) d\theta dy d\widehat{\eta}, \end{aligned}$$

we have

$$\begin{aligned} i[H_0, p_t(X,D)]f(x) &= i \iint e^{i(x-z)\xi} H_0(\xi) [p_t(z,\xi) - p_t(x,\xi)] f(z) dz d\widehat{\xi} \\ &\quad + i \iint e^{i(x-z)\xi} H_0(\xi) \sum_{j=1}^n \iint e^{iy\eta} \eta_j \int_0^1 \nabla_{\xi_j} p_t(z+y, \xi+\theta\eta) d\theta dy d\widehat{\eta} f(z) dz d\widehat{\xi}. \end{aligned}$$

By integration by parts we have the following.

$$\begin{aligned} & i[H_0, p_t(X,D)]f(x) \\ &= i \iint e^{i(x-z)\xi} H_0(\xi)(z-x) \cdot \int_0^1 \nabla_x p_t(x+\theta(z-x), \xi) d\theta f(z) dz d\widehat{\xi} \\ &\quad - \iint e^{i(x-z)\xi} H_0(\xi) \iint e^{iy\eta} \int_0^1 \sum_{j=1}^n \nabla_{y_j} \nabla_{\xi_j} p_t(z+y, \xi+\theta\eta) d\theta dy d\widehat{\eta} f(z) dz d\widehat{\xi} \\ &= i \int (-D_\xi)(e^{i(x-z)\xi}) \cdot \left[H_0(\xi) \int_0^1 \nabla_x p_t(x+\theta(z-x), \xi) d\theta \right] f(z) dz d\widehat{\xi} \\ &\quad - \iint e^{i(x-z)\xi} H_0(\xi) \iint e^{iy\eta} \int_0^1 \sum_{j=1}^n \nabla_{y_j} \nabla_{\xi_j} p_t(z+y, \xi+\theta\eta) d\theta dy d\widehat{\eta} f(z) dz d\widehat{\xi}. \end{aligned}$$

Further integration by parts gives

$$
\begin{aligned}
& i[H_0, p_t(X,D)]f(x) \\
&= \iint e^{i(x-z)\xi} \sum_{j=1}^{n} \nabla_{\xi_j} \left(H_0(\xi) \int_0^1 \nabla_{x_j} p_t(x+\theta(z-x),\xi)d\theta \right) f(z)dzd\widehat{\xi} \\
&\quad - \iint e^{i(x-z)\xi} H_0(\xi) \iint e^{iy\eta} \int_0^1 \sum_{j=1}^{n} \nabla_{y_j}\nabla_{\xi_j} p_t(z+y,\xi+\theta\eta)d\theta dy d\widehat{\eta} f(z)dzd\widehat{\xi} \\
&= \iint e^{i(x-z)\xi} \nabla_\xi H_0(\xi) \cdot \int_0^1 \nabla_x p_t(x+\theta(z-x),\xi)d\theta f(z)dzd\widehat{\xi} \\
&\quad + \iint e^{i(x-z)\xi} H_0(\xi) \int_0^1 \sum_{j=1}^{n} \nabla_{\xi_j}\nabla_{x_j} p_t(x+\theta(z-x),\xi)d\theta f(z)dzd\widehat{\xi} \\
&\quad - \iint e^{i(x-z)\xi} H_0(\xi) \iint e^{iy\eta} \int_0^1 \sum_{j=1}^{n} \nabla_{\xi_j}\nabla_{y_j} p_t(z+y,\xi+\theta\eta)d\theta dy d\widehat{\eta} f(z)dzd\widehat{\xi}.
\end{aligned}
$$

Noting that

$$
\int_0^1 \nabla_x p_t(x+\theta(z-x),\xi)d\theta = \nabla_x p_t(x,\xi) + \int_0^1 \int_0^1 \frac{d}{d\rho}\left(\nabla_x p_t(x+\rho\theta(z-x),\xi)\right) d\rho d\theta,
$$

$$
\begin{aligned}
& \int_0^1 \sum_{j=1}^{n} \nabla_{\xi_j}\nabla_{x_j} p_t(x+\theta(z-x),\xi)d\theta \\
&= \sum_{j=1}^{n} \nabla_{\xi_j}\nabla_{x_j} p_t(x,\xi) + \int_0^1 \int_0^1 \frac{d}{d\rho}\left(\sum_{j=1}^{n} \nabla_{\xi_j}\nabla_{x_j} p_t(x+\rho\theta(z-x),\xi) \right) d\rho d\theta
\end{aligned}
$$

and

$$
\begin{aligned}
& \iint e^{iy\eta} \int_0^1 \sum_{j=1}^{n} \nabla_{\xi_j}\nabla_{y_j} p_t(z+y,\xi+\theta\eta)d\theta dy d\widehat{\eta} \\
&= \sum_{j=1}^{n} \nabla_{\xi_j}\nabla_{x_j} p_t(z,\xi) + \iint e^{iy\eta} \int_0^1 \int_0^1 \frac{d}{d\rho}\left(\sum_{j=1}^{n} \nabla_{\xi_j}\nabla_{x_j} p_t(z+\rho y,\xi+\theta\eta) \right) d\rho d\theta dy d\widehat{\eta},
\end{aligned}
$$

we have

$$
\begin{aligned}
i[H_0, p_t(X,D)]f(x) &= \iint e^{i(x-z)\xi} \nabla_\xi H_0(\xi) \cdot \nabla_x p_t(x,\xi) f(z)dzd\widehat{\xi} \\
&\quad + \iint e^{i(x-z)\xi} H_0(\xi) \sum_{j=1}^{n} \left(\nabla_{\xi_j}\nabla_{x_j} p_t(x,\xi) - \nabla_{\xi_j}\nabla_{x_j} p_t(z,\xi) \right) f(z)dzd\widehat{\xi} \\
&\quad + r_t^1(X,D)f(x),
\end{aligned}
$$

where $r_t^1(x,\xi)$ satisfies (4.10). Hence we have

$$
\begin{aligned}
& (i[H_0, p_t(X,D)] + \partial_t p_t(X,D))f(x) \\
&= \iint e^{i(x-z)\xi} (\nabla_\xi H_0(\xi) \cdot \nabla_x p_t(x,\xi) + \partial_t p_t(x,\xi)) f(z)dzd\widehat{\xi} + r_t(X,D)f(x)
\end{aligned}
$$

for some function $r_t(x,\xi)$ which satisfies (4.10), which proves (4.9). The symbol of the first term is equal to

$$\begin{aligned}\nabla_\xi H_0(\xi)\cdot\nabla_x p_t(x,\xi)+\partial_t p_t(x,\xi) &= -\frac{2}{t}\phi'(|x/t-\nabla_\xi H_0(\xi)|^2<1)|x/t-\nabla_\xi H_0(\xi)|^2\chi(H_0(\xi))^2\\ &=: \frac{1}{t}u_t(x,\xi)\geq 0.\end{aligned}$$

Thus we have proved the following.

$$\begin{aligned}(i[H_0,P(t)]+\partial_t P(t))f(x) &= (i[H_0,p_t(X,D)]+\partial_t p_t(X,D))f(x)\\ &= \frac{1}{t}u_t(X,D)f(x)+L(t)f(x),\end{aligned}$$

where $L(t)$ satisfies

$$\|L(t)\|\leq Ct^{-2}\quad (t\geq 1).$$

As $u_t(x,\xi)\geq 0$ the function

$$q_t(x,\xi)=\sqrt{u_t(x,\xi)}.$$

is a C^∞ function of $x,\xi\in\mathbb{R}^n$ by our assumption $|\rho'(\lambda)|^{\frac{1}{2}}\in C^\infty(\mathbb{R})$ in section 2, and satisfies

$$\sup_{x,\xi\in\mathbb{R}^n}|\partial_x^\alpha\partial_\xi^\beta q_t(x,\xi)|\leq C_{\alpha\beta}t^{-|\alpha|}$$

for some constants $C_{\alpha\beta}>0$ for any multi-index α,β. Letting

$$Q(t)f(x)=q_t(X',D)f(x)=\iint e^{i(x-y)\xi}q_t(y,\xi)f(y)dyd\widehat{\xi}, \tag{4.11}$$

we set

$$S(t)=Q(t)^*Q(t)\geq 0.$$

Then we have

$$S(t)f(x)=s_t(X,D)f(x)=\iint e^{i(x-y)\xi}s_t(x,\xi)f(y)dyd\widehat{\xi},$$

where

$$s_t(x,\xi)=\iint e^{-iy\eta}q_t(x,\xi+\eta)q_t(x+y,\xi+\eta)dyd\widehat{\eta}.$$

The symbol $s_t(x,\xi)$ is expanded as follows.

$$s_t(x,\xi)=q_t(x,\xi)^2+s_t^1(x,\xi)=u_t(x,\xi)+s_t^1(x,\xi),$$

where $s_t^1(x,\xi)$ satisfies

$$|\partial_x^\alpha\partial_\xi^\beta s_t^1(x,\xi)|\leq C_{\alpha\beta}t^{-1-|\alpha|},$$

which yields

$$\|s_t^1(X,D)\|\leq Ct^{-1}\quad (t\geq 1).$$

Summing up we have proved that

$$i[H_0,P(t)]+\partial_t P(t)=\frac{1}{t}S(t)+R(t),$$

where $S(t)$ and $R(t) = -\frac{1}{t} s_t^1(X,D) + L(t)$ satisfy (4.6). □

To see the convergence (4.2) we note that the norm in (4.2) is equal to the following for $\tau > \sigma > 1$.

$$\sup_{\|g\|=1} |(e^{i\tau H_0} J^* P(\tau) e^{-\tau H} f - e^{i\sigma H_0} J^* P(\sigma) e^{-i\sigma H} f, g)|. \tag{4.12}$$

Calculating the inner product of this formula with using the fundamental theorem of calculus as in (4.3) and (4.4) and applying Lemma 4.1, we have

$$\begin{aligned}(e^{i\tau H_0} J^* P(\tau) e^{-i\tau H} f - e^{i\sigma H_0} J^* P(\sigma) e^{-i\sigma H} f, g)\\ = \int_\sigma^\tau \frac{1}{t}(Q(t)e^{-itH} f, Q(t) J e^{-itH_0} g) dt + \int_\sigma^\tau (M(t) f, g) dt,\end{aligned} \tag{4.13}$$

where $Q(t)$ is defined by (4.11) and $M(t)$ satisfies

$$\|M(t)\| \le C(1+|t|)^{-1-\delta}$$

for some constant $C > 0$.

By a similar computation we have

$$\begin{aligned}(e^{i\tau H_0} J^* P(\tau) J e^{-i\tau H_0} g - e^{i\sigma H_0} J^* P(\sigma) J e^{-i\sigma H_0} g, g)\\ = \int_\sigma^\tau \frac{1}{t} \|Q(t) J e^{-itH_0} g\|^2 dt + \int_\sigma^\tau (M_0(t) g, g) dt\end{aligned}$$

and

$$\begin{aligned}(e^{i\tau H} P(\tau) e^{-i\tau H} f - e^{i\sigma H} P(\sigma) e^{-i\sigma H} f, f)\\ = \int_\sigma^\tau \frac{1}{t} \|Q(t) e^{-itH} f\|^2 dt + \int_\sigma^\tau (M_1(t) f, f) dt,\end{aligned}$$

where $M_j(t)$ satisfies for some constant $C_j > 0$ $(j = 0, 1)$

$$\|M_j(t)\| \le C_j (1+|t|)^{-1-\delta}.$$

The left hand sides of these two inequalities are bounded by $C_0' \|g\|^2$, $C_1' \|f\|^2$ respectively for some constants $C_j' > 0$ $(j = 0, 1)$. Therefore we can find constants $M_0, M_1 > 0$ such that the following holds for any $\tau > \sigma > 1$.

$$\int_\sigma^\tau \frac{1}{t} \|Q(t) J e^{-itH_0} g\|^2 dt \le M_0^2 \|g\|^2, \tag{4.14}$$

$$\int_\sigma^\tau \frac{1}{t} \|Q(t) e^{-itH} f\|^2 dt \le M_1^2 \|f\|^2. \tag{4.15}$$

From (4.13), (4.14), (4.15) we obtain with using Schwarz inequality

$$\begin{aligned}&|(e^{i\tau H_0} J^* P(\tau) e^{-i\tau H} f - e^{i\sigma H_0} J^* P(\sigma) e^{-i\sigma H} f, g)|\\ &\le M_0 \left(\int_\sigma^\tau \frac{1}{t} \|Q(t) e^{-itH} f\|^2 dt \right)^{\frac{1}{2}} \|g\| + C(1+|\sigma|)^{-\delta} \|f\| \|g\|.\end{aligned}$$

Therefore together with (4.12) we have that the norm in (4.2) is estimated as follows.

$$\begin{aligned}&||e^{i\tau H_0}J^*P(\tau)e^{-i\tau H}f - e^{i\sigma H_0}J^*P(\sigma)e^{-i\sigma H}f||\\ &\leq M_0\left(\int_\sigma^\tau \frac{1}{t}||Q(t)e^{-itH}f||^2dt\right)^{\frac{1}{2}} + C(1+|\sigma|)^{-\delta}||f||.\end{aligned}$$

The inequality (4.15) yields that the right hand side converges to 0 as $\tau > \sigma \to \infty$. This proves (4.2) and the proof of the asymptotic completeness is complete.

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