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On the Stability and Stabilization of Parameter Dependent Perturbed Systems

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Abstract

We treat in this paper the problem of stability of a class of parameter dependent perturbed systems. Furthermore, the problem of stabilization using an estimated state feedback is developed. Under some sufficient conditions, we construct, first, an observer which provides an estimation of the state, then we consider the system in closed loop by the state estimated feedback. The results of this paper are illustrated by numerical examples.

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1 Introduction

Lyapunov stability of linear time-varying systems and applications to control theory have received considerable attention ([3], [7], [12], [17]). Parameter dependent systems ([8], [10], [14], [16]) are now established as one of important representations of these classes of systems . The use of Lyapunov functions is certainly the main tool for solving the stability problem ([1], [5], [6], [11], [13]). In order to provide less conservative results, parameter

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dependent Lyapunov functions have recently been employed and several techniques involving these functions have been proposed for the stability and stabilization ([2], [4], [16]). Moreover, observers have been a topic of interest and variety of methods has been developed for constructing nonlinear observers for some classes of systems ([9], [15], [18]).
In fact, since there is no way that we can measure the hole state x of a dynamical system and what we can really measure is a part of the system, then the problem of state estimate will be investigated and consequently we get an estimation $\hat{x}$ of x.

In this paper, we begin with the stability problem for a perturbed time-varying polytopic system. We continue with the problems of stabilization and the conception of a global exponential observer for the same type of systems. Finally, we establish a separation principle (stabilization by an estimated state feedback given by an observer). This investigation is done through parameter dependent Lyapunov functions and Lyapunov matrix inequalities (L_yMIs) conditions. Numerical examples in dimension two illustrating the results are given.

Consider the nonautonomous system

$$\dot{x} = g(t,x), \tag{1.1}$$

where $g : [0,\infty) \times D \longrightarrow \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $[0,\infty) \times D$, and $D \subset \mathbb{R}^n$ is a domain that contains the origin $x = 0$. The origin is an equilibrium point for (1.1), if

$$g(t,0) = 0, \quad \forall t \geq 0.$$

Definition 1.1. (Exponential stability) The equilibrium point $x = 0$ of (1.1) is exponentially stable if there exist positive constants c, k, and λ such that

$$\|x(t)\| \leq k\|x(t_0)\|e^{-\lambda(t-t_0)}, \quad \forall \ \|x(t_0)\| < c, \tag{1.2}$$

and it is globally exponentially stable if (1.2) is satisfied for any initial state $x(t_0)$.

Definition 1.2. (Convergence to a neighborhood) System (1.1) is said globally uniformly exponentially convergent to the following neighborhood

$$v = \{x \in \mathbb{R}^n, \|x\| \leq \eta\},$$

if there exist λ_1, λ_2 and η such that

$$\|x(t)\| \leq \lambda_1 \|x(t_0)\| \exp(-\lambda_2(t-t_0)) + \eta, \quad \forall t \geq t_0. \tag{1.3}$$

Definition 1.3. (Stabilization) An equilibrium point x^* of a dynamical system

$$\dot{x} = f(x,u),$$

with f a smooth function, x in $\mathbb{R}^n$ and u in $\mathbb{R}$ is said to be stabilizable if there exists a smooth function $\tilde{u}$ such that x^* is a globally asymptotically stable equilibrium point of $\dot{x} = f(x,\tilde{u}(x))$.

Definition 1.4. (Exponential observer) We consider the system

$$\begin{array}{rcl} \dot{x} & = & F(t,x,u) \\ y & = & Cx, \end{array} \tag{1.4}$$

where $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$.
The function $F : [0,+\infty[\times \mathbb{R}^n \times \mathbb{R}^p \longrightarrow \mathbb{R}^n$ is piecewise continuous in t and globally Lipschitz in x on $[0,+\infty] \times \mathbb{R}^n$ and C is a constant matrix $(q \times n)$.
An exponential observer for system (1.4) is a dynamical system which has the following form:

$$\dot{\hat{x}} = F(t,\hat{x},u) - L(C\hat{x} - y), \tag{1.5}$$

where L is the gain matrix and the origin of the error equation with $e = \hat{x} - x$, given by

$$\dot{e} = F(t,\hat{x},u) - F(t,x,u) - LCe, \tag{1.6}$$

is globally exponentially stable. It means that, there exist positive constants c, k, and λ, such that

$$\|e(t)\| \leq k\|e(t_0)\|e^{-\lambda(t-t_0)}, \quad \forall\, t \geq t_0. \tag{1.7}$$

2 Stability

Consider the system

$$\dot{x}(t) = A(\alpha(t))x(t) + f(t,\alpha(t),x). \tag{2.1}$$

The matrix $A(\alpha(t)) \in \mathbb{R}^{n\times n}$ is defined as

$$A(\alpha(t)) = \alpha_1(t)A_1 + \alpha_2(t)A_2, \tag{2.2}$$

where $\alpha_i(t), i = 1,2$ are continuous functions such that $\alpha_i(t) \geq 0,\ \alpha_1(t) + \alpha_2(t) = 1$, A_1, $A_2 \in \mathbb{R}^{n\times n}$ are constant matrices and $|\dot{\alpha}_1(t)| \leq \rho_1$ with $\rho_1 \in \mathbb{R}_+$.
The function $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the perturbation of the nominal system

$$\dot{x}(t) = A(\alpha(t))x(t), \tag{2.3}$$

which could result in general from modeling errors, aging of parameters, uncertainties or disturbances.

Consider the system (2.1)-(2.2), suppose that for all $t \geq 0,\ x \in \mathbb{R}^n$, there exist $\varepsilon > 0$ and $k > 0$, such that

$$(\mathcal{H}_1) \qquad \|f(t,\alpha(t),x)\| \leq k\|x\| + \varepsilon.$$

Let $V(x,\alpha) = x^T P(\alpha)x$, $P(\alpha) = P^T(\alpha) > 0$, a Lyapunov function candidate for (2.1). The time derivative of V along the trajectories of perturbed system is given by

$$\dot{V}(x,\alpha) = x^T[P(\alpha)A(\alpha) + A^T(\alpha)P(\alpha) + \dot{P}(\alpha)]x + 2x^T P(\alpha)f(t,\alpha(t),x).$$

Using Lyapunov matrix inequalities and by imposing some particular conditions on the nominal part of system (2.1), we will give some classes of perturbed dependent systems which can be globally uniformly exponentially convergent to a neighborhood of the origin.

Theorem 2.1. *Suppose that, for strict positive reals l_1, l_2, l_3, l_4 and a given parameter $\rho_4 \in \mathbb{R}^+$, there exist symmetric positive definite matrices $P_1 \in \mathbb{R}^{n\times n}$, $P_{12} \in \mathbb{R}^{n\times n}$, $P_2 \in \mathbb{R}^{n\times n}$ such that $(P_1 - P_2)$, $(2P_1 - P_{12})$, $(P_{12} - 2P_2)$ and $(P_1 - P_{12} + P_2)$ are symmetric positive definite matrices with $\lambda_{\min}(P_1) > \lambda_{\min}(P_2)$, $\lambda_{\min}(P_{12}) > 2\lambda_{\min}(P_2)$, that satisfying*

$$(A_1^T + \frac{l_1}{2}I)P_1 + P_1(A_1 + \frac{l_1}{2}I) + l_1(P_2 - P_{12}) + \rho_4(2P_1 - P_{12}) < 0 \tag{2.4}$$

$$\begin{aligned}(A_1^T - \frac{3}{2}l_2 I)P_{12} + P_{12}(A_1 - \frac{3}{2}l_2 I) \quad &+ \quad (A_2^T + \frac{3}{2}l_2 I)P_1 + P_1(A_2 + \frac{3}{2}l_2 I) + 3l_2P_2 \\ &+ \quad \rho_4(4P_1 - P_{12} - 2P_2) < 0\end{aligned} \tag{2.5}$$

$$\begin{aligned}(A_1^T + \frac{3}{2}l_3 I)P_2 + P_2(A_1 + \frac{3}{2}l_3 I) \quad &+ \quad (A_2^T - \frac{3}{2}l_3 I)P_{12} + P_{12}(A_2 - \frac{3}{2}l_3 I) + 3l_3P_1 \\ &+ \quad \rho_4(2P_1 + P_{12} - 4P_2) < 0\end{aligned} \tag{2.6}$$

and

$$(A_2^T + \frac{l_4}{2}I)P_2 + P_2(A_2 + \frac{l_4}{2}I) + l_4(P_1 - P_{12}) + \rho_4(P_{12} - 2P_2) < 0 \tag{2.7}$$

where ρ_4 satisfies $|\dot{\alpha}_1(t)| \le \rho_4$. Assume that $(\mathcal{H}_1)$ holds with

$$k < \frac{l}{2}\,\frac{\lambda_{\min}(P_1 - P_{12} + P_2)}{\lambda_{\max}(P_1) + \lambda_{\max}(P_{12}) + \lambda_{\max}(P_2)}, \qquad l = \inf(l_1, l_2, l_3, l_4). \tag{2.8}$$

Then, the solutions of system (2.1) converge globally uniformly exponentially to the following neighborhood of the origin

$$v_1 = \{x \in \mathbb{R}^n, \|x\| \le \eta_1\} \tag{2.9}$$

where

$$\eta_1 = \frac{2\,\varepsilon(\lambda_{\max}(P_1) + \lambda_{\max}(P_{12}) + \lambda_{\max}(P_2))^2}{\lambda_{\min}(P_2)(l\lambda_{\min}(P_1 - P_{12} + P_2) - 2k(\lambda_{\max}(P_1) + \lambda_{\max}(P_{12}) + \lambda_{\max}(P_2)))}.$$

Proof. Let $P(\alpha) = \alpha_1^2 P_1 + \alpha_1\alpha_2 P_{12} + \alpha_2^2 P_2$. On the one hand, $P(\alpha)$ satisfies

$$\begin{aligned} P(\alpha) \quad &\ge \quad \alpha_1^2\lambda_{\min}(P_1)I + \alpha_1\alpha_2\lambda_{\min}(P_{12})I + \alpha_2^2\lambda_{\min}(P_2)I \\ &\ge \quad \alpha_1^2\lambda_{\min}(P_1)I + \alpha_1\alpha_2\lambda_{\min}(P_{12})I + (1 - 2\alpha_1\alpha_2 - \alpha_1^2)\lambda_{\min}(P_2)I \\ &\ge \quad \alpha_1^2(\lambda_{\min}(P_1) - \lambda_{\min}(P_2))I + \alpha_1\alpha_2(\lambda_{\min}(P_{12}) - 2\lambda_{\min}(P_2))I + \lambda_{\min}(P_2)I \\ &\ge \quad \lambda_{\min}(P_2)I, \end{aligned}$$

and the time derivative of $P(\alpha)$ is given by

$$\dot{P}(\alpha) \quad = \quad 2\alpha_1\dot{\alpha}_1 P_1 + \dot{\alpha}_1\alpha_2 P_{12} - \alpha_1\dot{\alpha}_1 P_{12} - 2\alpha_2\dot{\alpha}_1 P_2.$$

Multiplying the last equality by $(\alpha_1+\alpha_2)^2$ which equal to 1, one gets

$$\begin{aligned}\dot P(\alpha) &= \alpha_1^3(\dot\alpha_1(2P_1-P_{12}))+\alpha_1^2\alpha_2(\dot\alpha_1(4P_1-P_{12}-2P_2))\\ &+ \alpha_1\alpha_2^2(\dot\alpha_1(2P_1+P_{12}-4P_2))+\alpha_2^3(\dot\alpha_1(P_{12}-2P_2)).\end{aligned}$$

On the other hand, the time-derivative of V along the trajectories of system (2.1) is given by

$$\begin{aligned}\dot V(x,\alpha) &= x^T(\ \alpha_1^3(A_1^TP_1+P_1A_1+\dot\alpha_1(2P_1-P_{12}))\\ &+ \alpha_1^2\alpha_2(A_1^TP_{12}+P_{12}A_1+A_2^TP_1+P_1A_2+\dot\alpha_1(4P_1-P_{12}-2P_2))\\ &+ \alpha_2^2\alpha_1(A_1^TP_2+P_2A_1+A_2^TP_{12}+P_{12}A_2+\dot\alpha_1(2P_1+P_{12}-4P_2))\\ &+ \alpha_2^3(A_2^TP_2+P_2A_2+\dot\alpha_1(P_{12}-2P_2))\ )x\\ &+ 2x^T(\alpha_1^2P_1+\alpha_1\alpha_2P_{12}+\alpha_2^2P_2)f(t,\alpha(t),x).\end{aligned}$$

Since $|\dot\alpha_1|\le\rho_4$, we have

$$\begin{aligned}\dot V(x,\alpha) &\le x^T(\ -l_1\alpha_1^3(P_1-P_{12}+P_2)-3l_2\alpha_1^2\alpha_2(P_1-P_{12}+P_2)\\ &- 3l_3\alpha_1\alpha_2^2(P_1-P_{12}+P_2)-l_4\alpha_2^3(P_1-P_{12}+P_2)\ )x\\ &+ 2(\|\alpha_1^2P_1+\alpha_1\alpha_2P_{12}+\alpha_2^2P_2\|)\ (k\|x\|+\varepsilon)\\ &\le -(\ l\lambda_{\min}(P_1-P_{12}+P_2)-2k(\lambda_{\max}(P_1)+\lambda_{\max}(P_{12})\\ &+ \lambda_{\max}(P_2))\ )\ \|x\|^2+2\,\varepsilon(\lambda_{\max}(P_1)+\lambda_{\max}(P_{12})+\lambda_{\max}(P_2))\|x\|\\ &\le -\frac{l\lambda_{\min}(P_1-P_{12}+P_2)-2k(\lambda_{\max}(P_1)+\lambda_{\max}(P_{12})+\lambda_{\max}(P_2))}{\lambda_{\max}(P_1)+\lambda_{\max}(P_{12})+\lambda_{\max}(P_2)}V(x,\alpha)\\ &+ \frac{2\,\varepsilon(\lambda_{\max}(P_1)+\lambda_{\max}(P_{12})+\lambda_{\max}(P_2))}{\sqrt{\lambda_{\min}(P_2)}}\sqrt{V(x,\alpha)}.\end{aligned}$$

Let

$$v(t)=\sqrt{V(x,\alpha)},$$

hence

$$\begin{aligned}\dot v(t) &\le -\frac{l\lambda_{\min}(P_1-P_{12}+P_2)-2k(\lambda_{\max}(P_1)+\lambda_{\max}(P_{12})+\lambda_{\max}(P_2))}{2(\lambda_{\max}(P_1)+\lambda_{\max}(P_{12})+\lambda_{\max}(P_2))}v(t)\\ &+ \frac{\varepsilon(\lambda_{\max}(P_1)+\lambda_{\max}(P_{12})+\lambda_{\max}(P_2))}{\sqrt{\lambda_{\min}(P_2)}}.\end{aligned}$$

Integrating between t_0 and t, one obtains

$$\begin{aligned}v(t) &\le v(t_0)e^{-\xi(t-t_0)}\\ &+ \frac{2\,\varepsilon(\lambda_{\max}(P_1)+\lambda_{\max}(P_{12})+\lambda_{\max}(P_2))^2}{\sqrt{\lambda_{\min}(P_2)}(l\lambda_{\min}(P_1-P_{12}+P_2)-2k(\lambda_{\max}(P_1)+\lambda_{\max}(P_{12})+\lambda_{\max}(P_2)))},\end{aligned}$$

where

$$\xi=\frac{l\lambda_{\min}(P_1-P_{12}+P_2)-2k(\lambda_{\max}(P_1)+\lambda_{\max}(P_{12})+\lambda_{\max}(P_2))}{2(\lambda_{\max}(P_1)+\lambda_{\max}(P_{12})+\lambda_{\max}(P_2))},$$

which implies that

$$\|x(t)\| \leq \sqrt{\frac{\lambda_{\max}(P_1)+\lambda_{\max}(P_{12})+\lambda_{\max}(P_2)}{\lambda_{\min}(P_2)}}\|x(t_0)\|e^{-\xi(t-t_0)}$$

$$+ \quad \frac{2\,\varepsilon(\lambda_{\max}(P_1)+\lambda_{\max}(P_{12})+\lambda_{\max}(P_2))^2}{\lambda_{\min}(P_2)(l\lambda_{\min}(P_1-P_{12}+P_2)-2k(\lambda_{\max}(P_1)+\lambda_{\max}(P_{12})+\lambda_{\max}(P_2)))}.$$

Then, the solutions of system (2.1) converge globally uniformly exponentially to v_1 given in (2.9). □

Remark We can use the same argument to prove the previous theorem by taking not four L_yMIs but just two or three. In fact, in the case of two L_yMIs

$$(A_1^T+\frac{l_1}{2}I)P+P(A_1+\frac{l_1}{2}I)<0 \tag{2.10}$$

$$(A_2^T+\frac{l_2}{2}I)P+P(A_2+\frac{l_2}{2}I)<0 \tag{2.11}$$

when $P(\alpha)=P$ and $(\mathcal{H}_1)$ holds with

$$k<\frac{l}{2}\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}, \qquad l=\inf(l_1,l_2), \tag{2.12}$$

where $l_1\in\mathbb{R}_+^*$ and $l_2\in\mathbb{R}_+^*$, solutions of system (2.1) converge globally uniformly exponentially to the following neighborhood of the origin

$$v_2 \quad = \quad \left\{x\in\mathbb{R}^n, \|x\|\leq\frac{2\,\varepsilon\lambda_{\max}^2(P)}{\lambda_{\min}(P)(l\lambda_{\min}(P)-2k\lambda_{\max}(P))}\right\}.$$

Now, the use of these three L_yMIs

$$(A_1^T+\frac{l_1}{2}I)P_1+P_1(A_1+\frac{l_1}{2}I)-l_1P_2+\rho_3(P_1-P_2)<0 \tag{2.13}$$

$$(A_2^T-\frac{l_2}{2}I)P_2+P_2(A_2-\frac{l_2}{2}I)+l_2P_1+\rho_3(P_1-P_2)<0 \tag{2.14}$$

and

$$(A_1^T-l_3I)P_2+P_2(A_1-l_3I)+(A_2^T+l_3I)P_1+P_1(A_2+l_3I)+2\rho_3(P_1-P_2)<0, \tag{2.15}$$

with $P(\alpha)=\alpha_1(t)P_1+\alpha_1(t)P_1$, l_1, l_2, l_3 are strict positive reals, ρ_3 satisfies
$|\dot{\alpha}_1(t)|\leq\rho_3$ and for $(\mathcal{H}_1)$ holding with

$$k<\frac{l}{2}\,\frac{\lambda_{\min}(P_1-P_2)}{\lambda_{\max}(P_1)+\lambda_{\max}(P_2)}, \qquad l=\inf(l_1,l_2,l_3), \tag{2.16}$$

ensures that solutions of system (2.1) converge globally uniformly exponentially to the following neighborhood of the origin

$$v_3 \quad = \quad \left\{x\in\mathbb{R}^n, \|x\|\leq\frac{2\,\varepsilon(\lambda_{\max}(P_1)+\lambda_{\max}(P_2))^2}{\lambda_{\min}(P_2)(l\lambda_{\min}(P_1-P_2)-2k(\lambda_{\max}(P_1)+\lambda_{\max}(P_2)))}\right\}.$$

Note that, if $\varepsilon\longrightarrow 0$, then the trajectories of system (2.1) converge globally uniformly exponentially to the origin when $t\longrightarrow+\infty$.

Example 2.2.

Consider the system

$$\dot{x}(t) = (\alpha_1(t)A_1 + \alpha_2(t)A_2)x(t) + \frac{kx}{1+t^2}(\alpha_1(t), \alpha_2(t)) + \varepsilon,$$

where $k = 5.\ 10^{-3}$, $\varepsilon = 10^{-4}$ and matrices A_1, A_2 are given by

$$A_1 = \begin{pmatrix} 2 & -6 \\ \\ 140 & -11 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} -75 & 3 \\ \\ 4 & -1 \end{pmatrix}.$$

It is clear that

$$\|f(t,\alpha(t),x)\|_2 \leq k\|x\|_2 + \varepsilon.$$

For $\rho_4 = 0.009$, $l_1 = 20$, $l_2 = 20$, $l_3 = 20.8$ and $l_4 = 20.6$, there exist symmetric positive definite matrices P_1, P_{12}, P_2 and $(P_1 - P_{12} + P_2)$ that satisfy (2.4)-(2.5)-(2.6)-(2.7). These solutions are given by

$$P_1 = \begin{pmatrix} 0.4229 & -0.0210 \\ \\ -0.0210 & 0.0202 \end{pmatrix}, \qquad P_{12} = \begin{pmatrix} 0.3161 & -0.0148 \\ \\ -0.0148 & 0.0343 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 0.0360 & -0.0010 \\ \\ -0.0010 & 0.0156 \end{pmatrix} \quad \text{and} \quad P_1 - P_{12} + P_2 = \begin{pmatrix} 0.1428 & -0.0072 \\ \\ -0.0072 & 0.0016 \end{pmatrix}.$$

Moreover, we have

$$\frac{l}{2}\frac{\lambda_{\min}(P_1 - P_{12} + P_2)}{\lambda_{\max}(P_1) + \lambda_{\max}(P_{12}) + \lambda_{\max}(P_2)} = 15.4 \times 10^{-3}.$$

When we consider (2.10)-(2.11), for $l_1 = 1.3$ and $l_2 = 0.6$, there exists a symmetric positive definite matrix P given by

$$P = \begin{pmatrix} 0.1083 & -0.0046 \\ \\ -0.0046 & 0.0048 \end{pmatrix},$$

and has the eigenvalues $\lambda_{\min}(P) = 0.0046$ and $\lambda_{\max}(P) = 0.1085$, which implies that

$$\frac{1}{2}\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)} = 12.7 \times 10^{-3}.$$

Moreover, the use of (2.13)-(2.14)-(2.15) for $\rho_3 = 2.8$, $l_1 = 28$, $l_2 = 20$ and $l_3 = 37$ ensures the existence of two symmetric positive definite matrices P_1, P_2 given by

$$P_1 = \begin{pmatrix} 0.4562 & -0.0242 \\ \\ -0.0242 & 0.0208 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 0.3489 & -0.0166 \\ \\ -0.0166 & 0.0192 \end{pmatrix},$$

with

$$P_1 - P_2 = \begin{pmatrix} 0.1073 & -0.0076 \\ -0.0076 & 0.0017 \end{pmatrix}.$$

Hence, one has

$$\frac{l}{2}\frac{\lambda_{\min}(P_1 - P_2)}{\lambda_{\max}(P_1) + \lambda_{\max}(P_2)} = 13.6 \times 10^{-3}.$$

Then, we get

$$k < \frac{1}{2}\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)} < \frac{l}{2}\frac{\lambda_{\min}(P_1 - P_2)}{\lambda_{\max}(P_1) + \lambda_{\max}(P_2)} < \frac{l}{2}\frac{\lambda_{\min}(P_1 - P_{12} + P_2)}{\lambda_{\max}(P_1) + \lambda_{\max}(P_{12}) + \lambda_{\max}(P_2)},$$

which shows the robustness of the algorithm.

3 Stabilization

Consider the non linear perturbed system

$$\begin{array}{rcl} \dot{x}(t) & = & A(\alpha(t))x(t) + B(\alpha(t))u(t) + f(t, \alpha(t), x) \\ y & = & Cx(t), \end{array} \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ is the control and $y(t) \in \mathbb{R}^q$ is the output, matrices $A(\alpha(t))$, $B(\alpha(t))$ belong to the convex envelope $\mathcal{D}$ defined by

$$\mathcal{D} = \left\{ \sum_{i=1}^{m} \alpha_i(t) D_i,\ \alpha_i(t) \in \mathbb{R}_+ \ and\ \sum_{i=1}^{m} \alpha_i(t) = 1 \right\},$$

where D_i are constant matrices and the function $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the perturbation such that $f(t, \alpha(t), 0) = 0$ for all $t \geq 0$ and

$(\mathcal{H})$ $\quad \|f(t, \alpha(t), x) - f(t, \alpha(t), y)\| \leq \theta_i \|x - y\|,\ \forall x, y \in \mathbb{R}^n,\ \forall t \geq 0,$

and $\theta_i > 0,\ for\ i = 1, 2, 3.$

Note that the nominal system of (3.1) in closed-loop with the linear feedback $u(x) = Kx$ is globally uniformly exponentially stable. In fact, it's sufficient to take the case where $A(\alpha(t)) = \alpha_1(t)A_1 + \alpha_2(t)A_2$, $B(\alpha(t)) = \alpha_1(t)B_1 + \alpha_2(t)B_2$ such that the pairs (A_1, B_1) and (A_2, B_2) are controllable, which implies the existence of two constant matrices $K_1 \in \mathbb{R}^{p\times n}$, $K_2 \in \mathbb{R}^{p\times n}$, and a symmetric positive definite matrix $Q \in \mathbb{R}^{n\times n}$, such that

$$Q(A_1 + B_1K_1) + (A_1 + B_1K_1)^T Q < 0$$

$$Q(A_2 + B_2K_2) + (A_2 + B_2K_2)^T Q < 0.$$

We suppose also that L_yMIs

$$Q(A_1 + B_1K) + (A_1 + B_1K)^T Q < 0$$

$$Q(A_2 + B_2K) + (A_2 + B_2K)^T Q < 0$$

are satisfied with $K = K_1 + K_2$.

Now, we consider system (3.1) with

$$A(\alpha(t)) = \alpha_1^2(t)A_1 + 2\alpha_1(t)\alpha_2(t)A_{12} + \alpha_2^2(t)A_2 \tag{3.2}$$

and

$$B(\alpha(t)) = \alpha_1^2(t)B_1 + 2\alpha_1(t)\alpha_2(t)B_{12} + \alpha_2^2(t)B_2. \tag{3.3}$$

Assume that

$(\mathcal{H}_1)$ The pairs (A_1, B_1), (A_{12}, B_{12}) and (A_2, B_2) are controllable, then there exist constant matrices $K_1 \in \mathbb{R}^{p\times n}$, $K_{12} \in \mathbb{R}^{p\times n}$, $K_2 \in \mathbb{R}^{p\times n}$ and a symmetric positive definite matrix Q_3 such that

$$Q_3(A_1 + B_1K_1) + (A_1 + B_1K_1)^T Q_3 < 0, \tag{3.4}$$

$$Q_3(A_{12} + B_{12}K_{12}) + (A_{12} + B_{12}K_{12})^T Q_3 < 0, \tag{3.5}$$

and

$$Q_3(A_2 + B_2K_2) + (A_2 + B_2K_2)^T Q_3 < 0. \tag{3.6}$$

$(\mathcal{H}_2)$ The following L_yMIs are satisfied

$$Q_3(A_1 + B_1K) + (A_1 + B_1K)^T Q_3 < -l_1Q_3, \tag{3.7}$$

$$Q_3(A_{12} + B_{12}K) + (A_{12} + B_{12}K)^T Q_3 < -l_{12}Q_3, \tag{3.8}$$

and

$$Q_3(A_2 + B_2K) + (A_2 + B_2K)^T Q_3 < -l_2Q_3, \tag{3.9}$$

where l_1, l_{12} and l_2 are strict positive reals and $K = K_1 + K_{12} + K_2$.

Theorem 3.1. *Assume that $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ are satisfied. Moreover, if $(\mathcal{H})$ holds with*

$$\theta_3 < \frac{l\lambda_{\min}(Q_3)}{2\lambda_{\max}(Q_3)} \; for \; l = \inf\{l_1, l_{12}, l_2\},$$

then, the system (3.1) in closed-loop with the linear feedback $u(x) = Kx$ is globally exponentially stable.

Proof. Let $V(t,x) = x^T Q_3 x$ a Lyapunov function candidate. The time derivative of V along the trajectories of system (3.1) is given by

$$\begin{aligned}
\dot V(t,x) &= x^T \big(\alpha_1^2(t)(Q_3(A_1 + B_1K) + (A_1 + B_1K)^T Q_3) \\
&+ 2\alpha_1(t)\alpha_2(t)(Q_3(A_{12} + B_{12}K) + (A_{12} + B_{12}K)^T Q_3) \\
&+ \alpha_2^2(t)(Q_3(A_2 + B_2K) + (A_2 + B_2K)^T Q_3)\big)x + 2x^T Q_3 f(t,\alpha(t),x) \\
&\le -(l\lambda_{\min}(Q_3) - 2\,\theta_3\lambda_{\max}(Q_3))\|x\|^2 \\
&\le -\frac{l\lambda_{\min}(Q_3) - 2\,\theta_3\lambda_{\max}(Q_3)}{\lambda_{\max}(Q_3)} V(t,x),
\end{aligned}$$

which implies that the system (3.1) in closed-loop with the linear feedback $u(x) = Kx$ is globally uniformly exponentially stable. □

Remark We can study the stabilization using just two or three Lyapunov matrix inequalities as we did in the previous section. In fact, when we consider constant matrices, it is sufficient to suppose that the pair (A,B) is controllable, so there exist a constant matrix $K \in \mathbb{R}^{p\times n}$ and a symmetric positive definite matrix Q_1 such that

$$Q_1(A+BK)+(A+BK)^T Q_1 < 0.$$

Also, if the L_yMI

$$Q_1(A+BK)+(A+BK)^T Q_1 < -lQ_1$$

is satisfied where l is a strict positive real and the assumption $(\mathcal{H})$ holds with

$$\theta_1 < \frac{l\lambda_{\min}(Q_1)}{2\lambda_{\max}(Q_1)},$$

we obtain the global uniform exponential stability of system (3.1) in closed-loop with the linear feedback $u(x) = Kx$.

Moreover, if we consider system (3.1) with $A(\alpha(t)) = \alpha_1(t)A_1 + \alpha_2(t)A_2$ and $B(\alpha(t)) = \alpha_1(t)B_1 + \alpha_2(t)B_2$ such that the pairs (A_1,B_1) and (A_2,B_2) are controllable, then there exist constant matrices $K_1 \in \mathbb{R}^{p\times n}$, $K_2 \in \mathbb{R}^{p\times n}$ and a symmetric positive definite matrix Q_2 satisfying

$$Q_2(A_1+B_1K_1)+(A_1+B_1K_1)^T Q_2 < 0, \tag{3.10}$$

$$Q_2(A_2+B_2K_2)+(A_2+B_2K_2)^T Q_2 < 0. \tag{3.11}$$

If the following L_yMIs are satisfied

$$Q_2(A_1+B_1K)+(A_1+B_1K)^T Q_2 < -l_1Q_2, \tag{3.12}$$

$$Q_2(A_2+B_2K)+(A_2+B_2K)^T Q_2 < -l_2Q_2, \tag{3.13}$$

where l_1, l_2 are strict positive reals, $K = K_1 + K_2$ and $(\mathcal{H})$ holds with

$$\theta_2 < \frac{l\lambda_{\min}(Q_2)}{2\lambda_{\max}(Q_2)}, \quad l = \inf\{l_1, l_2\},$$

then, the system (3.1) in closed-loop with the linear feedback $u(x) = Kx$ is globally uniformly exponentially stable.

4 Conception of the observer

Consider the system (3.1), to obtain an estimation of the state (we can reconstitute the state), we shall consider the following observer

$$\begin{aligned} \dot{\hat{x}} &= A(\alpha(t))\hat{x} + B(\alpha(t))u + f(t,\alpha(t),\hat{x}) - L(\alpha(t))C(\hat{x}-x) \\ \hat{y} &= C\hat{x} \end{aligned} \tag{4.1}$$

where $\hat{x}(t)$ is the state estimate of $x(t)$ and $L(\alpha(t)) \in \mathcal{D}$.

Note that the nominal system of (4.1) is a global exponential observer for the nominal part of system (3.1).To prove it, it is sufficient to take $A(\alpha(t)) = \alpha_1(t)A_1 + \alpha_2(t)A_2$ and $B(\alpha(t)) = \alpha_1(t)B_1 + \alpha_2(t)B_2$ and we suppose that the pairs (A_1,C) and (A_2,C) are observable, then there exist some gain matrices $L_1 \in \mathbb{R}^{n\times q}$, $L_2 \in \mathbb{R}^{n\times q}$ and a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$, such that

$$P(A_1 - L_1C) + (A_1 - L_1C)^T P < 0,$$
$$P(A_2 - L_2C) + (A_2 - L_2C)^T P < 0.$$

Now, in the same context, we consider systems (3.1) and (4.1) with $A(\alpha(t))$ is given by (3.2) and

$$L(\alpha(t)) = \alpha_1^2(t)L_1 + 2\alpha_1(t)\alpha_2(t)L_{12} + \alpha_2^2(t)L_2. \tag{4.2}$$

Suppose that

$(\mathcal{H}_1')$ The pairs (A_1,C), (A_{12},C) and (A_2,C) are observable, then there exist some gain matrices $L_1 \in \mathbb{R}^{n\times q}$, $L_{12} \in \mathbb{R}^{n\times q}$, $L_2 \in \mathbb{R}^{n\times q}$ and a symmetric positive definite matrix $P_3 \in \mathbb{R}^{n\times n}$, such that

$$P_3(A_1 - L_1C) + (A_1 - L_1C)^T P_3 < 0, \tag{4.3}$$
$$P_3(A_{12} - L_{12}C) + (A_{12} - L_{12}C)^T P_3 < 0, \tag{4.4}$$

and

$$P_3(A_2 - L_2C) + (A_2 - L_2C)^T P_3 < 0. \tag{4.5}$$

$(\mathcal{H}_2')$ The following L_yMIs

$$P_3(A_1 - L_1C) + (A_1 - L_1C)^T P_3 + l_1' P_3 < 0, \tag{4.6}$$
$$P_3(A_{12} - L_{12}C) + (A_{12} - L_{12}C)^T P_3 + l_{12}' P_3 < 0, \tag{4.7}$$

and

$$P_3(A_2 - L_2C) + (A_2 - L_2C)^T P_3 + l_2' P_3 < 0, \tag{4.8}$$

are satisfied, where l_1', l_{12}' and l_2' are strict positive reals.

Theorem 4.1. *Assume that $(\mathcal{H}_1')$ and $(\mathcal{H}_2')$ are satisfied. If assumption $(\mathcal{H})$ holds with*

$$\theta_3 < \frac{l'}{2}\frac{\lambda_{\min}(P_3)}{\lambda_{\max}(P_3)}, \text{ and } l' = \inf\{l_1, l_{12}, l_2\},$$

then, the system (4.1) is a global exponential observer for system (3.1).

Proof. Let $W(t,e) = e^T P_3 e$ a Lyapunov function candidate. The time derivative of W along the trajectories of system (3.1) is given by

$$\begin{aligned}
\dot{W}(t,e) &= e^T\left((A(\alpha(t)) - L(\alpha(t))C)^T P_3 + P_3(A(\alpha(t)) - L(\alpha(t))C)\right)e\\
&+ 2e^T P_3(f(t,\alpha(t),\hat{x}) - f(t,\alpha(t),x))\\
&= e^T\left(\alpha_1^2(t)(P_3(A_1 - L_1C) + (A_1 - L_1C)^T P_3)\right.\\
&+ 2\alpha_1(t)\alpha_2(t)(P_3(A_{12} - L_{12}C) + (A_{12} - L_{12}C)^T P_3)\\
&+ \left.\alpha_2^2(t)(P_3(A_2 - L_2C) + (A_2 - L_2C)^T P_3)\right)e\\
&+ 2e^T P_3(f(t,\alpha(t),\hat{x}) - f(t,\alpha(t),x))\\
&\leq -(l'\lambda_{\min}(P_3) - 2\theta_3\lambda_{\max}(P_3))\|e\|^2.
\end{aligned}$$

Since

$$\theta_3 < \frac{l'}{2}\frac{\lambda_{\min}(P_3)}{\lambda_{\max}(P_3)},$$

it follows that

$$\dot{W}(t,e) \leq -\frac{\gamma_3}{\lambda_{\max}(P_3)}W(t,e),$$

where $\gamma_3 = l'\lambda_{\min}(P_3) - 2\theta_3\lambda_{\max}(P_3) > 0$. Hence, the system (4.1) is a global exponential observer for system (3.1).

□

Remark As we did in the previous sections, we can establish an observer for system (3.1) using just two or three L_yMIs. For the case of constant matrices, where the pair (A,C) is observable, there exist a gain matrix $L \in \mathbb{R}^{n\times q}$ and a symmetric positive definite matrix $P_1 \in \mathbb{R}^{n\times n}$, such that

$$P_1(A-LC)+(A-LC)^T P_1 < 0.$$

If also the following L_yMI is satisfied

$$P_1(A-LC)+(A-LC)^T P_1 + l'P_1 < 0,$$

with l' is a strict positive real and $(\mathcal{H})$ holds with

$$\theta_1 < \frac{l'}{2}\frac{\lambda_{\min}(P_1)}{\lambda_{\max}(P_1)},$$

this allows to conclude that system (4.1) is a global exponential observer for system (3.1).

In the case where $A(\alpha(t)) = \alpha_1(t)A_1 + \alpha_2(t)A_2$ and $L(\alpha(t)) = \alpha_1(t)L_1 + \alpha_2(t)L_2$ such that the pairs (A_1,C) and (A_2,C) are observable, there exist some gain matrices $L_1 \in \mathbb{R}^{n\times q}$, $L_2 \in \mathbb{R}^{n\times q}$ and a symmetric positive definite matrix $P_2 \in \mathbb{R}^{n\times n}$, such that

$$P_2(A_1-L_1C)+(A_1-L_1C)^T P_2 < 0, \tag{4.9}$$

$$P_2(A_2-L_2C)+(A_2-L_2C)^T P_2 < 0. \tag{4.10}$$

If the L_yMIs given by

$$P_2(A_1-L_1C)+(A_1-L_1C)^T P_2 + l_1'P_2 < 0, \tag{4.11}$$

$$P_2(A_2-L_2C)+(A_2-L_2C)^T P_2 + l_2'P_2 < 0, \tag{4.12}$$

are satisfied with l_1' and l_2' are strict positive reals and $(\mathcal{H})$ holds with

$$\theta_2 < \frac{l'}{2}\frac{\lambda_{\min}(P_2)}{\lambda_{\max}(P_2)},$$

it follows, in this case ,that system (4.1) is a global exponential observer for system (3.1).

5 Stabilization with an estimated state feedback

In order to obtain a separation principle(stabilization with an estimated state feedback) for (3.1), we consider the system (3.1) controlled by the linear feedback $u(\hat{x}) = K\hat{x}$ and estimated with the observer (4.1).

Let us consider the system

$$\begin{array}{rcl} \dot{\hat{x}} & = & A(\alpha(t))\hat{x} + B(\alpha(t))K\hat{x} + f(t,\alpha(t),\hat{x}) - L(\alpha(t))\, Ce \\ \dot{e} & = & (A(\alpha(t)) - L(\alpha(t))C)e + f(t,\alpha(t),\hat{x}) - f(t,\alpha(t),x). \end{array} \tag{5.1}$$

Theorem 5.1. *Assume that $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_1')$, and $(\mathcal{H}_2')$ are satisfied. If $(\mathcal{H})$ holds with*

$$\theta_3 \leq \inf\left(\frac{l\lambda_{\min}(Q_3)}{2\lambda_{\max}(Q_3)}, \frac{l'\lambda_{\min}(P_3)}{2\lambda_{\max}(P_3)}\right),$$

then, the system (5.1) is globally exponentially stable.

Proof. In order to study the stabilization problem via an observer, we consider the equivalent system

$$\begin{array}{rcl} \dot{\hat{x}} & = & \psi(t,\alpha(t),\hat{x}) + g(t,\alpha(t),\hat{x})e \\ \dot{e} & = & h(t,\alpha(t),\hat{x},e), \end{array} \tag{5.2}$$

where

$$\psi(t,\alpha(t),\hat{x}) = A(\alpha(t))\hat{x} + B(\alpha(t))K\hat{x} + f(t,\alpha(t),\hat{x}),$$

$$g(t,\alpha(t),\hat{x}) = -L(\alpha(t))C,$$

and

$$h(t,\alpha(t),\hat{x},e) = (A(\alpha(t)) - L(\alpha(t))C)e + f(t,\alpha(t),\hat{x}) - f(t,\alpha(t),x).$$

Note that, $\dot{\hat{x}} = \psi(t,\alpha(t),\hat{x})$ is globally exponentially stable with a Lyapunov function associated to this system can be chosen as

$$V(t,\hat{x}) = \hat{x}^T Q_3 \hat{x},$$

which satisfies

$$\lambda_{\min}(Q_3)\|\hat{x}\|^2 \leq V(t,\hat{x}) \leq \lambda_{\max}(Q_3)\|\hat{x}\|^2$$

$$\dot{V}(t,\hat{x}) \leq -\frac{l\lambda_{\min}(Q_3) - 2\theta_3\lambda_{\max}(Q_3)}{\lambda_{\max}(Q_3)} V(t,\hat{x}).$$

Also, the following differential equation

$$\dot{e} = h(t,\alpha(t),\hat{x},e)$$

is globally exponentially stable with the following estimation on the trajectories

$$\|e(t)\| \leq \sqrt{\frac{\lambda_{\max}(P_3)}{\lambda_{\min}(P_3)}}\|e(t_0)\| \exp\left(-\frac{l'\lambda_{\min}(P_3) - 2\theta_3\lambda_{\max}(P_3)}{2\lambda_{\max}(P_3)}\ (t-t_0)\right).$$

Now, if we take the derivative of V along the trajectories of system (5.2), one obtains

$$\begin{aligned}
\dot{V}(t,\hat{x}) &= \frac{\partial V}{\partial t}(t,\hat{x}) + \frac{\partial V}{\partial \hat{x}}\dot{\hat{x}} \\
&= \frac{\partial V}{\partial t}(t,\hat{x}) + \frac{\partial V}{\partial \hat{x}}\psi(t,\alpha(t),\hat{x}) + \frac{\partial V}{\partial \hat{x}}g(t,\alpha(t),\hat{x})e \\
&\leq -\frac{l\lambda_{\min}(Q_3) - 2\theta_3\lambda_{\max}(Q_3)}{\lambda_{\max}(Q_3)}V(t,\hat{x}) + \left\|\frac{\partial V}{\partial \hat{x}}\right\| \|g(t,\alpha(t),\hat{x})\| \, \|e\| \\
&\leq -\frac{l\lambda_{\min}(Q_3) - 2\theta_3\lambda_{\max}(Q_3)}{\lambda_{\max}(Q_3)}V(t,\hat{x}) \\
&+ \frac{2\lambda_{\max}(Q_3)}{\sqrt{\lambda_{\min}(Q_3)}}(\|L_1\| + 2\|L_{12}\| + \|L_2\|)\|C\| \, \|e\|\sqrt{V(t,\hat{x})}.
\end{aligned}$$

Let

$$v(t) = \sqrt{V(t,\hat{x})},$$

hence,

$$\begin{aligned}
\dot{v} &\leq -\frac{l\lambda_{\min}(Q_3) - 2\theta_3\lambda_{\max}(Q_3)}{2\lambda_{\max}(Q_3)}v(t) + \frac{\lambda_{\max}(Q_3)}{\sqrt{\lambda_{\min}(Q_3)}}(\|L_1\| + 2\|L_{12}\| + \|L_2\|)\|C\| \, \|e\| \\
&\leq -\frac{l\lambda_{\min}(Q_3) - 2\theta_3\lambda_{\max}(Q_3)}{2\lambda_{\max}(Q_3)}v(t) + \frac{\lambda_{\max}(Q_3)}{\sqrt{\lambda_{\min}(Q_3)}}(\|L_1\| + 2\|L_{12}\| + \|L_2\|)\|C\| \\
&\left[\sqrt{\frac{\lambda_{\max}(P_3)}{\lambda_{\min}(P_3)}}\|e(t_0)\| \exp\left(-\frac{l'\lambda_{\min}(P_3) - 2\theta_3\lambda_{\max}(P_3)}{2\lambda_{\max}(P_3)}\,(t - t_0)\right)\right].
\end{aligned}$$

Setting

$$\lambda_1 = \frac{l\lambda_{\min}(Q_3) - 2\theta_3\lambda_{\max}(Q_3)}{2\lambda_{\max}(Q_3)},$$

$$\lambda_2 = \frac{\lambda_{\max}(Q_3)}{\sqrt{\lambda_{\min}(Q_3)}}(\|L_1\| + 2\|L_{12}\| + \|L_2\|)\|C\|\sqrt{\frac{\lambda_{\max}(P_3)}{\lambda_{\min}(P_3)}},$$

and

$$\lambda_3 = \frac{l'\lambda_{\min}(P_3) - 2\theta_3\lambda_{\max}(P_3)}{2\lambda_{\max}(P_3)},$$

it follows that

$$\dot{v} \leq -\lambda_1 v + \lambda_2\|e(t_0)\|e^{-\lambda_3(t-t_0)}.$$

We pose

$$y(t) = v(t)e^{\lambda_1(t-t_0)},$$

hence,

$$\begin{aligned}
\dot{y}(t) &= (\dot{v}(t) + \lambda_1 v(t))e^{\lambda_1(t-t_0)} \\
&\leq \lambda_2\|e(t_0)\|e^{-\lambda_3(t-t_0)}e^{\lambda_1(t-t_0)} \\
&\leq \lambda_2\|e(t_0)\|e^{(\lambda_1-\lambda_3)(t-t_0)}.
\end{aligned}$$

Similar approach toTheorem 2.1 implies that system (5.2) is globally exponentially stable.

□

Remark For the case of constant matrices, it is sufficient to take in assumption $(\mathcal{H})$

$$\theta_1 \leq \inf\left(\frac{l\lambda_{\min}(Q_1)}{2\lambda_{\max}(Q_1)}, \frac{l'\lambda_{\min}(P_1)}{2\lambda_{\max}(P_1)}\right),$$

to obtain the same result as in Theorem 5.1. Moreover, for

$$A(\alpha(t)) = \alpha_1(t)A_1 + \alpha_2(t)A_2, \tag{5.3}$$

$$B(\alpha(t)) = \alpha_1(t)B_1 + \alpha_2(t)B_2, \tag{5.4}$$

$$L(\alpha(t)) = \alpha_1(t)L_1 + \alpha_2(t)L_2, \tag{5.5}$$

and

$$K = K_1 + K_2,$$

we take the case where

$$\theta_2 \leq \inf\left(\frac{l\lambda_{\min}(Q_2)}{2\lambda_{\max}(Q_2)}, \frac{l'\lambda_{\min}(P_2)}{2\lambda_{\max}(P_2)}\right).$$

Example 5.2.

Consider system

$$\begin{array}{rcl} \dot{x}(t) & = & A(\alpha(t))x(t) + B(\alpha(t))u(t) + \dfrac{\theta x}{1+t^2}(\alpha_1(t), \alpha_2(t)) \\ y & = & Cx(t), \end{array} \tag{5.6}$$

where $\theta = 10^{-2}$ and $C = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

It is clear that,

$$\|f(t,\alpha(t),x) - f(t,\alpha(t),y)\| \leq \theta\|x-y\|, \quad \forall t \geq 0 \quad \forall x, y \in \mathbb{R}^n.$$

If we consider this system such that $A(\alpha(t))$, $B(\alpha(t))$ and $L(\alpha(t))$ satisfy respectively (3.2), (3.3) and (4.2)with

$$A_1 = \begin{pmatrix} 1 & -9 \\ 11 & -3 \end{pmatrix}, A_{12} = \begin{pmatrix} -90 & 2 \\ 2 & -3 \end{pmatrix}, A_2 = \begin{pmatrix} -75 & 3 \\ 4 & -1 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad B_{12} = \begin{pmatrix} 3 \\ 2.2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1.3 \\ 0.8 \end{pmatrix},$$

and for $l_1 = 2.2$, $l_{12} = 3.8$, and $l_2 = 1.8$, there exist some constant matrices

$$K_1 = \begin{pmatrix} -3.7760 \\ -5.7185 \end{pmatrix}, \quad K_{12} = \begin{pmatrix} 12.8427 \\ -7.9423 \end{pmatrix}$$

$$K_2 = \begin{pmatrix} -11.9518 \\ -6.0970 \end{pmatrix} \text{ and } Q_3 = \begin{pmatrix} 0.1246 & -0.0289 \\ -0.0289 & 0.2013 \end{pmatrix}$$

that satisfy (3.4), (3.5), (3.6), (3.7), (3.8) and (3.9). Furthermore, there exist some gain matrices L_1, L_{12}, L_2 and a symmetric positive definite matrix P_3 which are given respectively by

$$L_1 = \begin{pmatrix} 12.3652 \\ 20.3942 \end{pmatrix}, L_{12} = \begin{pmatrix} 44.0796 \\ 29.7807 \end{pmatrix}, L_2 = \begin{pmatrix} 46.2207 \\ 29.6407 \end{pmatrix}$$

and

$$P_3 = \begin{pmatrix} 0.6712 & -0.2281 \\ -0.2281 & 0.8930 \end{pmatrix}$$

that satisfy L_yMIs given in (4.3), (4.4), (4.5),(4.6) ,(4.7) and (4.8).

Moreover, assumption $(\mathcal{H})$ is satisfied since we have

$$\theta < \inf\left(\frac{l}{2}\frac{\lambda_{\min}(P_3)}{\lambda_{\max}(P_3)}, \frac{l}{2}\frac{\lambda_{\min}(Q_3)}{\lambda_{\max}(Q_3)}\right)$$

where

$$\frac{l}{2}\frac{\lambda_{\min}(P_3)}{\lambda_{\max}(P_3)} = 45.92 \times 10^{-2}$$

and

$$\frac{l}{2}\frac{\lambda_{\min}(Q_3)}{\lambda_{\max}(Q_3)} = 49.03 \times 10^{-2}.$$

Therefore, all assumptions of Theorem 5.1 are satisfied, it follows that system (5.6) is globally exponentially stable.

Now, if we take into account the case of constant matrices with

$$A = \begin{pmatrix} -75 & 3 \\ 4 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

then, for $l = 1.8$, there exist two constant matrices K, L and two symmetric positive definite matrices Q_1 and P_1 given respectively by

$$K = \begin{pmatrix} 108.3781 & -26.9856 \end{pmatrix}, \quad L = \begin{pmatrix} 53.4823 \\ 0.5613 \end{pmatrix}$$

$$Q_1 = \begin{pmatrix} 0.4036 & -0.0642 \\ -0.0642 & 0.0378 \end{pmatrix} \text{ and } P_1 = \begin{pmatrix} 0.0627 & 0.0361 \\ 0.0361 & 1.4802 \end{pmatrix}$$

that satisfy L_yMIs described before. Moreover, we get

$$\theta < \inf\left(\frac{l}{2}\frac{\lambda_{\min}(P_1)}{\lambda_{\max}(P_1)}, \frac{l}{2}\frac{\lambda_{\min}(Q_1)}{\lambda_{\max}(Q_1)}\right)$$

where

$$\frac{l}{2}\frac{\lambda_{\min}(P_1)}{\lambda_{\max}(P_1)} = 3.76 \times 10^{-2}$$

and

$$\frac{l}{2}\frac{\lambda_{\min}(Q_1)}{\lambda_{\max}(Q_1)} = 5.84 \times 10^{-2}.$$

Furthermore, if we consider system (5.6) for equalities (5.3), (5.4) and (5.5) with

$$A_1 = \begin{pmatrix} 1 & -9 \\ 11 & -3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -75 & 3 \\ 4 & -1 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 1.3 \\ 0.8 \end{pmatrix},$$

then, for $l_1 = 3.2$ and $l_2 = 1.8$, there exist two constant matrices K_1, K_2 and a symmetric positive definite matrix Q_2 that satisfy L_yMIs (3.10), (3.11), (3.12), (3.13) and two gain matrices L_1, L_2 and a symmetric positive definite matrix P_2 that satisfy L_yMIs (4.9), (4.10), (4.11) and (4.12). These solutions are given by

$$K_1 = \begin{pmatrix} 3.6154 \\ -7.5710 \end{pmatrix}, \; K_2 = \begin{pmatrix} 3.6220 \\ -12.7903 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 20.1847 \\ 29.5718 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 61.3860 \\ 31.9246 \end{pmatrix}$$

$$Q_2 = \begin{pmatrix} 0.1566 & -0.0765 \\ -0.0765 & 0.1757 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 0.6810 & -0.3190 \\ -0.3190 & 0.9488 \end{pmatrix}.$$

Moreover, assumption $(\mathcal{H})$ is satisfied since we get

$$\theta < \inf\left(\frac{l}{2}\frac{\lambda_{\min}(P_2)}{\lambda_{\max}(P_2)}, \frac{l}{2}\frac{\lambda_{\min}(Q_2)}{\lambda_{\max}(Q_2)}\right)$$

where

$$\frac{l}{2}\frac{\lambda_{\min}(P_2)}{\lambda_{\max}(P_2)} = 36.36 \times 10^{-2}$$

and

$$\frac{l}{2}\frac{\lambda_{\min}(Q_2)}{\lambda_{\max}(Q_2)} = 32.92 \times 10^{-2}.$$

6 conclusion

Based on Lyapunov techniques, it is shown in this paper that a class of perturbed systems with parameter dependence can be globally uniformly exponentially convergent to a neighborhood of the origin using l_yMIs. Moreover, the stabilization by an estimated state feedback given by an observer can be achieved, provided that the nominal system is globally uniformly exponentially stabilizable by a linear feedback and the perturbation term is subject to some conditions. The effectiveness of the proposed criteria is verified in numerical examples.

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