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ON A CLASS OF INFINITE HORIZON OPTIMAL CONTROL PROBLEMS

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Abstract

In this paper we establish the existence of solutions of infinite horizon optimal control problems with time-dependent and non-concave objective functions. We also consider an application of this problems to a forest management problem.

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1 Introduction and the main result

The study of the existence and the structure of solutions of optimal control problems defined on infinite intervals and on sufficiently large intervals has recently been a rapidly growing area of research. See, for example, [5-9, 12, 13, 36] and the references mentioned therein. These problems arise in engineering [1, 17], in models of economic growth [2, 3, 10, 11, 14, 15, 19, 21, 22, 24-30, 36], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [4, 31] and in the theory of thermodynamical equilibrium for materials [18, 20]. In this paper we study a general class of discrete-time optimal control problems which applications in a forest management problem studied in [10, 24-29].

Let (Δ,ρ) be a compact metric space and Ω be a nonempty closed subset of $\Delta\times\Delta$.

A sequence $\{x_t\}_{t=0}^{\infty}$ is called a program if $(x_t,x_{t+1})\in\Omega$ for all $t=0,1,\dots$

Let integers T_1,T_2 satisfy $0\le T_1<T_2$. A sequence $\{x_t\}_{t=T_1}^{T_1}$ is called a program if $(x_t,x_{t+1})\in\Omega$ for all integers t satisfying $T_1\le t<T_2$.

For each integer $t\ge 0$ let $w_t:\Omega\to R^1$ be a bounded upper semicontinuous function.

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For each pair of integers T_1, T_2 satisfying $0 \le T_1 < T_2$ and each $y, z \in \Delta$ we consider the optimization problems

$$\sum_{i=T_1}^{T_2-1} w_i(x_i, x_{i+1}) \to \min,$$

$$\{x_i\}_{i=T_1}^{T_2} \subset \Delta,\ x_{T_1} = y,\ x_{T_2} = z$$

and

$$\sum_{i=T_1}^{T_2-1} w_i(x_i, x_{i+1}) \to \min,$$

$$\{x_i\}_{i=T_1}^{T_2} \subset K,\ x_{T_1} = y.$$

The interest in discrete-time optimal problems of these types stems from the study of various optimization problems which can be reduced to it, e.g., continuous-time control systems which are represented by ordinary differential equations whose cost integrand contains a discounting factor [9], tracking problems in engineering [17], the study of Frenkel-Kontorova model related to dislocations in one-dimensional crystals [4, 31], the analysis of a long slender bar of a polymeric material under tension in [18, 20] and models of economic growth [3, 10, 12, 14, 15, 24-30]. See also [16, 32-35] where these problems were studied with $\Omega = \Delta \times \Delta$.

In this paper we suppose that the following assumptions hold.

(A1)

$$\lim_{t\to\infty} \sup\{|w_t(z)| :\ z \in \Omega\} = 0.$$

(A2) There exists a natural number $\bar{L}$ such that for each $y, z \in \Delta$ there is a program $\{x_t\}_{t=0}^{\bar{L}}$ such that $x_0 = y$ and $x_{\bar{L}} = z$.

Note that in [16, 32-35] it was studied the case where $\bar{L} = 1$.

For each $y \in \Delta$ and each natural number T put

$$U(y,T) = \sup\{\sum_{t=0}^{T-1} w_t(x_t, x_{t_{t+1}}) :\ \{x_t\}_{t=0}^{T} \text{ is a program and } x_0 = y\}. \tag{1.1}$$

By the upper semicontinuity of the functions w_t, $t = 0, 1, \ldots$ the following proposition holds.

Proposition 1.1. *For each $y \in \Delta$ and each natural number T there is a program $\{x_t^{(y,T)}\}_{t=0}^{T}$ such that*

$$x_0^{(y,T)} = y,$$

$$\sum_{t=0}^{T-1} w_t(x_t^{(y,T)}, x_{t+1}^{(y,T)}) = U(y,T). \tag{1.2}$$

In the sequel for each $y \in \Delta$ and each natural number T let $\{x_t^{(y,T)}\}_{t=0}^{T}$ be a program satisfying (1.2).

In this paper we prove the following result.

Theorem 1.2. *For any $y \in \Delta$ there exists a program $\{x_t^{(y)}\}_{t=0}^{\infty}$ such that $x_0^{(y)} = y$ and the following property holds:*

For each $\varepsilon > 0$ there exists a natural number τ such that for each $y \in \Delta$ and each integer $T \geq \tau$,

$$|\sum_{t=0}^{T-1} w_t(x_t^{(y)}, x_{t+1}^{(y)}) - U(y,T)| \leq \varepsilon.$$

The next corollary easily follows from Theorem 1.2.

Corollary 1.3. *Let $y \in \Delta$. Than for any program $\{x_t\}_{t=0}^{\infty}$ satisfying $x_0 = y$,*

$$\limsup_{T\to\infty} \sum_{t=0}^{T-1} [w_t(x_t, x_{t+1}) - w_t(x_t^{(y)}, x_{t+1}^{(y)})] \leq 0.$$

Note that the program $\{x_t\}_{t=0}^{\infty}$ which exists by Corollary 1.3 is called in the literature as an overtaking optimal program [3, 9, 11, 30, 36].

Example. Let $w : \Omega \to [0,\infty)$ be a bounded upper semicontinuous function, $\{\rho_t\}_{t=0}^{\infty} \subset (0,1)$ satisfy

$$\lim_{t\to\infty} \rho_t = 0 \tag{1.3}$$

and let $w_t = \rho_t w$, $t = 0,1,\ldots$. Then Assumption (A1) holds. In the literature it is also considered an optimality criterion with $\rho_t = \alpha^t$, $t = 0,1,\ldots$ where $\alpha \in (0,1)$. In this case for any program $\{x_t\}_{t=0}^{\infty}$, $\sum_{t=0}^{\infty} \alpha^t w(x_t, x_{t+1}) < \infty$. This convergence does not hold in the general case with $\{\rho_t\}_{t=0}^{\infty} \subset (0,1)$ satisfying (1.3). Therefore in the general case the existence problem of an overtaking optimal program is more difficult and less understood.

The paper is organized as follows. Section 2 contains auxiliary results. Theorem 1.2 is proved in Section 3. The forest management problem is discussed is Section 4.

2 An auxiliary result

For any integer $t \geq 0$ set

$$||w_t|| = \sup\{|w_t(z)| : z \in \Omega\}. \tag{2.1}$$

Lemma 2.1. *Let $\varepsilon > 0$. Then there exists a natural number τ such that for each $y \in \Delta$ and each pair of integers $T_1 \geq \tau$ and $T_2 \geq \tau + \bar{L}$,*

$$\sum_{t=0}^{T_1-1} w_t(x_t^{(y,T_2)}, x_{t+1}^{(y,T_2)}) \geq U(y,T_1) - \varepsilon.$$

Proof. By (A1) and (2.1) there exists a natural number τ such that

$$||w_t|| \leq \varepsilon(4\bar{L})^{-1} \text{ for all integers } t \geq \tau. \tag{2.2}$$

Assume that $y \in \Delta$ and that integers

$$T_1 \geq \tau,\ T_2 \geq T_1 + \bar{L}. \tag{2.3}$$

In view of (A2) there exists a program $\{x_t\}_{t=T_1}^{T_1+\bar{L}}$ such that

$$x_{T_1} = x_{T_1}^{(y,T_1)},\ x_{T_1+\bar{L}} = x_{T_1+\bar{L}}^{(y,T_2)}. \tag{2.4}$$

Put

$$x_t = x_t^{(y,T_1)},\ t = 0,\ldots,T_1-1, \tag{2.5}$$

$$x_t = x_t^{(y,T_2)} \text{ for all integers } t \text{ satisfying } T_1+\bar{L} < t \le T_2.$$

Clearly, $\{x_t\}_{t=0}^{T_2}$ is a program and

$$x(0) = y. \tag{2.6}$$

By (2.6), (1.2), (1.1), (2.3), (2.1) and (2.2),

$$0 \le \sum_{t=0}^{T_2-1} w_t(x_t^{(y,T_2)}, x_{t+1}^{(y,T_2)}) - \sum_{t=0}^{T_2-1} w_t(x_t, x_{t+1})$$

$$= \sum_{t=0}^{T_1+\bar{L}-1} w_t(x_t^{(y,T_2)}, x_{t+1}^{(y,T_2)}) - \sum_{t=0}^{T_1+\bar{L}-1} w_t(x_t, x_{t+1})$$

$$\le \sum_{t=0}^{T_1-1} w_t(x_t^{(y,T_2)}, x_{t+1}^{(y,T_2)}) - \sum_{t=0}^{T_1-1} w_t(x_t^{(y,T_1)}, x_{t+1}^{(y,T_1)}) + 2\sum_{t=T_1}^{T_1+\bar{L}-1} ||w_t||$$

$$\le \sum_{t=0}^{T_1-1} w_t(x_t^{(y,T_2)}, x_{t+1}^{(y,T_2)}) - U(y,T_1) + 2\bar{L}(\varepsilon(4\bar{L})^{-1})$$

and

$$\sum_{t=0}^{T_1-1} w_t(x_t^{(y,T_2)}, x_{t+1}^{(y,T_1)}) \ge U(y,T_1) - \varepsilon.$$

Lemma 2.1 is proved. □

3 Proof of Theorem 1.2

Let $y \in \Delta$. Using the diagonalization process and the compactness of Δ we obtain a strictly increasing sequence of natural numbers $\{T_k\}_{k=1}^{\infty}$ such that for any integer $t \ge 0$ there exists

$$x_t^{(y)} = \lim_{k\to\infty} x_t^{(y,T_k)}. \tag{3.1}$$

Clearly, $\{x_t^y\}_{t=0}^{\infty}$ is a program for all $y \in \Delta$.

Let $\varepsilon > 0$ and let a natural number τ be as guaranteed by Lemma 2.1. Assume that an integer $T \ge \tau$ and $y \in \Delta$. Then for all sufficiently large natural numbers k

$$\sum_{t=0}^{T-1} w_t(x_t^{(y,T_k)}, x_{t+1}^{(y,T_k)}) \ge U(y,T) - \varepsilon.$$

By the inequality above, (3.1) and upper semicontinuity of the functions w_t, $t = 0,1,\ldots,$

$$\sum_{t=0}^{T-1} w_t(x_t^{(y)}, x_{t+1}^{(y)}) \ge U(y,T) - \varepsilon.$$

Theorem 1.2 is proved.

4 The forest management problem

We consider a discrete time model for the optimal management of a forest of total area S occupied by k species $I = \{1,\ldots,k\}$ with maturity ages of $n_1,\ldots,n_k$ years respectively. This model was studied in [10, 24-28]. Mitra and Wan [24, 25] studied the problem of the optimal harvesting of a multi-aged single species forest. The optimal management of a one-species forest was also studied by Rapaport, Sraidi and Terreaux [28] using a model where only mature trees older than a certain age may be harvested, addressing some of the effects of delay in the management of natural resources. In [10, 27] and here it is studied the optimal harvesting of a mixed forest composed of multiple species, each one having a different maturity age, where only mature trees can be harvested.

For each period $t = 0,1,\ldots$ we denote $x_i^j(t) \geq 0$ the area covered by trees of species i that are j years old with $j = 1,\ldots,n_i$ and $\bar{x}_i(t) \geq 0$ the area occupied by over-mature trees (older than n_i). We must decide how much land $u_i(t) \geq 0$ to harvest and how to reallocate this land to new seedlings.

Assuming that only mature trees can be harvested we must have

$$u_i(t) \leq \bar{x}_i(t) + x_i^{n_i}(t), \tag{4.1}$$

and then the area not harvested in that period will comprise the over-mature trees at the next step, namely

$$\bar{x}_t(t+1) = \bar{x}_i(t) + x_i^{n_i}(t) - u_i(t). \tag{4.2}$$

The fact that immature trees cannot be harvested is represented by

$$x_i^{j+1}(t+1) = x_i^j(t),\ j = 1,\ldots,n_i - 1. \tag{4.3}$$

The total harvested area $\sum_{i\in I} u_i(t)$ is allocated to new seedlings which is expressed by the equation

$$\sum_{i\in I} x_i^1(t+1) = \sum_{i\in I} u_i(t). \tag{4.4}$$

In the sequel we use the notation

$$x_i^{n_i+1} = \bar{x}_i,\ i \in I. \tag{4.5}$$

A representation of the forest in terms of the age distribution at time t is provided by the state $x(t) = (x_1(t),\ldots,x_k(t))$ where $x_i(t) = (x_i^1(t),\ldots,x_i^{n_i}(t),x_i^{n_i+1}(t))$ describes the areas occupied in year t by trees of species i with ages $1,2,\ldots,n_i$ and over n_i. The first and last components of each vector $x_i(t)$ are controlled by the sowing and harvesting policies. Note that we do not control $x(0)$ which corresponds to the initial state reflecting the age class composition of the forest at time $t = 0$.

Let $R_+^m = \{x = (x_1,\ldots,x_m) \in R^m :\ x_i \geq 0,\ i = 1,\ldots,m\}$.

Let $N = \sum_{i\in I}(n_i + 1)$. Every vector $x \in R^N$ is represented as $x = (x_1,\ldots,x_k)$, where $x_i = (x_i^1,\ldots,x_i^{n_i},x_i^{n_i+1}) \in R^{n_i+1}$ for all integers $i = 1,\ldots,k$.

Denote by Δ the set of all $x \in R_+^N$ such that

$$\sum_{i\in I}[\sum_{j=1}^{n_i+1} x_i^j] = S. \tag{4.6}$$

Now we give a formal description of the model.

A sequence $\{x(t)\}_{t=0}^{\infty} \subset \Delta$ is called a program if for all integers $t \geq 0$ and all $i \in I$ (4.1)-(4.4) hold (see (4.5)) with some $u(t) = (u_1(t), \ldots, u_k(t)) \in R^k_+$.

Let integers T_1, T_2 satisfy $0 \leq T_1 < T_2$. A sequence $\{x(t)\}_{t=T_1}^{T_2}$ is called a program if (4.1)-(4.4) hold for all $i \in I$ and for all integers $t = T_1, \ldots, T_2 - 1$ (see (4.5)) with some $u(t) = (u_1(t), \ldots, u_k(t)) \in R^k_+$.

An alternative equivalent definition of a program can be give with the help of the transition possibility. Put

$$\Omega = \{(x,y) \in \Delta \times \Delta : \ y_i^{j+1} = x_i^j \text{ for all } i \in I \text{ and all } j = 1, \ldots, n_i - 1$$
$$\text{and for all } i \in I,\ x_i^{n_i+1} + x_i^{n_i} - y_i^{n_i+1} \geq 0\}. \tag{4.7}$$

Clearly, if $(x,y) \in \Omega$, then

$$\sum_{i \in I} y_i^1 = \sum_{i \in I} (x_i^{n(i)+1} + x_i^{n_i} - y_i^{n(i)+1}). \tag{4.8}$$

It is easy to see that a sequence $\{x(t)\}_{t=0}^{\infty}$ is a program if and only if $(x(t), x(t+1)) \in \Omega$ for all integers $t \geq 0$.

Let integers T_1, T_2 satisfy $0 \leq T_1 < T_2$. It is easy to see that a sequence $\{x(t)\}_{t=T_1}^{T_2} \subset \Delta$ is a program if and only if $(x(t), x(t+1)) \in \Omega$ for all $t = T_1, \ldots, T_2 - 1$.

For each $(x,y) \in \Omega$ put

$$U(x,y) = (u_1(x,y), \ldots, u_k(x,y)),$$

where for $i = 1, \ldots, k$,

$$u_i(x,y) = x_i^{n_i+1} + x_i^{n_i} - y_i^{n_i+1}.$$

Put

$$\Delta_0 = \{u \in R^k_+ : \ \sum_{i=1}^{k} u_i \leq S\}.$$

In the present paper we assume that a benefit at moment $t = 0, 1, \ldots$ is represented by an upper semicontinuous function $w_t : \Delta_0 \to R^1$ and at a moment $t = 0, 1, \ldots,$ $w_t(U(x,y))$ is the benefit obtained today if the forest today is x and the forest tomorrow is y, where $(x,y) \in \Omega$.

Remark. Note that usually in the literature it is assumed that for $t = 0, 1, \ldots,$

$$w_t(U(x,y)) = \alpha^t \sum_{i=1}^{k} W^{(i)}(u_i(x,y)),\ (x,y) \in \Omega$$

where $W^i : [0, \infty) \to R^1$, $i = 1, \ldots, k$ are strictly concave, smooth and increasing functions and $\alpha \in (0,1)$ [10].

Clearly, Δ is a compact set in R^N, Ω is a closed subset of $\Delta \times \Delta$ and $w_t \circ U : \Omega \to R^1$ is an upper semicontinuous function for all integers $t \geq 0$. Put

$$\bar{n} = \max\{n_i : \ i \in I\}. \tag{4.9}$$

It is known that for our model (A2) holds [23]. For the reader's convenience we prove here the following result.

Proposition 4.1. *Let $x, y \in \Delta$. Then there exists a program $\{x(t)\}_{t=0}^{N+\bar{n}+1}$ such that $x(0) = x$ and $x(N + +\bar{n} + 1) = y$.*

Proof. Put $x(0) = x$. For all integers $t = 0, \dots, N-1$ define

$$x_i^{j+1}(t+1) = x_i^j(t),\ i \in I,\ j = 1, \dots, n_i - 1, \tag{4.10}$$

$$x_i^1(t+1) = 0,\ i \in I,$$

$$x_i^{n_i+1}(t+1) = x_i^{n_i+1}(t) + x_i^{n_i}(t),\ i \in I.$$

It is easy to see that $x(t) \in \Delta$ for all $t = 0, \dots, N$, $\{x(t)\}_{t=0}^N$ is a program and

$$x_i^j(N) = 0,\ i \in I,\ j = 1, \dots, n_i,$$

$$\sum_{i \in I} x_i^{n_i+1}(N) = S. \tag{4.11}$$

For each $s = 1, \dots, \bar{n}$ put

$$I_s = \{i \in I :\ n_i = s\} \tag{4.12}$$

(Note that for some integers s we can have $I_s = \emptyset$.)

We assume that sum over empty set is zero.

Define $x(N+1) \in \Delta$ as follows. Set

$$x_i^1(N+1) = y_i^{n_i+1},\ i \in I_{\bar{n}}, \tag{4.13}$$

$$x_i^1(N+1) = 0,\ i \in I \setminus I_{\bar{n}}.$$

For $i \in I$ and all integers j satisfying $1 < j \le n_i$ set

$$x_i^j(N+1) = 0. \tag{4.14}$$

Clearly, there exist

$$u_i \in [0, x_i^{n_i+1}(N)],\ i \in I \tag{4.15}$$

such that

$$\sum_{i \in I} u_i = \sum_{i \in I_{\bar{n}}} x_i^1(N+1). \tag{4.16}$$

Put

$$x_i^{n_i+1}(N+1) = x_i^{n_i+1}(N) - u_i,\ i \in I. \tag{4.17}$$

By (4.13)-(4.17), $x(N+1) \in \Delta$ and

$$(x(N), x(N+1)) \in \Omega.$$

Assume that q is an integer, $1 \le q < \bar{n}$ and we have defined a program $\{x(t)\}_{t=0}^{N+q}$ such that the following properties hold:

(P1) If an integer

$$i \in \cup\{I_s :\quad \text{an integer } s \text{ satisfies } 1 \le s \le \bar{n} - q\},$$

then $x_i^j(N+q) = 0$ for all integers j satisfying $1 \le j \le n_i$;

(P2) If an integer s satisfies $\bar{n} \geq s > \bar{n} - q$ and $i \in I_s$, then

$$x_i^p(N+q) = y_i^{n_i+1+p-(q+s-\bar{n})},\ p = 1,\dots,q+s-\bar{n}, \tag{4.18}$$

$$x_i^p(N+q) = 0 \text{ for all integers } p \text{ satisfying } q+s-\bar{n} < p \leq n_i. \tag{4.19}$$

(Note that for $q = 1$ our assumptions holds.)

Define $x(N+q+1) \in \Delta$ as follows. Let $i \in I$. If $i \in I_s$, where $1 \leq s \leq \bar{n} - q - 1$, then set

$$x_i^j(N+q+1) = 0 \text{ for all integers } j \text{ satisfying } 1 \leq j \leq n_i. \tag{4.20}$$

If $i \in I_{\bar{n}-q}$, then set

$$x_i^1(N+q+1) = y^{n_i+1},\ x_i^p(N+q+1) = 0 \tag{4.21}$$

for all integers p satisfying $1 < p \leq n_i$.

If $i \in I_s$, where $\bar{n} \geq s > \bar{n} - q$, then set (see (4.18))

$$x_i^{p+1}(N+q+1) = x_i^p(N+q) = y_i^{n_i+1+p-(q+s-\bar{n})},\ p = 1,\dots,q+s-\bar{n}, \tag{4.22}$$

$$x_i^1(N+q+1) = y_i^{n_i+1-(q+s-\bar{n})}, \tag{4.23}$$

$$x_i^p(N+q+1) = 0 \text{ for all integers } p \text{ satisfying } q+1+s-\bar{n} < p \leq n_i. \tag{4.24}$$

It is not difficult to see that

$$\sum_{i \in I} x_i^1(N+q+1) \leq \sum_{i \in I} x^{n_i+1}(N+q) \tag{4.25}$$

Therefore there exists

$$x^{n_i+1}(N+q+1) \in [0, x^{n_i+1}(N+q)],\ i \in I. \tag{4.26}$$

such that

$$\sum_{i \in I} [x_i^{n_i+1}(N+q) - x_i^{n_i+1}(N+q+1)] = \sum_{i \in I} x_i^1(N+q+1). \tag{4.27}$$

Clearly,

$$x(N+q+1) \in \Delta,\ (x(N+q), x(N+q+1)) \in \Omega$$

and the assumption made for q holds also for $q+1$. Thus by induction we have constructed a program $\{x(t)\}_{t=0}^{N+\bar{n}}$ such that (P1) and (P2) hold for all $q = 1,\dots,\bar{n}$.

Consider the state $x(N+\bar{n})$. Let $i \in I_s$ where $1 \leq s \leq \bar{n}$. By (P1) and (P2),

$$x_i^p(N+\bar{n}) = y_i^{p+1},\ p = 1,\dots,n_i. \tag{4.28}$$

Put

$$x(N+\bar{n}+1) = y. \tag{4.29}$$

By (4.28) and (4.29),

$$x(N+\bar{n}), x(N+\bar{n}+1)) \in \Omega.$$

Proposition 4.1 is proved. □

Assume that

$$\sup\{|w_t(z)| : z \in \Delta_0\} \to 0 \text{ as } t \to \infty.$$

It is easy now to see that we can apply the results of Section 1 to our model.

For each $z \in \Delta$ and each natural number T put

$$U(z,T) = \sup\{\sum_{t=0}^{T-1} w_t(U(x(t),x(t+1))) : \{x(t)\}_{t=0}^{T} \text{ is a program and } x(0) = z\}.$$

Then Theorem 1.2 and Corollary 1.3 imply the following results.

Theorem 4.2. *For any $y \in \Delta$ there exists a program $\{x^{(y)}(t)\}_{t=0}^{\infty}$ such that $x^{(y)}(0) = y$ and the following property holds:*

For each $\varepsilon > 0$ there exists a natural number τ such that for each $y \in \Delta$ and each integer $T \geq \tau$,

$$|\sum_{t=0}^{T-1} w_t(U(x^{(y)}(t),x^{(y)}(t+1)) - U(y,T)| \leq \varepsilon.$$

Corollary 4.3. *Let $y \in \Delta$. Than for any program $\{x(t)\}_{t=0}^{\infty}$ satisfying $x(0) = y$,*

$$\limsup_{T\to\infty} \sum_{t=0}^{T-1} [w_t(U(x(t),x(t+1))) - w_t(x^{(y)}(t),x^{(y)}(t+1))] \leq 0.$$

Theorem 4.2 generalizes to the case of non-concave benefit function the existence theorem presented in [27]. Note that an analog of Theorem 4.2 for the Robinson-Solow-Srinivasan model was established in [15].

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