Communications in Mathematical Analysis
Special Volume in Honor of Prof. Peter D. Lax
Volume 8, Number 2, pp. 136–144 (2010)
ISSN 1938-9787
www.commun-math-anal.org

Linking Numbers of Modular Knots

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To: Peter Lax with admiration

(Communicated by Toka Diagana)

§1. In this note we examine some questions about closed geodesics on the modular surface X which are suggested by the spectacular pictures from Ghys' talk [G] as well as his paper with Leys [G-L]. The discussion below follows closely my letter to Mozzochi [Sa1] and as is done there we will only outline the proofs of the main results. Detailed proofs of the delicate estimates that are needed will appear in Mozzochi's article [Mo].

First we review the results from [G]. He shows that the non-compact 3-dimensional homogeneous quotient space $Y = SL_2(\mathbb{R})/SL_2(\mathbb{Z})$ is homeomorphic to the 3-sphere S^3 with the trefoil knot τ removed. Y carries a number of flows and corresponding non-vanishing vector fields and in particular the diagonal flow $\mathcal{G}_t$, for $t \in \mathbb{R}$,

$$\mathcal{G}_t(y\,SL_2(\mathbb{Z})) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} y\,SL_2(\mathbb{Z})\,. \tag{1}$$

This flow corresponds to the geodesic flow on the modular surface $X = \mathbb{H}/\Gamma$ with $\Gamma = PSL_2(\mathbb{Z})$ and the primitive (i.e. once around) closed orbits of $\mathcal{G}_t$ correspond to oriented primitive (or "prime") closed geodesics on X. Such a periodic orbit of $\mathcal{G}_t$ yields a knot in $S^3 - \tau$. It is known that these primitive closed orbits correspond to primitive hyperbolic conjugacy classes $\{A\}_\Gamma$ of elements A in Γ (see [He]). A is primitive if it is not a nontrivial power of an element B in Γ and A is hyperbolic means that $|\mathrm{trace}\,(A)| = t(A) > 2$. In this way to each such $\{A\}_\Gamma$ we get a knot k_A in $S^3 - \tau$.

Figure 1 is taken from [G-L]. It depicts six knots corresponding to the A's indicated and how these wind around the trefoil. The question raised in [G] is to understand the function which takes $\{A\}_\Gamma$ to k_A. For example, which knots k_A arise in this way and how do the linking numbers of k_A with τ vary with A? Ghys establishes some very interesting things about these knots. Firstly, that the set of such knots, dubbed "modular knots", coincides with the set of "Lorenz knots". The latter are the knots which are primitive periodic orbits of the non-linear

Lorenz flow in $\mathbb{R}^3$ (see [G-L] for definitions and pictures). Birman and Williams [B-W] give an in depth study of Lorenz knots and while characterizing them appears to be difficult, they establish a number of properties that these knots satisfy. Secondly, Ghys shows that if $\text{link}(k_A, \tau)$ is the usual linking number of these knots in $\mathbb{R}^3$ (see [G-L] for a friendly definition)

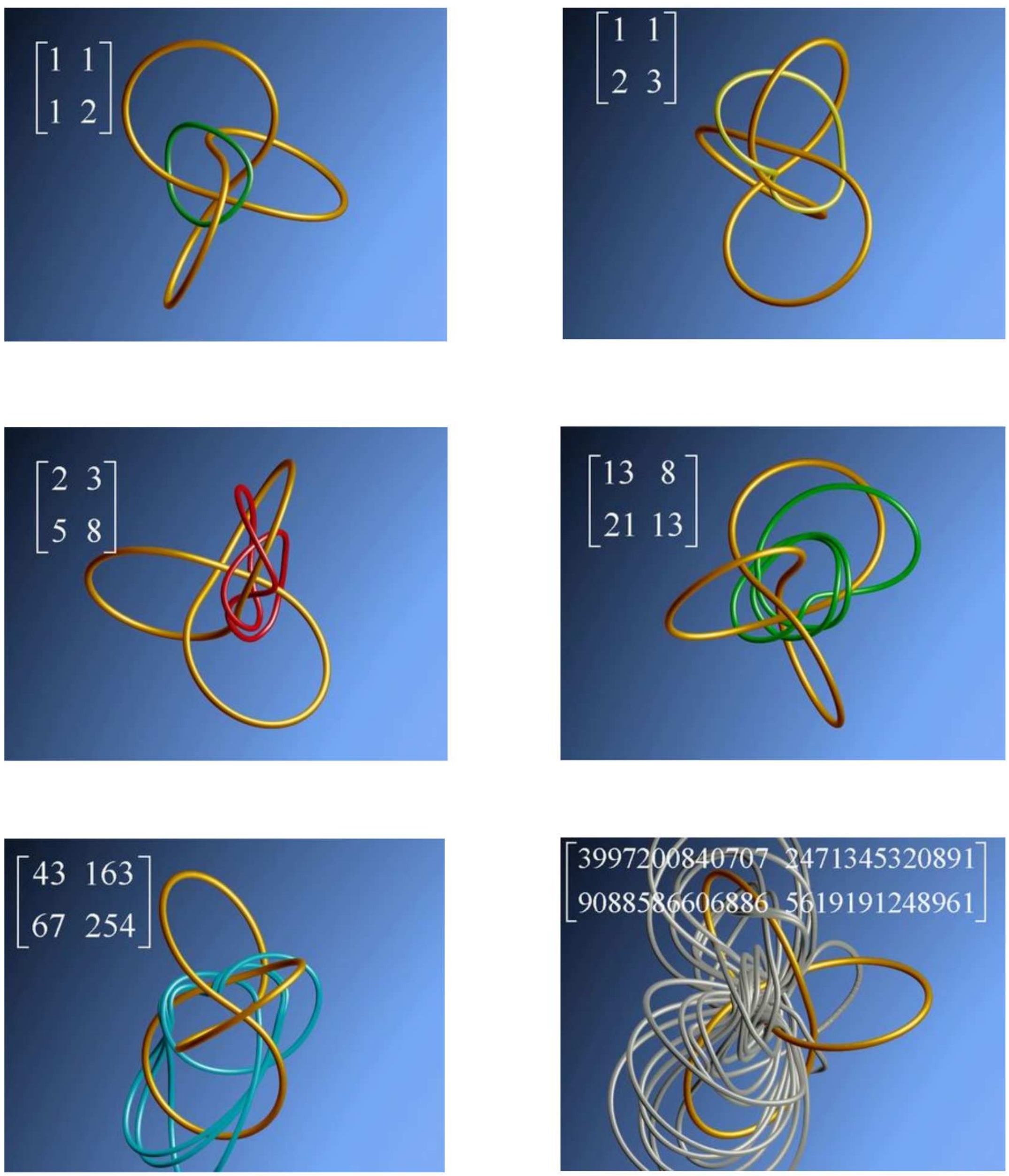

Figure 1: The orange knot is the trefoil the other knot is k_A for A as indicated.

then $\text{link}(k_A, \tau) = \psi(A)$ where $\psi : \Gamma \to \mathbb{Z}$ is the Rademacher function (see [R-G] page 54 for a definition and for various of its properties). Since $\psi(A)$ is easily calculated once A is expressed as a product of $U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $V = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ this allows one to compute $\text{link}(k_A, \tau)$ quickly. The first homology $\Gamma/[\Gamma, \Gamma]$ of Γ is finite and hence there are no non-trivial morphisms $f : \Gamma \to \mathbb{Z}$. There are however quasimorphisms, that is f's satisfying $|f(xy) - f(x) - f(y)| \leq c_f$ for some $c_f < \infty$ and ψ is such a function. It behaves like a morphism in a number of respects. The third point that is relevant for us that Ghys notes is that k_A is the trivial knot iff $\{A\}_\Gamma$ is of the form $\{(UV)^a(UV^{-1})^b\}_\Gamma$ for some $a, b \geq 1$.

Denote by Π the set of prime closed geodesics on X or as we have noted what is the same, the set of primitive hyperbolic conjugacy classes in Γ. If $\{A\}_\Gamma$ is such a class then A is conjugate in $SL_2(\mathbb{R})$ to $\pm \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ with $\lambda > 1$. Let $N(A) := \lambda^2$, then the length $\ell(A)$ of the corresponding closed geodesic, or the period of the corresponding periodic orbit of the geodesic flow, is $\log N(A)$. In terms of $t(A)$, $N(A) = \left(\frac{t(A)+\sqrt{t(a)^2-4}}{2}\right)^2$. In what follows we order the elements of Π according to their length. This is what is done in Figure 1 where the size of A is compared with the complexity of the knot k_A. The trefoil τ corresponds to the cusp at infinity of Y and we will see that this is the source of the linking of k_A with τ being singularly large. Using hyperbolic geometry it follows from the definition of ψ that

$$|\psi(A)| \ll e^{\ell(A)/2} \sim t(A)\,, \tag{2}$$

and this bound is sharp (for example take $A = (UV)^m(UV^{-1}))$. That $\psi(A)$ can be as large as indicated in (2) should be compared to what happens if we order the elements of Γ by word length ℓ_w relative to some generators (see [C] chapter 2 for a discussion). With respect to ℓ_w the cusp plays no special role and

$$|\psi(A)| \ll \ell_w(A)\,. \tag{3}$$

Our goal is to count prime geodesics which satisfy various conditions. For the full count set for $y \geq 1$,

$$\pi(y) = |\{C \in \Pi : \ell(C) \leq y\}|\,. \tag{4}$$

The prime geodesic theorem for X in its strongest presently known form [S-Y] asserts that

$$\pi(y) = \mathsf{Li}(e^y) + O(e^{\alpha y}) \tag{5}$$

for any fixed $\alpha > \frac{25}{36}$.

Here Li is the familiar logarithmic integral from the theory of prime numbers

$$\mathsf{Li}\,(x) = \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x} \text{ as } x \to \infty\,. \tag{6}$$

We will also need the following variation on Li. For $n \in \mathbb{Z}$ and $x \geq 2$ set

$$\mathsf{Li}(x;n) \;=\; \int_2^x \frac{\log t}{(\log t)^2 + \left(\frac{\pi n}{3}\right)^2}\, dt \tag{7}$$

As in the theory of prime numbers in progressions, Chebotarev's theorem, their generalizations and analogues for prime geodesics [Sa2], we define the counting function corresponding to a given linking number $n \in \mathbb{Z}$ by

$$\pi(y;n) \;:=\; \left|\{C \in \Pi : \ell(C) \leq y,\ \mathsf{link}(k_c,\tau) = n\}\right| \,. \tag{8}$$

According to (2), $\pi(y;n) = 0$ if $|n| \gg e^{y/2}$. Our main result is

Theorem 1. *For $y \geq 1$, $n \in \mathbb{Z}$*

$$\sum_{\substack{\ell(C) \leq y \\ \mathsf{link}\,(k_C,\tau) = n}} \ell(C) \;=\; \frac{1}{3}\,\mathsf{Li}\,(e^y;n) \;+\; O(e^{3y/4})\,.$$

Besides identifying the main term a key point in Theorem 1 is the uniformity in n (the implied constant is absolute) and in particular the main term is dominant for $|n|$ as large as $e^{y/8}$. As an immediate consequence we have:

Corollary 2. *For n fixed as $y \to \infty$*

$$\pi(y;n) \;\sim\; \frac{\pi(y)}{3y}\left(1 \;+\; \frac{2\left(1-\left(\frac{\pi n}{3}\right)^2\right)}{y^3} \;+\; O(y^{-3})\right) \;,$$

or in terms of $t(A)$

$$\sum_{\substack{\{A\}_\Gamma \in P \\ t(A) \leq x \\ \mathsf{link}(k_A,\tau) = n}} 1 \;\sim\; \frac{x^2}{12(\log x)^2},\ \ \textit{as } x \to \infty\,.$$

Thus, to leading order the number of prime geodesics with a given linking number is independent of n. However, the next order term ensures that $\pi(y;n) > \pi(y;m)$, if $|n| < |m|$ and y is large. Hence the most common linking number is 0. Among these elements of Π with $\mathrm{link}(k_c,\tau) = 0$ are the reciprocal geodesics (see [Sa3] for a discussion and the relation to elements of order 4 in Gauss' composition group) which are the fixed points of the involution r of Π given by $\{A\}_\Gamma \to \{A^{-1}\}_\Gamma$. As shown in [Sa3] the number of reciprocal geodesics whose length is at most y is asymptotically $3/4\, ye^{y/2}$, so these constitute roughly square root of the number of geodesics with zero linking number.

With the strong uniformity in Theorem 1 we determine the distribution of the numbers $\mathsf{link}(k_A,\tau)$ by summing over n with $|n| \ll y$. The normal order of $\mathsf{link}(k_A,\tau)$ turns out to be $\ell(A)$ and the corresponding distribution a Cauchy distribution.

Theorem 3. *For $-\infty \le a \le b \le \infty$, as $y \to \infty$*

$$\frac{\left|\{C \in \Pi : \ell(C) \le y, a \le \frac{\mathsf{link}(k_c,\tau)}{\ell(C)} \le b\}\right|}{\pi(y)} \to \frac{\mathsf{arctan}\left(\frac{\pi b}{3}\right) - \mathsf{arctan}\left(\frac{\pi a}{3}\right)}{\pi}.$$

The analogues of Corollary 2 and Theorem 3 are known in the setting of compact hyperbolic surfaces $\mathbb{H}/\Gamma$ and where in place of ψ we have a morphism $\Phi : \Gamma \to \mathbb{Z}$ (see [A-S], [P-S], [Sh], [P-R1]). In that case, we are counting the winding numbers of closed geodesics in homology classes. The notable differences in this compact morphism case are that $\Phi(C) \ll \ell(C)$, $\pi(y; n) \sim c(\Phi)\pi(y)/\sqrt{y}$ as $y \to \infty$, the normal order of $\Phi(C)$ is $\sqrt{\ell(C)}$ and the corresponding limiting distribution is Gaussian. If $\mathbb{H}/\Gamma$ is a non-compact but finite area hyperbolic surface and Φ a morphism of Γ to $\mathbb{Z}$ then the analogue of Corollary 2 is proven in [E]. If Φ is "noncuspidal" then the normal order of Φ is much larger due to the winding around the cusp. According to Theorem 1, Corollary 2 and Theorem 3 this effect persists for our quasi-morphism ψ. The non-local Cauchy distribution in Theorem 3 has appeared before in related contexts. In [V] it came up in connection with questions about the distribution of Dedekind sums while in [Gu-L] it appears in connection with the winding in homology of a generic (in measure) geodesic on the unit tangent bundle of a non-compact hyperbolic surface and in [F] in a similar analysis of the winding about τ of a generic orbit of $\mathcal{G}_t$ in Y. On the other hand, in the recent book ([C] Chapter 6) it is shown that when ones orders the values of a quite general integer valued quasimorphism (including our ψ!) by word length ℓ_w, the normal order is $\sqrt{\ell_w(B)}$ for B in the corresponding group and the limiting distribution is Gaussian. Thus ordering combinationally by word length removes the singular behavior associated with the cusp (see also [Ri] and [P-R2] for the case of morphisms and conjugacy classes).

We impose further conditions on our counting of prime geodesics. Fix a knot κ in S^3 and set

$$\pi(y; \kappa) = |\{C \in \Pi : k_c = \kappa,\ \ell(C) \le y\}| \tag{9}$$

and

$$\pi(y; \kappa, n) = |\{C \in \Pi : k_c = \kappa,\ \mathsf{link}(k_c, \tau) = n,\ \ell(C) \le y\}|\,. \tag{10}$$

As noted before it appears difficult to characterize the κ's for which $\pi(y; \kappa) \ne 0$ for some y however once κ is known to appear then the asymptotics in (9) is more approachable. For example, if κ_0 is the trivial knot then using the third of Ghys' results mentioned earlier, the following is deduced using elementary number theoretic arguments.

Proposition 4.

(i) $\pi(y; \kappa_0) \sim \frac{y}{2}\, e^{y/2}$ *as* $y \to \infty$.

(ii) $\pi(y; \kappa_0, n) \sim e^{y/4}$ *for* n *fixed as* $y \to \infty$.

(iii) *The normal order of* $\mathsf{link}(k_A, \tau)$ *when* $\{A\}$ *is conditioned to have* $k_A = \kappa_0$ *is exponential in* $\ell(A)$ *with exponent ranging in* $[1/4, 1/2]$ *and it has the following limit distribution: For* I *a subinterval of* $\mathbb{R}$

$$\frac{\left|\{C \in \Pi :\ \ell(C) \le y, k_c = \kappa_0,\ \frac{\log^+ \mathsf{link}(k_c,\kappa_0)}{2y} \in I\}\right|}{\pi(y;\kappa_0)} \to \lambda(I \cap I_0),\ \ as\ y \to \infty$$

where λ is Lebesgue measure, $I_0 = [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$ and $\log^+ m = \log m$ if $m > 0$, $-\log(-m)$ if $m < 0$ and is 0 if $m = 0$.

The fixed points of the involution of Π given by $\{A\}_\Gamma \to \{w^{-1}Aw\}_\Gamma$ where $w = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ are called ambiguous prime geodesics in [Sa3]. Via the natural identification of Π with primitive classes of indefinite integral binary quadratic forms (see (27) in [Sa3]) the ambiguous geodesics correspond to Gauss' ambiguous classes of forms, that is the elements of order 2 in his class group. One checks that any $\{A\}_\Gamma \in \Pi$ for which $k_A = \kappa_0$, is ambiguous. However, these don't account for all ambiguous geodesics since as shown in [Sa3] their number with length at most y is asymptotic to $c_1 y^2 e^{y/2}$, for a non-zero constant c_1.

Perhaps the simplest prime geodesics C in terms of their knots k_c, are those for which $k_c = \kappa_0$ and $\mathsf{link}(k_c, \tau) = 0$. The first knot in Figure 1 is such an example. Their count is given by (ii) in Proposition 4 and they have an algebraic description in terms of the class group. They correspond via the identification and notations in [Sa3] to $\overline{[1, a, -1]}$ with $a \geq 1$ and whose discriminant $d = a^2 + 4$. For each such d the above class is the identity class in Gauss' class group of that discriminant.

An interesting problem is to investigate the analogue of Proposition 4 for nontrivial knots $\mathcal{K}$.

§2. Outline of proofs:

To prove the main Theorem 1 we will use the Selberg trace formula for the group $SL_2(\mathbb{Z})$ with suitable multiplier systems whose weights are real numbers r. In order to relate ψ to these we use the allied function $\Phi : \Gamma \to \mathbb{Z}$ (see page 49 equation 60 of [R-G]) which is defined via the Dedekind eta function $\eta(z)$: For $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL_2(\mathbb{Z})$

$$\log \eta\left(\frac{az+b}{cz+a}\right) - \log \eta(z) \;=\; \frac{1}{2}\,\mathsf{sgn}^2(c)\,\log\left(\frac{cz+d}{i\mathsf{sign}(c)}\right) + \frac{\pi i}{12}\,\Phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right). \tag{11}$$

From this we can relate Φ to the multiplier system $v_{1/2}$ of the eta function that is;

$$\eta(Az) = v_{1/2}(A)(cz+d)^{1/2}\,\eta(z) \text{ for } A \in SL_2(\mathbb{Z}). \tag{12}$$

(Note that $v_{1/2}$ is defined on $SL_2(\mathbb{Z})$ and not $PSL_2(\mathbb{Z})$ unlike Φ and Ψ). From (11) and (12) and the relation between Φ and Ψ (see page 54 of [R-G]) we have that for $A \in SL_2(\mathbb{Z})$ and trace$(A) > 0$

$$v_{1/2}(A) \;=\; e^{i\pi\Psi(A)/12}. \tag{13}$$

Hence for any $r \in \mathbb{R}$ the multiplier system for $SL_2(\mathbb{Z})$ of weight r given by $(v_{1/2})^{r/2}$ satisfies

$$v_r(A) \;=\; e^{i\pi r\psi(A)/6} \tag{14}$$

for $A \in SL_2(\mathbb{Z})$ and trace$(A) > 0$.

Consider now the spectral problem for the Laplacian $\triangle_r$ on $L^2(\mathbb{H}/SL_2(\mathbb{Z}), v_r, r)$ (we use the definitions and notations of [He]) that is for functions on $\mathbb{H}$ transforming by

$$f(\gamma z) \;=\; v_r(\gamma)\left(\frac{cz+d}{|cz+d|}\right)^r f(z)\,,\ \text{for } \gamma \in SL_2(\mathbb{Z}). \tag{15}$$

We restrict to $-6 < r \leq 6$ with the critical interval being $-1 \leq r \leq 1$. In the last range the bottom eigenvalue $\lambda_0(r)$ of $\triangle_r$ is equal to $\frac{|r|}{2}(1-\frac{|r|}{2})$ and this is the only eigenvalue in $[0,1/4)$, see [Br]. This smallest eigenvalue and in particular its singular behavior near $r=0$ is responsible for the shape of the main term in Theorem 1 and for the large linking of the modular knots with the trefoil. For $r=-4,-2,2,4,6$ the multiplier system corresponds to each of the 5 non-trivial characters of Γ and we set up our multiplier system on intervals of width 2 about these and with adjusted weights in [-1,1]. For each of these it follows from [Br] that there are no exceptional eigenvalues, that is eigenvalues below 1/4. With this input and the general trace formula derived in [He] for this space for each fixed r, one derives the analogue of a "twisted" prime geodesic theorem. Using the technique in [Sa2] to derive these with a careful analysis of the dependence in r, one shows that uniformly for $-6 \leq r \leq 6$ and $x \geq 5$;

$$\sum_{\substack{\{\gamma\}_{SL_2(\mathbb{Z})} \\ \text{trace}(\gamma)>2 \\ N(\gamma)\leq x \\ \gamma \ \text{primitive}}} \log N(\gamma)\, v_r(\gamma) = \begin{cases} \frac{x^{1-\frac{|r|}{2}}}{1-\frac{|r|}{2}} + O\left(x^{3/4}\log\frac{1}{|r|}\right), & \text{if } |r| \leq \frac{1}{2} \\ O(x^{3/4}) & \text{otherwise} \end{cases} . \tag{16}$$

Integrating both sides of (16) against $e^{-i\pi nr/6}$ with respect to r over (- 6 , 6] and using (14) the left hand side becomes

$$12 \sum_{\substack{\{\gamma\}_\Gamma \in \Pi \\ N(\gamma)\leq x \\ \psi(\gamma)=n}} \log N(\gamma)\,. \tag{17}$$

The right hand side is equal to

$$\int_{-1/2}^{1/2} \frac{x^{1-\frac{|r|}{2}}}{1-\frac{|r|}{2}} e^{-i\pi nr/6}\, dr + O(x^{3/4}) = 4\,\mathsf{Li}\,(x;n) + O(x^{3/4}) \quad \text{uniformly in } x \geq 2 \text{ and } \in \mathbb{Z}\,, \tag{18}$$

or

$$\sum_{\substack{\ell(C)\leq y \\ \psi(c)=n}} \ell(C) = \frac{1}{3}\,\mathsf{Li}\,(e^y;n) + O(e^{3y/4})\,. \tag{19}$$

This proves Theorem 1. Corollary 2 is an immediate consequence of the theorem and Theorem 3 follows by summing over n as indicated above.

We turn to Proposition 4. The conjugacy classes $\{(UV)^a\,(UV^{-1})^b\}_\Gamma$ for $a,b \geq 1$ are all primitive and distinct elements of Π. This can be seen from the fact that $\Gamma = \langle U\rangle * \langle V\rangle$ where $\langle U\rangle = \mathbb{Z}/2\mathbb{Z}$ and $\langle V\rangle = \mathbb{Z}/3\mathbb{Z}$, so that for each $a,b \geq 1$ $(UVUV\cdots UV)\,(UV^{-1}\cdots UV^{-1})$ is cyclically reduced. Hence its length $2(a+b)$ and $\psi((UV)^a(UV^{-1})^b) = a-b$ are determined by $\{(UV)^a\,(UV^{-1})^b\}_\Gamma$ and hence so is (a,b). A similar consideration with the free product shows that these classes are primitive. Thus according to Ghys [G page 272] the map

$$(a,b) \longrightarrow \{(UV)^a\,(UV^{-1})^b\}_\Gamma = \left\{\begin{bmatrix} 1 & -b \\ -a & ab+1 \end{bmatrix}\right\}_\Gamma$$

is a bijection from $\mathbb{N} \times \mathbb{N}$ to the elements C of Π with $k_c = \kappa_0$. Hence

$$\sum_{\substack{\{A\}_\Gamma \in \Pi \\ t(A) \le x \\ k_A = \kappa_0}} 1 \;=\; \sum_{\substack{ab+2 \le x \\ a,b \ge 1}} 1\,. \tag{20}$$

It goes back at least to Dirichlet that the right hand side of (20) is asymptotic to $x \log x$ which proves part (i) of Proposition 4. (ii) is even easier as it the same count as in (20) but with $a - b = n$ fixed. As for (iii) we are counting say for $0 \le \alpha \le 1$

$$\sum_{\substack{ab \le x \\ a \le b \\ x^\alpha \le b-a}} 1 \;=\; \sum_{a \le \sqrt{x}} \;\; \sum_{x^\alpha + a \le b \le \frac{x}{a}} 1\,. \tag{21}$$

For $\alpha \le \frac{1}{2}$ this is asymptotic to $1/2x \log x$. While for $\frac{1}{2} \le \alpha \le 1$ it is

$$\sim \sum_{a \le x^{1-\alpha}} \frac{x}{a} \sim (1-\alpha)\, x \, \log x,$$

which proves part (iii) of the Proposition.

Finally, note that

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -b \\ -a & 1+ab \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+ab & -b \\ -a & 1 \end{bmatrix}$$

and hence

$$\left\{ \begin{bmatrix} 1 & -b \\ -a & 1+ab \end{bmatrix} \right\}_\Gamma = \left\{ w^{-1} \begin{bmatrix} 1 & -b \\ -a & 1+ab \end{bmatrix}^{-1} w \right\}_\Gamma ,$$

that is $\left\{ \begin{bmatrix} 1 & -b \\ -a & 1+ab \end{bmatrix} \right\}_\Gamma$ is ambiguous. When $a = b$, the elements $C_a = \left\{ \begin{bmatrix} 1 & -a \\ -a & 1+a^2 \end{bmatrix} \right\}_\Gamma$ are those for which $k_c = \kappa_0$ and $\mathsf{link}(k_{c_a}, \tau) = 0$ (these geodesics are both ambiguous and reciprocal) and the corresponding class of binary forms according to the identification (27) of [Sa3], is $\overline{[1, a, -1]}$. This is the identity class in the Gauss class group of discriminant $a^2 + 4$.

Acknowledgments:

Thanks to Zeev Rudnick for pointing me to the book [C] and to Danny Calegari for exchanges about the statistics of quasimorphisms.

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