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STABILITY OF THE PEXIDERIZED CAUCHY AND JENSEN'S EQUATIONS ON RESTRICTED DOMAINS

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Abstract

The generalized Hyers–Ulam stability for the Pexiderized Cauchy and Jensen's equations on restricted domains is investigated. As an application we study an asymptotic behavior of additive mappings.

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Keywords: generalized Hyers–Ulam stability, additive mapping, Banach space, Pexiderized Cauchy's equation, Pexiderized Jensen's equation.

1 Introduction

The following question concerning the stability of group homomorphisms was posed by Ulam [24]: *Under what conditions does there exist a group homomorphism near an approximate group homomorphism?*

Hyers [8] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \quad \text{for all} \ \ x, y \in E.$$

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In 1950, Aoki [2] provided a generalization of the Hyers' theorem for additive mappings and in 1978, Th.M. Rassias [19] generalized the Hyers' theorem for linear mappings by allowing the Cauchy difference to be unbounded (see also [3]). The result of Th.M. Rassias' theorem has been generalized Găvruta [7] who permitted the Cauchy difference to be bounded by a general control function. During the last three decades a number of papers have been published on the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [4], [6], [11], [15]–[18] and [20]). We also refer the readers to the books [1], [5], [10] and [14].

In [12] the authors investigated the stability of the generalized functional equation of Pexider type. In this paper, we investigate the stability problem for the Pexiderized Cauchy and Jensen's equations on restricted domains. As an application we study the asymptotic behavior of the additive mappings.

Throughout this paper, let $\mathbb{X}$ be a real normed space and $\mathbb{Y}$ be a real Banach space, respectively.

2 Generalized Hyers-Ulam stability on a restricted domain

Soon-Mo Jung investigated the Hyers–Ulam stability for Jensen's equation on a restricted domain. In the proof of the following theorem, we have used the method of S.-M. Jung [13].

Theorem 2.1. *Let $d > 0$ be given. If $f, g, h : \mathbb{X} \to \mathbb{Y}$ satisfy*

$$2f\Big(\frac{x+y}{2}\Big) = g(x) + h(y) \tag{2.1}$$

for all $x, y \in \mathbb{X}$ with $\|x\| + \|y\| \geq d$, then (2.1) holds true for all $x, y \in \mathbb{X}$.

Proof. Suppose $\|x\| + \|y\| < d$. If $\|x\| + \|y\| = 0$, let $z \in \mathbb{X}$ with $\|z\| = d$, otherwise

$$z := \begin{cases} (d + \|x\|)\frac{x}{\|x\|}, & \text{if } \|x\| \geq \|y\|; \\ (d + \|y\|)\frac{y}{\|y\|}, & \text{if } \|y\| \geq \|x\|. \end{cases}$$

It is easy to verify that

$$\begin{aligned} &\|x - z\| + \|y + z\| \geq d; \quad \|2z\| + \|x - z\| \geq d; \quad \|y\| + \|2z\| \geq d; \\ &\|y + z\| + \|z\| \geq d; \quad \|x\| + \|z\| \geq d. \end{aligned} \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\begin{aligned}2f\big(\frac{x+y}{2}\big)-g(x)-h(y) &= 2f\big(\frac{x+y}{2}\big)-g(y+z)-h(x-z)\\ &\quad-\Big[2f\big(\frac{x+z}{2}\big)-g(2z)-h(x-z)\Big]\\ &\quad+\Big[2f\big(\frac{y+2z}{2}\big)-g(2z)-h(y)\Big]\\ &\quad-\Big[2f\big(\frac{y+2z}{2}\big)-g(y+z)-h(z)\Big]\\ &\quad+\Big[2f\big(\frac{x+z}{2}\big)-g(x)-h(z)\Big]\\ &=0.\end{aligned} \tag{2.3}$$

□

The following theorem is another version of Theorem 2.1 for the Pexiderized Cauchy's equation and its proof is similar to the proof Theorem 2.1.

Theorem 2.2. *Let $d>0$ be given. If $f,g,h:\mathbb{X}\to\mathbb{Y}$ satisfy*

$$f(x+y)=g(x)+h(y) \tag{2.4}$$

for all $x,y\in\mathbb{X}$ with $\|x\|+\|y\|\geq d$, then (2.4) holds true for all $x,y\in\mathbb{X}$.

Theorem 2.3. *Let $d>0$, $p\in(0,1)$ and $\theta,\varepsilon\geq 0$ be given. Assume that $f,g,h:\mathbb{X}\to\mathbb{Y}$ satisfy the inequality*

$$\Big\|2f\big(\frac{x+y}{2}\big)-g(x)-h(y)\Big\|\leq\theta+\varepsilon(\|x\|^p+\|y\|^p) \tag{2.5}$$

for all $x,y\in\mathbb{X}$ with $\|x\|+\|y\|\geq d$. Then there exist a unique additive mapping $T:\mathbb{X}\to\mathbb{Y}$ and constants $C_1,C_2,C_3\geq 0$ such that

$$\begin{aligned}\|f(x)-T(x)\|&\leq C_1+\frac{6\times 2^p}{2-2^p}\varepsilon\|x\|^p,\\ \|g(x)-T(x)\|&\leq C_2+\frac{3(6-2^p)}{2-2^p}\varepsilon\|x\|^p,\\ \|h(x)-T(x)\|&\leq C_3+\frac{3(6-2^p)}{2-2^p}\varepsilon\|x\|^p\end{aligned} \tag{2.6}$$

for all $x\in\mathbb{X}$.

Proof. For the case $\|x\|+\|y\|<d$, let z be an element of $\mathbb{X}$ which is defined in the proof of

Theorem 2.1. It is clear that $\|z\| \le 2d$. Using (2.2), (2.3) and (2.5), we get

$$\begin{aligned}\Big\|2f\Big(\frac{x+y}{2}\Big)-g(x)-h(y)\Big\| &\le \Big\|2f\Big(\frac{x+y}{2}\Big)-g(y+z)-h(x-z)\Big\| \\ &+\Big\|2f\Big(\frac{x+z}{2}\Big)-g(2z)-h(x-z)\Big\| \\ &+\Big\|2f\Big(\frac{y+2z}{2}\Big)-g(2z)-h(y)\Big\| \\ &+\Big\|2f\Big(\frac{y+2z}{2}\Big)-g(y+z)-h(z)\Big\| \\ &+\Big\|2f\Big(\frac{x+z}{2}\Big)-g(x)-h(z)\Big\| \\ &\le 5\theta+(3+2^p)2^{p+1}\varepsilon+3\varepsilon(\|x\|^p+\|y\|^p)\end{aligned}$$

for all $x,y\in\mathbb{X}$ with $\|x\|+\|y\|<d$. Hence

$$\Big\|2f\Big(\frac{x+y}{2}\Big)-g(x)-h(y)\Big\| \le K+3\varepsilon(\|x\|^p+\|y\|^p) \tag{2.7}$$

for all $x,y\in\mathbb{X}$, where

$$K := 5\theta+(3+2^p)2^{p+1}\varepsilon.$$

By the result of [12], there exists a unique additive mapping $T:\mathbb{X}\to\mathbb{Y}$ given by

$$T(x)=\lim_{n\to\infty}2^{-n}f(2^nx)=\lim_{n\to\infty}2^{-n}g(2^nx)=\lim_{n\to\infty}2^{-n}h(2^nx)$$

such that

$$\begin{aligned}\|f(x)-T(x)\| &\le \frac{1}{2}\Big[3K+\|g(0)\|+\|h(0)\|\Big]+\frac{6\times 2^p}{2-2^p}\varepsilon\|x\|^p,\\ \|g(x)-T(x)\| &\le 4K+\|g(0)\|+2\|h(0)\|+\frac{3(6-2^p)}{2-2^p}\varepsilon\|x\|^p,\\ \|h(x)-T(x)\| &\le 4K+2\|g(0)\|+\|h(0)\|+\frac{3(6-2^p)}{2-2^p}\varepsilon\|x\|^p\end{aligned}$$

for all $x\in\mathbb{X}$. □

We apply the result of Theorem 2.3 to study the asymptotic behavior of additive mappings. An asymptotic property of additive mappings has been proved by F. Skof [22] (see also [9], [13]).

Corollary 2.4. *Let $f:\mathbb{X}\to\mathbb{Y}$ be mappings with $f(0)=g(0)=h(0)=0$ satisfying*

$$\Big\|2f\Big(\frac{x+y}{2}\Big)-g(x)-h(y)\Big\| \to 0 \quad as \quad \|x\|+\|y\|\to\infty. \tag{2.8}$$

Then f,g,h are additive and $f=g=h$.

Proof. It follows from (2.8) that there exists a sequence $\{\delta_n\}$, monotonically decreasing to zero, such that

$$\Big\|2f\Big(\frac{x+y}{2}\Big)-g(x)-h(y)\Big\| \le \delta_n \tag{2.9}$$

for all $x, y \in \mathbb{X}$ with $\|x\| + \|y\| \geq n$. Applying (2.9) and Theorem 2.3, we obtain a sequence $\{T_n : \mathbb{X} \to \mathbb{Y}\}$ of unique additive mappings satisfying

$$\begin{aligned} \|f(x) - T_n(x)\| &\leq \frac{15}{2}\delta_n, \\ \|g(x) - T_n(x)\| &\leq 20\delta_n, \\ \|h(x) - T_n(x)\| &\leq 20\delta_n \end{aligned} \tag{2.10}$$

for all $x \in \mathbb{X}$. Since the sequence $\{\delta_n\}$ is monotonically decreasing, we conclude

$$\begin{aligned} \|f(x) - T_m(x)\| &\leq \frac{15}{2}\delta_m \leq \frac{15}{2}\delta_n, \\ \|g(x) - T_m(x)\| &\leq 20\delta_m \leq 20\delta_n, \\ \|h(x) - T_m(x)\| &\leq 20\delta_m \leq 20\delta_n \end{aligned}$$

for all $x \in \mathbb{X}$ and all $m \geq n$. The uniqueness of T_n implies $T_m = T_n$ for all $m \geq n$. Hence letting $n \to \infty$ in (2.10), we obtain that f, g, h are additive and $f = g = h$. □

The following theorem is another version of Theorem 2.3 for the Pexiderized Cauchy's equation.

Theorem 2.5. *Let $d > 0$, $p \in (0,1)$ and $\theta, \varepsilon \geq 0$ be given. Assume that $f, g, h : \mathbb{X} \to \mathbb{Y}$ satisfy the inequality*

$$\|f(x+y) - g(x) - h(y)\| \leq \theta + \varepsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in \mathbb{X}$ with $\|x\| + \|y\| \geq d$. Then there exist a unique additive mapping $T : \mathbb{X} \to \mathbb{Y}$ and constants $C_1, C_2, C_3 \geq 0$ such that

$$\begin{aligned} \|f(x) - T(x)\| &\leq C_1 + \frac{12}{2-2^p}\varepsilon\|x\|^p, \\ \|g(x) - T(x)\| &\leq C_2 + \frac{3(6-2^p)}{2-2^p}\varepsilon\|x\|^p, \\ \|h(x) - T(x)\| &\leq C_3 + \frac{3(6-2^p)}{2-2^p}\varepsilon\|x\|^p \end{aligned}$$

for all $x \in \mathbb{X}$.

Proof. Similar to the proof of Theorem 2.3, we have

$$\|f(x+y) - g(x) - h(y)\| \leq K + 3\varepsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in \mathbb{X}$, where

$$K := 5\theta + (3 + 2^p)2^{p+1}\varepsilon.$$

By the result of [12], there exists a unique additive mapping $T : \mathbb{X} \to \mathbb{Y}$ given by

$$T(x) = \lim_{n\to\infty} 2^{-n} f(2^n x) = \lim_{n\to\infty} 2^{-n} g(2^n x) = \lim_{n\to\infty} 2^{-n} h(2^n x)$$

such that

$$\|f(x)-T(x)\| \le 3K+\|g(0)\|+\|h(0)\|+\frac{12}{2-2^p}\varepsilon\|x\|^p,$$
$$\|g(x)-T(x)\| \le 4K+\|g(0)\|+2\|h(0)\|+\frac{3(6-2^p)}{2-2^p}\varepsilon\|x\|^p,$$
$$\|h(x)-T(x)\| \le 4K+2\|g(0)\|+\|h(0)\|+\frac{3(6-2^p)}{2-2^p}\varepsilon\|x\|^p$$

for all $x \in \mathbb{X}$. □

Corollary 2.6. *Let $f: \mathbb{X} \to \mathbb{Y}$ be mappings with $f(0)=g(0)=h(0)=0$ satisfying*

$$\|f(x+y)-g(x)-h(y)\| \to 0 \quad as \quad \|x\|+\|y\| \to \infty. \tag{2.11}$$

Then f,g,h are additive and $f=g=h$.

The following theorem is another version of Theorem 2.3 for the case $p>1$.

Theorem 2.7. *Let $d>0$, $p>1$ and $\varepsilon \ge 0$ be given. Assume that $f,g,h: \mathbb{X} \to \mathbb{Y}$ with $g(0)=h(0)=0$ satisfy the inequality*

$$\left\|2f\Big(\frac{x+y}{2}\Big)-g(x)-h(y)\right\| \le \varepsilon(\|x\|^p+\|y\|^p) \tag{2.12}$$

for all $x,y \in \mathbb{X}$ with $\|x\|+\|y\| \le d$. Then there exist a unique additive mapping $\phi: \mathbb{X} \to \mathbb{Y}$ such that

$$\|\phi(x)-f(x)\| \le \frac{2^{p+1}}{2^p-2}\varepsilon\|x\|^p, \tag{2.13}$$
$$\|\phi(x)-g(x)\| \le \frac{2^p+2}{2^p-2}\varepsilon\|x\|^p, \tag{2.14}$$
$$\|\phi(x)-h(x)\| \le \frac{2^p+2}{2^p-2}\varepsilon\|x\|^p \tag{2.15}$$

for all $x \in \mathbb{X}$ with $\|x\| \le \frac{d}{2}$ and $\phi(x)=\lim_{n\to\infty} 2^n f(2^{-n}x)$ for all $x \in \mathbb{X}$.

Proof. Letting $x=y=0$ in (2.12), we get $f(0)=0$. Letting $y=x$ in (2.12), we get

$$\|2f(x)-g(x)-h(x)\| \le 2\varepsilon\|x\|^p \tag{2.16}$$

for all $x \in \mathbb{X}$ with $\|x\| \le \frac{d}{2}$. If we put $y=0$ in (2.12), we have

$$\left\|2f\Big(\frac{x}{2}\Big)-g(x)\right\| \le \varepsilon\|x\|^p \tag{2.17}$$

for all $x \in \mathbb{X}$ with $\|x\| \le d$. Similarly, we have

$$\left\|2f\Big(\frac{x}{2}\Big)-h(x)\right\| \le \varepsilon\|x\|^p \tag{2.18}$$

for all $x \in \mathbb{X}$ with $\|x\| \le d$. Hence it follows from (2.16), (2.17) and (2.18) that

$$\left\|2f\Big(\frac{x}{2}\Big)-f(x)\right\| \le 2\varepsilon\|x\|^p \tag{2.19}$$

for all $x \in \mathbb{X}$ with $\|x\| \le \frac{d}{2}$. We can replace x by $\frac{x}{2^n}$ in (2.19) for all non-negative integers n. Using a similar argument given in [19], we have

$$\left\|2^{n+1} f(2^{-n-1}x) - 2^n f(2^{-n}x)\right\| \le 2\varepsilon\left(\frac{2}{2^p}\right)^n \|x\|^p. \tag{2.20}$$

Hence we have the following inequality

$$\begin{aligned}\left\|2^{n+1} f(2^{-n-1}x) - 2^m f(2^{-m}x)\right\| &\le \sum_{k=m}^{n} \left\|2^{k+1} f(2^{-k-1}x) - 2^k f(2^{-k}x)\right\| \\ &\le 2\varepsilon \sum_{k=m}^{n} \left(\frac{2}{2^p}\right)^k \|x\|^p\end{aligned} \tag{2.21}$$

for all $x \in \mathbb{X}$ with $\|x\| \le \frac{d}{2}$ and all integers $n \ge m \ge 0$. Since Y is complete, (2.21) shows that the limit $T(x) = \lim_{n\to\infty} 2^n f(2^{-n}x)$ exists for all $x \in \mathbb{X}$ with $\|x\| \le \frac{d}{2}$. It follows from (2.17) and (2.18) that

$$T(x) = \lim_{n\to\infty} 2^n f(2^{-n}x) = \lim_{n\to\infty} 2^n g(2^{-n}x) = \lim_{n\to\infty} 2^n h(2^{-n}x)$$

for all $x \in \mathbb{X}$ with $\|x\| \le \frac{d}{2}$. Letting $m = 0$ and $n \to \infty$ in (2.21), we obtain that T satisfies the inequality (2.13) for all $x \in \mathbb{X}$ with $\|x\| \le \frac{d}{2}$. It follows from the definition of T that $T(0) = 0$ and we conclude from (2.12) that

$$T(x+y) = T(x) + T(y) \tag{2.22}$$

for all $x, y \in \mathbb{X}$ with $\|x\|, \|y\|, \|x+y\| \le \frac{d}{2}$. Hence

$$T\left(\frac{x}{2}\right) = \frac{1}{2} T(x) \tag{2.23}$$

for all $x \in \mathbb{X}$ with $\|x\| \le \frac{d}{2}$. We extend the additivity of T to the whole space $\mathbb{X}$ by using an extension method of Skof [23]. Let $\delta := \frac{d}{2}$ and $x \in \mathbb{X}$ be given with $\|x\| > \delta$. Let $k = k(x)$ be the smallest integer such that $2^{k-1}\delta < \|x\| \le 2^k\delta$. We define the mapping $\phi : \mathbb{X} \to \mathbb{Y}$ by

$$\phi(x) := \begin{cases} T(x) & \text{if } \|x\| \le \delta; \\ 2^k T(2^{-k}x) & \text{if } \|x\| > \delta. \end{cases}$$

Let $x \in \mathbb{X}$ be given with $\|x\| > \delta$ and let $k = k(x)$ be the smallest integer such that $2^{k-1}\delta < \|x\| \le 2^k\delta$. Then $k-1$ is the smallest integer satisfying $2^{k-2}\delta < \left\|\frac{x}{2}\right\| \le 2^{k-1}\delta$. If $k = 1$, we have $\phi(x/2) = T(x/2)$ and $\phi(x) = 2T(x/2)$. Therefore $\phi(x/2) = \frac{1}{2}\phi(x)$. For the case $k > 1$, it follows from the definition of ϕ that

$$\phi\left(\frac{x}{2}\right) = 2^{k-1} T\left(2^{-(k-1)}\frac{x}{2}\right) = \frac{1}{2} \cdot 2^k T(2^{-k}x) = \frac{1}{2}\phi(x).$$

From the definition of ϕ and (2.23), we get $\phi(x/2) = \frac{1}{2}\phi(x)$ holds true for all $x \in \mathbb{X}$. Let $x \in \mathbb{X}$ and let k be an integer such that $\|x\| \le 2^k\delta$. Then

$$\phi(x) = 2^k\phi(2^{-k}x) = 2^k T(2^{-k}x) = \lim_{n\to\infty} 2^{n+k} f(2^{-(n+k)}x) = \lim_{n\to\infty} 2^n f(2^{-n}x).$$

It remains to prove that ϕ is additive. Let $x,y \in \mathbb{X}$ and let n be a positive integer such that $\|x\|, \|y\|, \|x+y\| \leq 2^n\delta$. Since $\phi(x/2) = \frac{1}{2}\phi(x)$ for all $x \in \mathbb{X}$ and T satisfies (2.23), we have

$$\phi(x+y) = 2^n\phi\big(\frac{x+y}{2^n}\big) = 2^nT\big(\frac{x+y}{2^n}\big) = 2^n\Big[T\big(\frac{x}{2^n}\big) + T\big(\frac{y}{2^n}\big)\Big]$$
$$= 2^n\Big[\phi\big(\frac{x}{2^n}\big) + \phi\big(\frac{y}{2^n}\big)\Big] = \phi(x) + \phi(y).$$

Hence ϕ is additive. Also, ϕ satisfies the inequality (2.13) for all $x \in \mathbb{X}$ with $\|x\| \leq \frac{d}{2}$, by the definition of ϕ. To prove (2.14), we have from (2.13) and (2.17) that

$$\begin{aligned}\|\phi(x) - g(x)\| &\leq \Big\|2f\big(\frac{x}{2}\big) - g(x)\Big\| + \Big\|\phi(x) - 2f\big(\frac{x}{2}\big)\Big\| \\ &\leq \varepsilon\|x\|^p + \frac{4}{2^p-2}\varepsilon\|x\|^p \\ &= \frac{2^p+2}{2^p-2}\varepsilon\|x\|^p\end{aligned}$$

for all $x \in \mathbb{X}$ with $\|x\| \leq \frac{d}{2}$. Similarly, one can obtain (2.15). □

The following theorem is another version of Theorem 2.7 for the Pexiderized Cauchy's equation.

Theorem 2.8. *Let $d > 0$, $p > 1$ and $\varepsilon \geq 0$ be given. Assume that $f,g,h : \mathbb{X} \to \mathbb{Y}$ with $g(0) = h(0) = 0$ satisfy the inequality*

$$\|f(x+y) - g(x) - h(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

for all $x,y \in \mathbb{X}$ with $\|x\| + \|y\| \leq d$. Then there exist a unique additive mapping $\phi : \mathbb{X} \to \mathbb{Y}$ such that

$$\begin{aligned}\|\phi(x) - f(x)\| &\leq \frac{4}{2^p-2}\varepsilon\|x\|^p, \\ \|\phi(x) - g(x)\| &\leq \frac{2^p+2}{2^p-2}\varepsilon\|x\|^p, \\ \|\phi(x) - h(x)\| &\leq \frac{2^p+2}{2^p-2}\varepsilon\|x\|^p\end{aligned}$$

for all $x \in \mathbb{X}$ with $\|x\| \leq d$ and $\phi(x) = \lim_{n\to\infty} 2^n f(2^{-n}x)$ for all $x \in \mathbb{X}$.

Proof. Similar to the proof of Theorem 2.7, we have

$$\|f(2x) - 2f(x)\| \leq 4\varepsilon\|x\|^p$$

for all $x \in \mathbb{X}$ with $\|x\| \leq \frac{d}{2}$. So

$$\Big\|2f\big(\frac{x}{2}\big) - f(x)\Big\| \leq \frac{4}{2^p}\varepsilon\|x\|^p \tag{2.24}$$

for all $x \in \mathbb{X}$ with $\|x\| \le d$. Hence (2.24) implies that

$$\begin{aligned} \left\|2^{n+1}f(2^{-n-1}x) - 2^m f(2^{-m}x)\right\| &\le \sum_{k=m}^{n} \left\|2^{k+1}f(2^{-k-1}x) - 2^k f(2^{-k}x)\right\| \\ &\le \frac{4}{2^p}\varepsilon \sum_{k=m}^{n} \left(\frac{2}{2^p}\right)^k \|x\|^p \end{aligned} \tag{2.25}$$

for all $x \in \mathbb{X}$ with $\|x\| \le d$ and all integers $n \ge m \ge 0$. Since Y is complete, (2.25) shows that the limit $T(x) = \lim_{n\to\infty} 2^n f(2^{-n}x)$ exists for all $x \in \mathbb{X}$ with $\|x\| \le d$. The rest of the proof is similar to the proof of Theorem 2.7 and we omit the details. □

Remark 2.9. Th.M. Rassias and P. Šemrl [21] have constructed the continuous mapping $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x\log_2(x+1) & \text{for } x \ge 0; \\ x\log_2|x-1| & \text{for } x < 0. \end{cases}$$

It follows from [13] and [21] that the mapping f satisfies in the following inequalities

$$|f(x+y) - f(x) - f(y)| \le |x| + |y|,$$
$$\left|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right| \le 2(|x| + |y|)$$

for all $x, y \in \mathbb{R}$, and the range of $|f(x) - A(x)|/|x|$ for $x \ne 0$ is unbounded for each additive mapping $A : \mathbb{R} \to \mathbb{R}$. So the mapping f serves as a counterexample to Theorems 2.3, 2.5, 2.7 and 2.8 for the case $p = 1$.

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