Communications in **M**athematical **A**nalysis
Special Volume in Honor of Prof. Peter Lax
Volume 8, Number 1, pp. 12–25 (2010)
ISSN 1938-9787
www.commun-math-anal.org

Asymptotically Outgoing and Incoming Spaces and Quantum Scattering

Hitoshi Kitada*
Graduate School of Mathematical Sciences
University of Tokyo
Komaba, Meguro-ku, Tokyo 153-8914, Japan

(Communicated by Toka Diagana)

Abstract

The notion of outgoing and incoming spaces of Lax-Phillips [17] is generalized to *asymptotically* outgoing and incoming spaces. With this notion of asymptotically outgoing and incoming spaces, it is shown that the existence and asymptotic completeness of wave operators in quantum scattering theory is obtained by a slightly modified proof of Theorem 1.2 in [17].

AMS Subject Classification: Primary 35P25, 81U05 ; Secondary 47A40, 35J10, 35S30.

Keywords: Schrödinger equation, scattering theory, long-range potentials, Fourier integral operators, micro-local analysis.

1 Introduction

Scattering theory for hyperbolic equations like acoustic wave equations has been developed by P. D. Lax and R. S. Phillips since around 1960's at almost the same age as the development of quantum scattering theory for Schrödinger equations. With some initial works of considering the existence and construction of the solutions for those equations in Lax [13], Lax-Phillips [14], e.g., they developed an original abstract scattering theory and its applications in [15], [16], [17], [18], [19], ...

When the author was a graduate student, his supervisor recommended him to read their works, in addition to studying the literature of quantum scattering theory. He said that the time dependent method which they developed for hyperbolic equations might be able to be applied to the scattering theory for Schrödinger equations.

When the author first visited the US in 1980's to attend a conference, he had an opportunity to meet Professor Lax at a party during the conference. He asked Professor Lax if his method is applicable to quantum scattering theory. He answered "No." He felt sorry to have

*E-mail address: kitada@ms.u-tokyo.ac.jp

disturbed him with such an elementary question, so he apologized “I am sorry” with no knowledge of English speaking culture. In the author’s country, it was customary to say “I am sorry” often in daily life. However when he said the words to Professor Lax, it seemed that the situation might not have been appropriate in their custom. A friend explained that it is daily custom to say so in every situation in his country. Professor Lax seemed to have kindly understood it.

The author heard from the Editor-in-Chief Professor Toka Diagana of Communications in Mathematical Analysis that the journal will publish a special volume in honor of Professor Lax, and he asked the author if he would write a paper at this opportunity. He reread some of their works and noticed that it is possible to extend their method to accommodate quantum scattering theory with a slight modification. It has passed a long time since he questioned him. He hopes that this paper would be a compensation for his ignorance about English custom.

2 Asymptotically Outgoing and Incoming Spaces

We consider a Schrödinger operator of the form

$$H = H_0 + V(x) \tag{2.1}$$

defined in $\mathcal{H} = L^2(\mathbb{R}^n)$ $(n \geq 1)$. Here $H_0 = -\frac{1}{2}\Delta$, where

$$\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$$

is Laplacian with domain $\mathcal{D}(H_0) = \mathcal{D}(\Delta) = H^2(\mathbb{R}^n)$, the Sobolev space of order two. The potential $V(x)$ satisfies the following assumption. We use the notation: $\partial_x = (\partial/\partial_{x_1}, \cdots, \partial/\partial_{x_n})$, $\partial_x^\alpha = (\partial/\partial_{x_1})^{\alpha_1} \cdots (\partial/\partial_{x_n})^{\alpha_n}$ for a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $\alpha_j \geq 0$ being an integer, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $\langle y \rangle = (1 + |y|^2)^{1/2}$ for $y \in \mathbb{R}^d$ $(d \geq 1)$.

Assumption $V(x)$ is a real-valued C^∞ function of $x \in \mathbb{R}^n$ such that the derivatives $\partial_x^\alpha V(x)$ satisfy the condition:

There exists a constant ε $(1 > \varepsilon > 0)$ such that for any multi-index α

$$|\partial_x^\alpha V(x)| \leq C_\alpha \langle x \rangle^{-|\alpha| - \varepsilon}$$

with some constant $C_\alpha > 0$ independent of $x \in \mathbb{R}^n$.

H is considered a self-adjoint operator with $\mathcal{D}(H) = H^2(\mathbb{R}^n)$. The potentials satisfying the above assumption are called long-range potentials. It is of course possible to include short-range potential $V_S(x)$ with local singularities satisfying for example as in [2]

$$h(R) = \|V_S(H_0+1)^{-1}\chi_{\{x||x|>R\}}\| \in L^1((0,\infty)),$$

where $\chi_B(x)$ denotes the characteristic function of a set B, and $\|T\|$ denote the operator norm of an operator T from $\mathcal{H} = L^2(\mathbb{R}^n)$ into itself. For the sake of simplicity we here only consider the long-range potential $V(x)$ stated above.

Let $U_0(t)=e^{-itH_0}$ $(t\in\mathbb{R})$ be a unitary group generated by H_0.

Lax-Phillips ([17]) considers the hyperbolic equation like

$$\frac{\partial^2 u}{\partial t^2}(x,t)=\Delta u(x,t)$$

in free space $\mathbb{R}^n$ or in an exterior domain G with some boundary conditions. For the free case, the corresponding Hilbert space $\mathcal{H}_0^{LP}$ consists of all initial data $d=(u(x,0),u_t(x,0))$ having the energy norm:

$$\|d\|^2=\int_{\mathbb{R}^n}\left[|\partial_x u(x,0)|^2+|\partial_t u(x,0)|^2\right]dx.$$

For the propagator $U_0^{LP}(t)$ for such an equation, thanks to the finite speed of wave propagation, there is a natural choice of two closed subspaces D_+ and D_- of the Hilbert space $\mathcal{H}_0^{LP}$ called outgoing and incoming subspaces[1] with the following properties ([17]):

$$U_0^{LP}(t)D_\pm\subset D_\pm \quad \text{for} \quad \pm t\geq 0, \tag{2.2}$$

$$\bigcap_{t\in\mathbb{R}} U_0^{LP}(t)D_\pm=\{0\}, \tag{2.3}$$

$$\overline{\bigcup_{t\in\mathbb{R}} U_0^{LP}(t)D_\pm}=\mathcal{H}. \tag{2.4}$$

However in the case of our Hamiltonian in (2.1), the propagation speed of the wave function by the corresponding unitary propagator $U_0(t)$ is infinite. Therefore we cannot have such subspaces $D_\pm$ which exactly satisfy these properties. Nevertheless we can define subspaces $D_\pm$ which satisfy the properties (2.2)-(2.4) in an approximate sense as follows.

We fix constants a,b with $0<a<b<\infty$ arbitrarily and define a subspace $\mathcal{H}(a,b)$ of $\mathcal{H}$ by

$$\mathcal{H}(a,b)=E_0([a,b])\mathcal{H}, \tag{2.5}$$

where $E_0(B)$ is the spectral measure of the Hamiltonian H_0 for Borel sets $B\subset\mathbb{R}$.

Let $-1<\theta_1<\theta_2<1$ and let $\chi_\pm^{\theta_1,\theta_2}(\tau)\in C^\infty(\mathbb{R})$ satisfy $0\leq\chi_\pm^{\theta_1,\theta_2}(\tau)\leq 1$, $\chi_+^{\theta_1,\theta_2}(\tau)+\chi_-^{\theta_1,\theta_2}(\tau)\equiv 1$ and

$$\chi_+^{\theta_1,\theta_2}(\tau)=\begin{cases}1 & (\tau\geq\theta_2),\\ 0 & (\tau\leq\theta_1).\end{cases}$$

Further let $\psi(x)\in C^\infty(\mathbb{R}^n)$ with $0\leq\psi(x)\leq 1$ satisfy

$$\psi(x)=\begin{cases}0 & (|x|\leq\frac{1}{2})\\ 1 & (|x|\geq 1)\end{cases}$$

[1] As we will remark in footnote 3, the exact meaning of outgoing and incoming is slightly different in the context of [17] from what we mentioned here, whereas in [19] the present definition is adopted under the additional condition: $D_-\perp D_+$.

and $\phi(\xi) \in C^\infty(\mathbb{R}^n)$ satisfy $0 \le \phi(\xi) \le 1$, $\phi(\xi) = 1$ for $a \le |\xi|^2/2 \le b$ and supp $\phi \subset \{\xi \in \mathbb{R}^n |\ a/2 \le |\xi|^2/2 \le 2b\}$.

For $x, \xi \in \mathbb{R}^n \backslash \{0\}$ we set $\omega_x = x/|x|$ and $\omega_\xi = \xi/|\xi|$. We then define a real-valued C^∞ function $p_\pm^{\theta_1,\theta_2}(x,\xi)$ by

$$p_\pm^{\theta_1,\theta_2}(x,\xi) = \chi_\pm^{\theta_1,\theta_2}(\omega_x \cdot \omega_\xi)\psi(x)\phi(\xi). \tag{2.6}$$

We note that for $|x| \ge 1$ and $a \le |\xi|^2/2 \le b$

$$p_+^{\theta_1,\theta_2}(x,\xi) + p_-^{\theta_1,\theta_2}(x,\xi) = 1.$$

We denote by $\mathcal{S}$ the totality of rapidly decreasing functions on $\mathbb{R}^n$. Then the pseudodifferential operators $P_\pm^{\theta_1,\theta_2}$ with symbol functions $p_\pm^{\theta_1,\theta_2}(x,\xi)$ are defined by

$$P_\pm^{\theta_1,\theta_2} f(x) = P_\pm^{\theta_1,\theta_2}(X, D_x) f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} p_\pm^{\theta_1,\theta_2}(x,\xi) \hat{f}(\xi) d\xi \tag{2.7}$$

for $f \in \mathcal{S}$, where $\hat{f}(\xi) = \mathcal{F} f(\xi)$ denotes the Fourier transform of $f \in \mathcal{S}$:

$$\hat{f}(\xi) = \mathcal{F} f(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-iy\xi} f(y) dy.$$

If we use the notion of oscillatory integral,[2] this is equivalently expressed as follows.

$$P_\pm^{\theta_1,\theta_2} f(x) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i(x-y)\xi} p_\pm^{\theta_1,\theta_2}(x,\xi) f(y) dy d\xi. \tag{2.8}$$

It is well-known (Calderon-Vaillancourt theorem) that the pseudodifferential operators $P_\pm$ with those symbols are extended to bounded linear operators from $\mathcal{H} = L^2(\mathbb{R}^n)$ into itself. We note that the adjoint operators of $P_\pm^{\theta_1,\theta_2}$ are given by

$$(P_\pm^{\theta_1,\theta_2})^* f(x) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i(x-y)\xi} p_\pm^{\theta_1,\theta_2}(y,\xi) f(y) dy d\xi \tag{2.9}$$

for $f \in \mathcal{S}$. From the definition of the symbol functions we have

$$(P_+^{\theta_1,\theta_2} + P_-^{\theta_1,\theta_2}) f(x) = f(x) \tag{2.10}$$

for $|x| \ge 1$ and $f \in \mathcal{H}(a,b) = E_0([a,b])\mathcal{H}$.

Now we give our definition of asymptotically outgoing and incoming spaces.

Definition 2.1. Asymptotically outgoing and incoming spaces $D_+^{\theta_1,\theta_2}(a,b)$ and $D_-^{\theta_1,\theta_2}(a,b)$ for constants a, b with $0 < a < b < \infty$ are defined as follows.

$$D_\pm^{\theta_1,\theta_2}(a,b) = (P_\pm^{\theta_1,\theta_2})^* \mathcal{H} \cap \mathcal{H}(a,b). \tag{2.11}$$

We quote some estimate in Lemma 3.3 of [6] or Theorem 4.2 in [10] in the form given in Theorem 5.7 of [11].

[2] We should remark that the notion of oscillatory integral was first introduced by Lax [13].

Theorem 2.2. *Let* $0<\rho<1$, $-1<\theta_- -\rho<\theta_-<\theta_+<\theta_+ +\rho<1$. *Let* $P_+=P_+^{\theta_+,\theta_+ +\rho}$ *and* $P_-=P_-^{\theta_- -\rho,\theta_-}$ *be as above. Then we have for any* $s\ge 0$ *and* $\delta\ge 0$

$$\|\langle x\rangle^{\delta}P_-e^{-itH_0}P_+^*\langle x\rangle^{\delta}\|\le C_{s\delta}\langle t\rangle^{-s}\quad (t\ge 0), \tag{2.12}$$
$$\|\langle x\rangle^{\delta}P_+e^{-itH_0}P_-^*\langle x\rangle^{\delta}\|\le C_{s\delta}\langle t\rangle^{-s}\quad (t\le 0), \tag{2.13}$$

where the constant $C_{s\delta}>0$ *is independent of* t.

From (2.10) and (2.12) follows that for $f\in D_+(a,b)=D_+^{\theta_+,\theta_+ +\rho}(a,b)$

$$\|U_0(t)f-P_+^{\theta_- -\rho,\theta_-}U_0(t)f\|\to 0 \ \text{ as } \ t\to\infty. \tag{2.14}$$

Similarly we have for $f\in D_-(a,b)=D_-^{\theta_- -\rho,\theta_-}(a,b)$

$$\|U_0(t)f-P_-^{\theta_+,\theta_+ +\rho}U_0(t)f\|\to 0 \ \text{ as } \ t\to -\infty. \tag{2.15}$$

It is easy to see that $\|(P_+^*-P_+)U_0(t)f\|\to 0$ as $t\to\infty$. Therefore this implies that the state $U_0(t)f=E_0([a,b])U_0(t)f$ for $f\in D_+^{\theta_+,\theta_+ +\rho}(a,b)$ asymptotically equals an element $g(t)=E_0([a,b])(P_+^{\theta_- -\rho,\theta_-})^*U_0(t)f\in D_+^{\theta_- -\rho,\theta_-}(a,b)$ as $t\to\infty$, and similarly for the case of $D_-^{\theta_- -\rho,\theta_-}(a,b)$. Namely for $f\in D_+^{\theta_+,\theta_+ +\rho}(a,b)$

$$\|U_0(t)f-g(t)\|\to 0 \tag{2.16}$$

as $t\to+\infty$ for $g(t)=E_0([a,b])(P_+^{\theta_- -\rho,\theta_-})^*U_0(t)f\in D_+^{\theta_- -\rho,\theta_-}(a,b)$, and for $f\in D_-^{\theta_- -\rho,\theta_-}(a,b)$

$$\|U_0(t)f-g(t)\|\to 0 \tag{2.17}$$

as $t\to-\infty$ for $g(t)=E_0([a,b])(P_-^{\theta_+,\theta_+ +\rho})^*U_0(t)f\in D_-^{\theta_+,\theta_+ +\rho}(a,b)$. In this sense the condition (2.2) is satisfied as $t\to\pm\infty$ asymptotically for the spaces $D_\pm(a,b)$.

It is easy to see that the condition (2.3) holds.

To show a property analogous to (2.4), we quote Theorem 5.6 of [11].

Theorem 2.3. *Let* $P_\pm=P_\pm^{\theta_1,\theta_2}$ $(-1<\theta_1<\theta_2<1)$ *be as above. Then we have for any* $s\ge 0$ *and* $s\ge\delta\ge 0$

$$\|\langle x\rangle^{-s}e^{-itH_0}P_+^*\langle x\rangle^{\delta}\|\le C_{s\delta}\langle t\rangle^{-s+\delta}\quad (t\ge 0), \tag{2.18}$$
$$\|\langle x\rangle^{\delta}P_-e^{-itH_0}\langle x\rangle^{-s}\|\le C_{s\delta}\langle t\rangle^{-s+\delta}\quad (t\ge 0), \tag{2.19}$$

where the constant $C_{s\delta}>0$ *is independent of* t.

The relations (2.18) and (2.19) together with $\|(P_\pm^*-P_\pm)U_0(t)f\|\to 0$ as $t\to\pm\infty$ imply that for any $f\in\mathcal{H}(a,b)$

$$\lim_{t\to\pm\infty}P_\mp^*U_0(t)f=0. \tag{2.20}$$

This and (2.10) yield that for any $f\in\mathcal{H}(a,b)$

$$f=\lim_{t\to\pm\infty}U_0(-t)P_\pm^*U_0(t)f. \tag{2.21}$$

This means that any $f \in \mathcal{H}(a,b)$ is asymptotically equal to $U_0(-t)f(t)$ when $t \to \infty$ for an element $f(t) = E_0([a,b])P_+^* U_0(t) f \in D_+(a,b) = D_+^{\theta_1,\theta_2}(a,b)$, and similarly for the case $t \to -\infty$. Therefore (2.20) gives an asymptotic analogue to (2.4).

These justify our definition of asymptotically outgoing and incoming spaces.

For later use, we quote Theorem 5.5 from [11].

Theorem 2.4. *Let $q(\xi) \in C^\infty(\mathbb{R}^n)$ satisfy*

$$\sup_{\xi \in \mathbb{R}^n} |\partial_\xi^\alpha q(\xi)| < \infty \quad (\forall \alpha),$$
$$q(\xi) = 0 \quad (|\xi| < d) \quad (\exists d > 0).$$

Then for any $s \geq 0$ we have

$$\|\langle x\rangle^{-s} q(D_x) U_0(t) \langle x\rangle^{-s}\| \leq C_s \langle t\rangle^{-s} \quad (t \in \mathbb{R}), \tag{2.22}$$

where the constant $C_s > 0$ is independent of $t \in \mathbb{R}$.

3 Asymptotic Completeness

In Lax-Phillips [17], the perturbed system is the lossy wave equation in an exterior domain G. In section 8 of Part II in [17], they consider for simplicity a lossless medium with dissipative boundary conditions of the form

$$u_n + \alpha u_t = 0 \quad \text{on} \quad \partial G$$

for $\alpha \geq 0$, where n denotes the outward normal to G on ∂G. The corresponding Hilbert space $\mathcal{H}^{LP}$ consists of all initial data $d = (f_1, f_2)$ with the energy norm in G:

$$\|d\|^2 = \int_G \left[|\partial_x f_1(x)|^2 + |f_2(x)|^2\right] dx.$$

The propagator $T^{LP}(t)$ $(t \geq 0)$ for this system is a contraction and satisfies certain conditions which are similar to conditions (2.2)-(2.4).

Let $D_\pm^\rho = U_0^{LP}(\pm\rho) D_\pm$ for $\rho > 0$. Then it is shown for some ρ that $\mathcal{H}^{LP}$ contains $D_\pm^\rho$ as subspaces, $U_0^{LP}(-t)$ and $\left(T^{LP}\right)^*(t)$ coincide on D_-^ρ, and $U_0^{LP}(t)$ and $T^{LP}(t)$ coincide on D_+^ρ. The relations (2.2) and (2.3) when combined with those facts yield the following relations (3.1) and (3.2).

$$\left(T^{LP}\right)^*(t) D_-^\rho \subset D_-^\rho, \quad T^{LP}(t) D_+^\rho \subset D_+^\rho \quad (t \geq 0), \tag{3.1}$$

$$\bigcap_{t\geq 0} \left(T^{LP}\right)^*(t) D_-^\rho = \{0\}, \quad \bigcap_{t \geq 0} T^{LP}(t) D_+^\rho = \{0\}. \tag{3.2}$$

Those correspond to the properties (2.2) and (2.3) for $U_0^{LP}(t)$. For the property (2.4), Lax-Phillips [17] (p. 175) gives an analogous property for $T(t)$ and $T^*(t)$ as follows instead of giving a form similar to (2.4). Let P_+^ρ and P_-^ρ be orthogonal projections onto the orthogonal complements of D_+^ρ and D_-^ρ, respectively.[3] It is proved in Theorem 9.5 of [17] that the following limit relations hold for $f \in \mathcal{H}^{LP}$.

$$\lim_{t\to\infty} P_-^\rho \left(T^{LP}\right)^*(t)f = 0, \quad \lim_{t\to\infty} P_+^\rho T^{LP}(t)f = 0. \tag{3.3}$$

As easily seen, this is a property for $T^{LP}(t)$ and $(T^{LP})^*(t)$ analogous to (2.4) in the same sense that (2.20) is analogous to (2.4) for $U_0(t)$.

In the case of our Hamiltonian $H = H_0 + V$ in (2.1), the corresponding propagator is a unitary group $T(t) = e^{-itH}$.

Under these circumstances, the existence of wave operator

$$W_1 = \text{s-}\lim_{t\to\infty} W_1(t) \tag{3.4}$$

and the inverse wave operator

$$W_2 = \text{s-}\lim_{t\to\infty} W_2(t) \tag{3.5}$$

is shown in Theorem 1.2 of [17]. Here

$$W_1(t) = T^{LP}(t)J_0^{LP}U_0^{LP}(-t), \quad W_2(t) = U_0^{LP}(-t)J^{LP}T^{LP}(t),$$

where the operators $U_0^{LP}(t)$ and $T^{LP}(t)$ are considered to act on different Hilbert spaces $\mathcal{H}_0^{LP}$ and $\mathcal{H}^{LP}$ as in the above, and $J_0^{LP} : \mathcal{H}_0^{LP} \longrightarrow \mathcal{H}^{LP}$ and $J^{LP} : \mathcal{H}^{LP} \longrightarrow \mathcal{H}_0^{LP}$ are bounded linear operators which act as the identity on the common subspace $D_+^\rho + D_-^\rho$ of $\mathcal{H}_0^{LP}$ and $\mathcal{H}^{LP}$.

Let us see how they prove the existence of W_2 in Theorem 1.2 of [17] as the other case for W_1 is similar. They pick an arbitrary element f of $\mathcal{H}^{LP}$ and decompose $T^{LP}(t)f$ as a sum of elements of D_+^ρ and its orthogonal component

$$T^{LP}(t)f = d(t) + e(t), \tag{3.6}$$

where

$$d(t) \in D_+^\rho \ \text{ and } \ e(t) \perp D_+^\rho.$$

Then they use (3.6), $d(t) \in D_+^\rho$ and $J^{LP} = I$ on D_+^ρ to show that

$$W_2(t)f = U_0^{LP}(-t)J^{LP}T^{LP}(t)f = U_0^{LP}(-t)d(t) + U_0^{LP}(-t)J^{LP}e(t). \tag{3.7}$$

Similar argument with using (3.1) shows that for $s \geq 0$

$$\begin{aligned} W_2(t+s)f &= U_0^{LP}(-t-s)J^{LP}T^{LP}(t+s)f \\ &= U_0^{LP}(-t-s)T^{LP}(s)d(t) + U_0^{LP}(-t-s)J^{LP}T^{LP}(s)e(t). \end{aligned}$$

[3] We should note that the usage of signs $+$ and $-$ in the definition of our operators $P_\pm$ in the previous section is in a sense reverse to that of P_+^ρ and P_-^ρ in [17]. As announced in footnote 1, we state the exact definition of outgoing and incoming in the context of [17]: "outgoing" means "orthogonal to D_-^ρ," and "incoming" means "orthogonal to D_+^ρ." Namely the "outgoing space" is the space $P_-^\rho\mathcal{H}^{LP}$, and "incoming space" is the space $P_+^\rho\mathcal{H}^{LP}$.

As $U_0^{LP}(s)$ and $T^{LP}(s)$ coincide on D_+^ρ, it follows from this that

$$W_2(t+s)f = U_0^{LP}(-t)d(t) + U_0^{LP}(-t-s)J^{LP}T^{LP}(s)e(t). \tag{3.8}$$

Subtracting (3.7) from (3.8) yields

$$W_2(t+s)f - W_2(t)f = U_0^{LP}(-t-s)J^{LP}T^{LP}(s)e(t) - U_0^{LP}(-t)J^{LP}e(t). \tag{3.9}$$

By (3.3), $\|e(t)\|$ goes to 0 as t tends to ∞. This shows that the right hand side of (3.9) goes to 0 as $t \to \infty$ uniformly in $s \geq 0$, which proves that the limit (3.5) exists.

In the case of our Hamiltonian (2.1), the identification operator J is a bounded operator from $\mathcal{H} = L^2(\mathbb{R}^n)$ into itself, and is defined as follows. The original construction of the identification operator in [6] introduces two identification operators corresponding to the two cases $t \to \infty$ and $t \to -\infty$. We here follow the definition in [11], where only one identification operator J is defined as a Fourier integral operator as follows.

$$\begin{aligned} Jf(x) &= (2\pi)^{-n} \iint e^{i(\varphi(x,\xi)-y\xi)} f(y)dyd\xi \\ &= (2\pi)^{-n/2} \int e^{i\varphi(x,\xi)} \hat{f}(\xi)d\xi. \end{aligned} \tag{3.10}$$

Here the phase function $\varphi(x,\xi)$ is constructed as a solution of an eikonal equation for the Schrödinger equation corresponding to (2.1) and satisfies the following theorem (Theorem 2.5 [6], Theorem 6.6 [11]).

Theorem 3.1. *Let $d > 0$ and $-1 < \sigma_- < \sigma_+ < 1$ be fixed. Then there is $R = R_d = R_{d\sigma_\pm} > 1$ and a real-valued C^∞ function $\varphi(x,\xi)$ of $(x,\xi) \in \mathbb{R}^{2n}$ such that $R_d > 1$ increases as $d > 0$ decreases and the following holds.*

i) For $|\xi| \geq d$, $|x| \geq R$ and $\omega_x \cdot \omega_\xi \geq \sigma_+$ or $\omega_x \cdot \omega_\xi \leq \sigma_-$

$$\frac{1}{2}|\nabla_x \varphi(x,\xi)|^2 + V(x) = \frac{1}{2}|\xi|^2. \tag{3.11}$$

ii) For any multi-indices α, β there is a constant $C_{\alpha\beta} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta (\varphi(x,\xi) - x\cdot\xi)| \leq C_{\alpha\beta}\langle x\rangle^{1-\varepsilon-|\alpha|}\langle\xi\rangle^{-1}. \tag{3.12}$$

In particular for $|\alpha| \neq 0$, we have for $\varepsilon_0, \varepsilon_1 \geq 0$ with $\varepsilon_0 + \varepsilon_1 = \varepsilon$

$$|\partial_x^\alpha \partial_\xi^\beta (\varphi(x,\xi) - x\cdot\xi)| \leq C_{\alpha\beta} R^{-\varepsilon_0}\langle x\rangle^{1-\varepsilon_1-|\alpha|}\langle\xi\rangle^{-1}. \tag{3.13}$$

iii) Set

$$\begin{aligned} a(x,\xi) &= e^{-i\varphi(x,\xi)}\left(-\frac{1}{2}\Delta + V(x) - \frac{1}{2}|\xi|^2\right)e^{i\varphi(x,\xi)} \\ &= \frac{1}{2}|\nabla_x\varphi(x,\xi)|^2 + V(x) - \frac{1}{2}|\xi|^2 - \frac{i}{2}\Delta_x\varphi(x,\xi). \end{aligned} \tag{3.14}$$

Then $a(x,\xi)$ satisfies for $|\xi| \geq d$, $|x| \geq R$ and any α, β

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq \begin{cases} C_{\alpha\beta}\langle x\rangle^{-1-\varepsilon-|\alpha|}\langle\xi\rangle^{-1}, & \omega_x\cdot\omega_\xi \in [-1,\sigma_-]\cup[\sigma_+,1], \\ C_{\alpha\beta}\langle x\rangle^{-\varepsilon-|\alpha|}, & \omega_x\cdot\omega_\xi \in [\sigma_-,\sigma_+]. \end{cases} \tag{3.15}$$

Our J corresponds to J_0^{LP} of [17] and J is known to have a bounded inverse J^{-1} when the domain is restricted suitably. Thus we can define $W_1(t)$ and $W_2(t)$ as follows:

$$W_1(t) = T(-t)JU_0(t), \quad W_2(t) = U_0(-t)J^{-1}T(t),$$

where $U_0(t) = e^{-itH_0}$ and $T(t) = e^{-itH}$.

The element f which was assumed to belong to $\mathcal{H}^{LP}$ in the proof of the existence of the inverse wave operator $W_2 = \text{s-}\lim_{t\to\infty} W_2(t)$ in [17] will be an element of the continuous spectral subspace $\mathcal{H}_c$ for the Hamiltonian H in the present case. Here we assume that $f \in \mathcal{H}_c(a,b) = E_H([a,b])\mathcal{H}_c$ with $0 < a < b < \infty$, where $E_H(B)$ denotes the spectral measure for the Hamiltonian H. As our propagators $T(t)$ and $U_0(t)$ are unitary operators, we can consider the two limits

$$W_2^{\pm} = \text{s-}\lim_{t\to\pm\infty} W_2(t).$$

We consider the asymptotic behavior of $T(t)f$ for $f \in \mathcal{H}_c(a,b)$ to derive an asymptotic analogue to (3.1) for our $T(t) = e^{-itH}$. Let the pseudodifferential operators $P_{\pm} = P_{\pm}^{\theta_1,\theta_2}$ $(-1 < \theta_1 < \theta_2 < 1)$ be defined as in (2.7) or (2.8) with the same constants $0 < a < b < \infty$ as above. We calculate as follows for $t \in \mathbb{R}$.

$$\begin{aligned} T(t)P_{\pm}^* &= (T(t) - JU_0(t)J^{-1})P_{\pm}^* + JU_0(t)J^{-1}P_{\pm}^* \\ &= T(t)(I - T(-t)JU_0(t)J^{-1})P_{\pm}^* + JU_0(t)J^{-1}P_{\pm}^* \\ &= -T(t)\int_0^t \frac{d}{d\sigma}\left(T(-\sigma)JU_0(\sigma)J^{-1}\right)P_{\pm}^* d\sigma + JU_0(t)J^{-1}P_{\pm}^* \\ &= -iT(t)K_{\pm}(t) + JU_0(t)J^{-1}P_{\pm}^*, \end{aligned} \tag{3.16}$$

where

$$K_{\pm}(t) = \int_0^t T(-\sigma)(HJ - JH_0)U_0(\sigma)J^{-1}P_{\pm}^* d\sigma. \tag{3.17}$$

We note that we can write for $f \in \mathcal{S}$ with using the function $a(x,\xi)$ in Theorem 3.1-iii) (3.14)

$$(HJ - JH_0)f(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\varphi(x,\xi)}a(x,\xi)\hat{f}(\xi)d\xi. \tag{3.18}$$

Therefore, if we take $-1 < \theta_1 = \sigma_+ + \rho < \theta_2 < 1$ for some $\rho > 0$ and the constant $\sigma_+ \in (-1,1)$ of Theorem 3.1, $K_+(t)$ defines a compact operator on $\mathcal{H}$ and converges to a compact operator K_+ of $\mathcal{H} = L^2(\mathbb{R}^n)$ in operator norm when $t \to +\infty$ by Theorems 2.2 – 2.4, Theorem 3.1 and the factor $\phi(\xi)$ in the symbol $p_{\pm}(x,\xi)$ in (2.6) with some calculation of Fourier integral and pseudodifferential operators (section 6.3 [11]). Similarly if we take $-1 < \theta_1 < \theta_2 = \sigma_- - \rho < 1$ for some $\rho > 0$ and the constant $\sigma_- \in (-1,1)$ of Theorem 3.1, $K_-(t)$ converges to a compact operator K_- of $\mathcal{H} = L^2(\mathbb{R}^n)$ in operator norm when $t \to -\infty$ by the same reason. Therefore we have proved that for $t \in \mathbb{R}$

$$J^{-1}T(t)P_{\pm}^* = -iJ^{-1}T(t)K_{\pm}(t) + U_0(t)J^{-1}P_{\pm}^*, \tag{3.19}$$

where the first term is a compact operator on $\mathcal{H}$. This means that the operator $J^{-1}T(t)P_{\pm}^*$ behaves like $U_0(t)J^{-1}P_{\pm}^*$ except for a compact operator $-iJ^{-1}T(t)K_{\pm}(t)$. We will see that this fact plays a role similar to (3.1) in the proof of asymptotic completeness for quantum scattering.

The property (3.2) is easily seen to hold for our case also.

Finally we will see a correspondent to (3.3). If we could have a similar thing to (3.3), it would be that for any $f \in \mathcal{H}_c(a,b)$

$$\lim_{t\to\pm\infty} P_{\mp}^{*} T(t) f = 0. \tag{3.20}$$

In fact this is known to hold. We quote some result from section 6.2 [11] in the case of two body quantum scattering. Namely from Theorem 3.2 [11] follows that for any $f \in \mathcal{H}_c(a,b)$ with $\langle x\rangle^2 f \in \mathcal{H} = L^2(R^n)$, there exists a sequence $t_k \to \pm\infty$ as $k \to \pm\infty$ such that for any $\phi \in C_0^\infty(\mathbb{R})$ and $R > 0$

$$\|\chi_{\{x\in\mathbb{R}^n||x|<R\}} T(t_k) f\| \to 0, \tag{3.21}$$

$$\|(\phi(H) - \phi(H_0)) T(t_k) f\| \to 0, \tag{3.22}$$

$$\left\|\left(\frac{x}{t_k} - D_x\right) T(t_k) f\right\| \to 0 \tag{3.23}$$

as $k \to \pm\infty$, where $D_x = -i\partial_x$. In the two body case this is a consequence of Ruelle-Amrein-Georgescu theorem ([1], [20]). The relation (3.23) is proved in [3] by extending the result of [1], [20]. This relation in particular implies that the configuration x is proportional to momentum $\pm\xi$ in phase space asymptotically as $t \to \pm\infty$. As a consequence, the relation (3.20) holds for $f \in \mathcal{H}_c(a,b)$ when t tends to $\pm\infty$ along the sequence $t = t_k \to \pm\infty$ $(k \to \pm\infty)$ given above. The relation (3.21) implies that w-$\lim_{k\to\pm\infty} T(t_k) f = 0$.

Summarizing the arguments up to here we have proved the following theorem.

Theorem 3.2.

i) When $t \geq 0$, let $-1 < \theta_1 = \sigma_+ + \rho < \theta_2 < 1$ for some $\rho > 0$ and the constant $\sigma_+ \in (-1,1)$ of Theorem 3.1, and when $t \leq 0$ let $-1 < \theta_1 < \theta_2 = \sigma_- - \rho < 1$ for some $\rho > 0$ and the constant $\sigma_- \in (-1,1)$ of Theorem 3.1, and define $P_\pm = P_\pm^{\theta_1,\theta_2}$ by (2.7) or (2.8) with $0 < a < b < \infty$. Then for $t \in \mathbb{R}$

$$J^{-1} T(t) P_\pm^{*} = -iJ^{-1} T(t) K_\pm(t) + U_0(t) J^{-1} P_\pm^{*}, \tag{3.24}$$

where $K_\pm(t)$ in the first term on the right hand side is a compact operator on $\mathcal{H}$ and converges to a compact operator $K_\pm$ on $\mathcal{H}$ in operator norm as $t \to \pm\infty$.

ii)

$$\bigcap_{t\geq 0} T(-t) D_-(a,b) = \{0\}, \quad \bigcap_{t\geq 0} T(t) D_+(a,b) = \{0\}, \tag{3.25}$$

iii) For any $f \in \mathcal{H}_c(a,b)$ there is a sequence $t_k \to \pm\infty$ as $k \to \pm\infty$ such that

$$\lim_{k\to\pm\infty} P_\mp^{*} T(t_k) f = 0, \tag{3.26}$$

$$\text{w-}\lim_{k\to\pm\infty} T(t_k) f = 0, \tag{3.27}$$

and for any $\phi \in C_0^\infty(\mathbb{R})$

$$\|(\phi(H)-\phi(H_0))T(t_k)f\| \to 0 \ \ (k \to \pm\infty). \tag{3.28}$$

We now prove the asymptotic completeness of $W_2^\pm$. As the case $t \to -\infty$ is treated similarly to the case $t \to \infty$, we will consider the case $t \to \infty$ below. It thus suffices to prove the existence of the limit

$$W_2^+ = \text{s-}\lim_{t\to\infty} W_2(t) \tag{3.29}$$

on $\mathcal{H}_c(a,b)$ for any $0 < a < b < \infty$. For this purpose we will prove as in [17] that for $f \in \mathcal{H}_c(a,b)$

$$\begin{aligned} W_2(t+s)f - W_2(t)f &= U_0(-t-s)J^{-1}T(t+s)f - U_0(-t)J^{-1}T(t)f \\ &= \left\{U_0(-t-s)J^{-1}T(s) - U_0(-t)J^{-1}\right\}T(t)f \end{aligned} \tag{3.30}$$

converges to 0 uniformly in $s \geq 0$ as t goes to ∞ along the sequence $t = t_k \to \infty$ $(k \to \infty)$ specified in Theorem 3.2-iii). If we have shown this, we have proved the existence of W_2^+. To prove this we let $P_\pm = P_\pm^{\theta_1,\theta_2}$ for $-1 < \theta_1 = \sigma_+ + \rho < \theta_2 < 1$ for some $\rho > 0$ and the constant $\sigma_+ \in (-1,1)$ of Theorem 3.1. Then the state $T(t)f$ is decomposed

$$T(t)f = d(t) + e(t) + r(t),$$

where

$$d(t) = P_+^* T(t)f, \quad e(t) = P_-^* T(t)f, \quad r(t) = T(t)f - (P_+^* + P_-^*)T(t)f.$$

By (3.24) of Theorem 3.2-i), we have

$$J^{-1}T(s)P_+^* = -iJ^{-1}T(s)K_+(s) + U_0(s)J^{-1}P_+^*. \tag{3.31}$$

Thus

$$\begin{aligned} U_0(-t-s)J^{-1}T(s)d(t) &= U_0(-t-s)J^{-1}T(s)P_+^*T(t)f \\ &= -iU_0(-t-s)J^{-1}T(s)K_+(s)T(t)f + U_0(-t)J^{-1}P_+^*T(t)f. \end{aligned}$$

On the other hand we have

$$U_0(-t)J^{-1}d(t) = U_0(-t)J^{-1}P_+^*T(t)f.$$

The difference (3.30) is thus equal to

$$\begin{aligned} &W_2(t+s)f - W_2(t)f \\ &\quad = \widetilde{K}_+(t,s)T(t)f + \left\{U_0(-t-s)J^{-1}T(s) - U_0(-t)J^{-1}\right\}(e(t)+r(t)), \end{aligned} \tag{3.32}$$

where

$$\widetilde{K}_+(t,s) = -iU_0(-t-s)J^{-1}T(s)K_+(s). \tag{3.33}$$

By (3.26) of Theorem 3.2-iii)

$$e(t_k) \to 0 \ \text{as} \ k \to \infty. \tag{3.34}$$

From the definition of pseudodifferential operators $P_\pm$, it is easy to see that $P_\pm - P_\pm^*$ are compact operators on $\mathcal{H}$. From this fact and (3.27) in Theorem 3.2-iii) we have

$$\|r(t_k) - \{I - (P_+ + P_-)\}\, T(t_k) f\| \to 0 \ \text{as} \ k \to \infty. \tag{3.35}$$

From $f \in \mathcal{H}_c(a,b)$ and (3.28) in Theorem 3.2-iii), we have

$$\|T(t_k) f - E_0([a,b]) T(t_k) f\| \to 0 \quad \text{as} \quad k \to \infty. \tag{3.36}$$

By (3.35), (3.36), (2.10) and (3.27), we have

$$\|r(t_k)\| \to 0 \ \text{as} \ k \to \infty. \tag{3.37}$$

From (3.32), (3.34) and (3.37), we have uniformly in $s \geq 0$

$$\left\|W_2(t_k + s) f - W_2(t_k) f - \widetilde{K}_+(t_k, s) T(t_k) f\right\| \to 0 \ \text{as} \ k \to \infty. \tag{3.38}$$

Here by Theorem 3.2-i), (3.33) and (3.27), we have uniformly in $s \geq 0$

$$\left\|\widetilde{K}_+(t_k, s) T(t_k) f\right\| = \left\|J^{-1} T(s) K_+(s) T(t_k) f\right\| \to 0 \ \text{as} \ k \to \infty. \tag{3.39}$$

The relations (3.38) and (3.39) imply that uniformly in $s \geq 0$

$$\|W_2(t_k + s) f - W_2(t_k) f\| \to 0 \ \text{as} \ k \to \infty. \tag{3.40}$$

This proves that the inverse wave operator W_2^+ exists on $\mathcal{H}_c$, and concludes the proof of the asymptotic completeness for quantum mechanical scattering with Hamiltonian (2.1).

4 Concluding Remarks

We have shown that the usual quantum scattering theory can be incorporated into the framework of Lax-Phillips scattering theory with slightly extending their abstract framework. The concrete method of estimation used in applying the abstract framework is what has been done in (i) [1], [20] and (ii) [4], [5], [6], [7], [8], [10], [11], [12]. The first group (i) is concerned with the Ruelle-Amrein-Georgescu theorem which was used in proving that the incoming/outgoing part of the solution vanishes when $t \to \pm\infty$, i.e. $\left\|P_\mp^* T(t_k) f\right\| \to 0$ as $t_k \to \pm\infty$. The main point of the proof of their theorem, i.e. Proposition 3 on page 660 of [20] and Lemma 2 on page 641 of [1], is almost the same in essence as that of Theorem 9.1, especially as Lemma 9.3, Corollary 9.4 and the argument thereafter on page 216 of Lax-Phillips [17], utilizing which and the properties of wave equation, they prove their crucial result: Theorem 9.5 [17], i.e. (3.3) $\lim_{t\to\infty} P_+^\rho T^{LP}(t) f = 0$ and $\lim_{t\to\infty} P_-^\rho \left(T^{LP}\right)^* (t) f = 0$.

The works in the group (ii) were done in the atmosphere when the Enss method was fashionable and the trend at the age. Looking back upon those days, there were problems of many-body scattering as another fashion and trend. In either case, what is essential has been the micro-local analysis, the essential part of which was developed in [4]-[12].

While people are tending to be enthusiastic over some trendy fashion at each age and even if a seemingly new clever method looks having been invented, the important thing is not changed. In fact the essential part, both in abstract framework and concrete method of estimation, had been given in Lax-Phillips theory already.

An example of the flexibility of their framework is that we can adopt other operators instead of $P_\pm = P_\pm^{\theta_1,\theta_2}$ in (2.8). Jensen-Mourre-Perry [9] defined the notion of conjugate of a self-adjoint operator A with respect to Hamiltonian H and the n-smoothness of H with respect to A in definition 2.1 of [9]. As a substitute for our $P_\pm$ one can use the projection operators P_A^+ and P_A^- onto the subspaces $E_A((0,\infty))\mathcal{H}$ and $E_A((-\infty,0))\mathcal{H}$. The details are left to the reader.

References

[1] W. O. Amrein and V. Georgescu, *On the characterization of bound states and scattering states in quantum mechanics*, Helv. Physica Acta **46** (1973), 635-658.

[2] V. Enss, *Asymptotic completeness for quantum mechanical potential scattering I. Short-range potentials,* Commun. Math. Phys., **61** (1978), 285-291.

[3] V. Enss, *Introduction to asymptotic observables for multiparticle quantum scattering*, in "Schrödinger Operators, Aarhus 1985" edited by E. Balslev, Lect. Note in Math., vol. 1218, Springer-Verlag, 1986, 61-92.

[4] H. Isozaki and H. Kitada, *Asymptotic behavior of the scattering amplitude at high energies*, in "Differential Equations," ed. I. W. Knowles and R. T. Lewis, Elsevier Science Publishers B. V., North-Holland, 1984, pp. 327-334.

[5] H. Isozaki and H. Kitada, *Micro-local resolvent estimates for 2-body Schrödinger operators*, Journal of Functional Analysis, **57** (1984), 270-300.

[6] H. Isozaki and H. Kitada, *Modified wave operators with time-independent modifiers,* J. Fac. Sci., Univ. Tokyo, Sec. IA, **32** (1985), 77-104.

[7] H. Isozaki and H. Kitada, *Scattering matrices for two-body Schrödinger operators*, Scientific Papers of the College of Arts and Sciences, The University of Tokyo **35** (1985), 81-107.

[8] H. Isozaki and H. Kitada, *A remark on the microlocal resolvent estimates for two body Schrödinger operators*, Publ. RIMS Kyoto Univ. **21** (1985), 889-910.

[9] A. Jensen, É. Mourre, and P. Perry, *Multiple commutator estimates and resolvent smoothness in quantum scattering theory*, Ann. Inst. Henri Poincaré, **41** (1984), 207-225.

[10] H. Kitada, *Fourier integral operators with weighted symbols and micro-local resolvent estimates*, J. Math. Soc. Japan, **39** (1987), 455-476.

[11] H. Kitada, Quantum Mechanics, Lectures in Mathematical Sciences, The University of Tokyo, Vol. 23, Graduate School of Mathematical Sciences, 2005. (http://arxiv.org/abs/quant-ph/0410061)

[12] H. Kitada and K. Yajima, *A scattering theory for time-dependent long-range potentials*, Duke Math. J., **49** (1982), 341-376.

[13] P. D. Lax, *Asymptotic solutions of oscillatory initial value problems*, Duke Math. J. **24** (1957), 627-646.

[14] P. D. Lax and R. S. Phillips, *Local boundary conditions for dissipative symmetric linear differential operators*, Comm. Pure and Appl. Math. **XIII** (1960), 427-455.

[15] P. D. Lax and R. S. Phillips, Scattering Theory, Academic Press, New York, 1967.

[16] P. D. Lax and R. S. Phillips, *On the scattering frequencies of the Laplace operator for exterior domains*, Comm. Pure and Appl. Math. **XXV** (1972), 85-101.

[17] P. D. Lax and R. S. Phillips, *Scattering theory for dissipative hyperbolic systems*, J. Funct. Anal. **14** (1973), 172-235.

[18] P. D. Lax and R. S. Phillips, *A scattering theory for automorphic functions*, Seminaire Goulaouic-Schwartz 1973-1974, École Polytechnique, **XXIII** (1974), 1–7.

[19] P. D. Lax and R. S. Phillips, *The scattering of sound waves by an obstacle*, Comm. Pure and Appl. Math. **XXX** (1977), 195-233.

[20] D. Ruelle, *A remark on bound states in potential-scattering theory*, Il Nuovo Cimento **61** A (1969), 655-662.