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THE ADJOINT PROBLEM ON BANACH SPACES

TEPPER L. GILL[*]
Department of Mathematics and E & CE
Howard University
Washington, DC 20059, USA

WOODFORD W. ZACHARY[†]
Department of Electrical & Computer Engineering
Howard University
Washington, DC 20059, USA

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Abstract

In this paper we survey recent work on the existence of an adjoint for operators on Banach spaces and applications. In [GBZS] it was shown that each bounded linear operator A, defined on a separable Banach space $\mathcal{B}$, has a natural adjoint A^* defined on the space. Here, we show that, for each closed linear operator C defined on $\mathcal{B}$, there exists a pair of contractions A, B such that $C = AB^{-1}$. We also show that, if C is densely defined, then $B = (I - A^*A)^{-1/2}$. This result allows us to extend the results of [GBZS] (in a domain independent way) by showing that every closed densely defined linear operator on $\mathcal{B}$ has a natural adjoint. As an application, we show that our theory allows us to provide a natural definition for the Schatten class of operators in separable Banach spaces. In the process, we extend an important theorem due to Professor Lax.

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1 Introduction

In a previous paper [GBZS], we used the fact that every separable Banach space $\mathcal{B}$ may be continuously embedded in a separable Hilbert space $\mathcal{H}$ to show that each operator $A \in L[\mathcal{B}]$, the algebra of bounded linear operators on $\mathcal{B}$, has a natural adjoint operator $A^* \in L[\mathcal{B}]$. This means that, for example, every ideal is a star ideal in $L[\mathcal{B}]$, and such notions as unitary, selfadjoint, normal, etc, may be defined in (almost) the same manner as for a Hilbert space.

[*]E-mail address: tgill@access4less.net
[†]E-mail address: wwzachary@earthlink.net

Here, we show that the bounded linear operators $L[\mathcal{B}]$ are continuously embedded in $L[\mathcal{H}]$ provided that $\mathcal{B}' \subset \mathcal{H}$. (This extends a theorem of Professor Lax [LX].) Furthermore, if $\mathcal{B}$ has the approximation property, then the embedding is dense. This allows us to prove the existence of new classes of operators which naturally deserve to be called the Schatten classes over $\mathcal{B}$ in the sense that they are the restrictions of the Schatten classes on $\mathcal{H}$ to $\mathcal{B}$. (The importance of these results is they imply that the structure of separable Banach spaces, and the linear operators which act on them, are much closer to those of Hilbert spaces than perviously thought possible.)

2 Preliminaries

As above, $L[\mathcal{B}], L[\mathcal{H}]$ denote the bounded linear operators on $\mathcal{B}, \mathcal{H}$ respectively, and $\mathcal{B}$ is a continuous dense embedding in $\mathcal{H}$. The following is the major result in Gill et al [GBZS]. It generalizes the well-known result of von Neumann [VN] for bounded operators on Hilbert spaces.

Theorem 1. *Let $\mathcal{B}$ be a separable Banach space and let A be a bounded linear operator on $\mathcal{B}$. Then A has a well-defined adjoint A^* defined on $\mathcal{B}$ such that:*

1. *the operator $A^*A \geq 0$ (maximal accretive),*
2. *$(A^*A)^* = A^*A$, and*
3. *$I + A^*A$ has a bounded inverse.*

The proof depends on the fact that, given $\mathcal{B}$, there always exist Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\mathcal{H}_1 \subset \mathcal{B} \subset \mathcal{H}_2$, as continuous dense embeddings, with $\mathcal{H}_1$ determined by $\mathcal{H}_2$ (see [GBZS]). If A is any bounded linear operator on $\mathcal{B}$, we define A^* by

$$A^*x = J_1^{-1}[(A_1)']J_2|_{\mathcal{B}}(x), \tag{1}$$

where A_1 is A restricted to $\mathcal{H}_1$, $J_2|_{\mathcal{B}}$ maps $\mathcal{B}$ into $\mathcal{H}'_2$ and J_1^{-1} maps $\mathcal{H}'_1$ onto $\mathcal{H}_1$.

Remark 2. *Recall that, on any Hilbert space $\mathcal{H}$, the adjoint of a linear operator is defined as $A^*x = J^{-1}A'Jx$ for all $Jx \in D(A')$. Thus, we see that (1) is very close to the Hilbert space definition.*

Returning to Theorem 1, it is not clear that A need have a bounded extension to $\mathcal{H}_2$. On the other hand, the theorem by Lax [LX] states that:

Theorem 3. *If A is a bounded linear operator on $\mathcal{B}$ such that A is selfadjoint (i.e., $(Ax,y)_2 = (x,Ay)_2$ for all $x,y, \in \mathcal{B}$), then A is bounded on $\mathcal{H}_2$ and $\|A\|_{\mathcal{H}_2} \leq k\|A\|_{\mathcal{B}}$ with k constant.*

Since A^*A is selfadjoint on $\mathcal{B}$, it is natural to expect that the same is true on $\mathcal{H}_2$. However, this need not be the case. To get a simple counterexample, recall that, in standard notation, the simplest class of bounded linear operators on $\mathcal{B}$ is $\mathcal{B} \otimes \mathcal{B}'$, in the sense that:

$$\mathcal{B} \otimes \mathcal{B}' : \mathcal{B} \to \mathcal{B}, \text{ by } Ax = (b \otimes b')x = \langle x, b' \rangle b.$$

Thus, if b' is in $\mathcal{B}' \backslash \mathcal{H}'_2$, then $J_2\{J_1^{-1}[(A_1)']J_2|_{\mathcal{B}}\}$ is not in $\mathcal{H}'_2$, so that A^*A is not selfadjoint as an operator on $\mathcal{H}_2$. The next result is an extension of Theorem 1. (This also corrects an error in Theorem 6 of [GBZS].)

Theorem 4. *Let A be a bounded linear operator on $\mathcal{B}$. If $\mathcal{B}' \subset \mathcal{H}_2$, then A has a bounded extension to $L[\mathcal{H}]$, with $\|A\|_{\mathcal{H}_2} \leq k\|A\|_{\mathcal{B}}$ with k constant (i.e., $L[\mathcal{B}]$ is continuously embedded in $L[\mathcal{H}]$).*

Proof. : For any bounded linear operator A defined on $\mathcal{B}$, let $T = A^*A$. It is easy to see that T extends to a closed linear operator $\overline{T}$ on $\mathcal{H}_2$. As $\mathcal{B}' \subset \mathcal{H}_2$, we see that $\overline{T}$ is selfadjoint on $\mathcal{B}$. By Lax's theorem, $\overline{T}$ is bounded on $\mathcal{H}_2$ and $\|A^*A\|_{\mathcal{H}_2} = \|A\|^2_{\mathcal{H}_2} \leq \|A^*A\|_{\mathcal{B}} \leq C\|A\|^2_{\mathcal{B}}$, where $C = inf\{M|\ \|A^*A\|_{\mathcal{B}} \leq M\|A\|^2_{\mathcal{B}}\}$. □

Clearly, Theorem 4 is empty unless spaces with the above properties exist. The following theorem is a by-product of the work in [GZ].

Theorem 5. *Let $\mathcal{B}$ be a classical Banach space. Then there exists a Hilbert space $\mathcal{H}$ such that $\mathcal{B}' \subset \mathcal{H}$.*

2.1 Closed Linear operators on $\mathcal{B}$

In this section, we extend Theorem 1 to the class of closed densely defined linear operators on $\mathcal{B}$. For a single opeator, this is both direct and easy (see [GZ1]), but depends on the domain of A and hence, on $\mathcal{H}_1$. A result that is independent of $\mathcal{H}_1$ requires additional effort.

Definition 6. *If B is a bounded linear operator on $\mathcal{B}$, we define B^{-1} to be the inverse of the restriction of B to the closure of $B^*(\mathcal{B})$.*

Theorem 7. *Suppose that $\mathcal{S}$ is a subset of $(\mathcal{B}, \|\cdot\|)$, and $(\mathcal{S}, \|\cdot\|')$ is a Banach space with $\|\psi\|' \geq \|\psi\|$ for each $\psi \in \mathcal{S}$. Then $\mathcal{S}$ is the range of a nonnegative bounded linear operator in $\mathcal{B}$.*

Proof. Since $\mathcal{S}$ is a subset of $\mathcal{B}$, the inclusion map T from $(\mathcal{S}, \|\cdot\|')$ into $(\mathcal{B}, \|\cdot\|)$ is bounded. It follows that the adjoint of T, T^*, is bounded from $(\mathcal{B}, \|\cdot\|)$ to $(\mathcal{S}, \|\cdot\|')$. If $T^* = U[TT^*]^{1/2}$ is the polar decomposition of T^*, then U is a partial isometry mapping $\mathcal{B}$ onto $\mathcal{S}$. Since T is nonnegative, so is U. □

Theorem 8. *Let $R(\cdot)$ denote the range of an operator. If $A, B \in L(\mathcal{B})$, then*

$$R(A^*) + R(B^*) = R([A^*A + B^*B]^{1/2}).$$

Proof. Let T^* act on $\mathcal{B} \oplus \mathcal{B}$ in the normal way and represent it as $T^* = \begin{pmatrix} A^* & B^* \\ 0 & 0 \end{pmatrix}$, so that $T = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$, and $T^*T = \begin{pmatrix} A^*A + B^*B & 0 \\ 0 & 0 \end{pmatrix}$. It follows that:

$$\begin{aligned}
&[R(A^*) + R(B^*)] \oplus \{0\} = R(T^*) = R([T^*T]^{1/2}) = R\begin{pmatrix} [A^*A + B^*B]^{1/2} & 0 \\ 0 & 0 \end{pmatrix} \\
&= R([A^*A + B^*B]^{1/2}) \oplus \{0\}\,.
\end{aligned}$$

□

Theorem 9. *Let C be a closed linear operator on $\mathcal{B}$. Then there exists a pair of bounded linear contraction operators $A, B \in L[\mathcal{B}]$ such that $C = AB^{-1}$, with B nonnegative. Furthermore, $D(C) = R(B)$, $R(C) = R(A)$ and $P = A^*A + B^*B$ is the orthogonal projection $B^{-1}B$ onto $\bar{R}(B^*) = R(A^*) + R(B^*)$.*

Proof. Let $\mathcal{S} = D(C)$ be the domain of C and endow it with the graph norm, so that $\|\varphi\|' = \|\psi\| + \|C\psi\|$. Since C is linear and closed, $(\mathcal{S}, \|\cdot\|')$ is a Banach space and $\|\psi\|' \geq \|\psi\|$.

We will have use of the fact that, since $\mathcal{S}$ is a Banach subspace of $\mathcal{B}$, it is embedded in a Hilbert subspace $(\mathcal{S}', \langle\cdot,\cdot\rangle')$ of $\mathcal{H}$, where $\langle\varphi,\psi\rangle' = \langle\varphi,\psi\rangle + \langle C\varphi, C\psi\rangle$.

By Theorem 7, there is a bounded nonnegative contraction B with $B(\mathcal{B}) = \mathcal{S}$ and, for $\psi \in \mathcal{S}$, $\|\psi\|' = \left\|B^{-1}\psi\right\|$. Now let $A = CB$ so that, for $\psi \in \mathcal{B}$, we have:

$$\begin{aligned}\|A\psi\| &= \|CB\psi\| \leqslant \|B\psi\| + \|CB\psi\| \\ &= \|B\psi\|' = \left\|B^{-1}B\psi\right\| = \|P\psi\| \leqslant \|\psi\|\,.\end{aligned}$$

Hence, $\|A\varphi\| \leq \|\varphi\|$ so that A is a contraction and $A = CB = (AB^{-1})B = A(B^{-1}B) = AP$. Also, since A and B are bounded on $\mathcal{B}$, they have extensions to $\mathcal{H}$. With the same notation, we also have on $\mathcal{H}$:

$$\begin{aligned}&\langle\varphi, [A^*A + B^*B]\psi\rangle = \langle B\varphi, B\psi\rangle + \langle CB\varphi, CB\psi\rangle \\ &= \langle B\varphi, B\psi\rangle' = \left\langle B^{-1}B\varphi, B^{-1}B\psi\right\rangle = \langle P\varphi, P\psi\rangle = \langle\varphi, P\psi\rangle\,.\end{aligned}$$

Hence, $A^*A + B^*B = P$ and, since $R(A^*) + R(B^*) = R([A^*A + B^*B]^{1/2})$, $R(A^*) + R(B^*)$ is closed and equal to the closure of $R(B^*)$ on $\mathcal{H}$, the same is true for the restriction to $\mathcal{B}$ (note that B is selfadjoint). □

Let $\mathbf{V}(\mathcal{B})$ be the set of contractions and $\mathbf{C}(\mathcal{B})$ the set of closed densely defined linear operators on $\mathcal{B}$. The following improvement of Theorem 9 is possible when the operator C is also densely defined. This extends a theorem of Kaufman [KA] to Banach spaces.

Theorem 10. *The equation $\mathrm{K}(A) = A(I - A^*A)^{-1/2}$ defines a bijection from $\mathbf{V}(\mathcal{B})$ onto $\mathbf{C}(\mathcal{B})$, with inverse $\mathrm{K}^{-1}(C) = C(I + C^*C)^{-1/2}$.*

Proof. Let $A \in \mathbf{V}(\mathcal{B})$ and set $B = (I - A^*A)^{1/2}$, which is easily seen to be positive and in $\mathbf{V}(\mathcal{B})$. It follows that $\mathrm{K}(A) = AB^{-1}$ and $A^*A + B^2 = I$ so that, by the proof of Theorem 9, we see that $\mathrm{K}(A)$ is a closed linear operator on $\mathcal{B}$. Since the domain of $\mathrm{K}(A)$ is $B(\mathcal{B})$, which is dense in $\mathcal{B}$, $\mathrm{K}(A)$ is in $\mathbf{C}(\mathcal{B})$. For the opposite direction, if $C \in \mathbf{C}(\mathcal{B})$, using the same notation, let C be the extension to $\mathcal{H}$. Then, by Theorem 9 there exists a pair of bounded linear contraction operators $A, B \in L[\mathcal{H}]$ such that $C = AB^{-1}$ with B positive with range $D(C)$ and $A^*A + B^2 = I$. Furthermore, for each nonzero $\varphi \in \mathcal{H}$, $\|\varphi\|_{\mathcal{H}}^2 - \|A\varphi\|_{\mathcal{H}}^2 = \|B\varphi\|_{\mathcal{H}}^2 > 0$. Thus, $A \in \mathbf{V}(\mathcal{H})$ with $\mathrm{K}(A) = C$, so that the restriction of $A \in \mathbf{V}(\mathcal{B})$ and $\mathrm{K}(A) = C$ on $\mathcal{B}$.

Now, the graph of C in $\mathcal{H}$ is the set of all $\{(B\varphi, A\varphi), \varphi \in \mathcal{H}\}$, so that $\mathrm{C}^* = \{(\phi,\psi) \in \mathcal{H} \times \mathcal{H}\}$ such that $(\phi, A\varphi)_{\mathcal{H}} = (\psi, B\varphi)_{\mathcal{H}}$, or $(A^*\phi, \varphi)_{\mathcal{H}} = (B\psi, \varphi)_{\mathcal{H}}$ for all $\varphi \in \mathcal{H}$, so that $C^* = B^{-1}A^*$. Thus, the same is true for the restriction of C^* to $\mathcal{B}$. It is clear that $I + C^*C$ is an invertible linear operator with bounded inverse and, for each $\varphi \in \mathcal{B}$, we have that

$$\begin{aligned}\varphi &= B^2\varphi + B^{-1}(I - B^2)B^{-1}B^2\varphi \\ &= (I + B^{-1}A^*AB^{-1})B^2\varphi = (I + C^*C)B^2\varphi.\end{aligned}$$

It follows that $(I+C^*C)^{-1} = B^2$ and therefore, $A = CB = C(I+C^*C)^{-1/2} = K^{-1}(C)$. $\square$

Corollary 11. *Let $\mathcal{B}$ be a separable Banach space and let A be a closed densely defined linear operator on $\mathcal{B}$. Then A has a well-defined adjoint A^* defined on $\mathcal{B}$ such that:*

1. *the operator $A^*A \geq 0$ (maximal accretive),*
2. *$(A^*A)^* = A^*A$, and*
3. *$I + A^*A$ has a bounded inverse.*

2.2 Semigroups of Operators

In this section we introduce some basic results from the theory of semigroups of operators. Our purpose is to provide background material that will be required later. We use a fixed separable Banach space over $\mathbb{C}$, the complex numbers, and assume, when convenient, that $\mathcal{B} = \mathbf{KS}^2[\mathbb{R}^n]$ (see [GZ]). The basic references are Goldstein [GS] and Pazy [PZ], where complete proofs can be found.

Definition 12. *A family of linear operators $\{S(t), 0 \leq t < \infty\}$* (not necessarily bounded), *defined on $\mathcal{D} \subset \mathcal{B}$, is a semigroup if*

1. *$S(t+s)\varphi = S(t)S(s)\varphi$ for $\varphi \in \mathcal{D}$, the domain of the semigroup.*
2. *The semigroup is said to be strongly continuous if $\lim_{\tau \to 0} S(t+\tau)\varphi = S(t)\varphi$ for all $\varphi \in \mathcal{D}$, $t > 0$.*
3. *It is said to be a C_0-semigroup if it is strongly continuous, $S(0) = I$, and $\lim_{t \to 0} S(t)\varphi = \varphi$ for all $\varphi \in \mathcal{B}$.*
4. *$S(t)$ is a C_0-contraction semigroup if $\|S(t)\| \leqslant 1$.*

Definition 13. *The linear operator A defined by*

$$D(A) = \left\{ \varphi \in \mathcal{B} \left| \lim_{t \downarrow 0} \tfrac{1}{t}[S(t)\varphi - \varphi] \text{ exists} \right. \right\} \quad \text{and}$$

$$A\varphi = \lim_{t \downarrow 0} \tfrac{1}{t}[S(t)\varphi - \varphi] = \left. \frac{d^+S(t)\varphi}{dt} \right|_{t=0} \quad \text{for } \varphi \in D(A)$$

is the infinitesimal generator of the semigroup $S(t)$ and $D(A)$ is the domain of A.

Definition 14. *For each $\lambda > 0$, we define the Yosida approximator by: $A_\lambda = \lambda A R(\lambda, A) = \lambda^2 R(\lambda, A) - \lambda I$.*

Theorem 15. *Let A be a closed linear operator with $\overline{D(A)} = \mathcal{B}$. If the resolvent set $\rho(A)$ of A contains R^+ and, for every $\lambda > 0$, $\|R(\lambda, A)\|_{\mathcal{B}} \leqslant \lambda^{-1}$, then*

1. $\lim_{\lambda \to \infty} A_\lambda \varphi = A\varphi \quad$ for $\varphi \in D(A)$,

2. *A_λ is a bounded generator of a contraction semigroup and, for each $\varphi \in \mathcal{B}$, $\lambda, \mu > 0$, we have:*

$$\left\| e^{tA_\lambda}\varphi - e^{tA_\mu}\varphi \right\|_{\mathcal{B}} \leqslant t \left\| A_\lambda\varphi - A_\mu\varphi \right\|_{\mathcal{B}}.$$

Definition 16. *Let A be a linear operator on $\mathcal{B}$. A is said to be dissipative if*

$$\mathrm{Re}\left\langle A\varphi, f_\varphi^s \right\rangle \leqslant 0 \text{ for all } \varphi \in D(A).$$

Definition 17. *Let A be a closed dissipative linear operator with $D(A)$ dense in $\mathcal{B}$. If there is a λ_0 such that $Ran(\lambda_0 I - A) = \mathcal{B}$, then A is said to be m-dissipative.*

2.3 Generalized Yosida Approximator

If all we know is that A is the generator of a strongly continuous semigroup $T(t) = \exp(tA)$, $t > 0$, on $\mathcal{B}$, this is not enough to ensure that A has a Yosida Approximator. Unfortunately, for general strongly continuous semigroups, A may not have a bounded resolvent. The following (artificial example) shows what can (and will) happen in some real cases.

Example 18. *Let $\mathcal{H} = \mathbf{H}_0^0(\mathbb{R}^n)$ be the Hilbert space* (over $\mathbb{R}$) *of functions mapping $\mathbb{R}^n$ to itself, which vanish at infinity. Consider the Cauchy problem:*

$$\frac{d}{dt}\mathbf{u}(\mathbf{x},t) = a\left|\mathbf{x}\right|\mathbf{u}(\mathbf{x},t),\ \mathbf{u}(\mathbf{x},0) = \mathbf{f}(\mathbf{x}),$$

where $a = \prod_{i=1}^n sign(x_i)$. Let $T(t)\mathbf{f}(\mathbf{x}) = \mathrm{e}^{ta|\mathbf{x}|}\,\mathbf{f}(\mathbf{x})$, where $\mathbf{x} = [x_1, \cdots, x_n]^t$. It is easy to see that $T(t)$ is a semigroup on $\mathcal{H}$ with generator A such that $A\mathbf{f}(\mathbf{x}) = a\left|\mathbf{x}\right|\mathbf{f}(\mathbf{x})$. It follows that $u(\mathbf{x},t) = S(t)\mathbf{f}(\mathbf{x})$ solves the above initial-value problem. If we compute the resolvent, we get that:

$$R(\lambda,A)\mathbf{f}(\mathbf{x}) = \int_0^\infty e^{-\lambda t}\exp\{-t\left|\mathbf{x}\right|\}\mathbf{f}(\mathbf{x})dt = \frac{1}{\lambda - a\left|\mathbf{x}\right|}\mathbf{f}(\mathbf{x}).$$

It is clear that the spectrum of A is the real line, so that $R(\lambda,A)$ is an unbounded operator for all real λ. However, it can be checked that the bounded linear operator

$$A_\lambda = a\lambda|\mathbf{x}|\big/[\lambda + |\mathbf{x}|]$$

converges strongly to A (on $D(A)$) as $\lambda \to \infty$, and

$$\lim_{\lambda\to 0} T_\lambda(t)\mathbf{f}(\mathbf{x}) = T(t)\mathbf{f}(\mathbf{x}).$$

We do not prove this since it is a special case of the next theorem.

As an application of our extension theory, we will show that the Yosida approach can be generalized in such a way as to give a contractive approximator for all strongly continuous semigroups of operators on $\mathcal{B}$. For any closed densely defined linear operator A on $\mathcal{B}$, let $T = -[A^*A]^{1/2}$, $\bar{T} = -[AA^*]^{1/2}$. Since $-T(-\bar{T})$ is maximal accretive, $T(\bar{T})$ is m-dissipative and hence, generates a contraction semigroup. We can now write A as $A = UT$, where U is a partial isometry. Define A_λ by $A_\lambda = \lambda AR(\lambda,T)$. Note that $A_\lambda = \lambda UTR(\lambda,T) = \lambda^2 UR(\lambda,T) - \lambda U$ and, although A does not commute with $R(\lambda,T)$, we have $\lambda AR(\lambda,T) = \lambda R(\lambda,\bar{T})A$.

Theorem 19. *For every closed densely defined linear operator A on $\mathcal{B}$, we have that*

1. A_λ *is a bounded linear operator and* $\lim_{\lambda\to\infty} A_\lambda\varphi = A\varphi$, *for all* $\varphi \in D(A)$,
2. $\exp[tA_\lambda]$ *is a bounded contraction for* $t > 0$, *and*
3. *if A generates a strongly continuous semigroup* $T(t) = \exp[tA]$ *on D for* $t > 0$, $D(A) \subseteq D$, *then* $\lim_{\lambda\to\infty} \|\exp[tA_\lambda]\varphi - \exp[tA]\varphi\|_{\mathcal{B}} = 0$ *for all* $\varphi \in D$.

Proof. : To prove (1), let $\varphi \in D(A)$. Now use the fact that $\lim_{\lambda\to\infty} \lambda R(\lambda,\bar{T})\varphi = \varphi$ and $A_\lambda\varphi = \lambda R(\lambda,\bar{T})A\varphi$. To prove (2), use $A_\lambda = \lambda^2 UR(\lambda,T) - \lambda U$, $\|\lambda R(\lambda,T)\|_{\mathcal{B}} = 1$, and $\|U\|_{\mathcal{B}} = 1$ to get that $\|\exp[t\lambda^2 UR(\lambda,T) - t\lambda U]\|_{\mathcal{B}} \leq \exp[-t\lambda\|U\|_{\mathcal{B}}]\exp[t\lambda\|U\|_{\mathcal{B}}\|\lambda R(\lambda,T)\|_{\mathcal{B}}] \leq 1$.

To prove (3), let $t > 0$ and $\varphi \in D(A)$. Then

$$\begin{aligned}
\|\exp[tA]\varphi - \exp[tA_\lambda]\varphi\|_{\mathcal{B}} &= \|\int_0^t \frac{d}{ds}[e^{(t-s)A_\lambda}e^{sA}]\varphi ds\|_{\mathcal{B}} \\
&\leq \int_0^t \|[e^{(t-s)A_\lambda}(A - A_\lambda)e^{sA}\varphi]\|_{\mathcal{B}} \\
&\leq \int_0^t \|[(A - A_\lambda)e^{sA}\varphi]\|_{\mathcal{B}} ds.
\end{aligned}$$

Now use $\|[A_\lambda e^{sA}\varphi]\|_{\mathcal{B}} = \|[\lambda R(\lambda,\bar{T})e^{sA}A\varphi]\|_{\mathcal{B}} \leq \|[e^{sA}A\varphi]\|_{\mathcal{B}}$ to get $\|[(A - A_\lambda)e^{sA}\varphi]\|_{\mathcal{B}} \leq 2\|[e^{sA}A\varphi]\|_{\mathcal{B}}$. Now, since $\|[e^{sA}A\varphi]\|_{\mathcal{B}}$ is continuous, by the bounded convergence theorem we have $\lim_{\lambda\to\infty} \|\exp[tA]\varphi - \exp[tA_\lambda]\varphi\|_{\mathcal{B}} \leq \int_0^t \lim_{\lambda\to\infty} \|[(A - A_\lambda)e^{sA}\varphi]\|_{\mathcal{B}} ds = 0$. □

Theorem 20. *Every C_0-semigroup of contractions on* $\mathbf{L}^2[\mathbb{R}^n]$, $\{S(t), 0 \leqslant t < \infty\}$, *extends to a C_0-semigroup of contractions on* $\mathbf{KS}^2[\mathbb{R}^n]$.

2.4 Schatten Classes

In this section, we show how our approach allows us to provide a natural definition for the Schatten class of operators on $\mathcal{B}$.

Let $\mathbb{K}(\mathcal{B})$ be the class of compact operators on $\mathcal{B}$ and let $\mathbb{F}(\mathcal{B})$ be the set of operators of finite rank. Recall that, for separable Banach spaces, $\mathbb{K}(\mathbf{B})$ is an ideal that need not be the maximal ideal in $L[\mathcal{B}]$. If $\mathbb{M}(\mathcal{B})$ is the set of weakly compact operators and $\mathbb{N}(\mathcal{B})$ is the set of operators that map weakly convergent sequences into strongly convergent sequences, it is known that both are closed two-sided ideals in the operator norm, and, in general, $\mathbb{F}(\mathcal{B}) \subset \mathbb{K}(\mathcal{B}) \subset \mathbb{M}(\mathcal{B})$ and $\mathbb{F}(\mathcal{B}) \subset \mathbb{K}(\mathcal{B}) \subset \mathbb{N}(\mathcal{B})$ (see Dunford and Schwartz [DS], pg. 553). For reflexive Banach spaces $\mathbb{K}(\mathcal{B}) = \mathbb{N}(\mathcal{B})$ and $\mathbb{M}(\mathcal{B})$=$L[\mathcal{B}]$. For the space of continuous functions $\mathbf{C}[\Omega]$, on a compact Hausdorff space Ω, Grothendieck [GR] has shown that $\mathbb{M}(\mathcal{B})$=$\mathbb{N}(\mathcal{B})$. On the other hand, it is shown in Dunford and Schwartz [DS] that, for a positive measure space, (Ω,Σ,μ), on $\mathbf{L}^1(\Omega,\Sigma,\mu)$, $\mathbb{M}(\mathcal{B}) \subset \mathbb{N}(\mathcal{B})$.

We assume that $\mathcal{B}$ has the approximation property (i.e., every compact operator can be approximated by operators of finite rank). Recall that, given $\mathcal{B}$, there always exist $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\mathcal{H}_1 \subset \mathcal{B} \subset \mathcal{H}_2$, as continuous dense embeddings, with $\mathcal{H}_1$ determined by $\mathcal{H}_2$ (see [GBZS]). Let A be a compact operator on $\mathcal{B}$ and let $\bar{A}$ be its extension to $\mathcal{H}_2$. For each compact operator $\bar{A}$ on $\mathcal{H}_2$, there exists an orthonormal set of functions $\{\bar{\varphi}_n \,|n \geqslant 1\}$ such that

$$\bar{A} = \sum_{n=1}^\infty \mu_n(\bar{A})\,(\cdot\,,\bar{\varphi}_n)_2\,\bar{U}\bar{\varphi}_n,$$

where the μ_n are the eigenvalues of $[\bar{A}^*\bar{A}]^{1/2} = |\bar{A}|$, counted by multiplicity and in decreasing order, and $\bar{U}$ is the partial isometry associated with the polar decomposition of $\bar{A} = \bar{U}|\bar{A}|$. Without loss, we can assume that the set of functions $\{\bar{\varphi}_n \,|n \geqslant 1\}$ is contained in $\mathcal{B}$ and $\{\varphi_n \,|n \geqslant 1\}$ is the normalized version in $\mathcal{B}$. If $\mathbb{S}_p[\mathcal{H}_2]$ is the Schatten Class of order p in $L[\mathcal{H}_2]$, it is well-known that, if $\bar{A} \in \mathbb{S}_p[\mathcal{H}_2]$, its norm can be represented as:

$$\|\bar{A}\|_p^{\mathcal{H}_2} = \{Tr[\bar{A}^*\bar{A}]^{p/2}\}^{1/p} = \left\{\sum\nolimits_{n=1}^{\infty} \left(\bar{A}^*\bar{A}\bar{\varphi}_n, \bar{\varphi}_n\right)_{\mathcal{H}_2}^{p/2}\right\}^{1/p} = \left\{\sum\nolimits_{n=1}^{\infty} \left|\mu_n(\bar{A})\right|^p\right\}^{1/p}.$$

Definition 21. *We define the Schatten Class of order p in $L[\mathcal{B}]$ by:*

$$\mathbb{S}_p[\mathcal{B}] = \mathbb{S}_p[\mathcal{H}_2] \cap L[\mathcal{B}]\,|_{\mathcal{B}}$$

Since $\bar{A}$ is the extension of $A \in \mathbb{S}_p[\mathcal{B}]$, we can represent A on $\mathcal{B}$ as

$$A = \sum\nolimits_{n=1}^{\infty} \mu_n(A)\, \langle \cdot\,, f_n^s(\varphi)\rangle\, U\varphi_n,$$

where $f_n^s(\varphi) = J_2(\varphi_n) \big/ \|\varphi_n\|_2^2$ is the Steadman duality map associated with φ_n. The corresponding norm of A on $\mathbb{S}_p[\mathcal{B}]$ is defined by:

$$\|A\|_p^{\mathcal{B}} = \left\{\sum\nolimits_{n=1}^{\infty} \langle A^*A\varphi_n, f_n^s(\varphi)\rangle^{p/2}\right\}^{1/p}.$$

Theorem 22. *Let $A \in \mathbb{S}_p[\mathcal{B}]$, then $\|A\|_p^{\mathcal{B}} = \|\bar{A}\|_p^{\mathcal{H}_2}$.*

Proof. It is clear that $\{\varphi_n \,|n \geqslant 1\}$ is a set of eigenfunctions for A^*A on $\mathcal{B}$. Furthermore, by our extension of Lax's Theorem, A^*A is selfadjoint and the point spectrum of A^*A is unchanged by its extension to $\mathcal{H}_2$. It follows that $A^*A\varphi_n = \left|\mu_n(\bar{A})\right|^2 \varphi_n$,

$$\langle A^*A\varphi_n, f_n^s(\varphi)\rangle = \frac{1}{\|\varphi_n\|_2^2}\,(A^*A\varphi_n, \varphi_n)_2 = \frac{|\mu_n|^2}{\|\varphi_n\|_2^2}\,(\varphi_n, \varphi_n)_2 = |\mu_n|^2,$$

and

$$\|A\|_p^{\mathcal{B}} = \left\{\sum\nolimits_{n=1}^{\infty} \langle A^*A\varphi_n, f_n^s(\varphi)\rangle^{p/2}\right\}^{1/p} = \left\{\sum\nolimits_{n=1}^{\infty} |\mu_n|^p\right\}^{1/p} = \|\bar{A}\|_p^{\mathcal{H}_2}.$$

□

Lemma 23. *If $\mathcal{B}$ has the approximation property, the embedding of $L[\mathcal{B}]$ in $L[\mathcal{H}_2]$ is both continuous and dense.*

Proof. Recall that the embedding is continuous by Theorem 4. Since $\mathcal{B}$ has the approximation property, the finite rank operators $\mathbb{F}(\mathcal{B})$ on $\mathcal{B}$ are dense in the finite rank operators $\mathbb{F}(\mathcal{H}_2)$ on $\mathcal{H}_2$. It follows that $\mathbb{S}_p[\mathcal{B}]$ is dense in $\mathbb{S}_p[\mathcal{H}_2]$. In particular, $\mathbb{S}_1[\mathcal{B}]$ is dense in $\mathbb{S}_1[\mathcal{H}_2]$ and, since $\mathbb{S}_1[\mathcal{H}_2]^* = L[\mathcal{H}_2]$, we see that $\mathbb{S}_1[\mathcal{B}]^* = L[\mathcal{B}]$ must be dense in $L[\mathcal{H}_2]$. □

It is clear that much of the theory of operator ideals on Hilbert spaces extend to separable Banach spaces in a straightforward way. We state a few of the more important results to give a sense of the power provided by the existence of adjoints. The first result extends theorems due to Weyl [WY], Horn [HO], Lalesco [LE] and Lidskii [LI]. (The methods of proof for Hilbert spaces carry over without much difficulty.)

Theorem 24. *Let $\mathbf{A} \in \mathbb{K}(\mathcal{B})$, the set of compact operators on $\mathcal{B}$, and let $\{\lambda_n\}$ be the eigenvalues of $\mathbf{A}$ counted up to algebraic multiplicity. If Φ is a mapping on $[0,\infty]$ which is nonnegative and monotone increasing, then we have:*

1. *(Weyl)*
$$\sum_{n=1}^{\mathbf{N}} \Phi\left(|\lambda_n(\mathbf{A})|\right) \leqslant \sum_{n=1}^{\mathbf{N}} \Phi\left(\mu_n(\mathbf{A})\right)$$
and

2. *(Horn)*
$$\sum_{n=1}^{\mathbf{N}} \Phi\left(|\lambda_n(\mathbf{A}_1\mathbf{A}_2)|\right) \leqslant \sum_{n=1}^{\mathbf{N}} \Phi\left(\mu_n(\mathbf{A}_1)\mu_n(\mathbf{A}_2)\right).$$
In case $\mathbf{A} \in \mathbb{S}_1(\mathcal{B})$, we have:

3. *(Lalesco)*
$$\sum_{n=1}^{\mathbf{N}} |\lambda_n(\mathbf{A})| \leqslant \sum_{n=1}^{\mathbf{N}} \mu_n(\mathbf{A})$$
and

4. *(Lidskii)*
$$\sum_{n=1}^{\mathbf{N}} \lambda_n(\mathbf{A}) = Tr(\mathbf{A}).$$

2.5 Discussion

In a Hilbert space $\mathcal{H}$ the Schatten classes $\mathbb{S}_p(\mathcal{H})$ are the only ideals in $\mathbb{K}(\mathcal{H})$ and $\mathbb{S}_1(\mathcal{H})$ is minimal. In a Banach space, this is far from true. For a fairly complete review up to 1975, see Retherford [RE]. We limited this discussion to a few of the major topics in the history of the subject. First, Grothendieck [GR] defined an important class of nuclear operators as follows:

Definition 25. *If $\mathbf{A} \in \mathbb{F}(\mathcal{B})$ (the operators of finite rank), define the ideal $\mathbf{N}_1(\mathcal{B})$ by:*

$$\mathbf{N}_1(\mathcal{B}) = \{\mathbf{A} \in \mathbb{F}(\mathcal{B}) \mid \mathbf{N}_1(\mathbf{A}) < \infty\},$$

where

$$\mathbf{N}_1(\mathbf{A}) = \operatorname{glb}\left\{\sum_{n=1}^{m} \|f_n\| \, \|\phi_n\| \;\middle|\; f_n \in \mathcal{B}',\, \phi_n \in \mathcal{B},\, \mathbf{A} = \sum_{n=1}^{m} \phi_n \langle \cdot\,,\, f_n\rangle \right\}$$

and the greatest lower bound is over all possible representations for $\mathbf{A}$.

Grothendieck has shown that $\mathbf{N}_1(\mathcal{B})$ is the completion of the finite rank operators. $\mathbf{N}_1(\mathcal{B})$ is a Banach space with norm $\mathbf{N}_1(\cdot)$, and is a two-sided ideal in $\mathbb{K}(\mathcal{B})$. It is easy to show that:

Corollary 26. *$\mathbb{M}(\mathcal{B}), \mathbb{N}(\mathcal{B})$ and $\mathbf{N}_1(\mathcal{B})$ are two-sided *ideals.*

In order to compensate for the (apparent) lack of an adjoint for Banach spaces, Pietsch [PI] defined a number of classes of operator ideals for a given $\mathcal{B}$. Of particular importance for our discussion is the class $\mathbb{C}_p(\mathcal{B})$, defined by

$$\mathbb{C}_p(\mathcal{B}) = \left\{\mathbf{A} \in \mathbb{K}(\mathcal{B}) \;\middle|\; \mathbb{C}_p(\mathbf{A}) = \sum_{i=1}^{\infty} [s_i(\mathbf{A})]^p < \infty \right\},$$

where the singular numbers $s_n(\mathbf{A})$ are defined by:

$$s_n(\mathbf{A}) = \inf\{\|\mathbf{A}-\mathbf{K}\| \;|\text{rank of } \mathbf{K} \leqslant n\}.$$

Pietsch has shown that $\mathbb{C}_1(\mathcal{B}) \subset \mathbf{N}_1(\mathcal{B})$, while Johnson et al [JKMR] have shown that for each $\mathbf{A} \in \mathbb{C}_1(\mathcal{B})$, $\sum_{n=1}^{\infty} |\lambda_n(\mathbf{A})| < \infty$. On the other hand, Grothendieck [GR] has provided an example of an operator $\mathbf{A}$ in $\mathbf{N}_1(L^\infty[0,1])$ with $\sum_{n=1}^{\infty} |\lambda_n(\mathbf{A})| = \infty$ (see Simon [SI], pg. 118). Thus, it follows that, in general, the containment is strict. It is known that, if $\mathbb{C}_1(\mathcal{B}) = \mathbf{N}_1(\mathcal{B})$, then $\mathcal{B}$ is isomorphic to a Hilbert space (see Johnson et al). It is clear from the above discussion, that:

Corollary 27. *$\mathbb{C}_p(\mathcal{B})$ is a two-sided *ideal in $\mathbb{K}(\mathcal{B})$, and $\mathbb{S}_1(\mathcal{B}) \subset \mathbf{N}_1(\mathcal{B})$.*

For a given separable Banach space, it is not clear how the spaces $\mathbb{C}_p(\mathcal{B})$ of Pietsch relate to our Schatten Classes $\mathbb{S}_p(\mathcal{B})$ (clearly $\mathbb{S}_p(\mathcal{B}) \subseteq \mathbb{C}_p(\mathcal{B})$). Thus, one question is that of equality for $\mathbb{S}_p(\mathcal{B})$ and $\mathbb{C}_p(\mathcal{B})$. (There are many interesting research directions one can pursue from here.)

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