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ACCESSIBILITY OF A CLASS OF GENERALIZED DOUBLE-BRACKET FLOWS*

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Abstract. We investigate a generalization of Brockett's celebrated double bracket flow that is closely related to matrix Riccati differential equations. Using known results on the classification of transitive Lie group actions on homogeneous spaces, necessary and sufficient conditions for accessibility of the generalized double bracket flow on Grassmann manifolds are derived. This leads to sufficient Lie–algebraic conditions for controllability of the generalized double bracket flow. Accessibility on the Lagrangian Grassmann manifold is studied as well, with applications to matrix Riccati differential equations from optimal control.

Keywords: Double bracket flows, Grassmann manifolds, transitive Lie group actions, matrix Riccati equations.

1. Introduction.

In this paper, we study the *controlled* double bracket equation

$$\dot{X} = [\Omega(u), X] + \big[[S(u), X], X\big] \tag{1}$$

on adjoint orbits of real symmetric, or complex Hermitian matrices X. Here, $\Omega(u)$ and $S(u)$ denote arbitrary real skew–symmetric and real symmetric (or, complex skew–Hermitian and complex Hermitian) matrices, respectively, depending on real input functions u. For $\Omega(u) = 0$ and $S(u) = A$, system (1) coincides with Brockett's isospectral double bracket flow [7]

$$\dot{X} = \big[[A, X], X\big] \tag{2}$$

on real symmetric (Hermitian) matrices. Hence, (1) provides a straightforward extension of Brockett's equation and is thus called *generalized double bracket flow*, although such terminology might be in conflict with [5] (see also exercises in [16]), where a different class of generalizations has been considered.

The double bracket flow (2) has found numerous applications to diverse topics, such as e.g. linear programming, eigenvalue and singular value computations, model reduction, variational problems and Hamiltonian systems; see e.g. [16] and the references therein. For Lie–algebraic extensions we refer to [2, 3, 11]. Most of the prior research has focused on the uncontrolled double bracket equation (2), with an exception in [8], where (1) is considered as a means to simulate arbitrary finite-state automata. Here, as well as in further applications of isospectral flows to neural networks and subspace learning [9, 10], Brockett introduced a controlled variant of the

*Dedicated to Roger Brockett on the occasion of his 70th birthday.

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double bracket flow that is equivalent to the generalized double bracket system (1). Interest in (1) is also spurred through a mechanical systems interpretation. In fact, if one considers (1) as a flow on an adjoint orbit of the special unitary group SU_n acting via conjugation on its Lie algebra, then the first summand $[\Omega, X]$ is a Hamiltonian vector field, while for S skew-Hermitian, the second one $\big[[S,X],X\big]$ presents a gradient term.

Despite these scattered efforts towards a control theory of (1), a systematic analysis of the accessibility properties of generalized double bracket flows is missing. The purpose of this paper is to close this gap and to develop necessary and sufficient conditions for accessibility. For simplicity, we focus on the case, when the initial states are restricted to a Grassmannian, i.e. to the set of all selfadjoint (Hermitian) projection operators of a fixed rank. The general case of analyzing accessibility of (1) on an arbitrary adjoint orbit is much more involved and left out here. However, even the special case of considering (1) on the Grassmannian is of interest for various applications and thus deserves a detailed controllability analysis, as is done in this paper. Equation (2) on the Grassmannian has been first considered in [1, 15] and later in [4]. A characterization of the stable manifolds together with structural stability properties of the double bracket flow on Grassmannians was explored in [18, 24]. A Lie algebraic generalization in the context of parabolic subalgebras was treated in [22].

In addition to the above mentioned applications in neural networks and principal component analysis we mention two further examples for applying double bracket flows to numerical analysis and computer vision. In numerical analysis there is interest to consider continuous–time versions of the shifted QR-factorization as continuous–time eigenvalue methods, cf. [13]. For instance, if A denotes any cyclic real $n \times n$ matrix and if $\sum_{j=0}^{n-1} u_j x^j$ denotes any real polynomial of degree at most $n-1$, then $A(u) := \sum_{j=0}^{n-1} u_j A^j$ will be an arbitrary element of the centralizer of A. Here, the n coefficients of the polynomial act as control variables. Another important example from numerical analysis with scalar control u includes the case $A(u) := (A - u\,\mathrm{I}_n)^{-1}$, which corresponds to a continuous-time version of the celebrated shifted inverse power iteration. In any case, for $\Omega(u) + S(u) := A(u)$ denoting the decomposition of $A(u)$ into skew-symmetric and symmetric part Ω and S, respectively, the equation

$$\dot{P} = [\Omega(u), P] + \big[[S(u), P], P\big] \tag{3}$$

defines a control system on the $(n-1)$–dimensional real Grassmannian of rank 1 selfadjoint projection operators of $\mathbb{R}^n$. A natural question then is, whether one can always steer this system from any initial point to a desired target point, e.g. to a one-dimensional eigenspace of A. A partial answer towards this problem is given in this paper. However, in the special case, where $A(u) = (A - u\,\mathrm{I}_n)^{-1}$, the one-dimensional eigenspaces of A are fixed points of (3) so that neither controllability nor accessibility from any initial point can be expected. Thus, in the presence of fixed points, the subsequent results based on the classification of transitive Lie group actions are not powerful enough to capture the fine structure of the set of accessible points, e.g. they do not provide necessary conditions for accessibility from almost all initial points. For results in this direction further arguments on almost homogeneous spaces are needed.

Proceeding in a different direction, one can try to estimate the projective lines $[x(t)] = \mathbb{R}\, x(t)$ of non-zero state trajectories of bilinear control systems

$$\dot{x}(t) = \big(A + u(t)B\big)x(t) \tag{4}$$

from perspective output measurements $y(t) = [Cx(t)]$. This is a interesting problem in computer vision and observability conditions analogous to the celebrated Hautus-Popov criterion are available; see [14, 27]. The induced flow of (4) on the real projective space $\mathbb{P}^{n-1}$ is equivalent to the double bracket flow (3); see Section 3 for further details. Thus, the combination of control and perspective estimation problems leads to interesting control tasks for generalized double bracket flows, e.g. by finding controls that steer an initial state to a target space, defined by minimizing a cost function on the outputs.

In the sequel, we will not address these challenging issues and specific examples in any more detail, but rather focus on the characterization of controllability and accessibility properties of the general system (1) on Grassmannians. The paper is structured as follows. In Section 2, we present simple and well-known facts on the control of systems on homogeneous spaces. In Section 3, we derive a necessary and sufficient accessibility condition for system (1) in terms of the associated system Lie algebra. This main result exploits the known classification of matrix Lie groups that act transitively on Grassmann manifolds. The same line of reasoning has been used earlier by Brockett [6] to analyze controllability of bilinear systems on spheres. We deduce a simple controllability criterion for (1). An extension to generalized double bracket flows on Lagrangian Grassmannians is treated as well. In Section 4, we apply these results to characterize accessibility of *controlled* matrix Riccati differential equations. This part of our paper complements previous work by Rosenthal [27] on the dual problem of observability for the Riccati equation.

2. Preliminaries. In this section we recall some well-known definitions and facts on control system which are induced by a smooth Lie group action; see e.g. [20] for further details on nonlinear control. To simplify notation, we assume that G is a closed *matrix Lie group*, i.e. a closed Lie subgroup of the group $\mathrm{GL}_n(\mathbb{K})$ of real ($\mathbb{K} = \mathbb{R}$) or complex ($\mathbb{K} = \mathbb{C}$) invertible $n \times n$–matrices. Let $\mathfrak{g}$ denote its *matrix Lie algebra*, which is thus a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{K}) := \mathbb{K}^{n\times n}$ with Lie bracket $[A, B] := AB - BA$. Consider a right invariant control system on G

$$(\Sigma) \qquad \dot{g} = A(u)g, \quad g(0) = \mathrm{I}_n, \tag{5}$$

where $u \mapsto A(u) \in \mathfrak{g}$ denotes an arbitrary map from a control set $U \subset \mathbb{R}^m$ to the Lie algebra $\mathfrak{g}$. Let M be a connected smooth manifold and $\alpha : G \times M \to G$, $(g,p) \mapsto \alpha(g,p) = g \cdot p$ be a smooth left group action of G on M. Then, the *induced control system* on M is

$$(\widehat{\Sigma}) \qquad \dot{p} = f_u(p) := \mathrm{D}_1\, \alpha(\mathrm{I}_n, p) A(u). \tag{6}$$

Conversely, Σ is said to be a *group lift* of $\widehat{\Sigma}$. Clearly, one has the following relation between the solutions of Σ and $\widehat{\Sigma}$. We omit the straightforward proof.

LEMMA 1. *Let $u : [0,T] \to U$ be any piecewise constant control and let $g : [0,T] \to G$ be the corresponding unique solution of Σ. Then $p(t) := \alpha\big(g(t), p_0\big)$ is a solution of $\widehat{\Sigma}$. Moreover, any trajectory of $\widehat{\Sigma}$, with piecewise constant controls, can be obtained in this way.*

The *orbit* of Σ through I_n is called the *systems group* of Σ. It is defined by the subgroup

$$\mathcal{G} := \mathcal{O}(\mathrm{I}_n) := \langle \mathrm{e}^{tA(u)} \mid u \in U,\, t \in \mathbb{R} \rangle \subset G \tag{7}$$

generated by the one-parameter groups $t \mapsto \mathrm{e}^{tA(u)}$. It can be shown that $\mathcal{G}$ is actually a Lie subgroup of G with Lie algebra

$$\mathcal{L} := \mathrm{L}(\mathcal{G}) = \langle A(u) \mid u \in U \rangle \subset \mathfrak{g} \tag{8}$$

generated by $\{A(u) \mid u \in U\}$. Thus, $\mathcal{L}$ is called the *system algebra* of Σ. Similarly, for $\widehat{\Sigma}$ let

$$\widehat{\mathcal{G}} := \langle \Phi_{u,t} \mid u \in U,\, t \in \mathbb{R} \rangle, \tag{9}$$

be the group generated by all diffeomorphisms $\Phi_{u,t}$, where $\Phi_{u,t}$ denotes the flow which corresponds to the vector field f_u. Let

$$\mathcal{O}(p_0) := \{\Phi(p_0) \mid \Phi \in \widehat{\mathcal{G}}\} \tag{10}$$

denote the associated group *orbit*. Hence, $\widehat{\mathcal{G}}$ is a subgroup of the diffeomorphism group $\mathrm{Diff}(M)$. Furthermore, let $\mathcal{S}$ and $\widehat{\mathcal{S}}$ denote the *system semi-groups* obtained by restricting $t \geq 0$ in the definitions of $\mathcal{G}$ and $\widehat{\mathcal{G}}$, respectively. Thus the *reachable sets* $\mathcal{R}(\mathrm{I}_n) = \mathcal{S}$ and $\mathcal{R}(p_0) = \widehat{\mathcal{S}} \cdot p_0$ are semigroup orbits of I_n and p_0, respectively. A control system is called *accessible* if all reachable sets have non-empty interior, and *controllable* if all reachable sets coincide with the entire state space. In the sequel, let $VF(M)$ be the Lie–algebra of all smooth vector fields on M endowed with its standard Lie algebra structure, i.e. $[X,Y] := L_X Y$, where $L_X Y$ denotes the Lie derivative of vector fields $X, Y \in VF(M)$. Moreover, let

$$\Gamma_\alpha : G \to \mathrm{Diff}(M), \quad g \mapsto \alpha_g(\cdot) := \alpha(g, \cdot) \tag{11}$$

denote the canonical representation of G, associated with the Lie group action α.

PROPOSITION 2 ([21]). *One has* $\Gamma_\alpha(\mathcal{G}) = \widehat{\mathcal{G}}$ *and* $\Gamma_\alpha(\mathcal{S}) = \widehat{\mathcal{S}}$. *The map*

$$\xi : \mathfrak{g} \to VF(M), \quad A \mapsto \xi_A := -\,\mathrm{D}_1\,\alpha(I_n, \cdot\,)\,A \tag{12}$$

is a Lie algebra homomorphism. Moreover, the pull-back of ξ_A *under the diffeomorphism* α_g *satisfies* $\alpha_g^*(\xi_A) = \xi_{\mathrm{Ad}_{g^{-1}}(A)}$ *with* $\mathrm{Ad}_{g^{-1}}(A) := g^{-1}Ag$.

COROLLARY 3.

(a) The group $\widehat{\mathcal{G}}$ *acts transitively on* M *if and only if* $\mathcal{G}$ *acts transitively on* M.

(b) The induced system $\widehat{\Sigma}$ *is controllable if and only if* $\mathcal{S}$ *acts transitively on* M.

We briefly recall the so-called Lie algebra rank condition. A control system defined on a manifold M satisfies the *Lie algebra rank condition* if the Lie subalgebra $\mathcal{F} \subset VF(M)$, generated by the vector fields of the control system by taking iterated Lie derivatives, satisfies $\mathrm{T}_p\,M = \{f(p) \mid f \in \mathcal{F}\}$ for all $p \in M$.

COROLLARY 4. *The induced system* $\widehat{\Sigma}$ *satisfies the Lie algebra rank condition if and only if the map* $\xi_p : \mathcal{L} \to \mathrm{T}_p\,M$, $A \mapsto \xi_A(p)$ *is onto for all* $p \in M$. *Moreover, the Lie subalgebra* $\mathcal{F}$ *generated by the vector fields of* $\widehat{\Sigma}$ *has constant rank along* α*-orbits.*

Proof. The first assertion is a straightforward consequence of the homomorphism property of ξ. The second one follows from the identity $\mathrm{D}_1\,\alpha(\mathrm{I}_n, p)\,\mathfrak{g} = \mathrm{T}_p\,\mathcal{O}_\alpha(p)$ and the pull-back property $\alpha_g^*(\xi_A) = \xi_{\mathrm{Ad}_{g^{-1}}(A)}$. □

The following basic result relates accessibility of $\widehat{\Sigma}$ to transitivity of the system group action on M.

PROPOSITION 5. *The following statements are equivalent:*

(a) The induced system $\widehat{\Sigma}$ *is accessible.*

(b) The group $\mathcal{G}$ *acts transitively on* M.

(c) The induced system $\widehat{\Sigma}$ *satisfies the Lie algebra rank condition.*

Proof. First, assume that $\widehat{\Sigma}$ is accessible, i.e. $\mathcal{R}(p_0)$ contains an interior point for all $p_0 \in M$. Hence, each orbit $\mathcal{O}(p_0)$ has non-empty interior and thus $\mathcal{O}(p_0)$ is open for all $p_0 \in M$. Then, the connectedness of M implies $\mathcal{O}(p_0) = M$ for all $p_0 \in M$. Now, from Corollary 3 we conclude that $\mathcal{G}$ acts transitively on M. If $\mathcal{G}$ acts transitively on M, then by Corollary 3 one has $\mathcal{O}(p_0) = M$ for all $p_0 \in M$. Thus, Corollary 4 implies that the Lie subalgebra generated by the control vector fields of $\widehat{\Sigma}$ has constant rank on M. Thus, the Frobenius theorem guarantees the Lie algebra rank condition, as otherwise one would obtain the contradiction $\dim \mathcal{O}(p_0) < \dim M$. Finally, it is well-known that accessibility follows from the Lie algebra rank condition [20]. This completes the proof. □

COROLLARY 6. *If* $\mathcal{G}$ *is compact, then one has the equivalence:*

$$\widehat{\Sigma} \text{ is controllable} \quad \Longleftrightarrow \quad \mathcal{G} \text{ acts transitively on } M.$$

Proof. The compactness of $\mathcal{G}$ implies the equality $\mathcal{G} = \mathcal{S}$, cf. [19, 20]. Hence, the result follows immediately from Corollary 3. □

We emphasize, that Corollary 6 becomes false if compactness of $\mathcal{G}$ is replaced by compactness of M. A counter-example appears at the end of Section 3. For control affine systems, Corollary 6 can be strengthened significantly. Here, we give two different generalizations. A right-invariant system Σ is said to be *control affine*, if $A(u)$ is of the form

$$A(u) = A_0 + \sum_{k=1}^{m} u_k A_k, \quad u := (u_1, \ldots, u_m)^\top \in U \subset \mathbb{R}^m \tag{13}$$

with $A_0, A_1, \ldots, A_m \in \mathfrak{g}$. Here, A_0 is called the *drift term* of Σ and the $A_1, \ldots, A_m$ are referred to as *control terms*. Moreover, we assume that the following condition is satisfied:

(∗) The origin of $\mathbb{R}^m$ is an interior point of the convex hull of U.

The Lie subalgebra generated by the control terms is denoted by $\mathcal{L}_0$ and the corresponding Lie subgroup by $\mathcal{G}_0$. The following two results yield useful sufficient conditions for checking controllability.

Theorem 7. *Let Σ be control affine with drift term A_0 and $\xi : \mathfrak{g} \to VF(M)$ be given by* (12). *If there exists $K_0 \in \ker \xi$ such that the one-parameter group $\mathrm{e}^{t(A_0+K_0)}$, $t \in \mathbb{R}$, is contained in some compact subgroup of G, then the following statements are equivalent:*

(a) The induced system $\widehat{\Sigma}$ is controllable.
(b) The induced system $\widehat{\Sigma}$ is accessible.
(c) The group $\mathcal{G}$ acts transitively on M.

Proof. By Proposition 5, it is sufficient to show the equivalence (a) $\iff$ (b). Obviously, (a) implies (b), Therefore, we are left with proving the implication (b) $\Longrightarrow$ (a). By the above compactness assumption, the right-invariant vector field $g \mapsto (A_0 + K_0)g$ is (weakly) positively Poisson stable. Thus, the induced vector field $\xi_{A_0} = \xi_{A_0+K_0}$ is also (weakly) positively Poisson stable and therefore the Lie algebra rank condition, which holds by Proposition 5, together with condition (∗) implies controllability. □

From a previous remark we know that compactness of M together with accessibility of the system is not sufficient to guarantee controllability of $\widehat{\Sigma}$. Therefore, one needs an additional assumption to exploit the compactness of M for controllability. For instance, if the induced drift vector field is *Hamiltonian*, the compactness of M guarantees again positive Poisson stability and therefore controllability of $\widehat{\Sigma}$.

Proposition 8. *Let Σ be control affine with drift term A_0 and unbounded control set $U = \mathbb{R}^m$. If the closure of the convex hull of the adjoint orbit $\mathrm{Ad}_{\mathcal{G}_0}(A_0) := \{gA_0g^{-1} \mid g \in \mathcal{G}_0\}$ intersects the open half space $\mathbb{R}^- A_0 + \ker \xi := \{\lambda A_0 + K \mid \lambda < 0,\ K \in \ker \xi\}$, then accessibility of the induced system $\widehat{\Sigma}$ is equivalent to controllability.*

Proof. Since controllability implies accessibility it suffices to show the converse. Moreover, the kernel of ξ is the Lie subalgebra of the closed subgroup $\Gamma_\alpha^{-1}(\mathrm{id}_M) :=$

$\{g \in G \mid \alpha_g = \mathrm{id}_M\}$. Thus, $G' := G/\Gamma_\alpha^{-1}(\mathrm{id}_M)$ is a Lie group and α has a natural restriction to G' which acts effectively on M. Therefore, we can assume without loss of generality that the closure of the convex hull of the adjoint orbit $\mathrm{Ad}_{\mathcal{G}_0}(A_0)$ intersects the open half line $\mathbb{R}^- A_0$. Now, since the controls can be chosen arbitrarily in $\mathbb{R}^m$, a standard argument shows that the $\exp\big(t\mathrm{Ad}_g(A_0)\big)$, $t > 0$ is contained in the closure of $\mathcal{S}$ (relative to $\mathcal{G}$). The same applies to all A' in the closure of the convex hull of the adjoint orbit $\mathrm{Ad}_{\mathcal{G}_0}(A_0)$, i.e. $\exp\big(tA'\big) \in \mathrm{cl}_{\mathcal{G}}(\mathcal{S})$ for all $t > 0$. Thus, by the above assumption $\exp(-tA_0) \in \mathrm{cl}_{\mathcal{G}}(\mathcal{S})$ for $t > 0$. This, however, implies $\mathrm{cl}_{\mathcal{G}}(\mathcal{S}) = \mathcal{G}$ and therefore $\mathcal{S} = \mathcal{G}$, cf. [20]. $\square$

An illustrative application of the previous result is given after Corollary 15.

3. Generalized Double Bracket Flow. We now analyze the controllability properties of the *generalized double bracket equation* (GDBE)

$$\dot{X} = [\Omega(u), X] + \big[[S(u), X], X\big]. \tag{14}$$

Throughout the remaining sections the set of admissible controls is supposed to contain at least all piecewise constant controls with arbitrary values in $U := \mathbb{R}^n$. Moreover, $\Omega(u)$ and $S(u)$ are either real skew-symmetric and symmetric, respectively, or complex skew-Hermitian and Hermitian, respectively. Thus, $A(u) := \Omega(u) + S(u)$ denotes an arbitrary real or complex $n \times n$-matrix. For symmetric $A(u) = A$, Equation (14) reduces to Brockett's double bracket equation (DBE)

$$\dot{X} = \big[[A, X], X\big] \tag{15}$$

on symmetric matrices X, cf. [7, 8, 9, 10]. Thus, (14) constitutes a natural generalization of Brockett's equation. Both systems (15) and (14) define isospectral flows on the set of real symmetric (complex Hermitian) matrices. In this paper, we focus on the simplified situation, where X is an arbitrary selfadjoint or Hermitian projection operator of rank k. For the sake of clarity, we first discuss the complex, i.e. Hermitian case. Later on, we summarize the corresponding results for real selfadjoint projection operators. Finally, we also address the symplectic case.

Clearly, the GDBE evolves on the unitary similarity orbit of its initial value $X(0) = X_0$. Thus, if X_0 is a Hermitian projection operator of rank k, Equation (14) restricts to a control system on the complex Grassmannian $\mathrm{Grass}_{k,n}(\mathbb{C})$. Furthermore, it is straightforward to see that (14) can also be regarded as a control system on the complex Grassmann manifold of k-dimensional subspaces of $\mathbb{C}^n$. This relation will be heavily exploited in the sequel. Therefore, we briefly review some well-known facts on Grassmannians and the Grassmann manifold.

Let $\mathfrak{gl}_n(\mathbb{C}) := \mathbb{C}^{n\times n}$ be the set of all complex $n \times n$-matrices and let $\mathfrak{sl}_n(\mathbb{C}) \subset \mathfrak{gl}_n(\mathbb{C})$ be the Lie subalgebra of all complex $n\times n$-matrices with trace zero. Moreover, let $\mathrm{GL}_n(\mathbb{C})$ be the set of all invertible $n \times n$-matrices and let $\mathrm{SL}_n(\mathbb{C}) \subset \mathrm{GL}_n(\mathbb{C})$ be the closed Lie subgroup of all invertible $n \times n$-matrices with determinant one. The *Grassmannian* $\mathrm{Grass}_{k,n}(\mathbb{C})$ is defined as the set of all Hermitian $n \times n$-projection operators of rank k, i.e.

$$\mathrm{Grass}_{k,n}(\mathbb{C}) := \{X \in \mathbb{C}^{n\times n} \mid X = X^*,\, X^2 = X,\, \mathrm{tr}X = k\}. \tag{16}$$

Here, $M^* := \overline{M}^\top$ denotes the conjugate transpose of a matrix. In contrast, the *Grassmann manifold* $\mathrm{G}_{k,n}(\mathbb{C})$ is given as the set of all complex k-dimensional subspaces of $\mathbb{C}^n$, i.e.

$$\mathrm{G}_{k,n}(\mathbb{C}) := \{V \subset \mathbb{C}^n \mid V \text{ complex linear subspace}, \dim V = k\}. \tag{17}$$

Note, that $\mathrm{Grass}_{k,n}(\mathbb{C})$ and $\mathrm{G}_{k,n}(\mathbb{C})$ carry natural real analytic manifold structures. More precisely, $\mathrm{Grass}_{k,n}(\mathbb{C})$ can be viewed as a homogeneous orbit of a compact Lie group action on $\mathbb{C}^{n\times n}$ and thus constitutes a real analytic submanifold of $\mathbb{C}^{n\times n}$, whereas $\mathrm{G}_{k,n}(\mathbb{C})$ can be equipped with a quotient manifold structure via coordinate charts.

LEMMA 9 ([16]). *The manifolds* $\mathrm{Grass}_{k,n}(\mathbb{C})$ *and* $\mathrm{G}_{k,n}(\mathbb{C})$ *are diffeomorphic via the real analytic map* $\mu : \mathrm{Grass}_{k,n}(\mathbb{C}) \to \mathrm{G}_{k,n}(\mathbb{C})$, $X \mapsto \mathrm{Im}\, X$.

The GDBE as induced system. Clearly, any right invariant control system

$$\dot g = A(u)g, \quad g(0) = \mathrm{I}_n \tag{18}$$

on $\mathrm{GL}_n(\mathbb{C})$ induces by the left action $\beta : (g, V) \mapsto gV$ a control system on $\mathrm{G}_{k,n}(\mathbb{C})$. By Lemma 9, the β-action can be pulled–back to the Grassmannian $\mathrm{Grass}_{k,n}(\mathbb{C})$ via $\alpha := \mu^{-1}\big(\beta(\,\cdot, \mu(\cdot))\big)$ and therefore (18) induces a control system on $\mathrm{Grass}_{k,n}(\mathbb{C})$. Next, we show that this construction yields exactly the generalized double bracket flow. The proof of Lemma 10 below is obtained by a straightforward computation, using the fact, that (19) does not depend on the choice of the factors R and R^* in the decomposition $X = RR^*$.

LEMMA 10. *For any full rank factorization* $X = RR^* \in \mathrm{Grass}_{k,n}(\mathbb{C})$ *with* $R^*R = \mathrm{I}_k$, *the* $\mathrm{GL}_n(\mathbb{C})$*–action* $\alpha := \mu^{-1}\big(\beta(\,\cdot, \mu(\cdot))\big)$ *on* $\mathrm{Grass}_{k,n}(\mathbb{C})$ *is given by*

$$\alpha : (g, X) \mapsto gR\big(R^*g^*gR\big)^{-1}R^*g^*. \tag{19}$$

Lemma 10 leads to an explicit formula for the α-induced vector fields on $\mathrm{Grass}_{k,n}(\mathbb{C})$. It therefore shows how the GDBE is related with a linear induced flow on $\mathrm{G}_{k,n}(\mathbb{C})$.

PROPOSITION 11. *Let* $A := \Omega + S \in \mathfrak{gl}_n(\mathbb{C})$ *be a constant matrix with* Ω *skew Hermitian and* S *Hermitian and let* $g \mapsto Ag$ *be the corresponding right invariant vector field on* $\mathrm{GL}_n(\mathbb{C})$. *Then the vector field on* $\mathrm{Grass}_{k,n}(\mathbb{C})$ *induced by* α *is*

$$\mathrm{D}_1\, \alpha(\mathrm{I}_n, X)\, A = [\Omega, X] + \big[[S, X], X\big]. \tag{20}$$

Proof. Consider the solution $t \mapsto \mathrm{e}^{tA}$ of the linear system (18). Then, we know that $t \to \alpha(\mathrm{e}^{tA}, X)$ yields a solution of the induced vector field. Now, let $X = RR^*$ with $R^*R = \mathrm{I}_k$. Hence, from Lemma 10 we obtain

$$\begin{aligned}\mathrm{D}_1\,\alpha(\mathrm{I}_n, X)\,A &= \frac{\mathrm{d}}{\mathrm{d}\,t}\,\alpha(\mathrm{e}^{tA}, X)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}\,t}\,\mathrm{e}^{tA}R\big(R^*\mathrm{e}^{tA^*}\mathrm{e}^{tA}R\big)^{-1}R^*\mathrm{e}^{tA^*}\Big|_{t=0}\\ &= AX + XA^* - X(A^*+A)X\\ &= [\Omega, X] + SX + XS - 2XSX = [\Omega, X] + \big[[S,X],X\big]\end{aligned}$$

and thus the induced vector field satisfies (20). $\square$

The following result is therefore an immediate consequence of Lemma 1 and 9 together with Proposition 11.

COROLLARY 12. *Let $A := \Omega + S \in \mathfrak{gl}_n(\mathbb{C})$ with Ω skew Hermitian and S Hermitian. The unique solution $X(t)$ of the* GDBE *on the Grassmannian*

$$\dot{X} = [\Omega, X] + \big[[S,X],X\big], \quad X(0) = X_0 \in \mathrm{Grass}_{k,n}(\mathbb{C}) \tag{21}$$

with $X_0 = R_0R_0^*$ and $R_0^*R_0 = \mathrm{I}_k$, is given by

$$X(t) = \mathrm{e}^{tA}R_0\big(R_0^*\mathrm{e}^{tA^*}\mathrm{e}^{tA}R_0\big)^{-1}R_0^*\mathrm{e}^{tA^*}. \tag{22}$$

Moreover, $\mathrm{Im}\,\mathrm{X(t)} = \mathrm{e}^{\mathrm{tA}}\mathrm{Im}\,\mathrm{X}_0$ *holds for all $t \in \mathbb{R}$.*

To compute the Lie algebra homomorphism ξ of Proposition 2 induced by $-\,\mathrm{D}_1\,\alpha(\mathrm{I}_n, \cdot\,)$ we introduce the following notation. For Ω skew-Hermitian and S Hermitian, let ζ_Ω and η_S be vector fields on $\mathrm{Grass}_{k,n}(\mathbb{C})$ given by

$$\zeta_\Omega : \mathrm{Grass}_{k,n}(\mathbb{C}) \to \mathrm{T}\,\mathrm{Grass}_{k,n}(\mathbb{C}), \quad X \mapsto \zeta_\Omega(X) := [\Omega, X] \tag{23}$$

and

$$\eta_S : \mathrm{Grass}_{k,n}(\mathbb{C}) \to \mathrm{T}\,\mathrm{Grass}_{k,n}(\mathbb{C}), \quad X \mapsto \eta_S(X) := \big[[S,X],X\big]. \tag{24}$$

THEOREM 13. *The map $\xi : \mathfrak{gl}_n(\mathbb{C}) \to VF\big(\mathrm{Grass}_{k,n}(\mathbb{C})\big)$, $A \mapsto \xi_A = -\zeta_\Omega - \eta_S$ is a Lie algebra homomorphism, where Ω and S denote the skew-Hermitian and Hermitian part of A, respectively. In particular, for Ω_1, Ω_2 skew-Hermitian and S_1, S_2 Hermitian one has the Lie-brackets relations*

$$\begin{aligned}[\zeta_{\Omega_1}, \zeta_{\Omega_2}] &= -\zeta_{[\Omega_1,\Omega_2]},\\ [\zeta_{\Omega_1}, \eta_{S_1}] &= -\eta_{[\Omega_1,S_1]},\\ [\eta_{S_1}, \eta_{S_2}] &= -\zeta_{[S_1,S_2]},\end{aligned} \tag{25}$$

The kernel of ξ is given by $\mathbb{C}\cdot\mathrm{I}_n$. Thus, the restriction $\xi : \mathfrak{sl}_n(\mathbb{C}) \to VF\big(\mathrm{Grass}_{k,n}(\mathbb{C})\big)$ is a Lie algebra isomorphism onto the image of ξ.

Proof. Clearly, by Proposition 11 we have $\xi = -\mathrm{D}_1\,\alpha(\mathrm{I}_n,\cdot\,)$ and thus the first assertion follows immediately from Proposition 2. The identities in (25) are obtained by the usual commutator relations for Hermitian and skew-Hermitian matrices. Hence, we are left with the computation of the kernel of ξ. Let $A = \Omega + S \in \mathbb{C}^{n\times n}$ with Ω skew-Hermitian and S Hermitian and suppose $\xi_A = 0$, i.e.

$$[\Omega, X] + \big[[S, X], X\big] = 0 \quad \text{for all } X \in \mathrm{Grass}_{k,n}(\mathbb{C}). \tag{26}$$

Then, the identity $\mathrm{Grass}_{k,n}(\mathbb{C}) = \left\{\Theta \left[\begin{smallmatrix}\mathrm{I}_k & 0\\ 0 & 0\end{smallmatrix}\right]\Theta^* \;\middle|\; \Theta \in \mathrm{SU}_n\right\}$ implies that (26) is equivalent to

$$\left[\Theta^*\Omega\Theta, \begin{bmatrix}\mathrm{I}_k & 0\\ 0 & 0\end{bmatrix}\right] + \left[\left[\Theta^* S\Theta, \begin{bmatrix}\mathrm{I}_k & 0\\ 0 & 0\end{bmatrix}\right], \begin{bmatrix}\mathrm{I}_k & 0\\ 0 & 0\end{bmatrix}\right] = 0 \quad \text{for all } \Theta \in \mathrm{SU}_n, \tag{27}$$

where SU_n denotes the set of all special unitary $n \times n$-matrices. Now, define $\widehat{\Omega} := \left[\begin{smallmatrix}\widehat{\Omega}_{11} & \widehat{\Omega}_{12}\\ \widehat{\Omega}_{21} & \widehat{\Omega}_{22}\end{smallmatrix}\right] := \Theta^*\Omega\Theta$ and $\widehat{S} := \left[\begin{smallmatrix}\widehat{S}_{11} & \widehat{S}_{12}\\ \widehat{S}_{21} & \widehat{S}_{22}\end{smallmatrix}\right] := \Theta^* S\Theta$. Then (27) yields

$$\begin{bmatrix}0 & -\widehat{\Omega}_{12}\\ \widehat{\Omega}_{21} & 0\end{bmatrix} + \begin{bmatrix}0 & \widehat{S}_{12}\\ \widehat{S}_{21} & 0\end{bmatrix} = 0 \quad \text{for all } \widehat{\Omega}, \widehat{S}, \tag{28}$$

which further implies

$$\Theta^* A\Theta = \Theta^*\Omega\Theta + \Theta^* S\Theta = \begin{bmatrix}* & *\\ 0 & *\end{bmatrix} \quad \text{for all } \Theta \in \mathrm{SU}_n. \tag{29}$$

It follows that $\Theta\left[\begin{smallmatrix}\mathrm{I}_k\\ 0\end{smallmatrix}\right]$ is an A-invariant subspace for all $\Theta \in \mathrm{SU}_n$, i.e. all $V \in \mathrm{G}_{k,n}(\mathbb{C})$ are invariant subspaces of A. Thus $A = \lambda\,\mathrm{I}_n$ with $\lambda \in \mathbb{C}$ and hence $\ker\xi = \mathbb{C}\cdot\mathrm{I}_n$. The result follows. □

Accessibility and Controllability. Let $A_0, A_1, \ldots, A_m \in \mathfrak{gl}_n(\mathbb{C})$ and $A(u) := A_0 + \sum_{j=1}^m u_j A_j = \Omega(u) + S(u)$ with $u := (u_1, \ldots, u_m)^\top \in \mathbb{R}^m$, where $\Omega(u)$ and $S(u)$ denote the skew-Hermitian and Hermitian part of $A(u)$, respectively. Moreover, let $\mathcal{G}$ and $\mathcal{L}$ be the corresponding system group and system algebra, cf. (7) and (8). Thus, we are prepared to apply the accessibility and controllability results of Section 2 to the GDBE

$$\dot{X} = [\Omega(u), X] + \big[[S(u), X], X\big], \quad X(0) = X_0 \in \mathrm{Grass}_{k,n}(\mathbb{C}). \tag{30}$$

evolving on the Grassmannian $\mathrm{Grass}_{k,n}(\mathbb{C})$.

PROPOSITION 14. *The generalized double bracket equation* (30) *is accessible on* $\mathrm{Grass}_{k,n}(\mathbb{C})$ *if and only if the system group* $\mathcal{G}$ *acts transitively on the Grassmann manifold* $\mathrm{G}_{k,n}(\mathbb{C})$.

Proof. From Proposition 11 we readily concluded that the GDBE (30) coincides with the α-induced system on $\mathrm{Grass}_{k,n}(\mathbb{C})$. Thus, Proposition 5 says that accessibility is equivalent to transitive group action of $\mathcal{G}$ on $\mathrm{Grass}_{k,n}(\mathbb{C})$. By construction, however, the action of α on $\mathrm{Grass}_{k,n}(\mathbb{C})$ and the linear action of $\mathcal{G}$ on $\mathrm{G}_{k,n}(\mathbb{C})$ are related via the diffeomorphism μ, cf. Lemma 9 and 10. Thus, transitivity of $\mathcal{G}$ on $\mathrm{Grass}_{k,n}(\mathbb{C})$ is equivalent to transitivity of $\mathcal{G}$ on $\mathrm{G}_{k,n}(\mathbb{C})$. □

Now, combining Theorem 7 and 13 as well as Proposition 8 yields the following sufficient controllability conditions for the GDBE. For the definition of $\mathcal{G}_0$ see the paragraph before Theorem 7.

Corollary 15. *With the same notation as in Proposition 8 one has.*

(a) If the drift term A_0 is of the form $A_0 = \Omega_0 + \lambda\,\mathrm{I}_n$ with $\lambda \in \mathbb{C}$ and Ω_0 skew-Hermitian, then controllability of the generalized double bracket equation (30) *is equivalent to its accessibility.*

(b) If the closure of the convex hull of the adjoint orbit $\mathrm{Ad}_{\mathcal{G}_0}(A_0)$ intersects the half space $\mathbb{R}^- A_0 + \mathbb{C}\,\mathrm{I}_n$, then controllability of the generalized double bracket equation (30) *is equivalent to its accessibility.*

A similar result for the linear induced flow on the real Grassmann manifold $\mathrm{G}_{k,n}(\mathbb{R})$ can be found in [20], Ch. 6, Thm. 5. However, part (a) and (b) of the cited result is somewhat incomplete; a closedness assumption on the subgroup H is missing.

With regard to the classical DBE

$$\dot{X} = \big[[A(u), X], X\big] \tag{31}$$

with symmetric $A_0, \dots, A_m$, the condition of Corollary 15(b) can only be fulfilled if there is more than one control term A_j. Then, however, the Lie algebra $\mathcal{L}_0$ generated by the control terms is known to coincide generically with $\mathfrak{sl}_n(\mathbb{C})$ or $\mathfrak{gl}_n(\mathbb{C})$, cf. [20], and controllability follows straightforwardly without referring to Corollary 15(b). Therefore, it is of interest to see a non-trivial application of Corollary 15(b), i.e. an example where controllability holds, while $\mathcal{L}_0$ does not coincide with $\mathfrak{sl}_n(\mathbb{C})$ or $\mathfrak{gl}_n(\mathbb{C})$. Let

$$A_0 := H_0 + \mathrm{I}_4 \quad \text{with} \quad H_0 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \tag{32}$$

and let $\{A_1, A_2\}$ be any generating set of the Lie algebra

$$\{Z_1 \otimes \mathrm{I}_2 + \mathrm{I}_2 \otimes Z_1 \mid Z_1, Z_2 \in \mathfrak{sl}_2(\mathbb{C})\}, \tag{33}$$

where $\otimes$ denotes the Kronecker product of matrices. Then $\mathcal{G}_0 = \mathrm{SL}_2(\mathbb{C}) \otimes \mathrm{SL}_2(\mathbb{C}) \subsetneq \mathrm{SL}_4(\mathbb{C})$. However, $\mathcal{G}_0$ contains a subgroup Π of the Weyl group of $\mathrm{SL}_4(\mathbb{C})$ (i.e of the group of all 'signed' permutation matrices) such that the convex hull of the orbit $\{\Theta H_0 \Theta^* \mid \Theta \in \Pi\}$ intersects $\mathbb{R}^- H_0$. Thus, the adjoint orbit $\mathrm{Ad}_{\mathcal{G}_0}(A_0)$ satisfies the hypothesis of Corollary 15(b). Moreover, it is easy to check that the set $\{A_0, A_1, A_2\}$ generates $\mathfrak{gl}_4(\mathbb{C})$. Hence, the DBE

$$\dot{X} = \big[[A_0 + u_1 A_1 + u_2 A_2, X], X\big] \tag{34}$$

with $(u_1, u_2)^\top \in \mathbb{R}^2$ is controllable on $\mathrm{Grass}_{k,4}(\mathbb{C})$ for all $1 \leq k \leq 3$. Yet, $\mathcal{L}_0 \neq \mathfrak{sl}_4(\mathbb{C})$ and $\mathcal{L}_0 \neq \mathfrak{gl}_4(\mathbb{C})$.

Above, we have seen that accessibility of the GDBE is equivalent to transitivity of the system group action on a corresponding Grassmann manifold. Now, Völklein has completely characterized all connected linear matrix groups which act transitively on Grassmann manifolds. Here, we state only that part of Völklein's work which is relevant for us.

THEOREM 16 ([30]). *Let G be a connected Lie subgroup of $\mathrm{GL}_n(\mathbb{C})$.*

(a) If $n \in \mathbb{N}$ is odd, then G acts transitively on $\mathrm{G}_{k,n}(\mathbb{C})$ if and only if G is a direct product of the form $G = Z \cdot G_0$, where Z is a connected Lie subgroup of $\mathbb{C}^* \mathrm{I}_n$ and G_0 is conjugate to $\mathrm{SL}_n(\mathbb{C})$ or SU_n.

(b) If $n \in \mathbb{N}$ is even, then one has to distinguish two cases:

(i) For $1 < k < n$, transitivity holds if and only if Z and G_0 satisfy the conditions of (a).

(ii) For $k = 1$ or $k = n-1$, transitivity holds if and only if Z is a connected Lie subgroup of $\mathbb{C}^* \mathrm{I}_n$ and G_0 is conjugate to $\mathrm{SL}_n(\mathbb{C})$, SU_n, $\mathrm{SL}_{n/2}(\mathbb{H})$, $\mathrm{Sp}_{n/2}(\mathbb{C})$ or $\mathrm{Sp}_{n/2}$.

Here, SU_n and $\mathrm{Sp}_{n/2}$ denote the special unitary and the compact symplectic group, respectively. $\mathrm{Sp}_{n/2}(\mathbb{C})$ is the non-compact complex symplectic group and $\mathrm{SL}_{n/2}(\mathbb{H})$ the special linear group of $\mathbb{H}^{n/2}$ embedded in $\mathrm{SL}_n(\mathbb{C})$, where $\mathbb{H}$ denotes the quaternions [12]. By combining Völklein's result with Theorem 14 we conclude that the GDBE (30) is accessible if and only if the system group $\mathcal{G}$ is conjugate to one of the groups listed above. Hence, the subsequent theorem follows readily from the fact that the Lie algebra of $\mathcal{G}$ is given by the system Lie algebra $\mathcal{L}$.

THEOREM 17. *Let $\mathcal{L}$ denote the system Lie algebra generated by $A(u)$, $u \in \mathbb{R}^m$. The generalized double bracket equation* (30) *is accessible on the Grassmannian $\mathrm{Grass}_{k,n}(\mathbb{C})$ if and only if*

(a) $\mathcal{L} = \mathfrak{z} \oplus \mathcal{L}_0$, where $\mathfrak{z}$ is a Lie subalgebra of $\mathbb{C}\,\mathrm{I}_n$ and $\mathcal{L}_0$ is equal to $\mathfrak{sl}_n(\mathbb{C})$ or conjugate to $\mathfrak{su}_n$, if n is odd or $1 < k < n$.

(b) $\mathcal{L} = \mathfrak{z} \oplus \mathcal{L}_0$, where $\mathfrak{z}$ is a Lie subalgebra of $\mathbb{C}\,\mathrm{I}_n$ and $\mathcal{L}_0$ is equal to $\mathfrak{sl}_n(\mathbb{C})$ or conjugate to $\mathfrak{su}_n$, $\mathfrak{sl}_{n/2}(\mathbb{H})$, $\mathfrak{sp}_{n/2}(\mathbb{C})$ or $\mathfrak{sp}_{n/2}$, if n is even and $k = 1$ or $k = n-1$.

Observe, that Theorem 17 and Corollary 15 lead to a simple purely algebraic controllability test for the GDBE. Next, we specify the previous result to the double bracket equation. In this case, all compact candidates in Theorem 17 can be excluded. Note, however, that all remaining Lie algebras of Corollary 18 can indeed occur as system Lie algebra of the DBE.

COROLLARY 18. *Let $\mathcal{L}$ denote the system Lie algebra generated by the Hermitian matrices $A(u)$, $u \in \mathbb{R}^m$. The double bracket equation*

$$\dot{X} = \big[[A(u), X], X\big], \quad X(0) = X_0 \in \mathrm{Grass}_{k,n}(\mathbb{C}) \tag{35}$$

is accessible on the Grassmannian $\mathrm{Grass}_{k,n}(\mathbb{C})$ *if and only if*

(a) $\mathcal{L} = \mathfrak{z} \oplus \mathfrak{sl}_n(\mathbb{C})$, *where* $\mathfrak{z}$ *is either trivial or* $\mathbb{R}\,\mathrm{I}_n$, *if* n *is odd or* $1 < k < n$.

(b) $\mathcal{L} = \mathfrak{z} \oplus \mathcal{L}_0$, *where* $\mathfrak{z}$ *is either trivial or* $\mathbb{R}\,\mathrm{I}_n$ *and* $\mathcal{L}_0$ *is conjugate to* $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{sl}_{n/2}(\mathbb{H})$ *or* $\mathfrak{sp}_{n/2}(\mathbb{C})$, *if* n *is even and* $k=1$ *or* $k=n-1$.

Finally, we give an example of an accessible, but non-controllable DBE on the complex projective line $\mathbb{CP}^1 = \mathrm{Grass}_{1,2}(\mathbb{C})$, proving a previous remark of Section 2. For $a \in \mathbb{R}$ and $w \in \mathbb{C}$ with $\mathrm{Re}\, w \geq 0$ let

$$S_0 := \begin{bmatrix} a & w \\ \overline{w} & -a \end{bmatrix} \quad \text{and} \quad S_1 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{36}$$

Consider the system

$$\dot{z} = (S_0 + uS)z \tag{37}$$

with $u \in \mathbb{R}$ on $\mathbb{C}^2 \setminus \{0\}$ and the non-convex cone

$$\mathcal{C} := \{z = (z_1, z_2)^\top \in \mathbb{C}^2 \setminus \{0\} \mid \mathrm{Re}\, z_1\overline{z}_2 \geq 0\} \subsetneq \mathbb{C}^2 \setminus \{0\}.$$

Then any solution $z(t)$ of (37) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\Big(\mathrm{Re}\, z_1(t)\overline{z}_2(t)\Big) = \big(|z_1(t)|^2 + |z_2(t)|^2\big)\mathrm{Re}\, w \geq 0. \tag{38}$$

Hence, the set $\mathcal{C}$ is positively invariant under the flow of (37). Therefore, $\mathcal{R}(z) \subset \mathcal{C}$ for all $z \in \mathcal{C}$ and thus (37) is *not* controllable on $\mathbb{C}^2 \setminus \{0\}$. Note, that $\mathcal{C}$ is invariant under scalar multiplication by non-zero complex numbers and thus defines a proper subset of $\mathrm{Grass}_{1,2}(\mathbb{C})$. Then, controllability fails also for the induced system on $\mathrm{Grass}_{1,2}(\mathbb{C})$. Finally, it is straightforward to show that the Lie algebra generated by S_0 and S is equal to $\mathfrak{sl}_2(\mathbb{C})$ for a generic choice of a and w. A similar construction can be carried out for $\mathrm{Re}\, w \leq 0$ and $\widehat{\mathcal{C}} := \{z = (z_1, z_2)^\top \in \mathbb{C}^2 \setminus \{0\} \mid \mathrm{Re}\, z_1\overline{z}_2 \leq 0\}$. This implies that the double-bracket flow

$$\dot{X} = \big[[S_0 + uS, X], X\big] \tag{39}$$

is in general accessible but *never* controllable on $\mathrm{Grass}_{1,2}(\mathbb{C})$.

REMARK 19. It is known that the control system

$$\dot{X} = (A_0 + u_1 A_1)X, \quad u \in \mathbb{R}, \tag{40}$$

where A_0 and A_1 are symmetric with trace zero, is never controllable on $\mathrm{SL}_n(\mathbb{R})$ for $n = 2$ and $n = 3$, e.g. [28]. Hence, Theorem 20 below implies that (39) is never controllable on $\mathrm{Grass}_{k,n}(\mathbb{R})$ for $n = 2$, $n = 3$ and $1 \le k \le n-1$. Jurdjevic and Kupka conjectured that controllability of (40) fails for all $n \in \mathbb{N}$. According to Theorem 20 below, this is equivalent to the double bracket equation (39) being never controllable on $\mathrm{Grass}_{k,n}(\mathbb{R})$ for $1 < k < n-1$. (Note, that for symmetric A_0 and A_1, the compact Lie algebras $\mathfrak{so}_n$, $\mathfrak{g}_2$ and $\mathfrak{spin}_7$ can be excluded from the list in Theorem 20).

The real and symplectic case. In this subsection, we summarize the corresponding results for the real and symplectic GDBE. Thereby, the real case can be treated completely by the same techniques as before—yet the classification of all groups acting transitively on real Grassmann manifolds becomes much more involved [30]. For the symplectic GDBE, we need to classify all groups acting transitively on Lagrangian Grassmann manifolds; a case which was not studied by Völklein. The only reference for this case we are aware of is the unpublished diploma thesis by H. Kramer at the University Würzburg (2001). Therefore, we will sketch the proof after Theorem 21.

Real case. Let $A_0, A_1, \dots A_m \in \mathfrak{gl}_n(\mathbb{R})$ and $A(u) := A_0 + \sum_{j=1}^m u_j A_j = \Omega(u) + S(u)$ with $u := (u_1, \dots, u_m)^\top \in \mathbb{R}^m$, where $\Omega(u)$ and $S(u)$ denote the skew-symmetric and symmetric part of $A(u)$, respectively. Then the *real* GDBE is given by

$$\dot{X} = [\Omega(u), X] + \big[[S(u), X], X\big], \quad X(0) = X_0 \in \mathrm{Grass}_{k,n}(\mathbb{R}), \tag{41}$$

where $\mathrm{Grass}_{k,n}(\mathbb{R}) := \{X \in \mathbb{R}^{n\times n} \mid X = X^\top, X^2 = X, \mathrm{tr}X = k\}$ denotes the Grassmannian of all real selfadjoint projectors of rank k.

THEOREM 20. *Let $\mathcal{L}$ denote the system Lie algebra generated by $A(u)$, $u \in \mathbb{R}^m$.*

(a) The generalized double bracket equation (41) *is accessible on the real Grassmannian* $\mathrm{Grass}_{k,n}(\mathbb{R})$ *with $1 < k < n$ if and only if $\mathcal{L} = \mathfrak{z} \oplus \mathcal{L}_0$, where $\mathfrak{z}$ is a Lie subalgebra of the centralizer of $\mathcal{L}_0$ and $\mathcal{L}_0$ is conjugate to one of the following cases:*
   - *(i)* $\mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{so}_n$
   - *(ii)* $\mathfrak{g}_2$ *for $n = 7$ and $k \in \{2, 5\}$,*
   - *(iii)* $\mathfrak{spin}_7$ *for $n = 8$ and $k \in \{2, 3, 5, 6\}$.*

(b) The generalized double bracket equation (41) *is accessible on the real Grassmannian $Grass_{1,n}(\mathbb{R})$ (and $Grass_{n-1,n}(\mathbb{R})$) if and only if $\mathcal{L} = \mathfrak{z} \oplus \mathcal{L}_0$, where $\mathfrak{z}$ is a Lie subalgebra of the centralizer of $\mathcal{L}_0$ and $\mathcal{L}_0$ is conjugate to one of the following cases:*
   - *(i)* $\mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{sp}_n(\mathbb{R})$, $\mathfrak{so}_n$
   - *(ii)* $\mathfrak{sl}_{n/2}(\mathbb{C})$, $\mathfrak{su}_{n/2}$ *for n even,*
   - *(iii)* $\mathfrak{sl}_{n/4}(\mathbb{H})$, $\mathfrak{sp}_{n/2}(\mathbb{C})$, $\mathfrak{sp}_{n/4}$ *for $n \equiv 0 \mod 4$,*
   - *(iv)* $\mathfrak{g}_2$ *for $n = 7$,*

(v) $\mathfrak{spin}_7$ *for* $n = 8$,

(vi) $\mathfrak{spin}_9$, $\mathfrak{spin}_{9,1}(\mathbb{R})$ *for* $n = 16$.

Here, $\mathfrak{so}_n$ and $\mathfrak{sp}_n(\mathbb{R})$ denote the Lie algebras of the special orthogonal and real symplectic group, respectively. A matrix representation of $\mathfrak{g}_2$, the compact real form of $\mathfrak{g}_2^*$, cf. [17], can be obtained by intersecting $\mathfrak{g}_2^*$ with $\mathfrak{su}_7$. For the spin Lie algebras see e.g. [12, 26]. Note, that $\mathfrak{spin}_7$ and $\mathfrak{spin}_9$ are compact Lie algebras.

Symplectic case. Let $A_0, \dots, A_m \in \mathfrak{sp}_n(\mathbb{C})$ be arbitrary complex Hamiltonian matrices and let $A(u) := A_0 + \sum_{j=1}^m u_j A_j = \Omega(u) + S(u)$ with $u := (u_1, \dots, u_m)^\top \in \mathbb{R}^m$, where $\Omega(u)$ and $S(u)$ denote the skew-Hermitian and Hermitian part of $A(u)$, respectively. Moreover, let

$$\mathrm{J}_n := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

denote the standard symplectic form on $\mathbb{C}^{2n}$. Then the *complex symplectic* GDBE is given by

$$\dot{X} = [\Omega(u), X] + \big[[S(u), X], X\big], \quad X(0) = X_0 \in \mathrm{LGrass}_n(\mathbb{C}), \tag{42}$$

where $\mathrm{LGrass}_n(\mathbb{C}) := \{X \in \mathrm{Grass}_{n,2n}(\mathbb{C}) \mid X^\top \mathrm{J}_n X = 0\}$ denotes the *complex Lagrangian Grassmannian* of all Hermitian projectors onto Lagrangian subspaces. Similarly, for real Hamiltonian matrices $A_0, A_1, \dots A_m \in \mathfrak{sp}_n(\mathbb{R})$ one has the *real symplectic* GDBE on the *real Lagrangian Grassmannian*

$$\mathrm{LGrass}_n(\mathbb{R}) := \{X \in \mathrm{Grass}_{n,2n}(\mathbb{R}) \mid X^\top \mathrm{J}_n X = 0\}.$$

THEOREM 21. *Let $\mathcal{L}$ denote the system Lie algebra generated by $A(u)$, $u \in \mathbb{R}^m$.*

(a) The complex symplectic GDBE *is accessible on the complex Lagrangian Grassmannian* $\mathrm{LGrass}_n(\mathbb{C})$ *if and only if $\mathcal{L}$ is equal to $\mathfrak{sp}_n(\mathbb{C})$ or conjugate to $\mathfrak{sp}_n := \mathfrak{u}_{2n} \cap \mathfrak{sp}_n(\mathbb{C})$.*

(b) The real symplectic GDBE *is accessible on the real Lagrangian Grassmannian* $\mathrm{LGrass}_n(\mathbb{R})$ *if and only if $\mathcal{L}$ is equal to $\mathfrak{sp}_n(\mathbb{R})$ or conjugate to $\mathfrak{osp}_n := \mathfrak{so}_{2n} \cap \mathfrak{sp}_n(\mathbb{R})$.*

Proof. First, consider the complex case. Let $X = RR^* \in \mathrm{Grass}_{n,2n}(\mathbb{C})$ with $R^*R = \mathrm{I}_n$. Then, one has the equivalence

$$X^\top \mathrm{J}_n X = 0 \iff \overline{R}R^\top \mathrm{J}_n RR^* = 0 \iff R^\top \mathrm{J}_n R = 0. \tag{43}$$

Thus, the diffeomorphism μ of Lemma 9 restricts to a diffeomorphism from $\mathrm{LGrass}_n(\mathbb{C})$ to the complex Lagrangian Grassmann manifold

$$\mathrm{LG}_n(\mathbb{C}) := \{V \in \mathrm{G}_{n,2n}(\mathbb{C}) \mid v^\top \mathrm{J}_n v = 0 \text{ for all } v \in V\}. \tag{44}$$

According to the previous sections, one is therefore left with the problem of classifying all connected Lie subgroups of $\mathrm{Sp}_n(\mathbb{C})$ which act transitively on $\mathrm{LG}_n(\mathbb{C})$. Thus, part

(a) follows easily from Proposition 22(a) below. The same line of arguments applies to (b). □

PROPOSITION 22. *A connected Lie subgroup G of $\mathrm{Sp}_n(\mathbb{K})$ with $\mathbb{K}=\mathbb{C}$ or $\mathbb{K}=\mathbb{R}$ acts transitive on $\mathrm{LG}_n(\mathbb{K})$ if and only if*

(a) G is equal to $\mathrm{Sp}_n(\mathbb{C})$ or conjugate to Sp_n, if $\mathbb{K}=\mathbb{C}$.

(b) G is equal to $\mathrm{Sp}_n(\mathbb{R})$ or conjugate to $\mathrm{OSp}_n := \mathrm{SO}_{2n} \cap \mathrm{Sp}_n(\mathbb{R})$, if $\mathbb{K}=\mathbb{R}$.

Proof. (Sketch) First, it is straightforward to show that the compact symplectic groups Sp_n and OSp_n act transitively on $\mathrm{LGrass}_n(\mathbb{C})$ and $\mathrm{LGrass}_n(\mathbb{R})$, respectively. Then, exploiting the classification of all irreducible factorizations of compact connected simple Lie groups [25] one can derive that Sp_n and OSp_n are up to conjugation the only compact subgroups which act transitively. Finally, let $G \subset \mathrm{Sp}_n(\mathbb{C})$ be any connected Lie subgroup which acts transitively on $\mathrm{LG}_n(\mathbb{C})$. Then a result by Montgomery [23] implies that G contains a compact subgroup which already acts transitively. Thus, G contains a subgroup which is conjugate to Sp_n. Since Sp_n is the maximally compact connected subgroup of a Cartan decomposition of $\mathrm{Sp}_n(\mathbb{C})$ and $\mathrm{Sp}_n(\mathbb{C})$ itself is simple, we conclude that $G = \mathrm{Sp}_n(\mathbb{C})$. The same arguments apply to OSp_n and $\mathrm{Sp}_n(\mathbb{R})$. □

4. Accessibility of Riccati Equations. Due to the well–known connection between linear induced flows on the Grassmann manifold and matrix Riccati differential equations [29], the above results readily translate to accessibility conditions for Riccati equations. For $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$, let $A(u) := A_0 + \sum_{j=1}^m u_j A_j$ with $A_0, A_1, \dots A_m \in \mathfrak{gl}_n(\mathbb{K})$ as before. Given any integer $1 \le k \le n-1$ and a corresponding partitioning

$$A(u) = \begin{bmatrix} A_{11}(u) & A_{12}(u) \\ A_{21}(u) & A_{22}(u) \end{bmatrix} \in \mathfrak{gl}_n(\mathbb{K}) \tag{45}$$

with $A_{11}(u) \in \mathbb{K}^{k\times k}$, $A_{22}(u) \in \mathbb{K}^{(n-k)\times(n-k)}$ and $A_{12}(u) \in \mathbb{K}^{k\times(n-k)}, A_{21}(u) \in \mathbb{K}^{(n-k)\times k}$, then we associate with $A(u)$ a *matrix Riccati differential equation*

$$\dot K = -KA_{11}(u) + A_{22}(u)K - KA_{12}(u)K + A_{21}(u), \tag{46}$$

where $u \in \mathbb{R}^m$ are controls parameters. Similarly, for $2n\times 2n$-Hamiltonian matrices

$$A(u) = \begin{bmatrix} A_{11}(u) & A_{12}(u) \\ A_{21}(u) & -A_{11}^\top(u) \end{bmatrix} \in \mathfrak{sp}_n(\mathbb{K}) \qquad \text{with } A_{12}(u), A_{21}(u) \text{ symmetric}, \tag{47}$$

we consider the matrix Riccati differential equation from optimal control

$$\dot K = -KA_{11}(u) - A_{11}^\top(u)K - KA_{12}(u)K + A_{21}(u), \tag{48}$$

evolving on the space $\mathrm{S}(n)$ of (real and, respectively, complex) symmetric $n\times n$-matrices. Here, we are interested in necessary and sufficient conditions that guarantee accessibility for Riccati equations, i.e. conditions which guarantee that the reachable sets of (46) and (48), respectively, have non-empty interior. To this end, we relate the Riccati equation with the GDBE by the subsequent result.

LEMMA 23.

(a) Let $\rho : \mathbb{K}^{(n-k)\times k} \to \mathrm{Grass}_{k,n}(\mathbb{K})$ *be defined by*

$$\rho(K) := \begin{bmatrix} \mathrm{I}_k \\ K \end{bmatrix} (I_k + K^*K)^{-1} \begin{bmatrix} \mathrm{I}_k & K^* \end{bmatrix}. \tag{49}$$

Then, ρ *maps* $\mathbb{K}^{(n-k)\times k}$ *diffeomorphically onto an open and dense subset of* $\mathrm{Grass}_{k,n}(\mathbb{K})$. *Moreover, restricting* ρ *to symmetric matrices for* $k = \frac{n}{2}$ *yields a diffeomorphism onto an open dense subset of the Lagrangian Grassmannian* $\mathrm{LGrass}_n(\mathbb{K})$.

(b) The push-forward of the Riccati vector field $R(K) := -KA_{11} + A_{22}K - KA_{12}K + A_{21}$ *to* $\mathrm{Grass}_{k,n}(\mathbb{K})$ *yields the generalized double bracket equation, i.e.*

$$\mathrm{D}\,\rho(K)\,R(K) = (I_n - P)AP + PA^*(I_n - P) = [\Omega, P] + \big[[S,P],P\big] \tag{50}$$

with $K := \rho^{-1}(P)$, $\Omega := \frac{1}{2}(A - A^*)$ *and* $S := \frac{1}{2}(A + A^*)$.

Proof. The result follows either by a straightforward computation or from Corollary 12 together with the well-known relation between matrix Riccati differential equations and linear induced flows on Grassmann manifolds [16, 29]. □

Thus, the results of the previous section imply the following accessibility criterion for (46) and (48), respectively. We begin with the complex case of (46). For the real case of (46), we refer to Theorem 20.

THEOREM 24. *Let* $A(u) \in \mathfrak{gl}_n(\mathbb{C})$ *be partitioned as in (45) and let* $\mathcal{L}$ *be the system Lie algebra generated by* $A(u)$, $u \in \mathbb{R}^m$. *The Riccati equation (46) is accessible if and only if*

(a) $\mathcal{L} = \mathfrak{z} \oplus \mathcal{L}_0$, *where* $\mathfrak{z}$ *is a Lie subalgebra of* $\mathbb{C}\,\mathrm{I}_n$ *and* $\mathcal{L}_0$ *is equal to* $\mathfrak{sl}_n(\mathbb{C})$ *or conjugate to* $\mathfrak{su}_n$, *if* n *is odd or* $1 < k < n$.

(b) $\mathcal{L} = \mathfrak{z} \oplus \mathcal{L}_0$, *where* $\mathfrak{z}$ *is a Lie subalgebra of* $\mathbb{C}\,\mathrm{I}_n$ *and* $\mathcal{L}_0$ *is equal to* $\mathfrak{sl}_n(\mathbb{C})$ *or conjugate to* $\mathfrak{su}_n$, $\mathfrak{sl}_{n/2}(\mathbb{H})$, $\mathfrak{sp}_{n/2}(\mathbb{C})$ *or* $\mathfrak{sp}_{n/2}$, *if* n *is even and* $k = 1$ *or* $k = n - 1$.

For the Riccati equation (48) we obtain.

THEOREM 25. *Let* $A(u) \in \mathfrak{sp}_n(\mathbb{C})$ *be partitioned as in (47) and let* $\mathcal{L}$ *be the system Lie algebra generated by* $A(u)$, $u \in \mathbb{R}^m$.

(a) The complex Riccati equation (48) *is accessible if and only if* $\mathcal{L}$ *is equal to* $\mathfrak{sp}_n(\mathbb{C})$ *or conjugate to* $\mathfrak{sp}_n := \mathfrak{u}_{2n} \cap \mathfrak{sp}_n(\mathbb{C})$.

(b) The real Riccati equation (48) *is accessible if and only if* $\mathcal{L}$ *is equal to* $\mathfrak{sp}_n(\mathbb{R})$ *or conjugate to* $\mathfrak{osp}_n := \mathfrak{so}_{2n} \cap \mathfrak{sp}_n(\mathbb{R})$.

REFERENCES

[1] A. Bloch, *The Kähler structure of the total least squares problem, Brockett's steepest descent equations and constrained flows*. in: M. Kaashoek, J. van Schuppen and A.C. Ran (eds.), Realization and Modelling in Systems Theory, 83–88, Birkhäuser, Boston, 1990.

[2] A. Bloch, R.W. Brockett and T. Ratiu, *A new formulation of the generalized Toda lattice equations and their fixed point analysis via the moment map*. Bull. Amer. Math. Soc., 23:2(1990), pp. 477–486.

[3] A. Bloch, R.W. Brockett and T. Ratiu, *Completely integrable gradient flows*. Comm. Math. Phys., 147:1(1992), pp. 54–74.

[4] A. Bloch, R.W. Brockett and P. Crouch, *Double bracket equations and geodesic flows on symmetric spaces*. Comm. Math. Phys., 187:2(1997), pp. 357–373.

[5] A. Bloch and A. Iserles, *Optimality of double bracket and generalized double bracket flows*. in: Proc. of the 42nd IEEE Conference on Decision and Control, 528–532, Maui, Hawai, 2003.

[6] R.W. Brockett, *Lie theory and control systems defined on spheres*. SIAM J. Appl. Math., 25(1973), pp. 213–225.

[7] R.W. Brockett, *Dynamical systems that sort lists, diagonalize matrices and solve linear programming problems*. in: Proc. of the 27th IEEE Conference on Decision and Control, 799–803, Austin, Texas, 1988. See also Linear Algebra Appl., 146(1991), pp.79–91.

[8] R.W. Brockett, *Smooth dynamical systems which realize arithmetical and logical operations*. in: H. Nijmeijer and J. Schumacher (eds.), Three Decades of Mathematical System Theory, Lecture Notes in Control and Information Sciences, 135, 19–30, Springer, Berlin, 1989.

[9] R.W. Brockett, *An estimation theoretic basis for the design of sorting and classification networks*. in: R. Mammone and Y. Zeevi (eds.), Neural Networks, 23–41, Academic Press, 1991.

[10] R.W. Brockett, *Dynamical systems that learn subspaces*. in: A. Antoulas (ed.), Mathematical system theory. The influence of R.E. Kalman, 579–592, Springer, Berlin, 1991.

[11] R.W. Brockett, *Differential geometry and the design of gradient algorithms*. in: R. Greene (ed.) et al., Differential geometry. Part 1: Partial differential equations on manifolds, Proc. Symp. Pure Math., 54, 69–92, AMS, Providence, 1993.

[12] T. Bröcker and T. tom Dieck, *Representations of Compact Lie Groups*. Springer, New York, 1995.

[13] P. van Dooren and R. Sepulchre, *Shift policies in QR-like algorithms and feedback control of self-similar flows*. in: V. Bondel (ed.) etal., Open Problems in Mathematical Systems and Control Theory, 245-249, Springer, London, 1999.

[14] W. Dayawansa, B. Ghosh, C. Martin and X. Wang, *A necessary and sufficient condition for the perspective observability problem*. Systems and Control Letters. 25(1995), pp. 159–166.

[15] U. Helmke, *Isospectral flows on symmetric matrices and the Riccati equation*. System and Control Letters, 16(1991), pp. 159–166.

[16] U. Helmke and J. Moore, *Optimization and Dynamical Systems*. Springer, London, 1994.

[17] J. Humphreys, *Introduction to Lie Algebras and Representation Theory*. Springer, New York, 1972.

[18] D. Hurtubise, *The Floer homotopy type of height functions on complex Grassmann manifolds*. Trans. Amer. Math. Soc., 349:6(1997), pp. 2493–2505.

[19] V. Jurdevic and H. Sussmann, *Control systems on Lie groups*. J. Differ. Equations, 12(1972), pp. 313–329.

[20] V. Jurdevic, *Geometric Control Theory*. Cambridge University Press, Cambridge, 1997

[21] P. Libermann and C.-M. Marle, *Symplectic Geometry and Analytical Mechanics*. Reidel Publishing Company, Dordrecht, 1993.

[22] D. Mittenhuber, *Transitive Semigroup Actions and Controllability of Systems on Lie Groups: A Solvable and a Semisimple Problem*. Habil. Thesis, Tech. University of Darmstadt, 2002.

[23] D. Montgomery, *Simply connected homogeneous spaces*. Proc. Amer. Math. Soc., 1:4(1950), pp. 467–469.

[24] W. Luber, *Isospectral Flows on Grassmann Manifolds*, Diploma Thesis, University of Regensburg, 1996.

[25] A.L. Onishchik, *Topology of Transitive Transformation Groups*. Barth, Leipzig, 1994.

[26] I. Porteous, *Topological Geometry*. van Nostrand Reinhold, London, 1969.

[27] J. Rosenthal, *An observability criterion for dynamical systems governed by Riccati differential equations*. IEEE Trans. Autom. Control, 41:3(1996), pp. 434–436.

[28] Y. Sachkov, *On invariant orthants of bilinear systems*. J. Dyn. Control Syst., 4:1(1998), pp. 137–147.

[29] M.A. Shayman, *Phase portrait of the matrix Riccati equation*. SIAM J. Control Optim., 24(1986), pp. 1-65.

[30] H. Völklein, *Transitivitätsfragen bei linearen Gruppen*. Arch. Math., 36(1981), pp. 23-34.