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GEOMETRIC COMPRESSION USING RIEMANN SURFACE STRUCTURE*

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Abstract. This paper introduces a theoretic result that shows any surface in 3 dimensional Euclidean space can be determined by its conformal factor and mean curvature uniquely up to rigid motions. This theorem disproves the common belief that surfaces have three functional freedoms and immediately shows that one third of geometric data can be saved without loss of information.

The paper develops a practical algorithm to losslessly compress geometric surfaces based on Riemann surface structures. First we compute a global conformal parameterization of the surface. The surface can be segmented by holomorphic flows, where each segment can be conformally mapped to a rectangle on the parameter plane, which is guaranteed by circle-valued Morse theory. We construct a conformal geometry image for each segment, and record conformal factor and dihedral angle for each edge. In this way, we represent the surface using only two functions with canonical connectivity. We present the proofs of the theorems and the compression examples.

Keywords: Geometric Modeling, Lossless Compression, Conformal Parameterization, Riemann Surface.

1. Introduction.

Surfaces in R^3 are usually represented by their position vector $r(u,v)$,

$$r(u,v) = (x(u,v), y(u,v), z(u,v)),$$

under some parameterization (u,v). By differential geometry theories, the surface is determined by its first fundamental form and the second fundamental form uniquely up to rigid motions. The first and second fundamental form are represented as

$$I = E(u,v)du^2 + 2F(u,v)dudv + G(u,v)dv^2 \tag{1}$$

$$II = L(u,v)du^2 + 2M(u,v)dudv + N(u,v)dv^2. \tag{2}$$

The functions E, F, G, L, M, N are related by the Gauss equation and Codazzi equation.

Considering a curve in R^3, in general we use three functions to represent it

$$r(t) = (x(t), y(t), z(t)).$$

In order to reconstruct $r(t)$, at least two functions are needed, the curvature and the torsion. It is a common belief that in order to reconstruct a surface $r(u,v)$, at least three functions are needed. In this work, we introduce a surprising theorem that only two functions are enough to represent a surface without loss of information. The theorem is based on Riemann surface theories.

It is well known that all orientable surfaces are Riemann surfaces, and they admit global conformal parameterizations. Figure 1 illustrates a global conformal parameterization for the

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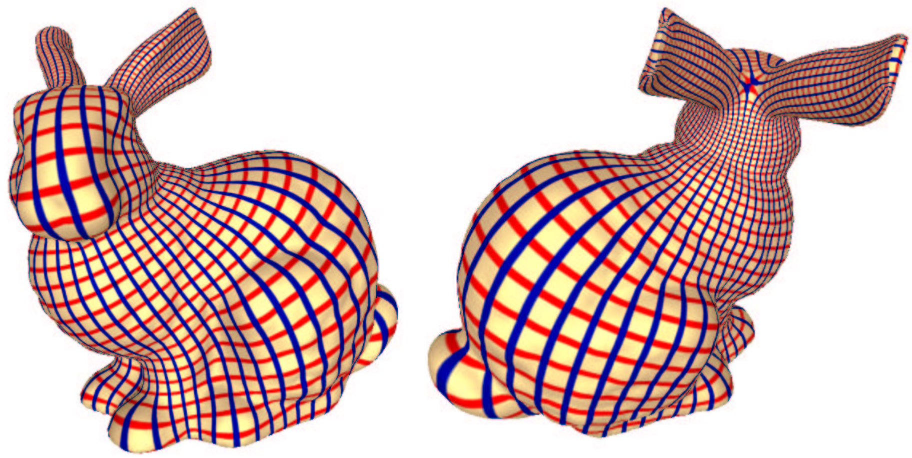


a. Front view. b. Back view.

FIG. 1. *An electric field on a surface is a global conformal parameterization of the surface. The darker curves are equi-potential lines, the lighter curves are electric field lines.*

Sanford bunny model. The first fundamental form of the surface can be formulated as

$$ds^2 = \lambda(u,v)(du^2 + dv^2), \tag{3}$$

where $\lambda(u,v)$ describes the stretching factor between the surface and the parameter plane, and is called *conformal factor*. The mean curvature of the surface can be computed as

$$H(u,v) = \frac{1}{\lambda^2(u,v)} \Delta r(u,v) \cdot n(u,v), \tag{4}$$

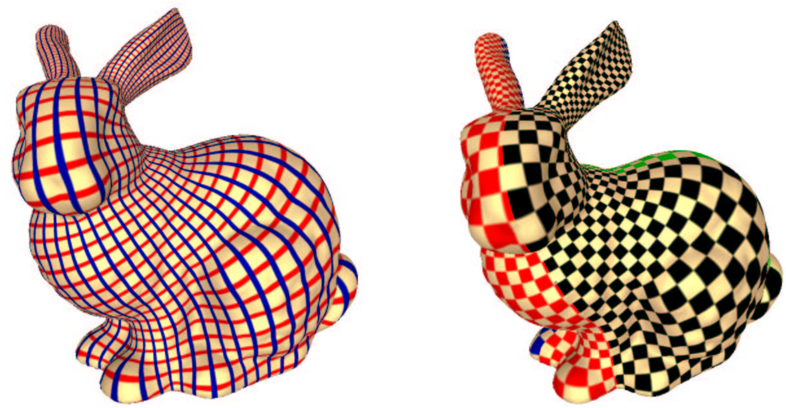
where $n(u,v)$ is the normal of the surface, Δ is the Laplacian operator defined as

$$\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}. \tag{5}$$

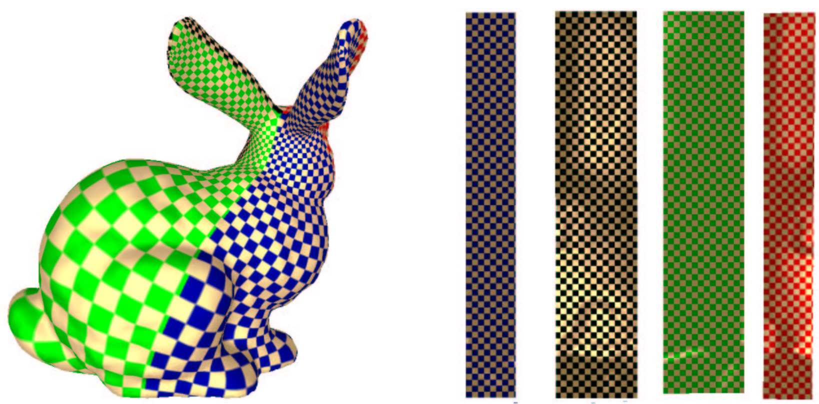
We will show that by $H(u,v)$ and $\lambda(u,v)$, the surface $r(u,v)$ can be reconstructed uniquely up to rigid motions.

By Riemann-Roch theorem, there are $2g-2$ zero points on a global conformal parameterization of a genus g closed surface. By circle-valued Morse theorem the iso-parametric curves through the zero points will partition surfaces to several patches. Each patch can be mapped to a rectangle on the uv plane.

In practice, triangular meshes are usually used to represent surfaces. We compute the global conformal parameterization first, locate zero the points, subdivide the mesh to patches and construct conformal geometry image for each patch. Then we compute the conformal factor and mean curvature for each geometry image, and record them as the representation of the surface.



a. Global conformal parameterization. b. Front view of the partition of the bunny.



c. Back view of the partition of the bunny. d. Patches on the parameter plane.

FIG. 2. *The bunny surface with 3 cuts, two on ear tips and one at the bottom. There is only one zero point, the iso-v curves through the zero point segment the whole surface to 4 patches, all of which can be mapped to rectangles on the parameter plane.*

2. Theoretic Background. In this section, we briefly prove the following main theorem,

THEOREM 1. *A closed surface $r(u,v)$ in R^3 with conformal parameter (u,v) is determined by its conformal factor $\lambda(u,v)$ and its mean curvature $H(u,v)$ uniquely up to rigid motions. A simply connected surface $r(u,v)$ with a boundary in R^3 with conformal parameter (u,v) is determined by its conformal factor $\lambda(u,v)$ and its mean curvature $H(u,v)$ and the boundary position.*

We denote $z = u + iv$, then $dz = du + idv$, $d\bar{z} = du - idv$, $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v})$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial u} +$

$i\frac{\partial}{\partial v}$). Because (u,v) is the conformal parameter, we can get

$$< r_z, r_z > = 0 \tag{4}$$

$$< r_{\bar{z}}, r_{\bar{z}} > = 0 \tag{5}$$

$$< r_z, r_{\bar{z}} > = \frac{\lambda^2}{2}. \tag{6}$$

Let $\mu = < r_{zz}, n >$. Then the motion equation for the natural frame $\{r_z, r_{\bar{z}}, n\}$ of the surface can be formulated as

$$r_{zz} = \frac{2}{\lambda}\lambda_z r_z + \mu n \tag{7}$$

$$r_{z\bar{z}} = \frac{\lambda^2}{2} H n \tag{8}$$

$$n_z = -H r_z - 2\lambda^{-2}\mu r_{\bar{z}} \tag{9}$$

with the Gauss equation and Codazzi equation

$$(\log\lambda)_{z\bar{z}} = \frac{\mu\bar{\mu}}{\lambda^2} - \frac{\lambda^2}{4}H^2 \tag{10}$$

$$\mu_{\bar{z}} = \frac{\lambda^2}{2} H_z. \tag{11}$$

The Codazzi equation is equivalent to the following Poisson problem:

$$\mu_{\bar{z}\bar{z}} = \frac{1}{2}\lambda(2\lambda_z H_z + \lambda H_{zz}). \tag{12}$$

We can solve μ easily using Fast Fourier Transformation. Then we can integrate the natural frame $\{r_z, r_{\bar{z}}, n\}$ and reconstruct the surface.

Given first fundamental form $ds^2 = Edu^2 + 2Fdudv + Gdv^2$, it is easy to reparameterize the surface and change the new parameterization to be conformal.

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = O^T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} O, \tag{13}$$

where O is a rotation matrix. Define new parameters

$$\begin{pmatrix} d\tilde{u} \\ d\tilde{v} \end{pmatrix} = \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix} O \begin{pmatrix} du \\ dv \end{pmatrix}, \tag{14}$$

here $(\tilde{u}, \tilde{v})$ are conformal parameters, namely $ds^2 = d\tilde{u}^2 + d\tilde{v}^2$. Hence we have proved the following theorem.

THEOREM 2. *A closed surface $r(u,v)$ in R^3 is determined by its first fundamental form $E(u,v), F(u,v), G(u,v)$ and the mean curvature $H(u,v)$ uniquely up to rigid motions. A simply connected surface $r(u,v)$ in R^3 with a boundary is determined by its first fundamental form $E(u,v), F(u,v), G(u, v)$ and the mean curvature $H(u,v)$ and the boundary position.*

Under conformal parameterization, the surface can also be determined by its principle curvatures.

THEOREM 3. *A closed surface $r(u,v)$ in R^3 with conformal parameter (u,v) is determined by its principle curvatures $k_1(u,v)$ and $k_2(u,v)$ uniquely up to rigid motions. A simply connected surface $r(u,v)$ with one boundary in R^3 with conformal parameter (u,v) is determined by its principle curvatures $k_1(u,v)$, $k_2(u,v)$ and the boundary position.*

Suppose the conformal factor is $\lambda(u,v)$, then $\lambda(u,v)$ can be solved by the following equation

$$\frac{1}{\lambda(u,v)^2}\Delta \ln \lambda(u,v) = k_1(u,v)k_2(u,v), \tag{15}$$

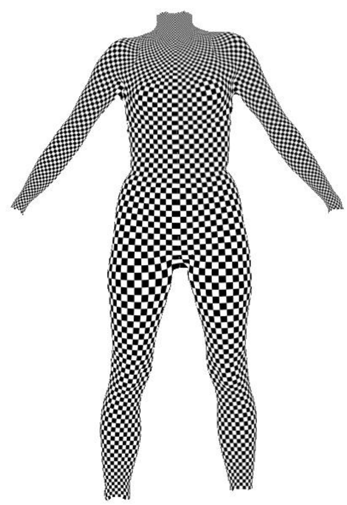
and mean curvature is $H(u,v) = \frac{1}{2}(k_1(u,v)+k_2(u,v))$. The global structure of conformal parameterization can be represented by the following theorem

THEOREM 4. *For a genus $g>1$ closed surface, a holomorphic one-form has $2g-2$ zero points. The iso-parametric curves will partition the surface into cylinders, topological disks, and each of them can be mapped to a parallelogram on (u,v) plane.*

The number of zero points is computed using Riemann-Roch theorem. The decomposition structure can be proved using circle-valued Morse theorem.

3. Previous Work. In the literature, many researchers have reported their work on geometric compression [8, 14, 12, 9, 7, 13]. Rossignac gave a nice review on this topic in [13]. There are two major tasks for compression: geometry compression and connectivity compression. The compression of vertex coordinates usually combines three steps: Quantization, prediction, and statistical coding of the residues. Among them, prediction is the most crucial step. The most popular vertex predictor for single-rate compression is based on the parallelogram construction [14]. Rossignac proposed Edgebreaker [12] algorithm which is arguably the simplest and one of the most effective single-rate compression approaches. When the exact geometry and connectivity of the mesh are not essential, the triangulated surface may be simplified or retiled. Hoppe [8] describes a progressive mesh method which repeatly uses edge collapse operations to simplify the mesh structure. While Khodakovsky et al. [9] introduce a semi-regular meshes for progressive geometry compression, Gu et al. [7] introduce a method which can remesh an arbitrary surface onto a completely regular structure, so called a geometry image. It captures geometry as a simple 2D array of quantized points. Surface signals like normals and colors are stored in similar 2D arrays using the same implicit surface parametrization.

Conformal geometry has been applied in computer graphics for texture mapping purpose. The algorithm for computing conformal maps from a topological disk to the plane have been studied in [1, 2, 10, 11]. For surfaces with arbitrary topologies, Gu and Yau introduce an algorithm based on Hodge theory. The algorithm for computing conformal structures of real surfaces has been introduced in [3]. Then the method is applied to brain mapping [6, 15] in medical imaging; surface classification in [5], and global surface parameterizations in [4].



a. Human Body surface. b. Global Conformal parametrization.

c. Canonical Partition. d. Conformal mapped to parameter plane.

FIG. 3. *The human body surface is global conformal parameterized. The iso-u, iso-v curves through zero points partition the surface to several patches. Each patch is conformally mapped to the plane, the planar images are rectangles.*

4. Algorithm. This section will explain the algorithm to construct $\lambda - h$ representation in details. The pipeline can be summarized as the following: first we compute the conformal structure of the surface, then we select one global conformal parameterization, and locate the zero points on the parameterization; by tracing iso-parametric curves through these zero points, the surfaces can be segmented to several patches, each patch can be conformally mapped to a rectangle. We resample each patch using regular grids, and construct a conformal geometry image on it. We compute the dihedral angle of each edge, and the conformal factor of each vertex. We predicate the edge length by the conformal factors on its vertices, and record the residue. By storing the conformal factors, residues and the dihedral angles, the mesh can be uniquely reconstructed.

Suppose K is a simplicial complex, and a mapping $f : |K| \to R^3$ embeds $|K|$ in R^3, then $M = (K, f)$ is called a *triangular mesh*. K_n where $n = 0, 1, 2$ are the sets of *n-simplicies*. We use σ^n to denote a n-simplex, $\sigma^n = [v_0, v_1, \cdots, v_n]$, where $v_i \in K_0$.

4.1. Construct Conformal Geometry Images. First we compute the conformal structure of the surface using the series of algorithms introduced in [4]. We compute a homology basis first, which are a set of closed curves $B = \{a_1, b_1, a_2, b_2, \cdots, a_g, b_g\}$. Then we compute a cohomology basis for the mesh, then we diffuse each cohomlogy base to be harmonic, denoted as $B^* = \{\omega_1, \omega_2, \cdots, \omega_{2g}\}$. Finally we compute a basis of holomorphic 1-forms, denoted as $\Omega = \{\omega_1 + \sqrt{-1}\omega_1^*, \omega_2 + \sqrt{-1}\omega_2^*, \cdots, \omega_{2g} + \sqrt{-1}\omega_{2g}^*\}$, where ω_i^* is obtained by applying the Hodge star operator on ω_i.

Then each holomorphic one-form is represented as a map $f : K_1 \to \mathbb{C}$. We associate a pair of real numbers (du, dv) with each oriented edge. Then we define the conformal factor with each edge as the ratio between the edge length in R^3 and the edge length in the parameter plane. The conformal factor for each vertex is defined as the average of the adjacent edge conformal factors.

After computing the conformal factor for all vertices, we locate the vertices with local minimal conformal factors, which are the approximation of zero points. Through each zero point, we trace the iso-parametric curves, these curves will partition the whole surface to patches, each patch is mapped to a rectangle in the *uv* plane by integrating f. We sample each surface patch by regular grids on the parameter plane, and construct a conformal geometry image.

Figure 3 illustrates the process. The human body surface in (a) is processed, its conformal structure is illustrated in (b). Then we locate the zero points and trace the iso-parametric curves through them as shown in (c), the surfaces are partitioned to several patches, each patch is conformally mapped to a rectangle to the parameter plane as shown in (d).

4.2. Surface Reconstruction. From the section 2, it is clear that the surface can be reconstructed from the first fundamental form and the mean curvature.

Edge Length and Dihedral Angle. For general triangle meshes, the first fundamental form is represented as the length of edges, the mean curvature is represented as dihedral angles of edges. By these two set of data, the surface can be reconstructed uniquely up to rigid motions.

First we pick an arbitrary triangle, and reconstruct the triangle in a plane with the given edge lengths. Suppose the vertices of the triangle are $\{v_1, v_2, v_3\}$, the corresponding positions are $\{r_1, r_2, r_3\} \subset R^2$, and the edge lengths are $\{l_{12}, l_{23}, l_{31}\}$, then the positions must satisfy the following equations

$$l_{ij}^2 = (r_i - r_j)\cdot(r_i - r_j), i, j = 1,2,3. \tag{16}$$

Then the normal of the face is

$$n_1 = \frac{(r_2 - r_1)\times(r_3 - r_1)}{|(r_2 - r_1)\times(r_3 - r_1)|}. \tag{17}$$

Then we can reconstruct the neighboring triangles using edge lengths and dihedral angles. Suppose triangle $[v_1, v_2, v_3]$ and triangle $[v_3, v_2, v_4]$ are adjacent, the normal of the first triangle is n_1, and the norm of the second triangle is n_2, the dihedral angle of edge $[v_2, v_3]$ is α_{23}, then the position of v_4 can be computed by the following equations:

$$\begin{cases} (r_4 - r_2)\cdot(r_4 - r_2) & = & l_{42}^2 \\ (r_4 - r_3)\cdot(r_4 - r_3) & = & l_{43}^2 \\ (r_4 - r_2)\cdot n_2 & = & 0 \\ (r_4 - r_3)\cdot n_2 & = & 0 \\ n_2\cdot n_2 & = & 1 \\ n_2\cdot n_1 & = & \cos\alpha_{23}. \end{cases} \tag{18}$$

This way, we can reconstruct the whole mesh one face by one face.

Conformal Factor. If we use conformal parameterization, the edge lengths can be encoded more efficiently by using conformal factors defined on each vertex. The conformal vector for vertex $v \in K_0$ is defined as

$$\lambda(v) = \frac{\sum_{[w,v]\in K_1} |r(w) - r(v)|}{\sum_{[v,w]\in K_1} |uv(w) - uv(v)|}. \tag{19}$$

Then we estimate the edgelength by the following approximation

$$\tilde{l}_{vw} = \frac{1}{2}(\lambda(v) + \lambda(w))|uv(w) - uv(v)|. \tag{20}$$

Then we only record the estimation error $l_{vw} - \tilde{l}_{vw}$ for each edge $[v, w]$. The relative error rate is less than one percent. Hence we only needs a few bits to encode.

Connectivity. The connectivity for each conformal geometry image is regular grids. We only need to record the dimensions of each geometry image. Different geometry images may share the same boundary vertices.

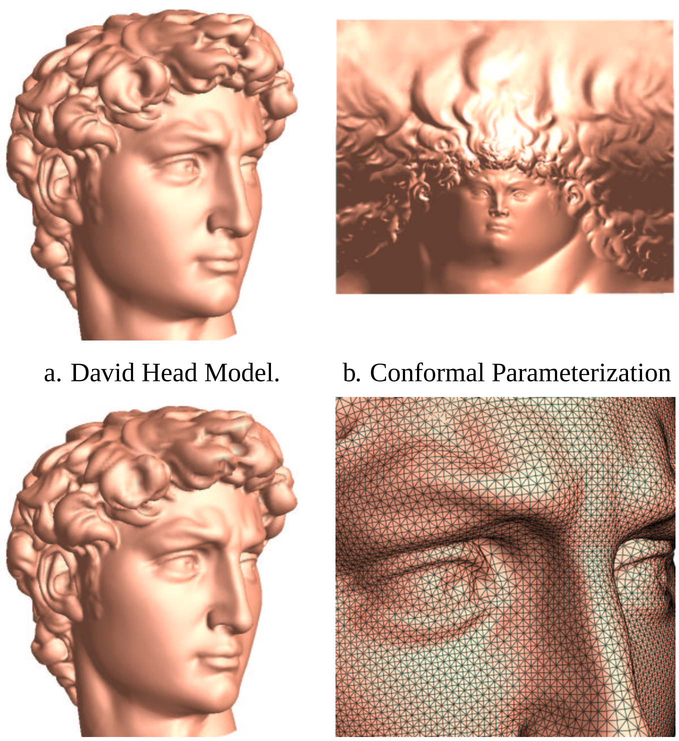
Codazzi Equation. From section 2, it is shown that the conformal factor λ and mean curvature H have to satisfy the Gauss equation and Codazzi equation. Therefore, there must be redundant information among them.

Suppose we reconstruct a triangle $[v_1, v_2, v_3]$ first, then for arbitrary vertex v_k, there are different ways to reconstruct it through different paths on the mesh. Suppose two paths are homotopic, then the Codazzi equation guarantees that the reconstructed position of v_k will be consistent.

The mesh can be treated as a graph, each node is a face. The link between two nodes is an edge shared by the two faces. We build a spanning tree for the graph and only keep the dihedral angles for the edges in the tree. The dihedral angles for non-tree edges are redundant and can be recovered via the Codazzi equation.

Normal. Because each patch is a conformal geometry image, conformal geometry guarantees the accuracy of reconstructed normal. Therefore, no normal information is necessary to be stored.

The conformal factors, edge length deviations and dihedral angles are further encoded using arithmetic encoder, and quantized for the purpose of compression.



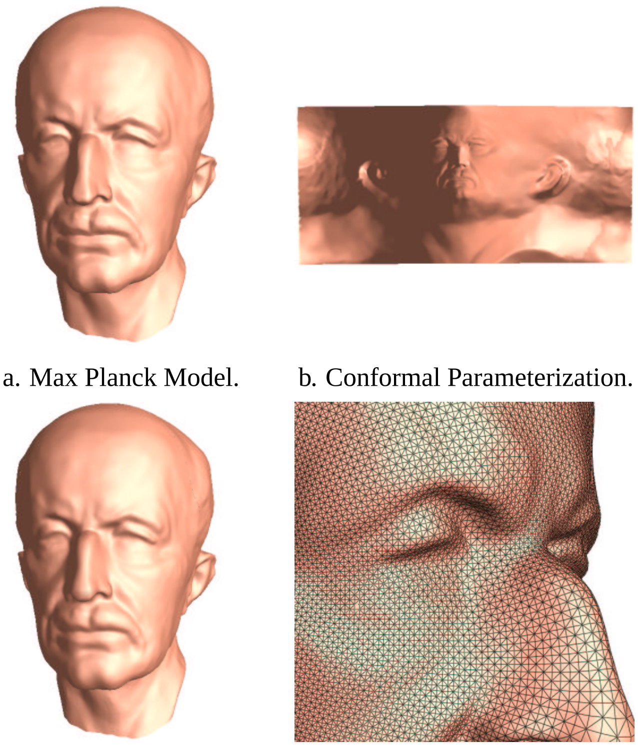
a. David Head Model. b. Conformal Parameterization

c. Reconstructed Surface. d. Regular Connectivity.

FIG. 4. *The sculpture of David head Model.*

5. Examples. We tested our algorithms on several surfaces obtained by laser scanning real sculptures. The David head sculpture model is a topological disk. We introduce a small slice on the top of the surface and conformally parameterized the resulted cylinder. The process is illustrated in figure 4. We represent the whole surface by only one conformal geometry image (b), the local connectivity is shown in (d). The Max Planck sculpture is processed in the similar way. The results are shown in figure 5.

The horse model is much more complicated. The original model is a topological sphere. We introduced five cuts, one is near the mouth, the other four are at the bottom of its feet.



a. Max Planck Model. b. Conformal Parameterization.

c. Reconstructed Surface. d. Regular Connectivity.

FIG. 5. *The sculpture of Max Planck head Model.*

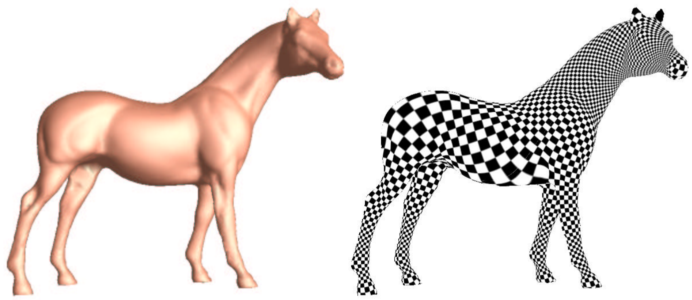
Then we computed the doubling of the surface, and get the conformal structure of the doubled surface. We choose a holomorphic one-form to parameterize the surface as shown in figure 6 (b). There are three zero points on the surface, then we partition the surface following the iso-parametric curves through the zero points, and represent each of them by a conformal geometry image. The regular connectivity is illustrated in (d).

We use arithmetic compression method to record the conformal factors, the edge length residues and the dihedral angles. Each vertex only needs 20 bits. For the original mesh file, each vertex requires more than 100 bits.

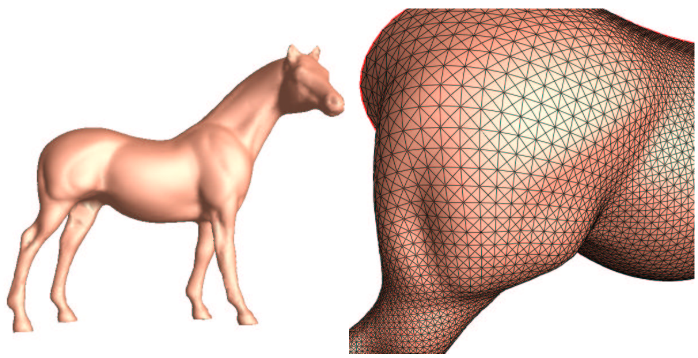
6. Future Work. Current compression method heavily depends on the accuracy of the conformality of the parameterization. In the ideal case, if the parameterization is accurate, then all edge length residues will be identically zero. The compression ratio can be improved by improving the conformality of the parameterization.

Uniform sampling geometry image will introduce errors. For regions with very high conformal factors, the surface will be under sampled. It is crucial to choose a holomorphic one-form such that the conformal factor is as uniform as possible.

We will explore along these two directions in the future.



a. Horse Surface. b. Conformal Parameterization.



c. Reconstructed Surface. d. Regular Connectivity.

FIG. 6. *Horse Surface Model.*

7. Conclusion. This paper introduces a theoretic result that says a surface in R^3 can be uniquely determined by its conformal factor and mean curvature. This result disproves the common belief that surfaces have three functional freedoms. Another theoretic result is the global structure of holomorphic one-forms. A holomorphic one-form gives a canonical way to partition a surface to patches, where each patch can be mapped to a parallelogram on the parameter plane.

Practical algorithms for geometric compression are introduced based on these two theorems. The compressed mesh connectivity information is implied by the global canonical structure of the conformal parameterizations. No connectivity information is needed to store explicitly. For each patch, we use conformal factors, edge length residues and dihedral angles to encode the geometry without loss of information. By utilizing the Codazzi equation, more redundant dihedral angles can be removed.

The algorithms are efficient, stable and practical for real applications.

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