Cox's factoring of regression model likelihoods for continuous-time processes

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Cox showed that the likelihood of regression models for discrete-time processes factors into a partial likelihood and a product of conditional laws for the covariates, given the history. Jacod constructed a partial likelihood for continuous-time regression models in terms of the predictable characteristics of the response process. Here we prove a factorization of the likelihood, analogous to Cox's, assuming both the response and the covariates to be semimartingales. The result is useful for counting process regression modelling and inference, and also for regression involving continuous processes and diffusions with jumps.

Keywords: partial likelihood; partially specified model; semimartingale; survival analysis

1. Introduction

Suppose that we observe a response process X and a vector V of covariate processes. A regression model specifies how the history of X and V affects the evolution of the response. In discrete time, one models the conditional densities $p_n^\vartheta(x)$ of X_n given the past observations $X_1, \ldots, X_{n-1}$ and $V_1, \ldots, V_{n-1}$. Cox (1975) suggested basing inference about the parameter ϑ on the *partial likelihood*

$$\bar{Z}^\vartheta = \prod_{n=1}^N p_n^\vartheta(X_n), \tag{1.1}$$

and showed that the full likelihood factors as

$$Z = \bar{Z}^\vartheta Z_*, \qquad Z_* = \prod_{n=1}^N p_{*n}(X_n, V_n), \tag{1.2}$$

where $p_{*n}(x, v)$ is the conditional density of V_n given the observations before time n and $X_n = x$. The notation is chosen for consistency with later sections. It is natural to number the

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covariates $V_1, \dots, V_{n-1}$ so that they are in the past of X_n. The second factor may be completely unspecified, or it may depend on ϑ and on further parameters. If the second factor does not depend on ϑ, the partial likelihood contains all the information about ϑ. Note that p_{*n} varies independently of p_n^ϑ. Oakes (1981) reviews applications of the partial likelihood in survival analysis. Wong (1986) gives an optimality result for estimators based on the partial likelihood. Slud and Kedem (1994) and Møller and Sørensen (1994) analyse specific models. We emphasize that the partial likelihood (1.1) is different from Cox's (1972) partial likelihood for the proportional hazards model, which is a factor of the partial likelihood considered here. The construction of partial likelihoods has been discussed extensively in Kalbfleisch and Prentice (1980) and Arjas (1989).

Does Cox's factorization have a version for continuous-time processes? Regression models involving such processes have become prominent in statistics. For example, survival regression models involving time-dependent covariates and censoring are conveniently described by counting processes. Numerous applications have been discussed in the monographs by Fleming and Harrington (1991) and Andersen *et al.* (1993). In these applications, X_t is a counting process with intensity λ_t^ϑ depending on the history of both X and a vector V of covariate processes. The first example of such a counting process regression model is Aalen's (1980) additive risk model, with intensity $\lambda_t = \alpha_t^{\mathrm{T}} V_{t-}$. Another example is Andersen and Gill's (1982) version of Cox's (1972) proportional hazards model, with intensity $\lambda_t = C_{t-} \alpha_t \exp(\beta^{\mathrm{T}} V_{t-})$, where V_t together with the censoring process C_t form the covariate process. Gill (1985) suggested using as a *partial likelihood*

$$\bar{Z}_t^\vartheta = \prod_{T_n \leqslant t} \lambda_{T_n}^\vartheta \exp\left(-\int_0^t \lambda_s^\vartheta \, \mathrm{d}s\right),$$

where T_n are the successive jump times of X. This continuous-time partial likelihood, like the discrete-time partial likelihood above, has the form of the full likelihood of X except for the dependence on V.

A partial likelihood for semimartingales X was defined by Jacod (1987). Again it has the form of the full likelihood except for the dependence on V. For multivariate point processes and diffusion processes, Slud (1992) approximates the partial likelihood by discrete-time partial likelihoods.

When does the partial likelihood contain all the information about ϑ? When and how can one use additional information about the model? For discrete-time processes, both questions are answered by Cox's factorization (1.2) of the likelihood. Gill (1985) has given a heuristic derivation of the factorization for multivariate point processes in terms of product integrals (Andersen *et al.* 1993, p. 107). We obtain such a factorization for continuous-time processes in full generality, with X and V semimartingales. Our explicit description of the second factor makes it possible to decide when optimal inference can be based on the partial likelihood. Having the second factor explicitly, we can now also give a non-asymptotic justification for the efficiency concept which we introduced earlier (Greenwood and Wefelmeyer 1990). A different (asymptotic) justification was also given by us (Greenwood and Wefelmeyer 1992).

2. Factoring the likelihood

This section is organized as follows. First we recall some notation from the general theory of processes, and give Jacod's construction of the partial likelihood for semimartingales and Jacod and Mémin's representation of the full likelihood in terms of the predictable characteristics. For a response process X and a vector V of covariate processes, we describe the consistency relations between the characteristics of X and those of (X, V). We find a 'parametrization' of the full model in terms of the characteristics of X and of additional predictable processes not involving them. The theorem expresses the second factor of the likelihood in terms of these additional processes.

We need the following notation from the theory of semimartingales. A more detailed description and associated results have been given in the monograph of Jacod and Shiryaev (1987). Let $\mathscr{P}$ denote the predictable σ-field on $\Omega \times [0, \infty)$. If X is a semimartingale and b a predictable process, we write $b \cdot X_t$ for the stochastic integral $\int_0^t b_s \, \mathrm{d}X_s$, and X^{c} for the continuous martingale part of X. If ν is a random measure on $[0, \infty) \times \mathbf{R}^{\mathrm{d}}$ and Y a $\mathscr{P} \otimes \mathscr{B}^{\mathrm{d}}$ measurable function, we write $Y * \nu_t$ for the stochastic integral $\int_0^t \int Y(s, x)\nu(\mathrm{d}s, \mathrm{d}x)$.

We observe a real-valued cadlag response process X and a d-dimensional cadlag covariate process V on a finite time interval in $[0, \infty)$. They generate the filtration $(\mathscr{F}_t)_{t \geqslant 0}$. The random jump measure of X is

$$\mu^X(\mathrm{d}t, \mathrm{d}x) = \sum_{s:\Delta X_s \neq 0} \epsilon_{(s, \Delta X_s)}(\mathrm{d}t, \mathrm{d}x).$$

Similarly, the random jump measure of (X, V) is

$$\mu(\mathrm{d}t, \mathrm{d}x, \mathrm{d}v) = \sum_{s:(\Delta X_s, \Delta V_s) \neq 0} \epsilon_{(s, \Delta X_s, \Delta V_s)}(\mathrm{d}t, \mathrm{d}x, \mathrm{d}v).$$

Likelihoods of continuous-time processes are conveniently written with respect to a base measure in the model. We fix ϑ and ϑ' and introduce the corresponding partial and full likelihoods (likelihood ratios). The labels ϑ and ϑ' are dropped. First we recall how the characteristics of X change under an absolutely continuous change of measure. Consider two probability measures P and P' under which X is a semimartingale, with characteristics $(\bar{B}, \bar{C}, \bar{\nu})$ and $\bar{B}', \bar{C}', \bar{\nu}')$, respectively, with respect to a truncation function, say $\bar{h}$. Assume that $P'_t \ll P_t$ for $t \in [0, \infty)$, where $P_t = P|\mathscr{F}_t$. To keep the notation simpler, we assume that $P'_0 = P_0$. Write

$$\bar{a}_t = \bar{\nu}(\{t\} \times \mathbf{R}), \qquad \bar{a}'_t = \bar{\nu}'(\{t\} \times \mathbf{R}).$$

Choose an increasing predictable process $\bar{F}$ such that $\bar{C} = \bar{c} \cdot \bar{F}$ with $\bar{c}$ a non-negative predictable process. By a Girsanov theorem (Jacod and Shiryaev 1987, p. 159, Theorem 3.24), there exist a $\mathscr{P} \times \mathbf{R}$-measurable function $\bar{Y}(t, x)$ and a predictable process $\bar{\beta}$ such that P' a.s.,

$$\bar{B}' = \bar{B} + \bar{c}\bar{\beta} \cdot \bar{F} + \bar{h}(x)(\bar{Y} - 1) * \bar{\nu},$$

$$\bar{C}' = \bar{C},$$

$$\bar{\nu}'(\mathrm{d}t, \mathrm{d}x) = \bar{Y}(t, x)\bar{\nu}(\mathrm{d}t, \mathrm{d}x).$$

We recall Jacod's (1987; 1990b) construction of the partial likelihood. Introduce

$$\bar{\Sigma}^c = \{t: \bar{c}\bar{\beta} \cdot \bar{F}_t < \infty\},$$

$$\bar{\Sigma}^d = \{t: (1 - \bar{Y}^{1/2})^2 * \bar{\nu}_t + \sum_{s \le t}[(1 - \bar{a}_s')^{1/2} - (1 - \bar{a}_s)^{1/2}]^2 < \infty\}.$$

On $\bar{\Sigma} = \bar{\Sigma}^c \cap \bar{\Sigma}^d$, define the local P-martingale $\bar{N} = \bar{N}^c + \bar{N}^d$ with

$$\bar{N}^c = \bar{\beta} \cdot X^c, \tag{2.1}$$

$$\bar{N}^d = \left(\bar{Y} - 1 - \frac{\bar{a} - \bar{a}'}{1 - \bar{a}} 1_{\{\bar{a} < 1\}}\right) * (\mu^X - \bar{\nu}). \tag{2.2}$$

The *partial likelihood process* is defined on $\bar{\Sigma}$ as the Doléans exponential

$$\bar{Z}_t = \mathscr{E}(\bar{N})_t = \exp(\bar{N}_t - \tfrac{1}{2}\bar{c}\bar{\beta}^2 \cdot \bar{F}_t) \prod_{s \le t}(1 + \Delta\bar{N}_s)\,e^{-\Delta\bar{N}_s}. \tag{2.3}$$

The partial likelihood is described through the pairs of 'parameters' $\bar{B}$, $\bar{\nu}$ and $\bar{B}'$, $\bar{\nu}'$. The change from $\bar{B}$ to $\bar{B}'$ and from $\bar{\nu}$ to $\bar{\nu}'$ is given through $\bar{\beta}$ and $\bar{Y}$. These would play the role of local parameters in asymptotic theory. We shall describe the full model in such a way that the second factor of the likelihood does not involve the parameters $\bar{\beta}$, $\bar{Y}$ of the partial likelihood.

We assume that P_t' is dominated by P_t for $t \in [0, \infty)$ and have introduced the partial likelihood with reference to P and P'. However, a partial likelihood can also be constructed when P_t' is not dominated by P_t. Jacod (1990b) gives a definition which is free from P and P' except for the formal dependence of the stochastic integrals on P.

In order to produce an explicit factoring, we assume that the likelihood of (X, V) admits a representation in terms of its characteristics. This is not a serious restriction. Let (X, V) be a semimartingale under P and P', with characteristics (B, C, ν) and (B', C', ν'), respectively, with respect to a truncation function h. Write

$$a_t = \nu(\{t\} \times \mathbf{R} \times \mathbf{R}^d), \qquad a_t' = \nu'(\{t\} \times \mathbf{R} \times \mathbf{R}^d).$$

As before, the Girsanov theorem allows us to write $C = c \cdot F$ with a c a non-negative definite predictable $(d + 1) \times (d + 1)$ matrix, and we can write P' a.s.,

$$B' = B + c\beta \cdot F + h(x, v)(Y - 1) * \nu,$$

$$C' = C,$$

$$\nu'(\mathrm{d}t, \mathrm{d}x, \mathrm{d}v) = Y(t, x, v)\nu(\mathrm{d}t, \mathrm{d}x, \mathrm{d}v).$$

Define the random time intervals Σ^c, Σ^d and Σ in the same way as before, and on Σ define the local P-martingale $N = N^c + N^d$ with

$$N^c = \beta \cdot (X^c, V^c), \tag{2.4}$$

$$N^d = \left(Y - 1 - \frac{a - a'}{1 - a} 1_{\{a < 1\}}\right) * (\mu - \nu). \tag{2.5}$$

If all P-martingales have the representation property relative to (X, V), the density process $Z_t = \mathrm{d}P'_t/\mathrm{d}P_t$ can be represented on Σ as the Doléans exponential

$$Z_t = \mathscr{E}(N)_t = \exp\left(N_t - \tfrac{1}{2}\beta^{\mathrm{T}}c\beta \cdot F_t\right)\prod_{s \leqslant t}(1 + \Delta N_s)\,\mathrm{e}^{-\Delta N_s}. \tag{2.6}$$

The result is due to Jacod and Mémin (1976) and Kabanov *et al.* (1979; 1980) (see Jacod and Shiryaev (1987, p. 180, Theorem 5.19)). One sees from the representation that the full likelihood is described through the pairs of parameters B, ν and B', ν'.

The characteristics of (X, V) must be consistent with the characteristics of X. We may choose $\bar{F} = F$ and $\bar{h} = h_1$, the first component of the truncation function h. For the quadratic characteristics, we have

$$c'_{11} = c_{11} = \bar{c}. \tag{2.7}$$

The first component of the 'drift' characteristic of (X, V) is the specified characteristic of X:

$$B_1 = \bar{B}, \qquad B'_1 = \bar{B}'. \tag{2.8}$$

To describe consistency of ν with $\bar{\nu}$, we partition the state space as

$$\mathbf{R} \times \mathbf{R}^{\mathrm{d}} = ((\mathbf{R}\backslash\{0\}) \times \mathbf{R}^{\mathrm{d}}) + (\{0\} \times \mathbf{R}^{\mathrm{d}})$$

and write ν as its sum on these two sets:

$$\nu(\mathrm{d}t, \mathrm{d}x, \mathrm{d}v) = \nu_{-0}(\mathrm{d}t, \mathrm{d}x, \mathrm{d}v) + \nu_0(\mathrm{d}t, \mathrm{d}v)\epsilon_0(\mathrm{d}x).$$

For consistency, the marginal of ν_{-0} must be $\bar{\nu}$,

$$\nu_{-0}(\mathrm{d}t, \mathrm{d}x, \mathrm{d}v) = \bar{\nu}(\mathrm{d}t, \mathrm{d}x)\nu_{-*}(t, x, \mathrm{d}v), \tag{2.9}$$

and similarly for ν'. Here $\nu_{-*}(t, x, \mathrm{d}v)$ is the regular conditional jump size distribution of V given that X has a jump of size x at time t and varies independently of $\bar{\nu}$ just as p_{*n} varies independently of p_n^{ϑ}. In particular, $\nu_{-*}(t, x, \mathbf{R}^{\mathrm{d}}) = 1$, and the total mass of μ at each fixed t is $a(t) = a_0(t) + \bar{a}(t)$, where $a_0(t) = \nu_0(\{t\} \times \mathbf{R}^{\mathrm{d}})$.

We reparametrize the full model by the parameters $\bar{B}$, $\bar{\nu}$ of the partial specification and additional parameters which vary independently of them. Let $B_{(1)}$ denote the vector consisting of all components of B except the first. Note that $B_{(1)}$ and ν_{-*} vary independently of $\bar{B}$ and $\bar{\nu}$. Define ν_* by

$$\nu_0(\mathrm{d}t, \mathrm{d}v) = (1 - \bar{a}_t)\nu_*(\mathrm{d}t, \mathrm{d}v), \tag{2.10}$$

and similarly for ν'. In particular, $a_0 = (1 - \bar{a})a_*$. Note that ν_* equals ν_0 unless both X and V have a positive probability of jumping at time t, $\bar{a}_t > 0$ and $a_{0t} > 0$. Then

$$\nu_*(\{t\}, \mathrm{d}v) = \frac{\nu_0(\{t\}, \mathrm{d}v)}{a_{0t}}\frac{a_{0t}}{1 - \bar{a}_t},$$

where $\nu_0(\{t\}, \mathrm{d}v)/a_{0t}$ is the jump size distribution of $V_t 1_{\{\Delta X_t = 0\}}$, and $a_0/(1 - \bar{a}) = a_*$ is the conditional probability that V jumps given $\Delta X = 0$. Because both these factors vary independently of $\bar{B}$ and $\bar{\nu}$, so does ν_*. The model is now described by $\bar{B}$, $\bar{\nu}$, $B_{(1)}$, ν_{-*} and ν_*.

The consistency relation (2.8) for B translates into a consistency relation for the change from B to B':

$$\overline{c}\overline{\beta} = (c\beta)_1. \tag{2.11}$$

Since $v' \ll v$, also $v'_{-0} \ll v_{-0}$, $v'_0 \ll v_0$ and $v'_{-*} \ll v_{-*}$, $v'_* \ll v_*$. Write Y_{-0}, Y_0 and Y_{-*}, Y_* for the corresponding relative densities. Then

$$Y(t, x, v) = Y_{-0}(t, x, v)1_{\{x \neq 0\}} + Y_0(t, v)1_{\{x=0\}},$$

and the consistency relations (2.9) and (2.10) translate into

$$Y_{-0}(t, x, v) = \overline{Y}(t, x)Y_{-*}(t, x, v), \tag{2.12}$$

$$Y_0(t, v) = \frac{1 - \overline{a}'_t}{1 - \overline{a}_t} Y_*(t, v). \tag{2.13}$$

Partition c and β as

$$c = \begin{pmatrix} \overline{c} & c^{\mathrm{T}}_{(1)1} \\ c_{(1)1} & c_{(11)} \end{pmatrix}, \qquad \beta = \begin{pmatrix} \beta_1 \\ \beta_{(1)} \end{pmatrix}.$$

Here $\beta_{(1)}$ is the vector consisting of all components of β except the first. Similarly, $c_{(11)}$ is obtained from c by deleting the first row and column, and $c_{(1)1}$ from the first column by deleting its first element.

The factorization is expressed in terms of the processes

$$N^{\mathrm{c}}_* = \beta_{(1)} \cdot (-\overline{c}^{-1}c_{(1)1} \cdot X^{\mathrm{c}} + V^{\mathrm{c}}), \tag{2.14}$$

$$N^{\mathrm{d}}_{-*} = (Y_{-*} - 1)1_{\{x \neq 0\}} * \mu, \tag{2.15}$$

$$N^{\mathrm{d}}_* = \left(Y_* - 1 - \frac{a_* - a'_*}{1 - a_*}1_{\{a_* < 1\}} \right)1_{\{x=0\}} * (\mu - v_0), \tag{2.16}$$

using the Doléans exponential (2.6) with ΔN defined by (2.23).

Theorem. *On the random time interval $\Sigma \cap \overline{\Sigma}$, the likelihood process Z factors as $\overline{Z}Z_*$, where $\overline{Z}$ is the partial likelihood, and*

$$Z_* = \mathscr{E}(N^{\mathrm{c}}_*)\mathscr{E}(N^{\mathrm{d}}_{-*})\mathscr{E}(N^{\mathrm{d}}_*).$$

Here N^{c}_* is a continuous local P-martingale, N^{d}_{-*} and N^{d}_* are purely discontinuous local P-martingales, and $\mathscr{E}(N^{\mathrm{d}}_*)$ depends on v_0 only through v_*.

A more explicit description of the factorization $Z = \overline{Z}Z_*$ is

$$\overline{Z} = \mathscr{E}(\overline{N}^{\mathrm{c}})\mathscr{E}(\overline{N}^{\mathrm{d}})$$

with factors given by (2.18) and the limit in t of (2.26), and

$$Z_* = \mathscr{E}(N^{\mathrm{c}}_*)\mathscr{E}(N^{\mathrm{d}}_{-*})\mathscr{E}(N^{\mathrm{d}}_*),$$

with the first two factors given by (2.19) and (2.27), and the last by the limit of (2.28). Here we have

$$\mathscr{E}(N_{-*}^{d} + N_{*}^{d}) = \mathscr{E}(N_{-*}^{d})\mathscr{E}(N_{*}^{d}),$$

because N_{-*}^{d} and N_{*}^{d} jump on different times sets, $\Delta X \neq 0$ and $\Delta X = 0$.

Proof. By (2.6), the full likelihood on Σ is

$$Z = \mathscr{E}(N) = \mathscr{E}(N^{c})\mathscr{E}(N^{d}),$$

with N^{c} and N^{d} the continuous and purely discontinuous martingale parts of N, (2.4) and (2.5). We factor $\mathscr{E}(N^{c})$ first.

The consistency conditions (2.7) and (2.11) give

$$\beta = \begin{pmatrix} \bar{\beta} \\ 0 \end{pmatrix} + \begin{pmatrix} \beta_1 - \bar{\beta} \\ \beta_{(1)} \end{pmatrix}$$

$$= \begin{pmatrix} \bar{\beta} \\ 0 \end{pmatrix} + \begin{pmatrix} -\bar{c}^{-1}c_{(1)1}^{T}\beta_{(1)} \\ \beta_{(1)} \end{pmatrix}$$

$$= \begin{pmatrix} \bar{\beta} \\ 0 \end{pmatrix} + c^{-1}\begin{pmatrix} 0 \\ (c_{(11)} - \bar{c}^{-1}c_{(1)1}c_{(1)1}^{T})\beta_{(1)} \end{pmatrix},$$

a sum of two vectors orthogonal with respect to the inner product xcy. This implies that

$$\beta^{T}c\beta = \bar{c}\bar{\beta}^{2} + \beta_{(1)}^{T}(c_{(11)} - \bar{c}^{-1}c_{(1)1}c_{(1)1}^{T})\beta_{(1)}.$$

Hence $\bar{c}\bar{\beta}^{2} < \infty$ and $\beta_{(1)}^{T}(c_{(11)} - \bar{c}^{-1}c_{(1)1}c_{(1)1}^{T})\beta_{(1)} < \infty$ on Σ^{c}. Also

$$\beta \cdot (X^{c}, V^{c}) = \bar{\beta} \cdot X^{c} + \beta_{(1)} \cdot (-\bar{c}^{-1}c_{(1)1} \cdot X^{c} + V^{c}).$$

We obtain on Σ^{c}

$$\mathscr{E}(N^{c}) = \exp\{\beta \cdot (X^{c}, V^{c}) - \tfrac{1}{2}\beta^{T}c\beta \cdot F\} = \mathscr{E}(\bar{N}^{c})\mathscr{E}(N_{*}^{c}), \tag{2.17}$$

with

$$\mathscr{E}(\bar{N}^{c}) = \exp(\bar{\beta} \cdot X^{c} - \tfrac{1}{2}\bar{c}\bar{\beta}^{2} \cdot F) \tag{2.18}$$

and

$$\mathscr{E}(N_{*}^{c}) = \exp\{\beta_{(1)} \cdot (-\bar{c}^{-1}c_{(1)1} \cdot X^{c} + V^{c}) - \tfrac{1}{2}\beta_{(1)}^{T}(c_{(11)} - \bar{c}^{-1}c_{(1)1}c_{(1)1}^{T})\beta_{(1)} \cdot F\}. \tag{2.19}$$

Now we factor $\mathscr{E}(N^{d})$. Assume first that the jumps of X and V are bounded away from 0 by ϵ. Then

$$\mathscr{E}(N^{d})_{t} = \exp\{-(Y-1) * \nu_{t}^{c}\} \prod_{\substack{s \leqslant t \\ (\Delta X_{s}, \Delta V_{s}) \neq 0}} (1 + \Delta N_{s}^{d}).$$

By the consistency relations (2.12) and (2.13),

$$Y(s, x, v) = \bar{Y}(s, x)Y_{-*}(s, x, v)1_{\{x \neq 0\}} + \frac{1 - \bar{a}'_s}{1 - \bar{a}_s}Y_*(s, v)1_{\{x = 0\}}. \tag{2.20}$$

Since $v_{-*}(t, x, \mathbf{R}) = v'_{-*}(t, x, \mathbf{R}) = 1$,

$$\int \{Y_{-*}(t, x, v) - 1\}v_{-*}(t, x, dv) = 0. \tag{2.21}$$

Furthermore, $1/(1 - \bar{a}) = 1$ a.s. under v_0^c. Using (2.20), (2.21) and $v_{-0}^c(dt, dx, dv) = v_{-*}(t, x, dv)\bar{v}^c(dt, dx)$, we can compute

$$(Y - 1) * v^c = (\bar{Y}Y_{-*} - 1)1_{\{x \neq 0\}} * v_{-0}^c + (Y_* - 1) * v_0^c$$

$$= (\bar{Y} - 1) * \bar{v}^c + (Y_* - 1) * v_*^c. \tag{2.22}$$

As in the work of Jacod and Shiryaev (1987, p. 180, (5.13)),

$$\Delta N_s = \{Y(s, \Delta X_s, \Delta V_s) - 1\}1_{\{(\Delta X_s, \Delta V_s) \neq 0\}} + \frac{a_s - a'_s}{1 - a_s}1_{\{(\Delta X_s, \Delta V_s) = 0\}}. \tag{2.23}$$

By the consistency relation (2.10), integrated on v,

$$1 - a = 1 - \bar{a} - a_0 = (1 - \bar{a})\left(1 - \frac{a_0}{1 - \bar{a}}\right) = (1 - \bar{a})(1 - a_*), \tag{2.24}$$

and similarly for a'. With (2.20) and (2.22)–(2.24), we can write $\mathscr{E}(N^d)$ as

$$\mathscr{E}(N^d)_t = \exp\{-(Y - 1) * v_t^c\} \prod_{\substack{s \leqslant t \\ (\Delta X_s, \Delta V_s) \neq 0}} Y(s, \Delta X_s, \Delta V_s) \prod_{\substack{s \leqslant t \\ (\Delta X_s, \Delta V_s) = 0}} \frac{1 - a'_s}{1 - a_s}$$

$$= \exp\{-(\bar{Y} - 1) * \bar{v}_t^c - (Y_* - 1) * v_{*t}^c\} \prod_{\substack{s \leqslant t \\ \Delta X_s \neq 0}} \bar{Y}(s, \Delta X_s)Y_{-*}(s, \Delta X_s, \Delta V_s)$$

$$\times \prod_{\substack{s \leqslant t \\ \Delta V_s \neq 0, \Delta X_s = 0}} \frac{1 - \bar{a}'_s}{1 - \bar{a}_s}Y_*(s, \Delta V_s) \prod_{\substack{s \leqslant t \\ (\Delta X_s, \Delta V_s) = 0}} \frac{1 - \bar{a}'_s}{1 - \bar{a}_s}\frac{1 - a'_{*s}}{1 - a_{*s}}. \tag{2.25}$$

From (2.23) with N replaced by $\bar{N}$, we have

$$\mathscr{E}(\bar{N}^d)_t = \exp\{-(\bar{Y} - 1) * \bar{v}_t^c\} \prod_{\substack{s \leqslant t \\ \Delta X_s \neq 0}} \bar{Y}(s, \Delta X_s) \prod_{\substack{s \leqslant t \\ \Delta X_s = 0}} \frac{1 - \bar{a}'_s}{1 - \bar{a}_s}. \tag{2.26}$$

By relation (2.21),

$$N_{-*}^d = (Y_{-*} - 1)1_{\{x \neq 0\}} * \mu$$

$$= (Y_{-*} - 1)1_{\{x \neq 0\}} * (\mu - v)$$

is seen to be a purely discontinuous local P-martingale, and

$$\mathscr{E}(N^{\mathrm{d}}_{-*})_t = \prod_{\substack{s \leq t \\ \Delta X_s \neq 0}} Y_{-*}(s, \Delta X_s, \Delta V_s). \tag{2.27}$$

Further, from (2.23) with N replaced by N_* we have

$$\mathscr{E}(N^{\mathrm{d}}_*)_t = \exp\left\{-(Y_* - 1) * \nu^{\mathrm{c}}_{*t}\right\} \prod_{\substack{s \leq t \\ \Delta V_s \neq 0, \Delta X_s = 0}} Y_*(s, \Delta V_s) \prod_{\substack{s \leq t \\ \Delta V_s = 0, \Delta X_s = 0}} \frac{1 - a'_{*s}}{1 - a_{*s}}. \tag{2.28}$$

Applying (2.26)–(2.28) to (2.25), we obtain

$$\mathscr{E}(N^{\mathrm{d}}) = \mathscr{E}(\bar{N}^{\mathrm{d}})\mathscr{E}(N^{\mathrm{d}}_{-*})\mathscr{E}(N^{\mathrm{d}}_*). \tag{2.29}$$

Relations (2.17) and (2.29) give us the asserted factorization for $\epsilon > 0$.

The assertion that the jumps of X and V are of size greater than ϵ can be removed since the limit of each factor in (2.29) as $\epsilon \to 0$ is the corresponding factor with $\epsilon = 0$. In particular, the limit of $\mathscr{E}(N^{\mathrm{d}}_*)$ in (2.28) depends on ν_0 only through ν_*. □

3. Discussion

The theorem in Section 2 allows us to evaluate the relative efficiency of statistial procedures based on certain factors of the likelihood. Here we discuss some aspects of the factorization.

(i) Efficiency of the partial likelihood

The partial likelihood $\bar{Z}$ depends only on the partial specification $(\bar{B}, \bar{\nu})$. The second factor depends on $B_{(1)}$, ν_{-*} and ν_*, and through the continuous martingale part X^{c} in $\mathscr{E}(N^{\mathrm{c}}_*)$ it depends also on $\bar{B}$, but not on $\bar{\beta}$. The information about parameters of the model does not depend on the parameters (B, ν) of the base measure P, but only on the parameters (β, Y) of the likelihoood Z. Therefore, if we have a parametric model for $(\bar{B}, \bar{\nu})$, the partial likelihood leads to efficient inference as long as $\beta_{(1)}$, Y_{-*} and Y_* do not involve the parameter of interest.

(ii) The partial likelihood is a likelihood

If $B_{(1)}$, ν_{-*} and ν_* do not depend on the parameter of interest,

$$B'_{(1)} = B_{(1)}, \qquad \nu'_{-*} = \nu_{-*}, \qquad \nu'_* = \nu_*, \tag{3.1}$$

then $\beta_2 = 0$, and the ν density of ν' is

$$Y(s, x, v) = \bar{Y}(s, x)1_{\{x \neq 0\}} + \frac{1 - \bar{a}'_s}{1 - \bar{a}_s}1_{\{x = 0\}}.$$

Hence the representation (2.6) of the full likelihood Z reduces to the partial likelihood $\bar{Z}$. For multivariate point processes, this observation is due to Arjas and Haara (1984). Their condition A is contained in (3.1). They use Jacod's (1975) representation of the full likelihood. The reduction of Z to $\bar{Z}$ under (3.1) also follows from our factorization.

(iii) The partial likelihood is a projection

Suppose that we are given a parametric model for the partial specification $\bar{B}$, $\bar{\nu}$, but leave the model completely unspecified otherwise. If the full likelihood admits a representation (2.6), our factorization shows that the partial likelihood leads to efficient inference about the parameter. If we do not have the representation (2.6), it is still possible to prove this. Following Greenwood and Wefelmeyer (1990), introduce $\bar{P}$ by $\mathrm{d}\bar{P} = \bar{Z}\,\mathrm{d}P$, and call an estimator efficient if it is efficient in this model. Greenwood and Wefelmeyer (1992) show that this is equivalent to efficiency in the full model; $\bar{P}$ is in the full model because X has characteristics $\bar{B}$, $\bar{\nu}$ under $\bar{P}$ by the Girsanov theorem, and $\bar{P}$ is least favourable since $\bar{Z}$ is the projection of any full likelihood $Z_t = \mathrm{d}P'_t/\mathrm{d}P_t$ such that X has characteristics $\bar{B}'$, $\bar{\nu}'$ under P', by the converse of the Girsanov theorem (Jacod and Shiryaev 1987, p. 160, (3.28)). The latter argument has already been used by Jacod (1990a) (see also Jacod (1990b)) to prove that the partial Fisher information is smaller than the full Fisher information.

(iv) Factoring the partial likelihood

The partial likelihood factors as $\mathscr{E}(\bar{N}^{\mathrm{c}})\mathscr{E}(\bar{N}^{\mathrm{d}})$, with factors defined in (2.18) and (2.26). Suppose that we model X as $X_{\mathrm{c}} + X_{\mathrm{d}}$, where X_{c} is a continuous semimartingale with characteristics $(\bar{B}_{\mathrm{c}}, \bar{C}, 0)$, and X_{d} is a pure jump process with jump characteristic $\bar{\nu}$. Then $X^{\mathrm{c}} = X_{\mathrm{c}}^{\mathrm{c}} = X_{\mathrm{c}} - \bar{B}_{\mathrm{c}}$. Furthermore, the Girsanov theorem says that

$$\bar{B}'_{\mathrm{c}} = \bar{B}_{\mathrm{c}} + \bar{c}\bar{\beta}_{\mathrm{c}} \cdot \bar{F}.$$

Hence

$$\mathscr{E}(\bar{N}^{\mathrm{c}}) = \exp\left\{\bar{\beta}_{\mathrm{c}} \cdot (X_{\mathrm{c}} - \bar{B}_{\mathrm{c}}) - \tfrac{1}{2}\bar{c}\bar{\beta}_{\mathrm{c}}^2 \cdot \bar{F}\right\}$$

depends only on $\bar{B}_{\mathrm{c}}$, and $\mathscr{E}(\bar{N}^{\mathrm{d}})$ depends only on $\bar{\nu}$. We arrive at a factorization of $\bar{Z}$ with factors depending on independently varying parameters $\bar{B}_{\mathrm{c}}$ and $\bar{\nu}$.

(v) Fixed fixed jumps

In most applications, the probabilities $\bar{a}_t$ and a_{*t} of jumping at a fixed time t do not depend on the parameter of interest, $\bar{a}'_t = \bar{a}_t$ and $a'_{*t} = a_{*t}$. For example, they are simultaneously either 1 or 0, or all 0. In this case, $Y_* = Y_0$, and (2.26) and (2.28) reduce to

$$\mathscr{E}(\bar{N}^{\mathrm{d}})_t = \exp\left\{-(\bar{Y}-1)*\bar{\nu}_t^{\mathrm{c}}\right\} \prod_{\substack{s\leqslant t \\ \Delta X_s\neq 0}} \bar{Y}(s, \Delta X_s),$$

$$\mathscr{E}(N_*^{\mathrm{d}})_t = \exp\left\{-(Y_0-1)*\nu_{*t}^{\mathrm{c}}\right\} \prod_{\substack{s\leqslant t \\ \Delta V_s\neq 0, \Delta X_s=0}} Y_0(s, \Delta V_s).$$

4. Examples

The factorization simplifies for more specific processes. We give some examples.

(i) Discrete-time processes

For discrete-time processes, we have already described Cox's factorization in (1.2). It is instructive to derive a comparable factorization from our theorem, now for likelihood *ratios* rather than likelihoods. The observations will be jumps of a semimartingale, and we shall use small letters for them. Let $x_0, x_1, \ldots$ be real-valued responses, and $v_0, v_1, \ldots$ d-dimensional covariates. They generate a filtration $(\mathscr{F}_n)_{n\geqslant 0}$. As in Section 2 we assume, for simplicity, that the distribution of (x_0, v_0) is known. For $n = 1, 2, \ldots$, we specify a model for the regular conditional distributions $\bar{p}_n(\mathrm{d}x)$ of x_n given $\mathscr{F}_{n-1}$. The continuous-time process $X_t = \sum_{n\leqslant t} x_n$, $t \geqslant 0$, is a (special) semimartingale with characteristics $\bar{B} = 0$, $\bar{C} = 0$, and

$$\bar{\nu}(\mathrm{d}t, \mathrm{d}x) = \sum_{n\geqslant 1} \bar{\nu}_n(\mathrm{d}x)\epsilon_n(\mathrm{d}t),$$

where $\bar{\nu}_n$ is defined by

$$\bar{p}_n(\mathrm{d}x) = \bar{\nu}_n(\mathrm{d}x) + \bar{p}_n(\{0\})\epsilon_0(\mathrm{d}x).$$

The distinction between $x_n = 0$ and $x_n \neq 0$ is necessary because the jump measure of X does not charge $x_n = 0$. This will lead to a more complicated factorization than (1.2), namely f_{*n} has the form (4.9). Note that

$$\bar{p}_n(\{0\}) = 1 - \bar{\nu}_n(\mathbf{R}) = 1 - \bar{a}_n. \tag{4.1}$$

To introduce the partial likelihood process, let $\bar{p}_n'$ be another regular conditional distribution in the model, with $\bar{p}_n' \ll \bar{p}_n$, and write $\bar{Y}_n$ for the $\bar{\nu}_n$ density of $\bar{\nu}_n'$. As in (2.26), the partial likelihood process (2.23) can be written

$$\bar{Z}_t = \mathscr{E}(\bar{N}^{\mathrm{d}})_t = \prod_{\substack{n\leqslant t \\ x_n\neq 0}} \bar{Y}_n(x_n) \prod_{\substack{n\leqslant t \\ x_n=0}} \frac{1-\bar{a}_n'}{1-\bar{a}_n}. \tag{4.2}$$

Now let $p_n(\mathrm{d}x, \mathrm{d}v)$ be a regular conditional distribution of (x_n, v_n) given $\mathscr{F}_{n-1}$. Define

$V_t = \sum_{n \geqslant t} v_n$. Then (X, V) is a (special) semimartingale with characteristics $B = 0$, $C = 0$, and

$$\nu(dt, \, dx, \, dv) = \sum_{n \geqslant 1} \nu_n(dx, \, dv)\epsilon_n(dt),$$

where ν_n is defined by

$$p_n(dx, \, dv) = \nu_n(dx, \, dv) + p_n(\{(0, \, 0)\})\epsilon_{(0,0)}(dx, \, dv).$$

As in (4.1),

$$p_n(\{(0, \, 0)\}) = 1 - \nu_n(\mathbf{R} \times \mathbf{R}^d) = 1 - a_n. \qquad (4.3)$$

To introduce the full likelihood process, let p'_n be another regular conditional distribution of $(x_n, \, v_n)$ given $\mathscr{F}_{n-1}$, and assume that $p'_n \ll p_n$. With notation analogous to the above, the density process (2.6) can be written

$$Z_t = \prod_{\substack{n \leqslant t \\ (x_n, v_n) \neq 0}} Y_n(x_n, \, v_n) \prod_{\substack{n \leqslant t \\ (x_n, v_n) = 0}} \frac{1 - a'_n}{1 - a_n}. \qquad (4.4)$$

To describe the factorization, write

$$\nu_n(dx, \, dv) = \nu_{-0,n}(dx, \, dv) + \nu_{0n}(dv)\epsilon_0(dx).$$

The consistency relations (2.9) and (2.10) are then written

$$\nu_{-0,n}(dx, \, dv) = \overline{\nu}_n(dx)\nu_{-*,n}(x, \, dv),$$

$$\nu_{0n}(dv) = (1 - \overline{a}_n)\nu_{*n}(dv),$$

where $\nu_{-*,n}(x, \, dv)$ is the conditional jump size distribution of v_n given $x_n = x$. In particular (compare (2.24)),

$$1 - a_n = 1 - \overline{a}_n - a_{0n} = (1 - \overline{a}_n)(1 - a_{*n}). \qquad (4.5)$$

Then the consistency relations (2.12) and (2.13) for the relative densities are

$$Y_{-0,n}(x, \, v) = \overline{Y}_n(x)Y_{-*,n}(x, \, v), \qquad (4.6)$$

$$Y_{0n}(v) = \frac{1 - \overline{a}'_n}{1 - \overline{a}_n} Y_{*n}(v). \qquad (4.7)$$

The factorization $Z = \overline{Z}Z_*$ can now be obtained from the theorem in Section 2 and (2.26)–(2.28) or directly from (4.2) and (4.4) and the consistency relations (4.5) to (4.7), with

$$Z_{*t} = \prod_{\substack{n \leqslant t \\ x_n \neq 0}} Y_{-*,n}(x_n, \, v_n) \prod_{\substack{n \leqslant t \\ v_n \neq 0, x_n = 0}} Y_{*n}(v_n) \prod_{\substack{n \leqslant t \\ v_n = 0, x_n = 0}} \frac{1 - a'_{*n}}{1 - a_{*n}}. \qquad (4.8)$$

To compare with Cox's discrete-time factorization (1.2), write $\overline{f}_n$ for the $\overline{p}_n$ density of $\overline{p}'_n$. Then $\overline{f}_n$ equals $\overline{Y}_n$ on $\mathbf{R}\backslash\{0\}$ and, with (4.1),

$$\overline{f}_n(0) = \frac{\overline{p}'_n(\{0\})}{\overline{p}_n(\{0\})} = \frac{1 - \overline{a}'_n}{1 - \overline{a}_n}.$$

Hence the partial likelihood process (4.2) is $\bar{Z}_t = \prod_{n \leqslant t} \bar{f}_n(x_n)$. This is the partial likelihood ratio obtained from the usual discrete-time partial likelihood. Now factor

$$p_n(\mathrm{d}x, \mathrm{d}v) = \overline{p}_n(\mathrm{d}x) p_{*n}(x, \mathrm{d}v).$$

We have $p_{*n}(x, \mathrm{d}v) = \nu_{-*,n}(x, \mathrm{d}v)$ for $x \neq 0$, and $p_{*n}(0, \mathrm{d}v) = \nu_{*n}(\mathrm{d}v)$ on $\mathbf{R}^d \backslash \{0\}$. Further, $p_{*n}(0, \{0\}) = 1 - a_{*n}$. Write $f_{*n}(x, v)$ for the $p_{*n}(x, \mathrm{d}v)$ density of $p'_{*n}(x, \mathrm{d}v)$. By (1.2), the factor (4.8) must equal $\prod_{n \leqslant t} f_{*n}$. Indeed,

$$f_{*n}(x, v) = \begin{cases} Y_{-*,n}(x, v), & x \neq 0, \\ Y_{*n}(v), & x = 0, v \neq 0, \\ \dfrac{1 - a'_{*n}}{1 - a_{*n}}, & x = 0, v = 0. \end{cases} \tag{4.9}$$

(ii) Continuous processes

Suppose that (X, V) is a continuous semimartingale, with characteristics $(B, C, 0)$. Then the factorization reduces to $Z = \bar{Z} Z_* = \mathscr{E}(\bar{N}^c) \mathscr{E}(N_*^c)$ with factors defined in (2.18) and (2.19). In this case, $X^c = X - \bar{B}$ and $V^c = V - B_{(1)}$ with

$$B = \begin{pmatrix} \bar{B} \\ B_{(1)} \end{pmatrix} = cb \cdot F, \qquad b = \begin{pmatrix} b_1 \\ b_{(1)} \end{pmatrix}.$$

In particular,

$$\bar{Z} = \exp \{\bar{\beta} \cdot (X - \bar{B}) - \tfrac{1}{2}\bar{c}\bar{\beta}^2 \cdot F\}.$$

The compensator of $\beta_{(1)} \cdot (-\bar{c}^{-1} c_{(1)1} \cdot X + V)$ is

$$(-\bar{c}^{-1} c_{(1)1}^{\mathrm{T}} \beta_{(1)}, \beta_{(1)}^{\mathrm{T}}) cb \cdot F = \beta_{(1)}^{\mathrm{T}} (c_{(11)} - \bar{c}^{-1} c_{(1)1} c_{(1)1}^{\mathrm{T}}) b_{(1)} \cdot F.$$

Hence

$$Z_* = \exp \{\beta_{(1)} \cdot (-\bar{c}^{-1} c_{(1)1} \cdot X + V) - \beta_{(1)}^{\mathrm{T}} (c_{(11)} - \bar{c}^{-1} c_{(1)1} c_{(1)1}^{\mathrm{T}}) b_{(1)} \cdot F$$

$$- \tfrac{1}{2}\beta_{(1)}^{\mathrm{T}} (c_{(11)} - \bar{c}^{-1} c_{(1)1} c_{(1)1}^{\mathrm{T}}) \beta_{(1)} \cdot F\},$$

which does not depend on $\bar{\beta}$.

(iii) Jump processes

Let X and V be pure jump processes, and let $\bar{\nu}$ and ν be the compensators of the random jump measures of X and (X, V), respectively. Then the factorization reduces to $Z = \bar{Z} Z_*$ with

$$\bar{Z}_t = \exp \{-(\bar{Y} - 1) * \overline{\nu}_t^c\} \prod_{\substack{s \leqslant t \\ \Delta X_s \neq 0}} \bar{Y}(s, \Delta X_s) \prod_{\substack{s \leqslant t \\ \Delta X_s = 0}} \frac{1 - \overline{a}'_s}{1 - \overline{a}_s}$$

and

$$Z_{*t} = \prod_{\substack{s \leqslant t \\ \Delta X_s \neq 0}} Y_{-*}(s, \Delta X_s, \Delta V_s) \exp\left\{-(Y_* - 1) * \nu^{c}_{*t}\right\} \prod_{\substack{s \leqslant t \\ \Delta V_s \neq 0, \Delta X_s = 0}} Y_*(s, \Delta V_s)$$

$$\times \prod_{\substack{s \leqslant t \\ \Delta V_s = 0, \Delta X_s = 0}} \frac{1 - a'_{*s}}{1 - a_{*s}}.$$

An analogous factorization holds for multivariate point processes with more general state spaces; the partial likelihood $\bar{Z}$ is described by Arjas and Haara (1984).

(iv) Diffusions with jumps

Let (X, V) be a diffusion with jumps, with characteristics

$$\mathrm{d}B_t = b_t(X_{t-}, V_{t-}) \,\mathrm{d}t,$$

$$\mathrm{d}C_t = c_t(X_{t-}, V_{t-}) \,\mathrm{d}t,$$

$$\nu(\mathrm{d}t, \mathrm{d}x, \mathrm{d}v) = \mathrm{d}t \, K_t(X_{t-}, V_{t-}, \mathrm{d}x, \mathrm{d}v).$$

Specify a model for the characteristics of X:

$$\mathrm{d}\bar{B}_t = \bar{b}_t(X_{t-}, V_{t-}) \,\mathrm{d}t,$$

$$\mathrm{d}\bar{C}_t = \bar{c}_t(X_{t-}, V_{t-}) \,\mathrm{d}t,$$

$$\bar{\nu}(\mathrm{d}t, \mathrm{d}x) = \mathrm{d}t \, \bar{K}_t(X_{t-}, V_{t-}, \mathrm{d}x).$$

Write

$$K_t(X_{t-}, V_{t-}, \mathrm{d}x, \mathrm{d}v) = K_{-0,t}(X_{t-}, V_{t-}, \mathrm{d}x, \mathrm{d}v) + K_{0t}(X_{t-}, V_{t-}, \mathrm{d}v)\epsilon_0(\mathrm{d}x).$$

Consistency of ν with $\bar{\nu}$ gives

$$K_{-0,t}(X_{t-}, V_{t-}, \mathrm{d}x, \mathrm{d}v) = \bar{K}_t(X_{t-}, V_{t-}, \mathrm{d}x)K_{-*,t}(X_{t-}, V_{t-}, x, \mathrm{d}v).$$

Let P' be another distribution with $P'_t \ll P_t$ for $t \in [0, \infty)$, so that $C' = C$. For simplicity, assume again $P'_0 = P_0$. Write

$$\beta_t = c_t(X_{t-}, V_{t-})^{-1}\{b'_t(X_{t-}, V_{t-}) - b_t(X_{t-}, V_{t-})\},$$

$$\bar{\beta}_t = \bar{c}_t(X_{t-}, V_{t-})^{-1}\{\bar{b}'_t(X_{t-}, V_{t-}) - \bar{b}_t(X_{t-}, V_{t-})\}.$$

As in Section 2, the consistency relation for β and $\bar{\beta}$ is $\bar{c}_t(X_{t-}, V_{t-})\bar{\beta}_t = (c_t(X_{t-}, V_{t-})\beta_t)_1$. Write $Y_t(X_{t-}, V_{t-}, x, v)$ for the $K_t(X_{t-}, V_{t-}, \mathrm{d}x, \mathrm{d}v)$ density of $K'_t(X_{t-}, V_{t-}, \mathrm{d}x, \mathrm{d}v)$, and define Y_0, $\bar{Y}$ and Y_{-*} correspondingly. Since (X, V) does not have fixed jumps, we have

$a = 0$ and hence $\overline{a} = a_* = 0$ and $Y_*(x, v) = Y_0(X_{t-}, V_{t-}, x, v)$. We suppress (X_{s-}, V_{s-}). The partial likelihood process is

$$\overline{Z}_t = \exp\left(\int_0^t \overline{\beta}_s \, \mathrm{d}X_s^{\mathrm{c}} - \tfrac{1}{2}\overline{c}_s\overline{\beta}_s^2 \, \mathrm{d}s\right) \exp\left(-\int_0^t\!\!\int \{\overline{Y}_s(x) - 1\}\overline{K}_s(\mathrm{d}x)\,\mathrm{d}s\right) \prod_{\substack{s \leqslant t \\ \Delta X_s \neq 0}} \overline{Y}_s(\Delta X_s).$$

The likelihood factors, $Z = \overline{Z}Z_*$, where Z_* is obtained from (2.19), (2.27) and (2.28) as

$$Z_{*t} = \exp\left(\int_0^t \beta_{(1)s}(-\overline{c}_s^{-1}c_{(1)1s}\,\mathrm{d}X_s^{\mathrm{c}} + \mathrm{d}V_s^{\mathrm{c}}) - \tfrac{1}{2}\int_0^t \beta_{(1)s}^{\mathrm{T}}(c_{(11)s} - \overline{c}_s^{-1}c_{(1)1s}c_{(1)1s}^{\mathrm{T}})\beta_{(1)s}\,\mathrm{d}s\right)$$

$$\times \prod_{\substack{s \leqslant t \\ \Delta X_s \neq 0}} Y_{-*,s}(\Delta X_s, \Delta V_s) \exp\left(-\int_0^t\!\!\int \{Y_{0s}(v) - 1\}K_0(\mathrm{d}v)\,\mathrm{d}s\right) \prod_{\substack{s \leqslant t \\ \Delta V_s \neq 0, \Delta X_s = 0}} Y_{0s}(\Delta V_s).$$

Here $\overline{Z}$ depends on $\overline{b}$ through $\overline{\beta}$, and on $\overline{K}$ through $\overline{Y}$. The second factor Z_* depends on b through $\beta_{(1)}$, and on K through Y_{-*}, K_0 and Y_0. As in the continuous processes case, we argue that the first term of Z_* does not depend on $\overline{b}$ even though it contains X^{c}.

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