Bernoulli **12**(1), 2006, 169–189

Which multivariate gamma distributions are infinitely divisible?

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We define a multivariate gamma distribution on $\mathbb{R}^n$ by its Laplace transform $(P(-\boldsymbol{\theta}))^{-\lambda}$, $\lambda > 0$, where

$$P(\boldsymbol{\theta}) = \sum_{T\subset\{1,\ldots,n\}} p_T \prod_{i\in T} \theta_i.$$

Under $p_{\{1,\ldots,n\}} \neq 0$, we establish necessary and sufficient conditions on the coefficients of P, such that the above function is the Laplace transform of some probability distribution, for all $\lambda > 0$, thus characterizing all infinitely divisible multivariate gamma distributions on $\mathbb{R}^n$.

Keywords: Bell polynomials; exponential families; Frullani integral; generalized hypergeometric series; infinitely divisible distribution; Laplace transform; multivariate gamma distribution; Stirling numbers of the second kind

1. Introduction

In the literature, the multivariate gamma distributions on $\mathbb{R}^n$ have several non-equivalent definitions. Many authors require only that the marginal distributions are ordinary gamma distributions (Johnson *et al.* 1997). In the present paper we extend the classical one-dimensional definition to $\mathbb{R}^n$ as follows: we consider an affine polynomial $P(\boldsymbol{\theta})$ in $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ where 'affine' means that, for $j = 1, \ldots, n$, $\partial^2 P/\partial\theta_j^2 = 0$. We also assume that $P(\mathbf{0}) = 1$. For instance, for $n = 2$, we have $P(\theta_1, \theta_2) = 1 + p_1\theta_1 + p_2\theta_2 + p_{12}\theta_1\theta_2$. We fix $\lambda > 0$. If a probability distribution μ on $\mathbb{R}^n$ is such that $\mathrm{E}(\mathrm{e}^{\theta_1 x_1+\ldots+\theta_n x_n}) = (P(-\boldsymbol{\theta}))^{-\lambda}$ for a set of $\boldsymbol{\theta}$ with non-empty interior, then μ will be called the *multivariate gamma distribution associated with* (P, λ). Barndorff-Nielsen (1980) and Seshadri (1987) consider the case $n = 2$ and find that for all $\lambda > 0$, $(1 - p_1\theta_1 - p_2\theta_2 + p_{12}\theta_1\theta_2)^{-\lambda}$ is the Laplace transform of a probability distribution on $[0, \infty)^2$ if $p_1 > 0$, $p_2 > 0$, $p_{12} > 0$ and $-p_{12} + p_1 p_2 > 0$. Griffiths (1984), Moran and Vere-Jones (1969) and Vere-Jones (1967) consider the case where $P(-\boldsymbol{\theta} = |\mathbf{I}_n + \mathbf{V}\boldsymbol{\Theta}|$, where $\mathbf{V}$ is a symmetric positive definite or positive semi-definite matrix, $|\mathbf{A}| = \det(\mathbf{A})$, and $\boldsymbol{\Theta} = \mathrm{diag}(\theta_1, \ldots, \theta_n)$, another instance of an affine polynomial. These multivariate gamma distributions occur naturally in the classification of natural exponential families in $\mathbb{R}^n$ (Bar-Lev *et al.* 1994).

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Not all affine polynomials give rise to a valid Laplace transform. For instance, Griffiths' result for $n = 3$ implies that for $0 < b < 1/2$ there exists $\lambda < 1$ such that

$$(1 - \theta_1 - \theta_2 - \theta_3 + \tfrac{1}{2}\theta_1\theta_2 + \tfrac{1}{2}\theta_2\theta_3 + (1 - b^2)\theta_1\theta_3 - b(1 - b)\theta_1\theta_2\theta_3)^{-\lambda}$$

is not a Laplace transform. Finding all couples (P, λ) for which we obtain a multinomial gamma distribution is a difficult problem that we will not consider here. Instead, we address the simpler problem of characterizing the affine polynomials P on $\mathbb{R}^n$ with $P(\mathbf{0}) = 1$ such that, *for any positive* λ, there exists a multivariate gamma distribution associated with (P, λ). In other words, we wish to describe *all* multivariate gamma distributions, in our sense, that are infinitely divisible. For $n = 2$, the problem has been solved by Vere-Jones (1967). Griffiths (1984) also gives a necessary and sufficient condition on $\mathbf{V}$, a square symmetric matrix, such that $P(-\boldsymbol{\theta}) = |\mathbf{I}_n + \mathbf{V}\boldsymbol{\Theta}|$, $\boldsymbol{\Theta} = \mathrm{diag}(\theta_1, \ldots, \theta_n)$, is associated with infinite divisibility. The present paper considers a more general class of affine polynomials than Griffiths, with the sole restriction that the coefficient $p_{[n]}$, $[n] = \{1, \ldots, n\}$, of $\theta_1 \cdots \theta_n$ in P is non-zero. The paper finds a necessary and sufficient condition on the coefficients of P such that P is associated with infinite divisibility. This necessary and sufficient condition is expressed as a finite set of polynomial inequalities with respect to the coefficients of P, and relies on a previous paper by the author which solves the analogous problem for the negative multinomial distributions (Bernardoff 2003).

Section 2 gives definitions, and explains why the condition $p_{[n]} \neq 0$ is essential. Section 3 states the main result and applies it to the particular case of a symmetric polynomial $P(\theta_1, \ldots, \theta_n)$. Section 4 proves the main result. Section 5 develops a particular example. Section 6 makes the link with the necessary and sufficient condition obtained by Griffiths in the particular case $P(-\boldsymbol{\theta}) = |\mathbf{I}_n + \mathbf{V}\boldsymbol{\Theta}|$. Section 7 comments on the unsolved case $p_{[n]} = 0$.

2. Multivariate gamma distributions

Let us give some definitions (Letac 1991). Let $n \in \mathbb{N}$, the set of positive integers. Let μ be a positive Radon measure on $\mathbb{R}^n$. The support of μ, that is, the smallest closed set F such that $\mu(\mathbb{R}^n \backslash \mathrm{F}) = 0$, is denoted by $\mathrm{Supp}(\mu)$. We consider the Laplace transform of μ, $L_\mu(\theta) = \int_{\mathbb{R}^n} \exp\langle\boldsymbol{\theta}, \mathbf{x}\rangle\mu(\mathrm{d}\mathbf{x})$, where $\langle\boldsymbol{\theta}, \mathbf{x}\rangle$ denotes the scalar product. We denote by $\Theta(\mu)$ the interior of the convex set $D(\mu) = \{\boldsymbol{\theta} \in \mathbb{R}^n, L_\mu(\boldsymbol{\theta}) < \infty\}$. We denote by $\mathcal{M}_n$ the set of μs such that $\mathrm{Supp}(\mu)$ is not included in a strict affine subspace of $\mathbb{R}^n$, and such that $\Theta(\mu)$ is not empty. If $\mu \in \mathcal{M}_n$ and $\boldsymbol{\theta} \in \Theta(\mu)$, then $\mathbf{P}(\boldsymbol{\theta}, \mu)(\mathrm{d}\mathbf{x}) = L_\mu(\boldsymbol{\theta})^{-1} \exp\langle\boldsymbol{\theta}, \mathbf{x}\rangle\mu(\mathrm{d}\mathbf{x})$ is a probability measure on $\mathbb{R}^n$, and $F(\mu) = \{\mathbf{P}(\boldsymbol{\theta}, \mu), \boldsymbol{\theta} \in \Theta(\mu)\}$ is called the natural exponential family generated by μ. We denote $k_\mu : \Theta(\mu) \to \mathbb{R}$, $\boldsymbol{\theta} \mapsto k_\mu(\boldsymbol{\theta}) = \log L_\mu(\boldsymbol{\theta})$. The function k_μ is called the cumulant transform of μ.

We denote by $\mathfrak{B}_n = \mathfrak{B}([n])$ the family of all subsets of $[n]$ and $\mathfrak{B}_n^*$ the family of non-empty subsets of $[n]$. For simplicity, if n is fixed and if there is no ambiguity, we denote these families by $\mathfrak{B}$ and $\mathfrak{B}^*$, respectively.

We denote by $\mathbb{N}_0$ the set of non-negative integers. If $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{R}^n$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, then $\boldsymbol{\alpha}! = \alpha_1! \ldots \alpha_n!$, $|\boldsymbol{\alpha}| = \alpha_1 + \ldots + \alpha_n$, $a_{\boldsymbol{\alpha}} = a_{\alpha_1,\ldots,\alpha_n}$, and

$$\mathbf{z}^{\boldsymbol{\alpha}} = \prod_{i=1}^{n} z_i^{\alpha_i} = z_1^{\alpha_1} \ldots z_n^{\alpha_n}. \tag{2.1}$$

For T in $\mathfrak{B}_n$, we simplify the above notation by writing $\mathbf{z}^T = \prod_{t\in T} z_t$ instead of $\mathbf{z}^{\mathbf{1}_T}$ where

$$\mathbf{1}_T = (\alpha_1, \ldots, \alpha_n), \qquad \text{with } \alpha_i = 1 \text{ if } i \in T \text{ and } \alpha_i = 0 \text{ if } i \notin T. \tag{2.2}$$

We also write $\mathbf{z}^{-T}$ for $\prod_{t\in T} 1/z_t$. For a mapping $a : \mathfrak{B} \to \mathbb{R}$, we shall use the notation $a : \mathfrak{B} \to \mathbb{R}$, $T \mapsto a_T$. In this notation an affine polynomial with constant term equal to 1 is $P(\boldsymbol{\theta}) = \sum_{T\in\mathfrak{B}} p_T \boldsymbol{\theta}^T$, with $p_\varnothing = 1$. For simplicity, if $T = \{t_1, \ldots, t_k\}$, we denote $a_{\{t_1 \ldots, t_k\}} = a_{t_1 \ldots t_k}$. The indicator function of a set S is denoted by $\mathbb{1}_S$, that is, $\mathbb{1}_S(x) = 1$ for $x \in S$ and 0 for $x \notin S$.

Definition 1. *A probability distribution μ on $\mathbb{R}^n$ is called a multivariate gamma distribution associated with (P, λ), and is denoted by $\boldsymbol{\gamma}_{P,\lambda}$, if μ is in $\mathcal{M}_n$ and is such that*

$$L_\mu(\boldsymbol{\theta}) = (P(-\boldsymbol{\theta}))^{-\lambda}, \qquad \boldsymbol{\theta} \in \Theta(\mu), \tag{2.3}$$

where $P(\boldsymbol{\theta}) = \sum_{T\in\mathfrak{B}} p_T \boldsymbol{\theta}^T$ is an affine polynomial with constant term equal to 1 and where $\lambda > 0$.

Proposition 1. *Let μ be a multivariate gamma distribution on $\mathbb{R}^n$ associated with (P, λ). Assume that μ is not concentrated on a linear subspace of $\mathbb{R}^n$ of the form $\{x \in \mathbb{R}^n;\ x_k = 0\}$ for some k in $\{1, \ldots, n\}$. Then:*

(i) *for all $i \in [n]$, $p_i \neq 0$;*
(ii) *if $p_1, \ldots, p_k < 0$ and $p_{k+1}, \ldots, p_n > 0$, then* $\mathrm{Supp}(\mu) \subset (-\infty, 0]^k \times [0, \infty)^{n-k}$*;*
(iii) *if $p_1, \ldots, p_n > 0$ then $p_{[n]} \geqslant 0$.*

Proof. If $p_1 = 0$ (say) and if μ exists then for $\theta_2 = \ldots = \theta_n = 0$ we obtain that $L_\mu(\theta_1, 0, \ldots, 0) = 1$. We conclude that μ is concentrated on $\{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 = 0\}$ and we obtain a contradiction. A similar argument applies if any $p_i = 0$, $1 < i \leqslant n$.

Let $\gamma_{a,\lambda}(\mathrm{d}x) = |x|^{\lambda-1}|a|^{-\lambda}/\Gamma(\lambda)\exp(-|x/a|)\mathbb{1}_{(0,\infty)}(x/a)\,\mathrm{d}x$ be the ordinary gamma distribution on $(0, \infty)$ and with parameters $a \neq 0$ and $\lambda > 0$. For $i \in [n]$, we denote by φ_i the natural projection of $\mathbb{R}^n$ onto the ith coordinate and by μ_i the image measure of μ by φ_i. We have

$$\begin{aligned} L_{\mu_i}(\theta_i) &= \int_{\mathbb{R}} \mathrm{e}^{\theta_i x_i} \mu_i(\mathrm{d}x_i) = L_\mu((0, \ldots, 0, \theta_i, 0, 0)) \\ &= L_\mu(\varphi_i(\boldsymbol{\theta})) = (1 - p_i\theta_i)^{-\lambda} = L_{\gamma_{p_i,\lambda}}(\theta_i). \end{aligned}$$

We obtain that for all $i \in [k]$, $\mathrm{Supp}(\mu_i) = (-\infty, 0]$, and for all $i \in \{k+1, \ldots n\}$, $\mathrm{Supp}(\mu_i) = [0, \infty)$. Since $\mathrm{Supp}(\mu_i) = \mathrm{Supp}(\varphi_i(\mu)) = \varphi_i(\mathrm{Supp}(\mu))$, we have $\mathrm{Supp}(\mu) \subset (-\infty, 0]^k \times [0, \infty)^{n-k}$. If $\mathbf{b} = (b_1, \ldots, b_n) \in \Theta(\mu)$ then $\Theta(\mu) \supset (-\infty, b_1] \times \cdots \times (-\infty, b_n]$. Indeed, if $\theta_i \leqslant b_i$ for all $i \in [n]$, we have

$$
\begin{aligned}
L_\mu(\boldsymbol{\theta}) &= \int_{\mathbb{R}^n} \exp\langle \boldsymbol{\theta}, \mathbf{x}\rangle \mu(\mathrm{d}\mathbf{x}) \\
&= \int_{[0,\infty)^n} \exp\langle \boldsymbol{\theta}, \mathbf{x}\rangle \mu(\mathrm{d}\mathbf{x}) \\
&\leqslant \int_{[0,\infty)^n} \exp\langle \mathbf{b}, \mathbf{x}\rangle \mu(\mathrm{d}\mathbf{x}) < \infty.
\end{aligned}
$$

If $p_{[n]} < 0$ then we have $\lim_{t\to-\infty} P(-t, \ldots, -t) = \lim_{t\to-\infty} p_{[n]}(-t)^n = -\infty$, and $P(-\boldsymbol{\theta}) < 0$ for some $\boldsymbol{\theta} \in \Theta(\mu)$, thus $P(-\boldsymbol{\theta})^{-\lambda}$ cannot be the Laplace transform of a positive measure. □

From this proposition we now may assume, without loss of generality, that for all $i \in [n]$, $p_i > 0$ and $p_{[n]} \geqslant 0$. Let us recall the following Lévy-Khinchine result (Sato, 1999, p. 39): $\mu \in \mathcal{M}_n$ is infinitely divisible if and only if there exist $(\mathbf{A}, \boldsymbol{\gamma}, \nu)$, where $\mathbf{A}$ is a symmetric non-negative definite $n \times n$ real matrix, $\boldsymbol{\gamma} \in \mathbb{R}^n$, ν is a positive Radon measure on $\mathbb{R}^n \backslash \{0\}$ satisfying

$$\int_{\mathbb{R}^n\backslash\{0\}} \min(\|\mathbf{x}\|^2, 1)\nu(\mathrm{d}\mathbf{x}) < \infty, \tag{2.4}$$

and

$$
\begin{aligned}
k_\mu(\boldsymbol{\theta}) = \frac{1}{2}\langle \boldsymbol{\theta}, \mathbf{A}\boldsymbol{\theta}\rangle + \langle \boldsymbol{\gamma}, \boldsymbol{\theta}\rangle \\
+ \int_{\mathbb{R}^n\backslash\{0\}} (\mathrm{e}^{\langle \boldsymbol{\theta}, \mathbf{x}\rangle} - 1 - \langle \boldsymbol{\theta}, \mathbf{x}\rangle \mathbb{1}_{\|\mathbf{x}\|\leqslant 1}(\mathbf{x}))\nu(\mathrm{d}\mathbf{x})
\end{aligned}
\tag{2.5}
$$

for all $\boldsymbol{\theta} \in \Theta(\mu)$. The measure ν is called the Lévy measure of μ and $(\mathbf{A}, \boldsymbol{\gamma}, \nu)$ is called the generating triplet of μ. If ν satisfies the additional condition

$$\int_{\mathbb{R}^n\backslash\{0\}} \min(\|\mathbf{x}\|, 1)\nu(\mathrm{d}\mathbf{x}) < \infty, \tag{2.6}$$

then we can replace (2.5) by

$$k_\mu(\boldsymbol{\theta}) = \frac{1}{2}\langle \boldsymbol{\theta}, \mathbf{A}\boldsymbol{\theta}\rangle + \langle \boldsymbol{\gamma}_0, \boldsymbol{\theta}\rangle + \int_{\mathbb{R}^n\backslash\{0\}} (\mathrm{e}^{\langle \boldsymbol{\theta}, \mathbf{x}\rangle} - 1)\nu(\mathrm{d}\mathbf{x}) \tag{2.7}$$

where $\boldsymbol{\gamma}_0 \in \mathbb{R}^n$. Suppose now that μ is a multivariate gamma distribution associated with (P, λ), infinitely divisible or not. We show in Lemma 9 below that there exists a signed measure ν_P on $\mathbb{R}^n\backslash\{0\}$ such that $\lambda^{-1} k_\mu(\boldsymbol{\theta}) = \int_{\mathbb{R}^n\backslash\{0\}} (\mathrm{e}^{\langle \boldsymbol{\theta}, \mathbf{x}\rangle} - 1)\nu_P(\mathrm{d}\mathbf{x})$. Of course, μ will be infinitely divisible if and only if such a ν_P is positive. In this case the triplet $(\mathbf{A}, \boldsymbol{\gamma}_0, \nu)$ is $(\mathbf{0}, \mathbf{0}, \nu)$. Theorem 4 will give a necessary and sufficient condition for infinite divisibility of the multivariate gamma distribution associated with (P, λ) in terms of the signs of $2^n - 1$ polynomials $\tilde{b}_T$ in the variables $(p_S)_{S\in\mathfrak{B}^*}$ where $P(\boldsymbol{\theta}) = 1 + \sum_{S\in\mathfrak{B}^*} p_S \boldsymbol{\theta}^S$.

3. Main results

Recall first the Lévy measure of the ordinary gamma distribution.

Proposition 2. *For* $n = 1$, $p_1 > 0$, *and* $\lambda > 0$, *let* $\mu = \gamma_{p_1,\lambda}$ *be the gamma distribution on* $(0, \infty)$ *with parameters* p_1 *and* λ. *Then, for* $\theta_1 < 0$,

$$k_\mu(\theta_1) = \lambda \int_0^{+\infty} (\mathrm{e}^{\theta_1 x} - 1)\nu(\mathrm{d}x), \qquad \text{where } \nu(\mathrm{d}x) = \mathrm{e}^{-x/p_1} \mathbb{1}_{(0,\infty)}(x) \frac{\mathrm{d}x}{x}. \tag{3.1}$$

The measure ν *satisfies* (2.6) *and* (2.7). *Therefore* $\lambda\nu$ *is the Lévy measure of* μ.

Proof. See Sato (1999, p. 45, Example 8.10). □

To state our results in the general case, we use the following notation. If S is a *non-empty* set, $\prod_S^k$ denotes the set of all partitions of S into k *non-empty* subsets of S. We call the elements of $\prod_S^k$ k-partitions and $\prod_S = \bigcup_{k \geqslant 1} \prod_S^k$. If $S = [n]$, we write $\prod_{[n]}^k = \prod_n^k$ and $\prod_n = \bigcup_{k=1}^n \prod_n^k$ is the set of all partitions of $[n]$. For $\mathcal{T} = \{T_1, \dots T_k\} \in \prod_S$, we write

$$a_\mathcal{T} = \prod_{i=1}^k a_{T_i}. \tag{3.2}$$

Let $P(\mathbf{z}) = \sum_{T \in \mathfrak{B}_n^*} p_T \mathbf{z}^T$ and let $d_{\boldsymbol{\alpha}}(P)$ be the coefficient of $\mathbf{z}^{\boldsymbol{\alpha}}$ in the Taylor expansion

$$\log \frac{1}{1 - P(\mathbf{z})} = \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^n \setminus \{\mathbf{0}\}} d_{\boldsymbol{\alpha}}(P) \mathbf{z}^{\boldsymbol{\alpha}}. \tag{3.3}$$

The number $d_{\mathbf{1}_S}(P)$ will have a special importance and will be denoted $b_S(P)$.

Proposition 3. *For* $S \in \mathfrak{B}_n^*$, *let* $b_S(P)$ *denote the number* $d_{\mathbf{1}_S}(P)$ *as defined by* (3.3). *Then*

$$b_S(P) = \sum_{l=1}^{|S|} (l-1)! \sum_{\mathcal{T} \in \Pi_S^l} p_\mathcal{T} \tag{3.4}$$

where $|S|$ *is the cardinal of* S.

Proof. See Bernardoff (2003, Proposition 3). □

For simplicity, if P is fixed and if there is no ambiguity, we denote $d_{\boldsymbol{\alpha}}(P)$ and $b_S(P)$ by $d_{\boldsymbol{\alpha}}$ and b_S, respectively. Let $P(\boldsymbol{\theta}) = \sum_{T \in \mathfrak{B}_n} p_T \boldsymbol{\theta}^T$, with $p_\varnothing = 1$, an affine polynomial such that $p_{[n]} \neq 0$, and let $\tilde{P}(\boldsymbol{\theta}) = \sum_{T \in \mathfrak{B}_n} \tilde{p}_T \boldsymbol{\theta}^T$ where

$$\tilde{p}_T = -p_{\overline{T}} / p_{[n]} \tag{3.5}$$

with $\overline{T} = [n] \setminus T$. We denote, in particular,

$$\boldsymbol{\theta}_P = (\tilde{p}_1, \dots, \tilde{p}_n). \tag{3.6}$$

Thus

$$P(\boldsymbol{\theta}) = p_{[n]}\boldsymbol{\theta}^{[n]}(-\tilde{P}(\boldsymbol{\theta}^{-1})). \tag{3.7}$$

We shall use the Lévy-Khinchine result in Proposition 2 to establish our main result:

Theorem 4. *Let $\mu = \boldsymbol{\gamma}_{P,\lambda}$ be a gamma distribution associated with (P, λ), where $\lambda > 0$ and $P(\boldsymbol{\theta}) = \sum_{T\in\mathfrak{B}_n} p_T\boldsymbol{\theta}^T$ is such that $p_i > 0$ for all $i \in [n]$, and $p_{[n]} > 0$. Let $\tilde{P}(\boldsymbol{\theta}) = \sum_{T\in\mathfrak{B}_n} \tilde{p}_T\boldsymbol{\theta}^T$ be the affine polynomial such that $\tilde{p}_T = -p_{\overline{T}}/p_{[n]}$ for all $T \in \mathfrak{B}_n$. Let*

$$\tilde{b}_S = b_S(\tilde{P}) = \sum_{k=1}^{|S|}(k-1)! \sum_{\mathcal{T}\in\Pi_S^k} \prod_{T\in\mathcal{T}} \tilde{p}_T.$$

Then the measure μ is infinitely divisible if and only if

$$\tilde{p}_i < 0 \qquad \textit{for all } i \in [n], \tag{3.8}$$

and

$$\tilde{b}_S \geqslant 0 \qquad \textit{for all } \mathrm{S} \in \mathfrak{B}_n^* \textit{ such that } |S| \geqslant 2. \tag{3.9}$$

Before proving this theorem in Section 5, it is worthwhile to apply it to the particular case of a polynomial $P(\boldsymbol{\theta})$ which is affine symmetric in $\theta_1, \ldots, \theta_n$. Then p_T depends only on $|T|$. We also use the notation $s_k = p_T$ where $T \in \mathfrak{B}_n^*$ is such that $|T| = k$. Hence

$$P(\boldsymbol{\theta}) = 1 + \sum_{k=1}^{n} s_k\sigma_k(\boldsymbol{\theta}), \tag{3.10}$$

where $\sigma_k(\boldsymbol{\theta}) = \sum_{i_1<\ldots<i_k} \theta_{i_1} \cdots \theta_{i_k}$ is the elementary symmetric polynomial in $\theta_1, \ldots, \theta_n$ of degree k. Let $\mathbf{L}_n$ be the logarithmic polynomial defined by

$$\mathbf{L}_n(x_1, \ldots, x_n) = \sum_{k=1}^{n}(-1)^{k-1}(k-1)!\mathbf{B}_{n,k}(x_1, x_2, \ldots), \tag{3.11}$$

where $\mathbf{B}_{n,k}$ is the Bell partial exponential polynomial of order k, homogeneous of degree k and of weight n, and which is defined by

$$\mathbf{B}_{n,k}(x_1, \ldots, x_{n-k+1}) = \sum_{\substack{\mathbf{c}=(c_1,c_2,\ldots):\\ c_1+2c_2+\ldots=n,\, c_1+c_2+\ldots=k}} \frac{n!}{c_1!c_2!\cdots(1!)^{c_1}(2!)^{c_2}\cdots} x_1^{c_1}x_2^{c_2}\cdots. \tag{3.12}$$

These polynomials are defined, for instance, in Comtet (1970a). A table of these polynomials is given on pp. 184–185 of that reference.

Theorem 5. *Let $P(\boldsymbol{\theta}) = 1 + \sum_{k=1}^{n} s_k\sigma_k(\boldsymbol{\theta})$ be an affine polynomial where $s_1 > 0$ and $s_n > 0$. Suppose that there exists a gamma distribution $\mu = \boldsymbol{\gamma}_{P,\lambda}$ associated with (P, λ). Then $\tilde{p}_{\{i\}} = -s_{n-1}/s_n$ for all $i = 1, \ldots, n$ and*

$$\tilde{b}_S = -\mathbf{L}_{|S|}\left(\frac{s_{n-1}}{s_n}, \frac{s_{n-2}}{s_n} \ldots, \frac{s_{n-|S|}}{s_n}\right) \tag{3.13a}$$

$$= \frac{1}{s_n^{|S|}} \sum_{k=1}^{|S|} (-1)^k (k-1)! s_n^{|S|-k} \mathbf{B}_{|S|,k}(s_{n-1}, \ldots, s_{n-1+k-|S|}). \tag{3.13b}$$

Further, μ is infinitely divisible if and only if

$$s_{n-1} > 0 \tag{3.14}$$

and

$$-\mathbf{L}_\ell\left(\frac{s_{n-1}}{s_n}, \frac{s_{n-2}}{s_n} \ldots, \frac{s_{n-\ell}}{s_n}\right) \geqslant 0 \qquad \textit{for all } \ell = 2, \ldots, n. \tag{3.15}$$

Proof. These results are deduced from the following computations. For any $T \in \mathfrak{B}_n^*$,

$$\tilde{p}_T = -\frac{p_{\overline{T}}}{p_{[n]}} = -\frac{s_{n-|T|}}{s_n}; \tag{3.16}$$

in particular

$$\tilde{p}_{\{i\}} = -\frac{s_{n-1}}{s_n}, \qquad i = 1, \ldots, n. \tag{3.17}$$

Let N be a set of cardinality n. Let $\mathcal{T}$ be a partition of $\prod_N$ whose elements are called blocks of $\mathcal{T}$, and i-blocks if they have cardinality i. Let $c_1, \ldots, c_n$ be non-negative integers satisfying the condition $c_1 + 2c_2 + \ldots + nc_n = n$. The partition $\mathcal{T}$ is said to be of type $\mathbf{c} = (c_1, \ldots, c_n)$ if, for all $i = 1, \ldots, n$, $\mathcal{T}$ has c_i i-blocks. Noting that $\mathbf{c}$ then satifies the additional constraint $c_1 + \ldots + c_n = |\mathcal{T}|$, it follows from Comtet (1970b, p. 40), that the number of partitions of type $\mathbf{c}$ is

$$\frac{n!}{c_1! c_2! \cdots c_n! (1!)^{c_1} (2!)^{c_2} \cdots (n!)^{c_n}}. \tag{3.18}$$

If $p_T = s_{|T|}$ for any $T \in \mathfrak{B}_n$, then

$$
\begin{aligned}
b_S &= \sum_{k=1}^{|S|}(k-1)! \sum_{\mathcal{T}\in\Pi_S^k} \prod_{T\in\mathcal{T}} p_T \\
&= \sum_{k=1}^{|S|}(k-1)! \sum_{\substack{\mathbf{c}=(c_1,c_2,\ldots):\\ c_1+2c_2+\ldots=|S|, c_1+c_2+\ldots=k}} \sum_{\mathcal{T}\text{ of type }\mathbf{c}} s_1^{c_1} s_2^{c_2}\cdots \\
&= \sum_{k=1}^{|S|}(k-1)! \sum_{\substack{\mathbf{c}=(c_1,c_2,\ldots):\\ c_1+2c_2+\ldots=|S|, c_1+c_2+\ldots=k}} \frac{|S|!}{c_1!c_2!\cdots(1!)^{c_1}(2!)^{c_2}\cdots} s_1^{c_1} s_2^{c_2}\cdots \\
&= \sum_{k=1}^{|S|}(k-1)! B_{|S|,k}(s_1, s_2, \ldots) \\
&= -\sum_{k=1}^{|S|}(k-1)!(-1)^{k-1} B_{|S|,k}(-s_1, \ldots, -s_{|S|-k+1}) \\
&= -L_{|S|}(-s_1, \ldots, -s_{|S|})
\end{aligned}
\tag{3.19}
$$

and for any $S \in \mathfrak{B}_n^*$, $|S| \geqslant 2$, (3.16) implies

$$
\tilde{b}_S = -L_{|S|}\left(\frac{s_{n-1}}{s_n}, \ldots, \frac{s_{n-|S|}}{s_n}\right) \tag{3.20}
$$

where $s_0 = 1$. Inequalities (3.8) and (3.9) reduce to (3.14) and (3.15). □

Let us now apply Theorem 5 to the particular case in which $s_k = p^{k-1}$ for all $k \in [n]$, and $\lambda = 1$. We show in the next proposition, by application of the previous theorem, that this case corresponds to an infinitely divisible distribution μ. Further, Section 7 will provide a different proof of this fact by explicitly computing the λ powers of convolution of μ for any $\lambda > 0$, that is, the measure μ_λ such that $L_{\mu_\lambda}(\theta) = (L_\mu(\theta))^\lambda$. We utilize certain generalized hypergeometric functions (see Slater 1966), namely

$$
{}_0F_q(b_1, \ldots, b_q; z) = \sum_{k=0}^{\infty} \frac{1}{\langle b_1\rangle_k \cdots \langle b_q\rangle_k} \frac{z^k}{k!} \tag{3.21}
$$

where $\langle a\rangle_k = \Gamma(a+k)/\Gamma(a)$ is the Pochhammer symbol for $a > 0$ and $k \in \mathbb{N}_0$. We use here the notation of combinatorialists (see Comtet 1970a, pp. 15–16) rather than the notation $(a)_k$ of special functions. For the next proposition we only need ${}_0F_{n-1}(1, \ldots, 1; z)$, while Section 5 will make use of ${}_0F_{n-1}$ for more general parameters.

Proposition 6. *Let* $n \in \mathbb{N}$, $p = 1 - q \in (0, 1)$; *let*

$$\varphi_{n,p,1}(\mathrm{d}\mathbf{x}) = p^{-(n-1)}\exp\{-(x_1 + \ldots + x_n)/p\} \times {}_0F_{n-1}(1, \ldots, 1; qp^{-n}x_1 \ldots x_n)\mathbb{1}_{(0,\infty)^n}(\mathbf{x})\mathrm{d}\mathbf{x} \tag{3.22}$$

and the polynomial

$$P(\boldsymbol{\theta}) = \frac{-q}{p} + \frac{1}{p}\prod_{i=1}^{n}(1 + p\theta_i). \tag{3.23}$$

Then $\varphi_{n,p,1}$ *is an infinitely divisible multivariate gamma distribution with Laplace transform*

$$\int_{\mathbb{R}^n} \mathrm{e}^{\langle\boldsymbol{\theta},\mathbf{x}\rangle}\varphi_{n,p,1}(\mathrm{d}\mathbf{x}) = (P(-\boldsymbol{\theta}))^{-1} \tag{3.24}$$

defined for $\theta_i < 1/p$, $i = 1, \ldots, n$ *and* $\prod_{i=1}^{n}(1 - p\theta_i) > q$.

Proof. The proof consists of checking that the conditions of Theorem 5 are fulfilled. First, we compute $L_{\varphi_{n,p,1}}(\boldsymbol{\theta})$ for $\theta_i < 1/p$, $i = 1, \ldots, n$, and for $\prod_{i=1}^{n}(1 - p\theta_i) > q$. We obtain

$$\begin{aligned} L_{\varphi_{n,p,1}}(\boldsymbol{\theta}) &= \int_{(0,\infty)^n} \mathrm{e}^{\langle\boldsymbol{\theta},x\rangle} p^{-(n-1)} \exp\{-(x_1 + \ldots + x_n)/p\} \\ &\qquad \times \sum_{k=0}^{\infty} \frac{(qp^{-n}x_1 \ldots x_n)^k}{(k!)^n} \,\mathrm{d}\mathbf{x} \\ &= p^{-(n-1)} \sum_{k=0}^{\infty} (qp^{-n})^k \prod_{i=1}^{n} \int_0^{\infty} \exp\{-(1/\mathrm{p} - \theta_i)x_i\} \frac{x_i^k}{k!} \,\mathrm{d}x_i \\ &= p^{-(n-1)} \sum_{k=0}^{\infty} (qp^{-n})^k \prod_{i=1}^{n} (1/p - \theta_i)^{-(k+1)} = (P(-\boldsymbol{\theta}))^{-1}. \end{aligned}$$

Second, we apply Theorem 5 to $\varphi_{n,p,1}$. By (3.23), $s_k = p_T = p^{k-1}$. So $s_{n-k}/s_n = p^{-k}$, and by (3.13a) and (3.11) we have

$$\begin{aligned} \tilde{b}_S &= -\mathbf{L}_\ell(p^{-1}, p^{-2}, \ldots, p^{-\ell}) \\ &= -\sum_{k=1}^{\ell}(-1)^{k-1}(k-1)!\mathbf{B}_{\ell,k}(p^{-1}, p^{-2}, \ldots, p^{-\ell}) \\ &= \sum_{k=1}^{\ell}(-1)^k(k-1)!p^{-\ell}\mathbf{B}_{\ell,k}(1, \ldots, 1) = p^{-\ell}\sum_{k=1}^{\ell}(-1)^k(k-1)!\mathbf{S}_{\ell,k}, \end{aligned}$$

where $\mathbf{S}_{\ell,k}$ is the Stirling number of the second kind, that is, the number of k-partitions of a set with ℓ elements (Comtet 1970a, p. 146). By Stanley (1999, p. 34) $\tilde{b}_T = 0$ for $2 \leqslant |T| \leqslant n - 1$. Similar arguments show that $\tilde{b}_{[n]} = qp^{-n}$. By Theorem 5, $\varphi_{n,p,1}$ is infinitely divisible, and it is also a multivariate gamma distribution according to our Definition 1. □

4. The Lévy measures and the proof of Theorem 4

We will need the following result.

Theorem 7. *For $n \in \mathbb{N}$, let $P(\mathbf{z}) = \sum_{T \in \mathfrak{B}_n^*} p_T \mathbf{z}^T$. Then the coefficient $d_{\boldsymbol{\alpha}}$ of $\mathbf{z}^{\boldsymbol{\alpha}}$ in the Taylor expansion of $\log(1 - P(\mathbf{z}))^{-1}$ is a polynomial $Q_{\boldsymbol{\alpha}}$ in the $2^n - 1$ variables b_S, $S \in \mathfrak{B}_n^*$, and the coefficients of $Q_{\boldsymbol{\alpha}}$ are non-negative.*

Proof. See Bernardoff (2003, Theorem 1). □

We now construct certain measures on $[0, \infty)^n$ indexed by $I \in \mathfrak{B}^*$. For $i \in [n]$, define $\ell_i^I(\mathrm{d}x_i) = \mathbb{1}_{(0,\infty)}(x_i)\mathrm{d}x_i$ if $i \in I$ and $\ell_i^I(\mathrm{d}x_i) = \delta_0(\mathrm{d}x_i)$ if $i \notin I$. We define the following measure on $[0, \infty)^n$:

$$h_I(\mathrm{d}\mathbf{x}) = \bigotimes_{i=1}^{n} \ell_i^I(\mathrm{d}x_i). \tag{4.1}$$

For instance, if $n = 3$ and $I = \{2, 3\}$, then

$$h_{\{2,3\}}(\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3) = \delta_0(\mathrm{d}x_1)\mathbb{1}_{(0,\infty)^2}(x_2, x_3)\mathrm{d}x_2\, \mathrm{d}x_3.$$

For $I \in \mathfrak{B}_n^*$, we write $\mathbb{N}_i^I = \mathbb{N}$ if $i \in I$, $\mathbb{N}_i^I = \{0\}$ if $i \notin I$, and $\mathbb{N}^I = \times_{i=1}^n \mathbb{N}_i^I$. For instance, if $n = 3$ and $I = \{2\}$ then $\mathbb{N}^I = \{0\} \times \mathbb{N} \times \{0\}$. We introduce the notation $\mathbb{N}_{0,i}^I = \mathbb{N}_0$ if $i \in I$, $\mathbb{N}_{0,i}^I = \{0\}$ if $i \notin I$, and $\mathbb{N}_0^I = \times_{i=1}^n \mathbb{N}_{0,i}^I$. We denote by $\mathbf{1}$ the vector $(1, \ldots, 1)$ in $\mathbb{R}^n$. For $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ with $\theta_i \neq 0$ for all i in $[n]$, recall the notation $\boldsymbol{\theta}^{-1} = (\theta_1^{-1}, \ldots, \theta_n{}^{-1})$ and, for $\boldsymbol{\alpha} \in \mathbb{N}^n$, $\boldsymbol{\theta}^{-\boldsymbol{\alpha}} = (\boldsymbol{\theta}^{-1})^{\boldsymbol{\alpha}}$. For all $I \in \mathfrak{B}_n^*$, and $\boldsymbol{\alpha} \in \mathbb{N}^I$, let

$$\mu_{\boldsymbol{\alpha},I}(\mathrm{d}\mathbf{x}) = \frac{\mathbf{x}^{\boldsymbol{\alpha}-\mathbf{1}_I}}{(\boldsymbol{\alpha} - \mathbf{1}_I)!} h_I(\mathrm{d}\mathbf{x}). \tag{4.2}$$

Thus, for $\theta_1 < 0, \ldots, \theta_n < 0$, the Laplace transform of $\mu_{\boldsymbol{\alpha},I}$ is $L_{\mu_{\boldsymbol{\alpha},I}}(\boldsymbol{\theta}) = (-\boldsymbol{\theta})^{-\boldsymbol{\alpha}}$. More generally, for $a_1 + \theta_1 < 0, \ldots, a_n + \theta_n < 0$, if $\boldsymbol{a} = (a_1, \ldots, a_n)$ then we have

$$L_{\exp\langle \boldsymbol{a},\mathbf{x}\rangle \mu_{\boldsymbol{\alpha},I}}(\boldsymbol{\theta}) = (-\boldsymbol{a} - \boldsymbol{\theta})^{-\boldsymbol{\alpha}}. \tag{4.3}$$

The latter is still true if we replace $(\boldsymbol{\alpha} - \mathbf{1}_I)!$ in (4.2) by $\prod_{i \in I}\Gamma(\alpha_i)$ if $\alpha_i > 0$, $i = 1, \ldots, n$. The following lemma shows that the cumulant of a multivariate gamma distribution is represented by a signed measure.

Lemma 8. *Let $P(\boldsymbol{\theta}) = \sum_{T \in \mathcal{P}_n} p_T \boldsymbol{\theta}^T$, with $p_\varnothing = 1$, be an affine polynomial such that $p_i > 0$ for all $i \in [n]$ and $p_{[n]} > 0$. Let $\lambda > 0$ be such that $\boldsymbol{\gamma}_{P,\lambda}$ exists. Let $\tilde{P}(\boldsymbol{\theta}) = \sum_{T \in \mathfrak{B}_n} \tilde{p}_T \boldsymbol{\theta}^T$ where $\tilde{p}_T = -p_{\overline{T}}/p_{[n]}$. Let $\boldsymbol{\gamma}_{P,\lambda}$ be the gamma distriution associated with (P, λ). Then, for $\boldsymbol{\theta}_0$ in $\Theta(\boldsymbol{\gamma}_{P,\lambda})$,*

$$\lambda^{-1} k_{\mathbf{P}(\boldsymbol{\theta}_0,\boldsymbol{\gamma}_{P,\lambda})}(\boldsymbol{\theta}) = \int_{\mathbb{R}^n \setminus \{\mathbf{0}\}} (\mathrm{e}^{\langle \boldsymbol{\theta},\mathbf{x}\rangle} - 1)\nu_{P,\boldsymbol{\theta}_0}(\mathrm{d}\mathbf{x}) \tag{4.4}$$

with

$$\nu_{p,\boldsymbol{\theta}_0}(\mathrm{d}\mathbf{x}) = \mathrm{e}^{\langle \boldsymbol{\theta}_0, \mathbf{x}\rangle} \times \left\{ \mathrm{e}^{\langle \boldsymbol{\theta}_P, \mathbf{x}\rangle} \sum_{i=1}^{n} \frac{1}{x_i} h_{\{i\}} + \sum_{k=2}^{n} \sum_{I \in \mathfrak{B}_n : |I|=k} \left(\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^I} \tilde{d}_{\boldsymbol{\alpha}+\mathbf{1}_I} \frac{\mathbf{x}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} \right) h_I \right\} (\mathrm{d}\mathbf{x}) \tag{4.5}$$

where the coefficients $\tilde{d}_\alpha = d_\alpha(\tilde{P})$ *are defined by* (3.3).

Proof. If $\theta_i < 0$ for all $i \in [n]$ then, using (3.7), we have

$$\begin{aligned}
\lambda^{-1} \log L_\mu(\boldsymbol{\theta}) &= \lambda^{-1} \log(P(-\boldsymbol{\theta}))^{-\lambda} \\
&= \lambda^{-1} \log[p_{[n]}(-\boldsymbol{\theta})^{[n]} \{-\tilde{P}(-\boldsymbol{\theta}^{-1})\}]^{-\lambda} \\
&= \log \frac{1}{p_{[n]}} + \sum_{i=1}^{n} \log \frac{1}{-\theta_i} + \log\{-\tilde{P}(-\boldsymbol{\theta}^{-1})\}^{-1} \\
&= \log \frac{1}{p_{[n]}} + \sum_{i=1}^{n} \log \frac{1}{-\theta_i} + \sum_{\boldsymbol{\alpha}\in\mathbf{N}_0^n\setminus\{\mathbf{0}\}} \tilde{d}_{\boldsymbol{\alpha}} (-\boldsymbol{\theta}^{-1})^{-\boldsymbol{\alpha}} \\
&= \log \frac{1}{p_{[n]}} + \sum_{i=1}^{n} \log \frac{1}{-\theta_i} + \sum_{k=1}^{n} \sum_{I\in\mathfrak{B}^*:|I|=k} \sum_{\boldsymbol{\alpha}\in\mathbb{N}^I} \tilde{d}_{\boldsymbol{\alpha}} (-\boldsymbol{\theta})^{-\boldsymbol{\alpha}}.
\end{aligned}$$

Using (4.3), the latter expression becomes

$$\begin{aligned}
\lambda^{-1} k_\mu(\boldsymbol{\theta}) &= \log \frac{1}{p_{[n]}} + \sum_{i=1}^{n} \log \frac{1}{-\theta_i} + \sum_{k=1}^{n} \sum_{I\in\mathfrak{B}^*:|I|=k} \sum_{\boldsymbol{\alpha}\in\mathbb{N}^I} \tilde{d}_{\boldsymbol{\alpha}} L_{\mu_{\boldsymbol{\alpha}},I}(\boldsymbol{\theta}) \\
&= \log \frac{1}{p_{[n]}} + \sum_{i=1}^{n} \log \frac{1}{-\theta_i} + L_{\mu_1}(\boldsymbol{\theta})
\end{aligned}$$

where

$$\mu_1(\mathrm{d}\mathbf{x}) = \sum_{k=1}^{n} \sum_{I\in\mathfrak{B}^*:|I|=k} \left(\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^I} \frac{\tilde{d}_{\boldsymbol{\alpha}+\mathbf{1}_I}}{\boldsymbol{\alpha}!} \mathbf{x}^{\boldsymbol{\alpha}} \right) h_I(\mathrm{d}\mathbf{x}). \tag{4.6}$$

We observe that for all $\boldsymbol{\theta}_0$ in $\Theta(\mu)$ and all $\boldsymbol{\theta}$ in $\Theta(\mu) - \boldsymbol{\theta}_0$,

$$\lambda^{-1} k_{\mathbf{P}(\boldsymbol{\theta}_0,\mu)}(\boldsymbol{\theta}) = \log \frac{L_\mu(\boldsymbol{\theta}_0 + \boldsymbol{\theta})}{L_\mu(\boldsymbol{\theta}_0)} = \lambda^{-1}\{k_\mu(\boldsymbol{\theta}_0 + \boldsymbol{\theta}) - k_u(\boldsymbol{\theta}_0)\}. \tag{4.7}$$

By the Frullani integral (Berndt 1985), we have

$$\int_0^\infty \frac{\mathrm{e}^{\theta x} - \mathrm{e}^{-x}}{x} \,\mathrm{d}x = -\log(-\theta), \qquad \theta < 0.$$

We use this to represent (4.7) in the integral form

$$\lambda^{-1}k_{\mathbf{P}(\boldsymbol{\theta}_0,\mu)}(\boldsymbol{\theta}) = \sum_{i=1}^{n}\int_0^{\infty}(\mathrm{e}^{\theta_i x_i}-1)\frac{\mathrm{e}^{\theta_{0i}x_i}}{x_i}\,\mathrm{d}x_i + \int_{\mathbb{R}^n\setminus\{\mathbf{0}\}}(\mathrm{e}^{\langle\boldsymbol{\theta},\mathbf{x}\rangle}-1)\mathrm{e}^{\langle\boldsymbol{\theta}_0,\mathbf{x}\rangle}\mu_1(\mathrm{d}\mathbf{x}).$$

Finally, we obtain

$$\begin{aligned}\lambda^{-1}k_{\mathbf{P}(\boldsymbol{\theta}_0,\mu)}(\boldsymbol{\theta}) &= \int_{\mathbb{R}^n\setminus\{\mathbf{0}\}}(\mathrm{e}^{\langle\boldsymbol{\theta},\mathbf{x}\rangle}-1)\mathrm{e}^{\langle\boldsymbol{\theta}_0,\mathbf{x}\rangle}\sum_{i=1}^{n}\frac{1}{x_i}h_{\{i\}}(\mathrm{d}\mathbf{x})\\ &\quad+\int_{\mathbb{R}^n\setminus\{\mathbf{0}\}}(\mathrm{e}^{\langle\boldsymbol{\theta},\mathbf{x}\rangle}-1)\mathrm{e}^{\langle\boldsymbol{\theta}_0,\mathbf{x}\rangle}\sum_{i=1}^{n}\left(\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^{\{i\}}}\frac{\tilde{d}_{\boldsymbol{\alpha}+\mathbf{1}_{\{i\}}}}{\boldsymbol{\alpha}!}\mathbf{x}^{\boldsymbol{\alpha}}\right)h_{\{i\}}(\mathrm{d}\mathbf{x})\\ &\quad+\int_{\mathbb{R}^n\setminus\{\mathbf{0}\}}(\mathrm{e}^{\langle\boldsymbol{\theta},\mathbf{x}\rangle}-1)\mathrm{e}^{\langle\boldsymbol{\theta}_0,\mathbf{x}\rangle}\mu_1(\mathrm{d}\mathbf{x}).\end{aligned}\tag{4.8}$$

We remark that for $\boldsymbol{\alpha}\in\mathbb{N}_0^{\{i\}}$,

$$\begin{aligned}\tilde{d}_{\boldsymbol{\alpha}+\mathbf{1}_{\{i\}}} &= \tilde{d}_{(0,\dots0,\alpha_i+1,0,\dots0)}\\ &= \left(\frac{\mathrm{d}}{\mathrm{d}\theta_i}\right)^{\alpha_i+1}\log(1-\tilde{P}(0,\dots0,\theta_i,0,\dots0))^{-1}\Big|_{\theta_i=0}\\ &= \left(\frac{\mathrm{d}}{\mathrm{d}\theta_i}\right)^{\alpha_i+1}\log(1-\tilde{p}_i\theta_i)^{-1}\Big|_{\theta_i=0}\\ &= \frac{\tilde{p}_i^{\alpha_i+1}}{\alpha_i+1} = \frac{(\boldsymbol{\theta}_P)_i^{\alpha_i+1}}{\alpha_i+1};\end{aligned}\tag{4.9}$$

therefore

$$\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^{\{i\}}}\frac{\tilde{d}_{\boldsymbol{\alpha}+\mathbf{1}_{\{i\}}}}{\boldsymbol{\alpha}!}\mathbf{x}^{\boldsymbol{\alpha}} = \sum_{\alpha_i=0}^{\infty}\frac{(\boldsymbol{\theta}_P)_i^{\alpha_i+1}}{(\alpha_i+1)!}x_i^{\alpha_i} = \frac{\mathrm{e}^{(\boldsymbol{\theta}_P)_i x_i}-1}{x_i}.\tag{4.10}$$

By substituting (4.10) in (4.8), we obtain

$$\lambda^{-1}k_{\mathbf{P}(\boldsymbol{\theta}_0,\mu)}(\boldsymbol{\theta}) = \int_{\mathbb{R}^n\setminus\{\mathbf{0}\}}(\mathrm{e}^{\langle\boldsymbol{\theta},\mathbf{x}\rangle}-1)\mathrm{e}^{\langle\boldsymbol{\theta}_0,\mathbf{x}\rangle}\left(\mu_1+\mathrm{e}^{\langle\boldsymbol{\theta}_P,\mathbf{x}\rangle}\sum_{i=1}^{sn}\frac{1}{x_i}h_{\{i\}}\right)(\mathrm{d}\mathbf{x}),$$

according to (4.5). The proof of the lemma is complete. □

Let us now set $\boldsymbol{\theta}_0=\mathbf{0}$ in (4.5). We will give a different proof that $\nu_{P,\mathbf{0}}=\nu_P$. For all $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}_0^n$, we introduce the notation

$$\left(\frac{\partial}{\partial\boldsymbol{\theta}}\right)^{\alpha} = \frac{\partial^{|\alpha|}}{\partial\theta_1^{\alpha_1}\cdots\partial\theta_n^{\alpha_n}}.$$

For all $T \in \mathfrak{B}_n$, we also define $(\partial/\partial\boldsymbol{\theta})^T = (\partial/\partial\boldsymbol{\theta})^{\mathbf{1}_T}$. Now, we apply Taylor's formula to P at the point $-\boldsymbol{\theta}_P$ defined in (3.6). We write $\boldsymbol{\phi} = \boldsymbol{\theta}_P + \boldsymbol{\theta}$; then by Taylor's formula,

$$\begin{aligned} P(-\boldsymbol{\theta}) &= P(\boldsymbol{\theta}_P - \boldsymbol{\phi}) \\ &= \sum_{T\in\mathfrak{B}_n} \left(\frac{\partial}{\partial\boldsymbol{\theta}}\right)^T P(\boldsymbol{\theta}_P)(-\boldsymbol{\phi})^T \\ &= p_{[n]}(-\boldsymbol{\phi})^{[n]} \sum_{T\in\mathfrak{B}_n} \frac{1}{p_{[n]}}(-\boldsymbol{\phi})^{-\overline{T}}\left(\frac{\partial}{\partial\boldsymbol{\theta}}\right)^T P(\boldsymbol{\theta}_P) \\ &= -p_{[n]}(-\boldsymbol{\phi})^{[n]} R((-\boldsymbol{\phi})^{-1}) \end{aligned}$$

where

$$R(\boldsymbol{\phi}) = \sum_{T\in\mathfrak{B}_n} r_T \boldsymbol{\phi}^T, \qquad r_T = -\frac{1}{p_{[n]}}\left(\frac{\partial}{\partial\boldsymbol{\theta}}\right)^{\overline{T}} P(\boldsymbol{\theta}_P). \tag{4.11}$$

In particular, we have $r_{\{i\}} = 0$ for all $i = 1, \ldots, n$. Thus

$$P(-\boldsymbol{\theta}) = p_{[n]}(-\boldsymbol{\phi})^{[n]}\left\{1 - \sum_{T\in\mathfrak{P}_n:|T|\geqslant 2} r_T(-\boldsymbol{\phi})^{-T}\right\}. \tag{4.12}$$

Lemma 9. *Let $\mu = \boldsymbol{\gamma}_{P,\lambda}$ be a gamma distribution associated with (P, λ), where $P(\boldsymbol{\theta}) = \sum_{T\in\mathfrak{B}_n} p_T\boldsymbol{\theta}^T$ such that $p_i > 0$ for all $i \in [n]$ and $p_{[n]} > 0$. Consider the affine polynomials R and $\tilde{P}$ defined, respectively, by* (4.11) *and* (3.5). *For $\boldsymbol{\theta} \in \Theta(\boldsymbol{\gamma}_{P,\lambda})$, we have*

$$\lambda^{-1} k_\mu(\boldsymbol{\theta}) = \int_{\mathbb{R}^n\setminus\{\mathbf{0}\}} (\mathrm{e}^{\langle\boldsymbol{\theta},\mathbf{x}\rangle} - 1)\nu_P(\mathrm{d}\mathbf{x}) \tag{4.13}$$

with

$$\nu_P(\mathrm{d}\mathbf{x}) = \mathrm{e}^{\langle\boldsymbol{\theta}_P,\mathbf{x}\rangle}\left\{\sum_{i=1}^n \frac{1}{x_i} h_{\{i\}} + \sum_{k=2}^n \sum_{I\in\mathfrak{B}:|I|=k}\left(\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^I} d_{\boldsymbol{\alpha}+\mathbf{1}_I}(R)\frac{\mathbf{x}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}\right) h_I\right\}(\mathrm{d}\mathbf{x}). \tag{4.14}$$

Furthermore, $b_T(R) = \tilde{b}_T$ for $|T| \geqslant 2$, and $b_{\{i\}}(R) = 0$ for $i = 1, \ldots, n$. Finally, $d_{\boldsymbol{\alpha}+\mathbf{1}_I}(R)$ is a polynomial in the $2^n - n - 1$ variables $\tilde{b}_T$, $T \in \mathfrak{B}_n^*$, $|T| \geqslant 2$, with non-negative coefficients.

Proof. Using (4.12), we write

$$\lambda^{-1}\log L_\mu(\boldsymbol{\theta}) = -\log p_{[n]} - \sum_{i=1}^n \log(-\phi_i) + \sum_{k=2}^n \sum_{I\in\mathfrak{B}:|I|=k} \sum_{\boldsymbol{\alpha}\in\mathbb{N}^I} d_{\boldsymbol{\alpha}}(R)(-\boldsymbol{\phi})^{-\boldsymbol{\alpha}}.$$

We now apply (4.3) and deduce that

$$\lambda^{-1}\log L_\mu(\boldsymbol{\theta}) = -\log p_{[n]} - \sum_{i=1}^{n}\log(-\tilde{p}_i - \theta_i) + L_{\mu_2}(\boldsymbol{\theta}) \tag{4.15}$$

where

$$\mu_2(\mathrm{d}\mathbf{x}) = \mathrm{e}^{\langle \boldsymbol{\theta}_P, \mathbf{x}\rangle}\sum_{k=2}^{n}\sum_{I\in\mathfrak{B}:|I|=k}\left(\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^I} d_{\boldsymbol{\alpha}+\mathbf{1}_I}(R)\frac{\mathbf{x}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}\right)h_I(\mathrm{d}\mathbf{x}). \tag{4.16}$$

Applying the Frullani integral, we obtain

$$\lambda^{-1}\log L_\mu(\boldsymbol{\theta}) = -\log p_{[n]} + \sum_{i=1}^{n}\int_0^\infty \frac{\mathrm{e}^{(\tilde{p}_i+\theta_i)x_i} - \mathrm{e}^{-x_i}}{x_i}\,\mathrm{d}x_i + L_{\mu_2}(\boldsymbol{\theta}). \tag{4.17}$$

For $\boldsymbol{\theta} = \mathbf{0}$, this reduces to

$$0 = -\log p_{[n]} + \sum_{i=1}^{n}\int_0^\infty \frac{\mathrm{e}^{\tilde{p}_i x_i} - \mathrm{e}^{-x_i}}{x_i}\mathrm{d}x_i + L_{\mu_2}(\mathbf{0}). \tag{4.18}$$

We deduce that

$$\lambda^{-1}\log L_\mu(\boldsymbol{\theta}) = \int_{\mathbb{R}^n\setminus\{\mathbf{0}\}}(\mathrm{e}^{\langle\boldsymbol{\theta},\mathbf{x}\rangle} - 1)\nu_R(\mathrm{d}\mathbf{x}), \tag{4.19}$$

where

$$\nu_R(\mathrm{d}\mathbf{x}) = \mathrm{e}^{\langle \boldsymbol{\theta}_P, \mathbf{x}\rangle}\left\{\sum_{i=1}^{n}\frac{h_{\{i\}}}{x_i} + \sum_{k=2}^{n}\sum_{I\in\mathfrak{B}:|I|=k}\left(\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^I} d_{\boldsymbol{\alpha}+\mathbf{1}_I}(R)\frac{\mathbf{x}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}\right)h_I\right\}(\mathrm{d}\mathbf{x}) \tag{4.20}$$

according to (4.14).

We apply Lemma 8 for $\boldsymbol{\theta}_0 = 0$ and Lemma 9 to obtain

$$\begin{aligned}\nu_P(\mathrm{d}\mathbf{x}) &= \mathrm{e}^{\langle \boldsymbol{\theta}_P, \mathbf{x}\rangle}\sum_{i=1}^{n}\frac{1}{x_i}h_{\{i\}} + \sum_{k=2}^{n}\sum_{I\in\mathfrak{B}:|I|=k}\left(\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^I}\tilde{d}_{\boldsymbol{\alpha}+\mathbf{1}_I}\frac{\mathbf{x}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}\right)h_I(\mathrm{d}\mathbf{x})\\ &= \mathrm{e}^{\langle \boldsymbol{\theta}_P, \mathbf{x}\rangle}\left\{\sum_{i=1}^{n}\frac{1}{x_i}h_{\{i\}} + \sum_{k=2}^{n}\sum_{I\in\mathfrak{B}:|I|=k}\left(\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^I} d_{\boldsymbol{\alpha}+\mathbf{1}_I}(R)\frac{\mathbf{x}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}\right)h_I\right\}(\mathrm{d}\mathbf{x}).\end{aligned}$$

This leads to

$$\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^I}\tilde{d}_{\boldsymbol{\alpha}+\mathbf{1}_I}\frac{\mathbf{x}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} = \mathrm{e}^{\langle \boldsymbol{\theta}_P, \mathbf{x}\rangle}\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^I} d_{\boldsymbol{\alpha}+\mathbf{1}_I}(R)\frac{\mathbf{x}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}$$

for all $I \in \mathfrak{B}_n$ such that $|I| > 1$. Substituting $\mathbf{x} = \mathbf{0}$, we obtain $d_{\mathbf{1}_I}(R) = \tilde{d}_{\mathbf{1}_I}$, $|I| > 1$, that is, $\tilde{b}_I = \mathrm{b}_I(R)$, $|I| > 1$, and $b_{\{i\}}(R) = R_{\{i\}} = 0$ for all $i = 1, \ldots, n$. By Theorem 7, $d_{\boldsymbol{\alpha}+\mathbf{1}_I}(R)$ is a polynomial in $b_T(R) = \tilde{b}_T$, $|T| \geqslant 2$, with non-negative coefficients because $b_{\{i\}}(R) = 0$,

$i = 1, \ldots, n$. Since $\tilde{b}_T = b_T(R)$ for $T \in \mathfrak{B}_n$ with $|T| \geqslant 2$, we conclude that $d_{\boldsymbol{\alpha}+\mathbf{1}_I}(R)$ is a polynomial in $\tilde{b}_T$, $|T| \geqslant 2$, with non-negative coefficients.

It is important to compare (4.14) to (4.5) where we have set $\boldsymbol{\theta}_0 = \mathbf{0}$. Note that the term $\mathrm{e}^{\langle \boldsymbol{\theta}_P, \mathbf{x} \rangle}$ factorizes the whole $\nu_P = \nu_{P,\mathbf{0}}$. □

Proof of Theorem 4. We prove the 'only if' part. Since ν_P is a Lévy measure, it follows from (2.4) that ν_P is finite on $]1, \infty[^n$. Therefore $\tilde{p}_i$ is negative for all $i \in [n]$. All the measures on the right-hand side of (4.14) are mutually singular. Then ν_P is positive if and only if all these measures are positive. This implies $\tilde{b}_T \geqslant 0$ for all $T \in \mathfrak{B}_n^*$ such that $|T| \geqslant 2$ owing to the fact that $d_{\mathbf{1}_T}(R) = \tilde{b}_T$ and

$$\sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^I} d_{\boldsymbol{\alpha}+\mathbf{1}_I}(R) \frac{\mathbf{x}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} = d_{\mathbf{1}_T}(R) + \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^I \setminus \{\mathbf{0}\}} d_{\boldsymbol{\alpha}+\mathbf{1}_I}(R) \frac{\mathbf{x}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}.$$

Conversely, according to Theorem 9, (3.8) and (3.9) imply that $\lambda \nu_P$ is the Lévy measure of $\boldsymbol{\gamma}_{P,\lambda}$. □

5. An explicit case

Proposition 6 provides a particular example of an infinitely divisible multivariate gamma distribution. This section computes the densities of the convolution powers and the Lévy measure of this example.

Proposition 10. *Let P be the affine polynomial defined by* (3.23)*; let $\mu = \boldsymbol{\gamma}_{P,1} = \varphi_{n,p,1}$ be the infinitely divisible gamma distribution associated with* $(P, 1)$*. Let $\boldsymbol{\gamma}_{P,\lambda} = \varphi_{n,p,\lambda}$ be the gamma distribution associated with* (P, λ)*. Then we have:*

(i) *For* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$\boldsymbol{\gamma}_{P,\lambda}(\mathrm{d}\mathbf{x}) = \frac{p^{-(n-1)\lambda}}{(\Gamma(\lambda))^n} \exp\{-(x_1 + \ldots + x_n)/p\}(\mathbf{x}^{[n]})^{\lambda-1}$$
$$\times {}_0F_{n-1}(\lambda, \ldots, \lambda; qp^{-n}\mathbf{x}^{[n]}) \mathbb{1}_{(0,\infty)^n}(\mathbf{x})\mathrm{d}\mathbf{x}. \tag{5.1}$$

(ii) *The Lévy measure of $\boldsymbol{\gamma}_{P,\lambda}$ is $\lambda\nu_P$ with*

$$\nu_P(\mathrm{d}\mathbf{x}) = \exp\{-(x_1 + \ldots + x_n)/p\}$$
$$\times \left(qp^{-n} {}_0F_{n-1}(1, \ldots, 1, 2; qp^{-n}\mathbf{x}^{[n]}) \mathbb{1}_{(0,\infty)^n}(\mathbf{x}) + \sum_{i=1}^n \frac{h_{\{i\}}}{x_i} \right) \mathrm{d}\mathbf{x}. \tag{5.2}$$

Proof. Recall $p_T = p^{|T|-1}$ and $\tilde{p}_T = -p^{-|T|}$, for all $T \in \mathfrak{B}_n^*$. Then $\boldsymbol{\theta}_P = (-p^{-1}, \ldots, -p^{-1}) = -p^{-1}\mathbf{1}$. Let $\boldsymbol{\phi} = \boldsymbol{\theta} + \boldsymbol{\theta}_P$, that is, $\phi_i = \theta_i - p^{-1}$. For $\prod_{i=1}^n (p^{-1} - \theta_i) > qp^{-n}$ and $p^{-1} - \theta_i > 0$, $i = 1, \ldots, n$, we obtain

$$\begin{aligned}
L_\mu(\boldsymbol{\theta}) &= (P(-\boldsymbol{\theta}))^{-\lambda} = [L_\mu(\boldsymbol{\phi} - \boldsymbol{\theta}_P)]^{-\lambda} \\
&= p^{-(n-1)\lambda}\left(\prod_{i=1}^{n}(p^{-1} - \theta_i)^{-\lambda}\right)\left(1 - qp^{-n}\prod_{i=1}^{n}(p^{-1} - \theta_i)^{-1}\right)^{-\lambda} \\
&= p^{-(n-1)\lambda}\sum_{k=0}^{\infty}\langle\lambda\rangle_k(qp^{-n})^k\prod_{i=1}^{n}(p^{-1} - \theta_i)^{-k-\lambda} \\
&= p^{-(n-1)\lambda}\sum_{k=0}^{\infty}\langle\lambda\rangle_k(qp^{-n})^k L_{(\Pi_{i=1}^{n}\mathrm{e}^{-x_i/p}x_i^{\lambda+k-1}/\Gamma(\lambda+k))\mathbb{1}_{(0,\infty)^n}(\mathbf{x})\mathrm{d}\mathbf{x}}(\boldsymbol{\theta}) \\
&= L_{\varphi_{n,p},\lambda}(\boldsymbol{\theta}),
\end{aligned}$$

where

$$\begin{aligned}
\varphi_{n,p,\lambda}(\mathrm{d}\mathbf{x}) = &\frac{p^{-(n-1)\lambda}}{(\Gamma(\lambda))^n}\exp\{-(x_1 + \ldots + x_n)/p\}(x_1 \cdots x_n)^{\lambda-1} \\
&\times\sum_{k=0}^{\infty}\frac{(qp^{-n}x_1 \cdots x_n)^k}{(\langle\lambda\rangle_k)^{n-1}k!}\mathbb{1}_{(0,\infty)^n}(\mathbf{x})\mathrm{d}\mathbf{x} \\
= &\frac{p^{-(n-1)\lambda}}{(\Gamma(\lambda))^n}\exp\{-(x_1 + \ldots + x_n)/p\}(x_1 \cdots x_n)^{\lambda-1} \\
&\times {}_0F_{n-1}(\lambda, \ldots, \lambda; qp^{-n}x_1 \cdots x_n)\mathbb{1}_{(0,\infty)^n}(\mathbf{x})\mathrm{d}\mathbf{x}.
\end{aligned}$$

Finally, we obtain (5.1). Note that we have just obtained a second proof of the infinite divisibility of μ.

We now use (4.14) to compute the Lévy measure of $\mu = \mu_{P,\lambda}$. We write

$$L_\mu(\boldsymbol{\theta}) = \frac{1}{p}\left\{-q + (-p)^n\prod_{i=1}^{n}\phi_i\right\} = p^{n-1}(-\boldsymbol{\phi})^{[n]}\left\{1 - \sum_{|T|=2}^{n} r_T(-\boldsymbol{\phi})^{-T}\right\},$$

where $r_T = 0$, $1 \leqslant |T| \leqslant n - 1$ and $r_{[n]} = qp^{-n}$. Since

$$\log(1 - qp^{-n}\boldsymbol{\phi}^{[n]}) = \sum_{\ell=1}^{\infty}\frac{1}{\ell}(qp^{-n})^\ell(\boldsymbol{\phi}^{[n]})^\ell = \sum_{k=1}^{\infty}\frac{1}{\ell}(qp^{-n})^\ell\phi_1^\ell \cdots \phi_n^\ell,$$

we obtain $d_{\boldsymbol{\alpha}}(R) = 0$ if $\boldsymbol{\alpha} \neq \ell\mathbf{1}$, and $d_{\ell\mathbf{1}}(R) = \ell^{-1}(qp^{-n})^\ell$, $\ell \in \mathbb{N}$. Therefore, from (4.14), we have

$$\nu_P(\mathrm{d}\mathbf{x}) = \exp\langle -p^{-1}\mathbf{1}, \mathbf{x}\rangle \left\{ \sum_{i=1}^{n} \frac{1}{x_i} h_{\{i\}} + \left(\sum_{\ell=0}^{\infty} d_{(\ell+1)\mathbf{1}}(R) \frac{\mathbf{x}^{\ell\mathbf{1}}}{(\ell\mathbf{1})!} \right) h_{[n]} \right\}(\mathrm{d}\mathbf{x})$$

$$= \exp\{-(x_1 + \ldots + x_n)/p\}$$

$$\times \left\{ \sum_{i=1}^{n} \frac{1}{x_i} h_{\{i\}} + \left(qp^{-n} \sum_{\ell=\mathbf{0}}^{\infty} \frac{(qp^{-n}x_1 \cdots x_n)^{\ell}}{(\ell!)^{n-1}(\ell+1)!} \right) h_{[n]} \right\}(\mathrm{d}\mathbf{x})$$

$$= \exp\{-(x_1 + \ldots + x_n)/p\}$$

$$\times \left(\sum_{i=1}^{n} \frac{1}{x_i} h_{\{i\}} + qp^{-n} {}_0F_{n-1}(1, \ldots, 1, 2; qp^{-n}\mathbf{x}^{[n]}) \mathbb{1}_{(0,\infty)^n}(\mathbf{x}) \right)(\mathrm{d}\mathbf{x}),$$

according to (5.2). □

6. Application to Griffiths' result

Griffiths (1984, p. 14, Theorem 1) has proved the following result. Let μ be a probability distribution on $[0, \infty)^n$ such that

$$L_\mu(\boldsymbol{\theta}) = L_\mu(\theta_1, \ldots, \theta_n) = |\mathbf{I}_n - \mathbf{V}\boldsymbol{\Theta}|^{-1},$$

where $\mathbf{V}$ is a symmetric positive definite or positive semi-definite $n \times n$ $(n \geqslant 3)$ matrix, $\boldsymbol{\Theta} = \mathrm{diag}(\theta_1, \ldots, \theta_n)$ with $\theta_i < 0$ for all $i \in [n]$, and $\mathbf{I}_n$ is the $n \times n$ identity matrix. Denote by V_{ij} the cofactor of (i, j). Then μ is infinitely divisible if and only if, for all $3 \leqslant k \leqslant n$ and for all $\{i_1, \ldots, i_k\} \in [n]$, we have

$$(-1)^k V_{i_1 i_2} V_{i_2 i_3} \cdots V_{i_{k-1} i_k} V_{i_k i_1} \geqslant 0. \tag{6.1}$$

Furthermore, Griffiths obtains the corollary that when the matrix of cofactors $(V_{ij})_{i,j=1}^{n}$ of the matrix $\mathbf{V}$ has no zero elements, then μ is infinitely divisible if and only if for all distinct $i, j, \ell \in [n]$,

$$V_{ij} V_{j\ell} V_{\ell i} < 0. \tag{6.2}$$

Since the polynomial $P(\boldsymbol{\theta}) = |\mathbf{I}_n + \mathbf{V}\boldsymbol{\Theta}|$, where $\boldsymbol{\Theta} = \mathrm{diag}(\theta_1, \ldots, \theta_n)$, is affine, a natural question is: whether Theorem 4 yields Griffiths' result. Actually not, but Theorem 15 below offers another necessary and sufficient condition close to Griffiths' one. The next proposition matches Griffiths' notation with ours.

Proposition 11. *Let* $P(\boldsymbol{\theta}) = |\mathbf{I}_n + \mathbf{V}\boldsymbol{\Theta}|$, *where* $\mathbf{V}$ *is a symmetric positive definite* $n \times n$ $(n \geqslant 3)$ *matrix,* $\boldsymbol{\Theta} = \mathrm{diag}(\theta_1, \ldots, \theta_n)$. *For* $T \in \mathfrak{B}_n^*$, *let* $\mathbf{V}_T = (v_{ij})_{i,j \in T}$ *and* $\mathbf{V}_\varnothing = 1$. *Then we have:*

(i) *For all* $T \in \mathfrak{B}_n$, $p_T = |\mathbf{V}_T|$ *and* $\tilde{p}_T = -|\mathbf{V}_{\bar{T}}|/|\mathbf{V}|$.
(ii) *For all* $S \in \mathfrak{B}_n^*$, $\tilde{b}_S = \sum_{\ell=1}^{|S|}(\ell-1)!\sum_{\mathcal{T}\in\Pi_S^\ell} \prod_{T\in\mathcal{T}}(-|\mathbf{V}_{\bar{T}}|/|\mathbf{V}|)$ *with* $|\mathbf{V}_\varnothing| = 1$.

Proof. Let $T \in \mathfrak{B}_n^*$. The coefficient of $\boldsymbol{\theta}^T$ in

$$P(\boldsymbol{\theta}) = |\mathbf{I}_n + \mathbf{V}\boldsymbol{\Theta}| = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \prod_{i=1}^{n} (\delta_{\sigma(i),i} + v_{\sigma(i),i}\theta_i)$$

is

$$p_T = \sum_{\sigma \in \mathfrak{S}_n : \sigma(j)=j, j \notin T} \varepsilon(\sigma) \prod_{i \in T} v_{\sigma(i),i} = \sum_{\sigma \in \mathfrak{S}_T} \varepsilon(\sigma) \prod_{i \in T} v_{\sigma(i),i} = |\mathbf{V}_T|.$$

For $T \in \mathfrak{B}_n$, we obtain

$$\tilde{p}_T = -\frac{|\mathbf{V}_{\bar{T}}|}{|\mathbf{V}|}.$$

Then we have

$$\tilde{b}_S = \sum_{\ell=1}^{|S|} (\ell-1)! \sum_{\mathcal{T} \in \Pi_S^\ell} \sum_{T \in \mathcal{T}} \tilde{p}_T = \sum_{\ell=1}^{|S|} (\ell-1)! \sum_{\mathcal{T} \in \Pi_S^\ell} \prod_{T \in \mathcal{T}} \left(-\frac{|\mathbf{V}_{\bar{T}}|}{|\mathbf{V}|} \right).$$

□

Let us now recall a crucial result. For an $n \times n$ matrix $\boldsymbol{Q} = (q_{ij})$ define

$$p_T = (-1)^{|T|-1} |\boldsymbol{Q}_T|. \tag{6.3}$$

Theorem 12. *Let T be a non-empty subset of $[n]$ and $\mathfrak{C}_T$ be the set of all circular permutations of T. Then*

$$b_T = \sum_{c \in \mathfrak{C}_T} \prod_{t \in T} q_{tc(t)} = |T|^{-1} \sum_{\{i_1,\ldots,i_k\}=T} q_{i_1 i_2} \cdots q_{i_{k-1} i_k} q_{i_k i_1}.$$

Proof. See Bernardoff (2003, Theorem 3). □

The next theorem provides the link with Griffiths' result recalled in (6.1).

Theorem 13. *With the above notation, we have for S in $\mathfrak{B}_n^*$,*

$$(-|\mathbf{V}|)^{|S|} \tilde{b}_S = |S|^{-1} \sum_{\{i_1,\ldots,i_k\}=S} V_{i_1 i_2} \cdots V_{i_{k-1} i_k} V_{i_k i_1}. \tag{6.4}$$

Proof. The proof relies on a formula due to Jacobi. Let $S = \{s_1, \ldots, s_k : s_1 < \ldots < s_k\}$ and $T = \{t_1, \ldots, t_k : t_1 < \ldots < t_k\}$ be subsets of $[n]$. We denote $\bar{S} = [n] \backslash S = \{s_{k+1}, \ldots, s_n : s_{k+1} < \ldots < s_n\}$ and $\bar{T} = [n] \backslash T = \{t_{k+1}, \ldots, t_n : t_{k+1} < \ldots < t_n\}$. If $\mathbf{A} = (a_{ij})_{(i,j) \in [n]^2}$ is an $n \times n$ invertible matrix, let us use the notation $\mathbf{A}_{P,R} = (a_{ij})_{(i,j) \in P \times R}$. Then the minor of the inverse $\mathbf{A}^{-1}$ of $\mathbf{A}$ with respect to S and T is given by Jacobi's identity (Krob and Legros 1999, pp. 349–350)

$$|(\mathbf{A}^{-1})_{S,T}| = \varepsilon(\sigma\tau^{-1})|\mathbf{A}_{\overline{T},\overline{S}}|\,|\mathbf{A}|^{-1},$$

where ε indicates the signature and where σ and τ denote the permutations of $\mathfrak{S}_n$ defined respectively by $\sigma(i) = s_i$ and $\tau(i) = t_i$ for all i in $[n]$.

Consider the particular case in which $S = T$, and where $\mathbf{A} = \mathbf{V}$ is an $n \times n$ symmetric positive definite matrix. We obtain

$$|(\mathbf{V}^{-1})_T| = \frac{|\mathbf{V}_{\overline{T}}|}{|\mathbf{V}|}. \tag{6.5}$$

We substitute (6.5) in (6.3) and obtain

$$p_T = (-1)^{|T|-1}|(-\mathbf{V}^{-1})_T| = (-1)^{|T|-1}(-1)^{|T|}|(-\mathbf{V}^{-1})_T| = -\frac{|\mathbf{V}_{\overline{T}}|}{|\mathbf{V}|} = \tilde{p}_T$$

and $b_S = \tilde{b}_S$. We now apply Theorem 12 to $\boldsymbol{Q} = -\mathbf{V}^{-1} = (-V_{ij}/|\mathbf{V}|)_{(i,j)\in[n]^2}$ where $\mathbf{V}$ is symmetric positive definite. We then obtain

$$\begin{aligned}
\tilde{b}_S &= |S|^{-1} \sum_{\{i_1,\ldots,i_k\}=S} q_{i_1 i_2} \cdots q_{i_{k-1} i_k} q_{i_k i_1} \\
&= |S|^{-1} \sum_{\{i_1,\ldots i_k\}=S} \left(-\frac{V_{i_1 i_2}}{|\mathbf{V}|}\right) \cdots \left(-\frac{V_{i_{k-1} i_k}}{|\mathbf{V}|}\right)\left(-\frac{V_{i_k i_1}}{|\mathbf{V}|}\right) \\
&= |S|^{-1}(-|\mathbf{V}|)^{-|S|} \sum_{\{i_1,\ldots,i_k\}=S} V_{i_1 i_2} \cdots V_{i_{k-1} i_k} V_{i_k i_1},
\end{aligned}$$

according to (6.4). □

Corollary 14. *If $n \geqslant 3$, $S = \{i, j, \ell\}$, and $|\mathbf{V}| \neq 0$, then*

$$(-|\mathbf{V}|)^3 \tilde{b}_{i,j,\ell} = 2V_{ij}V_{j\ell}V_{\ell i}.$$

Proof. In this case $V_{i_1 i_2}V_{i_2 i_3}V_{i_3 i_1} = V_{ij}V_{j\ell}V_{\ell i}$ for all $\{i_1, i_2, i_3\} = S$ and

$$(-|\mathbf{V}|)^3 \tilde{b}_{i,j,\ell} = \frac{1}{3} \sum_{\{i_1,i_2,i_3\}=\{i,j,\ell\}} V_{i_1 i_2}V_{i_2 i_3}V_{i_3 i_1} = 2V_{ij}V_{j\ell}V_{\ell i}.$$

Theorem 15. *Let $\mathbf{V}$ be a positive semi-definite symmetric matrix. Then the following statements are equivalent:*

(i) $|\mathbf{I}_n - \mathbf{V}\boldsymbol{\Theta}|^{-1}$, *with* $\boldsymbol{\Theta} = \mathrm{diag}(\theta_1, \ldots, \theta_n)$, *is the Laplace transform of an infinitely divisible distribution.*
(ii) $(-1)^k V_{i_1 i_2} \cdots V_{i_{k-1} i_k} V_{i_k i_1} \geqslant 0$ *for any sequence of elements* $i_1, \ldots, i_k \in [n]$ *and for* $k \in \{3, \ldots, n\}$.
(iii) *For all* $S \subset [n]$, *the sign of* $\sum_{\{i_1,\ldots,i_k\}=S} V_{i_1 i_2} \cdots V_{i_{k-1} i_k} V_{i_k i_1}$ *is* $(-1)^{|S|}$.

Proof. (i) $\Leftrightarrow$ (ii) is due to Griffiths (1984). (ii) $\Rightarrow$ (iii) is trivial. Let us show (iii) $\Rightarrow$ (i). If $|\mathbf{V}| > 0$ then this is Theorem 13. Assume now that $|\mathbf{V}| = 0$. Let $\varepsilon > 0$, and consider $\mathbf{V}_\varepsilon = \mathbf{V} + \varepsilon \mathbf{I}_n$. Then $|\mathbf{I}_n - \mathbf{V}_\varepsilon \mathbf{\Theta}|^{-1}$ satisfies (iii) and $|\mathbf{V}_\varepsilon| > 0$. Then by the first case, $|\mathbf{I}_n - \mathbf{V}_\varepsilon \mathbf{\Theta}|^{-1}$ is infinitely divisible. As the limit of an infinitely divisible distribution is infinitely divisible, (i) is valid also for $\varepsilon = 0$. □

7. The case $p_{[n]} = 0$

Theorem 4 requires $p_{[n]} \neq 0$. In the particular case considered by Griffiths (1984), we have $p_{[n]} = |\mathbf{V}|$ and Theorem 15 ignores the condition $p_{[n]} \neq 0$. However, finding necessary and sufficient conditions for infinite divisibility in the case $p_{[n]} = 0$ seems to be a difficult problem. To illustrate this point we consider the classical Wishart distribution on positive definite symmetric 2×2 matrices

$$\boldsymbol{X} = \begin{bmatrix} X_1 & X_3 \\ X_3 & X_2 \end{bmatrix}$$

with

$$\mathrm{E}(\mathrm{e}^{s_1 X_1 + s_2 X_2 + 2 s_3 X_3}) = \left| \mathbf{I}_2 - \begin{bmatrix} s_1 & s_3 \\ s_3 & s_2 \end{bmatrix} \right|^{-p} = (1 + s_1 - s_2 + s_1 s_2 - s_3^2)^{-p},$$

$p \geqslant 2$. It is known (Bar-Lev *et al.* 1994) that there is no distribution in $\mathbb{R}^3$ having such a Laplace transform for $0 < p < 1/2$. Note that $R(s_1, s_2, s_3) = s_1 + s_2 - s_1 s_2 + s_3^2$ is not an affine polynomial. However, let $s_1 = \theta_1 + \theta_3$, $s_2 = \theta_2 + \theta_3$ and $s_3 = \theta_3$. Then R becomes

$$P(\boldsymbol{\theta}) = 1 + \theta_1 + 2\theta_3 + \theta_2 + \theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_1.$$

This polynomial satisfies $p_{[3]} = 0$. Therefore, for $Y_1 = X_1$, $Y_2 = X_2$ and $Y_3 = X_1 + X_2 + 2X_3$ we have $\mathrm{E}(\mathrm{e}^{\langle \boldsymbol{\theta}, \boldsymbol{Y} \rangle}) = (P(-\boldsymbol{\theta})^{-p}$, which is a Laplace transform if and only if $p \geqslant 1/2$. Therefore, infinite divisibility may or may not exist in the case $p_{[n]} = 0$, and even the case $n = 3$ is a challenge.

Acknowledgement

I thank Gérard Letac and Evelyne Bernadac for many helpful conversations and an anonymous referee for insightful comments.

References

Bar-Lev, S.K., Bshouty, D., Enis, P., Letac, G., Lu, I. and Richards, D. (1994) The diagonal multivariate natural exponential families and their classification. *J. Theoret. Probab.*, **7**, 883–929.

Barndorff-Nielsen, O.E. (1980) Conditionality resolutions. *Biometrika*, **67**, 293–310.

Bernardoff, P. (2003) Which negative multinomial distributions are infinitely divisible? *Bernoulli*, **9**, 877–893.

Berndt, B.C. (1985) *Ramanujan's Notebooks*, Part I. New York: Springer-Verlag.

Comtet, L. (1970a) *Analyse Combinatoire*, Vol. 1. Paris: PUF.

Comtet, L. (1970b) *Analyse Combinatoire*, Vol. 2. Paris: PUF.

Griffiths, R.C. (1984) Characterization of infinitely divisible multivariate gamma distribution. *J. Multivariate Anal.*, **15**, 13–20.

Johnson, N., Kotz, S. and Balakrishnan, N. (1997) *Continuous Multivariate Distributions*. New York: Wiley.

Krob, D. and Legros, S. (1999) *Algèbre Générale et Linéaire. Introduction au Calcul Symbolique et aux Mathématiques Expérimentales*, Vol 2. Paris: Vuibert.

Letac, G. (1991) *Lectures on Natural Exponential Families and Their Variance Functions*. Rio de Janeiro: Instituto de Matemática Pura e Aplicada.

Moran, P.A.P. and Vere-Jones, D. (1969) The infinite divisibility of multivariate gamma distributions. *Sankhyā Ser. A*, **40**, 393–398.

Sato, K.-I. (1999) *Lévy Processes and Infinitely Divisible Distributions*. Cambridge: Cambridge University Press.

Seshadri, V. (1987) Contribution to discussion of the paper by B. Jørgensen: Exponential dispersion models. *J. Roy. Statist. Soc. Ser. B*, **49**, 156.

Slater, L.J. (1966) *Generalized Hypergeometric Functions*. Cambridge: Cambridge University Press.

Stanley, R.P. (1999) *Enumerative Combinatorics*, Vol. 1, 3rd edn. Cambridge: Cambridge University Press.

Vere-Jones, D. (1967) The infinite divisibility of a bivariate gamma distribution. *Sankhyā Ser. A*, **29**, 421–422.

Received June 2004 and revised May 2005