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On Kesten's counterexample to the Cramér–Wold device for regular variation

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In 2002 Basrak, Davis and Mikosch showed that an analogue of the Cramér–Wold device holds for regular variation of random vectors if the index of regular variation is not an integer. This characterization is of importance when studying stationary solutions to stochastic recurrence equations. In this paper we construct counterexamples showing that for integer-valued indices, regular variation of all linear combinations does not imply that the vector is regularly varying. The construction is based on unpublished notes by Harry Kesten.

Keywords: heavy-tailed distributions; linear combinations; multivariate regular variation

1. Introduction

For $\mathbb{R}^d$-valued random vectors $\mathbf{X}_n$ and $\mathbf{X}$, the well-known Cramér–Wold theorem says that a necessary and sufficient condition for $\mathbf{X}_n \xrightarrow{d} \mathbf{X}_n$ is that $\mathbf{x}^{\mathrm{T}}\mathbf{X}_n \xrightarrow{d} \mathbf{x}^{\mathrm{T}}\mathbf{X}$ for every $\mathbf{x} \in \mathbb{R}^d$. We use the convention that $\mathbf{x} \in \mathbb{R}^d$ is a column vector and $\mathbf{x}^{\mathbf{T}}$ its transpose. In Basrak *et al.* (2002a) it was shown that, for non-integer-valued indices of regular variation, there is a similar characterization of regular variation for a random vector in terms of regular variation of its linear combinations, meaning that for some $\alpha > 0$ and some function L which is slowly varying at infinity,

$$\begin{array}{ll} \text{for every } \mathbf{x} \neq 0, & \lim_{t\to\infty} t^{\alpha} L(t) \mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X} > t) = w(\mathbf{x}) \text{ exists,} \\ w(\mathbf{x}) > 0, & \text{for some } \mathbf{x} \neq 0. \end{array} \tag{1}$$

If (1) holds, then necessarily $w(u\mathbf{x}) = u^{\alpha} w(\mathbf{x})$ for all $\mathbf{x} \neq 0$ and $u > 0$. The interest in this condition originates from a classical result by Kesten (1973) which (briefly) says that, under mild conditions, the stationary solution $\mathbf{X}$ of a multivariate stochastic recurrence equation $\mathbf{X}_n = \mathbf{A}_n \mathbf{X}_{n-1} + \mathbf{B}_n$ satsifies (1), where $L(t) = 1$ and α is the unique solution to

$$\lim_{n\to\infty} \frac{1}{n} \mathrm{E} \log \|\mathbf{A}_n \cdots \mathbf{A}_1\|^{\alpha} = 0.$$

Here $\|\cdot\|$ denotes the operator norm, $\|\mathbf{A}\| = \sup_{|\mathbf{x}|=1} |\mathbf{A}\mathbf{x}|$, and $|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$. A popular example is the stationary GARCH model which can be embedded in a stochastic recurrence equation (see Basrak *et al.*, 2002b). Other examples where condition (1)

1350–7265

appears are the stochastic recurrence equations with heavy-tailed innovations studied in Konstantinides and Mikosch (2005) and the random coefficient AR(q) models of Klüppelberg and Pergamenchtchikov (2004). Important and accessible papers on the extremal behaviour of solutions to univariate stochastic recurrence equations are de Haan *et al*. (1989), Goldie (1991) and Borkovec and Klüppelberg (2001).

A random vector $\mathbf{X}$ is said to be regularly varying if there exist an $\alpha > 0$ and a probability measure σ on $\mathcal{B}(\mathbb{S}^{d-1})$, the Borel σ-field of $\mathbb{S}^{d-1} = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| = 1\}$, such that, for every $x > 0$, as $t \to \infty$,

$$\frac{\mathrm{P}(|\mathbf{X}| > tx, \mathbf{X}/|\mathbf{X}| \in \cdot\,)}{\mathrm{P}(|\mathbf{X}| > t)} \xrightarrow{v} x^{-\alpha}\sigma(\cdot) \quad \text{on } \mathcal{B}(\mathbb{S}^{d-1}). \tag{2}$$

Here $\xrightarrow{v}$ denotes vague convergence, and α and σ are respectively called the index of regular variation and spectral measure of $\mathbf{X}$. For $\alpha \in (0, 2)$ this formulation of multivariate regular variation is a necessary and sufficient condition for the convergence in distribution of normalized partial sums of independent and identically distributed random vectors to a stable random vector; see Rvačeva (1962). It is also used for the characterization of the maximum domain of attraction of extreme value distributions and for weak convergence of point processes; see, for example, Resnick (1987).

In Basrak *et al.* (2002a, Theorem 1.1) it was proved that (2) implies (1) and the following statements hold:

(A) If $\mathbf{X}$ satisfies (1), where α is positive and non-integer, then (2) holds and the spectral measure σ is uniquely determined.

(B) If $\mathbf{X}$ assumes values in $[0, \infty)^d$ and satisfies (1) for $\mathbf{x} \in [0, \infty)^d \backslash \{0\}$, where α is positive and non-integer, then (2) holds and the spectral measure σ is uniquely determined.

(C) If $\mathbf{X}$ assumes values in $[0, \infty)^d$ and satisfies (1), where α is an odd integer, then (2) holds and the spectral measure σ is uniquely determined.

In Section 2 we construct a counterexample which shows that (A) cannot be extended to integer-valued indices of regular variation without additional assumptions on the distribution of $\mathbf{X}$. In Section 3 we construct a counterexample which shows that (B) cannot be extended to integer-valued indices of regular variation without additional assumptions on the distribution of $\mathbf{X}$. Whether (C) is true or not in the case of α belonging to the set of even integers is, to the best of the knowledge of the authors, still an open problem.

Let us point out that there are several equivalent formulations of (2); many of them are documented in Basrak (2000) and Resnick (2004). See also Basrak *et al*. (2002a), Hult (2003), Lindskog (2004) and Resnick (1987) for more on multivariate regular variation. For a detailed treatment of the concept of regularly varying functions, see Bingham *et al*. (1987).

2. Construction of the counterexamples

The constructions of the counterexamples corresponding to (A) and (B) for integer-valued indices of regular variation are rather similar and consist of two steps. First, note that due to the scaling property the limit function w in (1) is determined by its values on $\mathbb{S}^{d-1}$. Therefore it is sufficient to consider linear combinations $\mathbf{x}^{\mathrm{T}}\mathbf{X}$ with $\mathbf{x} \in \mathbb{S}^{d-1}$.

The first step consists of finding two bivariate regularly varying random vectors $\mathbf{X}_0$ and $\mathbf{X}_1$ with index of regular variation $\alpha > 0$ and spectral measures σ_0 and σ_1, with $\sigma_0 \neq \sigma_1$, such that for every $\mathbf{x} \in \mathbb{S}$ and $t > 1$,

$$\mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X}_0 > t) = \mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X}_1 > t). \tag{3}$$

For the example corresponding to (B) we restrict $\mathbf{x}$ to $\mathbb{S} \cap [0, \infty)^2$.

In the second step we will construct the counterexamples by finding a random vector $\mathbf{X}$ such that, for every $\mathbf{x} \in \mathbb{S}$ and $\mathbf{x} \in \mathbb{S} \cap [0, \infty)^2$, respectively,

$$\lim_{t\to\infty} t^{\alpha}\mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X} > t) = \lim_{t\to\infty} t^{\alpha}\mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X}_0 > t) = \lim_{t\to\infty} t^{\alpha}\mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X}_1 > t) =: w(\mathbf{x}) \tag{4}$$

and such that there are subsequences (u_n), (v_n), $u_n \uparrow \infty$, $v_n \uparrow \infty$, with the property that for every $S \in \mathcal{B}(\mathbb{S})$ with $\sigma_0(\partial S) = \sigma_1(\partial S) = 0$,

$$\lim_{n\to\infty} \frac{\mathrm{P}(|\mathbf{X}| > u_n, \mathbf{X}/|\mathbf{X}| \in S)}{\mathrm{P}(|\mathbf{X}| > u_n)} = \sigma_0(S), \tag{5}$$

$$\lim_{n\to\infty} \frac{\mathrm{P}(|\mathbf{X}| > v_n, \mathbf{X}/|\mathbf{X}| \in S)}{\mathrm{P}(|\mathbf{X}| > v_n)} = \sigma_1(S). \tag{6}$$

The counterexamples are easily extended to $\mathbb{R}^d$-valued random vectors. Take $\mathbf{X} = (X^{(1)}, X^{(2)})^{\mathrm{T}}$ as above and $\mathbf{Y} = (Y^{(1)}, \ldots, Y^{(d-2)})^{\mathrm{T}}$ independent of $\mathbf{X}$ with $\mathbf{Y}$ satisfying (1) with the same α as $\mathbf{X}$, $L(t) = 1$, and limit function $w_{\mathbf{Y}}$. Put $\mathbf{Z} = (X^{(1)}, X^{(2)}, Y^{(1)}, \ldots, Y^{(d-2)})^{\mathrm{T}}$. Then, by independence (cf. Davis and Resnick 1996, Lemma 2.1),

$$\begin{aligned}
&\lim_{t\to\infty} t^{\alpha}\mathrm{P}(\mathbf{z}^{\mathrm{T}}\mathbf{Z} > t) \\
&= \lim_{t\to\infty} t^{\alpha}\mathrm{P}((z^{(1)}, z^{(2)})\mathbf{X} > t) + \lim_{t\to\infty} t^{\alpha}\mathrm{P}((z^{(3)}, \ldots, z^{(d)})\mathbf{Y} > t) \\
&= w(z^{(1)}, z^{(2)}) + w_{\mathbf{Y}}(z^{(3)}, \ldots, z^{(d)}).
\end{aligned}$$

Hence, $\mathbf{Z}$ satisfies (1). However, $\mathbf{Z}$ does not satisfy (2). Indeed, assume on the contrary that $\mathbf{Z}$ satisfies (2) with spectral measure $\sigma_{\mathbf{Z}}$. Then since $\mathbf{X} = T(\mathbf{Z})$ with $T: \mathbb{R}^d \to \mathbb{R}^2$ is given by $T(\mathbf{z}) = (z^{(1)}, z^{(2)})^{\mathrm{T}}$ it follows that $\mathbf{X}$ satisfies (2) for some spectral measure σ (Basrak *et al.*, 2002b, Proposition A.1). This is a contradiction.

2.1. Construction of $\mathbf{X}_0$ and $\mathbf{X}_1$

We will now focus on the construction of $\mathbf{X}_0$ and $\mathbf{X}_1$ in the counterexample corresponding to (A) when α is a positive integer.

We will construct two regularly varying random vectors $\mathbf{X}_0$ and $\mathbf{X}_1$ with index of regular variation α and different spectral measures such that (3) is satisfied. A different construction in the case $\alpha = 1$ can be found in Meerschaert and Scheffler (2001, Example 6.1.35).

Let Θ_0 be a $[0, 2\pi)$-valued random variable with density f_0 satisfying, for some $\varepsilon > 0$, $f_0(\theta) > \varepsilon$ for all $\theta \in [0, 2\pi)$. Take $v \in (0, \varepsilon)$ and let Θ_1 have density f_1 given by

$$f_1(\theta) = f_0(\theta) + v\sin((\alpha+2)\theta), \qquad \theta \in [0, 2\pi).$$

Let $R \sim \text{Pareto}(\alpha)$, that is, $\mathrm{P}(R > x) = x^{-\alpha}$ for $x \geqslant 1$, be independent of Θ_i, $i = 0, 1$, and put

$$\mathbf{X}_i \stackrel{d}{=} (R\cos\Theta_i,\ R\sin\Theta_i)^{\mathrm{T}}.$$

Obviously $\mathbf{X}_i$ is regularly varying with $\sigma_i(\cdot) = \mathrm{P}((\cos\Theta_i,\ \sin\Theta_i) \in \cdot)$. Take $\mathbf{x} \in \mathbb{S}$ and let $\beta \in [0, 2\pi)$ be given by $\mathbf{x} = (\cos\beta,\ \sin\beta)^{\mathrm{T}}$. Then, for $t > 1$,

$$\begin{aligned}
&\mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X}_1 > t) - \mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X}_0 > t) \\
&\quad = v\int_t^\infty \int_{\beta-\arccos(t/r)}^{\beta+\arccos(t/r)} \alpha r^{-\alpha-1}\sin((\alpha+2)\theta)\mathrm{d}\theta\mathrm{d}r \\
&\quad = -\frac{v\alpha}{\alpha+2}\int_t^\infty r^{-\alpha-1}(\cos\{(\alpha+2)(\beta+\arccos(t/r))\} - \cos\{(\alpha+2)(\beta-\arccos(t/r))\})\mathrm{d}r \\
&\quad = \frac{2v\alpha}{\alpha+2}\sin((\alpha+2)\beta)\int_t^\infty r^{-\alpha-1}\sin\{(\alpha+2)\arccos(t/r)\}\mathrm{d}r.
\end{aligned}$$

Using standard variable substitutions and trigonometric formulae the integral can be rewritten as follows:

$$\begin{aligned}
&\int_t^\infty r^{-\alpha-1}\sin\{(\alpha+2)\arccos(t/r)\}\mathrm{d}r \\
&\quad = t^{-\alpha}\int_0^1 r^{\alpha-1}\sin\{(\alpha+2)\arccos(r)\}\mathrm{d}r \\
&\quad = t^{-\alpha}\int_0^{\pi/2} \cos^{\alpha-1}(r)\sin((\alpha+2)r)\sin(r)\mathrm{d}r \\
&\quad = t^{-\alpha}\int_0^{\pi/2} \cos^{\alpha-1}(r)\cos((\alpha+1)r)\mathrm{d}r - t^{-\alpha}\int_0^{\pi/2} \cos^{\alpha}(r)\cos((\alpha+2)r)\mathrm{d}r.
\end{aligned}$$

The two last integrals equal zero for every $\alpha \in \{1, 2, \ldots\}$; see Gradshteyn and Ryzhik (2000, p. 392). Hence, for $t > 1$, $\mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X}_1 > t) = \mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X}_0 > t)$, which proves (3).

The following construction of $\mathbf{X}$ satisfying (4)–(6) is based on unpublished notes by Harry Kesten relating to Kesten (1973, Remark 4, p. 245).

2.2. The counterexample

Consider the random vectors $\mathbf{X}_0$ and $\mathbf{X}_1$ above. Let g_0 and g_1 denote their densities. We construct a random vector $\mathbf{X}$ satisfying (1) which is not regularly varying; we will show that it satisfies (4)–(6).

Take $\mathbf{y} \in \mathbb{R}^2$ with $|\mathbf{y}| > 1$. There exist unique integers $j, n \geqslant 1$ such that

$$|\mathbf{y}| \in (j!, (j+1)!] \quad \text{and} \quad j \in \left\{ \sum_{k=1}^{n-1} 2^k + 1, \dots, \sum_{k=1}^{n} 2^k \right\}.$$

Let $\mathbf{X}$ be an $\mathbb{R}^2$-valued random vector with density g given by

$$\left(1 - \left(j - \sum_{k=1}^{n-1} 2^k\right) 2^{-n}\right) g_{b(n)}(\mathbf{y}) + \left(j - \sum_{k=1}^{n-1} 2^k\right) 2^{-n} g_{b(n+1)}(\mathbf{y}),$$

for $|\mathbf{y}| \in (j!, (j+1)!]$, where

$$b(n) = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ 1, & \text{if } n \text{ is even}. \end{cases}$$

That is, the density g is given by

$$g(\mathbf{y}) = \begin{cases} g_0(\mathbf{y}) = 0, & |\mathbf{y}| \in (0, 1], \\ \frac{1}{2} g_0(\mathbf{y}) + \frac{1}{2} g_1(\mathbf{y}), & |\mathbf{y}| \in (1, 2], \\ g_1(\mathbf{y}), & |\mathbf{y}| \in (2, 3!], \\ \frac{1}{4} g_0(\mathbf{y}) + \frac{3}{4} g_1(\mathbf{y}), & |\mathbf{y}| \in (3!, 4!], \\ \frac{1}{2} g_0(\mathbf{y}) + \frac{1}{2} g_1(\mathbf{y}), & |\mathbf{y}| \in (4!, 5!], \\ \frac{3}{4} g_0(\mathbf{y}) + \frac{1}{4} g_1(\mathbf{y}), & |\mathbf{y}| \in (5!, 6!], \\ \text{etc.} \end{cases}$$

Note that in each disc $|\mathbf{y}| \in (j!, (j+1)!]$ the density g is a convex combination of the densities g_0 and g_1. Therefore,

$$\begin{aligned} \int_{\mathbb{R}^2} g(\mathbf{y}) \mathrm{d}\mathbf{y} &= \sum_{j=1}^{\infty} \int_{|\mathbf{y}| \in (j!, (j+1)!]} g(\mathbf{y}) \mathrm{d}\mathbf{y} \\ &= \sum_{j=1}^{\infty} \int_{j!}^{(j+1)!} \alpha r^{-\alpha-1} \, \mathrm{d}r = \int_1^{\infty} \alpha r^{-\alpha-1} \, \mathrm{d}r = 1, \end{aligned}$$

so g is indeed a probability density. Take $\mathbf{x} \in \mathbb{S}$ and $t \in ((j-1)!, j!]$. Then there are two possibilities:

(i) $j - 1 \in \left\{ \sum_{k=1}^{n-1} 2^k + 1, \dots, \sum_{k=1}^{n} 2^k - 1 \right\}$ or

(ii) $j - 1 = \sum_{k=1}^{n-1} 2^k$.

Suppose (i) holds. Then, with $\gamma = (j - \sum_{k=1}^{n-1} 2^k) 2^{-n} \in [0, 1]$, we have

$$\begin{aligned}
P(\mathbf{x}^T\mathbf{X} > t) = (1-\gamma+2^{-n})&P(\mathbf{x}^T\mathbf{X}_{b(n)} > t,\ |\mathbf{X}_{b(n)}| \in (t, j!])\\
&+ (\gamma - 2^{-n})P(\mathbf{x}^T\mathbf{X}_{b(n+1)} > t,\ |\mathbf{X}_{b(n+1)}| \in (t, j!])\\
&+ (1-\gamma)P(\mathbf{x}^T\mathbf{X}_{b(n)} > t,\ |\mathbf{X}_{b(n)}| \in (j!, (j+1)!])\\
&+ \gamma P(\mathbf{x}^T\mathbf{X}_{b(n+1)} > t,\ |\mathbf{X}_{b(n+1)}| \in (j!, (j+1)!])\\
&+ P(\mathbf{x}^T\mathbf{X} > t,\ |\mathbf{X}| > (j+1)!).
\end{aligned}$$

Hence,

$$\begin{aligned}
P(\mathbf{x}^T\mathbf{X} > t) = 2^{-n}&\big\{P(\mathbf{x}^T\mathbf{X}_{b(n)} > t,\ |\mathbf{X}_{b(n)}| \in (t, j!]) - P(\mathbf{x}^T\mathbf{X}_{b(n+1)} > t,\ |\mathbf{X}_{b(n+1)}| \in (t, j!])\big\}\\
&+ \underbrace{(1-\gamma)P(\mathbf{x}^T\mathbf{X}_{b(n)} > t) + \gamma P(\mathbf{x}^T\mathbf{X}_{b(n+1)} > t)}_{B_n}\\
&- (1-\gamma)P(\mathbf{x}^T\mathbf{X}_{b(n)} > t,\ |\mathbf{X}_{b(n)}| > (j+1)!)\\
&- \gamma P(\mathbf{x}^T\mathbf{X}_{b(n+1)} > t,\ |\mathbf{X}_{b(n+1)}| > (j+1)!)\\
&+ P(\mathbf{x}^T\mathbf{X} > t,\ |\mathbf{X}| > (j+1)!).
\end{aligned}$$

We have

$$\begin{aligned}
&2^{-n}\big\{P(\mathbf{x}^T\mathbf{X}_{b(n)} > t,\ |\mathbf{X}_{b(n)}| \in (t, j!]) - P(\mathbf{x}^T\mathbf{X}_{b(n+1)} > t,\ |\mathbf{X}_{b(n+1)}| \in (t, j!])\big\}\\
&\quad \leqslant 2^{-n}\big\{P(|\mathbf{X}_{b(n)}| > t) + P(|\mathbf{X}_{b(n+1)}| > t)\big\} = 2^{-n+1}t^{-\alpha}.
\end{aligned}$$

Moreover, since $P(\mathbf{x}^T\mathbf{X}_1 > t) = P(\mathbf{x}^T\mathbf{X}_0 > t)$ we have $B_n = P(\mathbf{x}^T\mathbf{X}_0 > t)$. The absolute value of each of the remaining terms is less than or equal to $((j+1)!)^{-\alpha} \leqslant (j\,j!)^{-\alpha} \leqslant (j\,t)^{-\alpha}$, so we conclude that

$$t^\alpha|P(\mathbf{x}^T\mathbf{X} > t) - P(\mathbf{x}^T\mathbf{X}_0 > t)| \leqslant 2^{-n+1} + 3j^{-\alpha}.$$

Since $j = j(t) \to \infty$ and $n = n(t) \to \infty$ as $t \to \infty$, we have

$$\lim_{t\to\infty} t^\alpha|P(\mathbf{x}^T\mathbf{X} > t) - P(\mathbf{x}^T\mathbf{X}_0 > t)| = 0.$$

That is,

$$\lim_{t\to\infty} t^\alpha P(\mathbf{x}^T\mathbf{X} > t) = \lim_{t\to\infty} t^\alpha P(\mathbf{x}^T\mathbf{X}_0 > t) = \lim_{t\to\infty} t^\alpha P(\mathbf{x}^T\mathbf{X}_1 > t).$$

Suppose now that (ii) holds. Then

$$\begin{aligned}
\mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X} > t) &= \mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X}_{b(n)} > t,\, |\mathbf{X}_{b(n)}| \in (t, j!]) \\
&\quad + (1-2^{-n})\mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X}_{b(n)} > t,\, |\mathbf{X}_{b(n)}| \in (j!, (j+1)!]) \\
&\quad + 2^{-n}\mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X}_{b(n+1)} > t,\, |\mathbf{X}_{b(n+1)}| \in (j!, (j+1)!]) \\
&\quad + \mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X} > t,\, |\mathbf{X}| > (j+1)!) \\
&= \mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X}_{b(n)} > t) - \mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X}_{b(n)} > t,\, |\mathbf{X}_{b(n)}| > (j+1)!) \\
&\quad + 2^{-n}\{\mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X}_{b(n+1)} > t,\, |\mathbf{X}_{b(n+1)}| \in (j!, (j+1)!]) \\
&\quad - \mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X}_{b(n)} > t,\, |\mathbf{X}_{b(n)}| \in (j!, (j+1)!])\} \\
&\quad + \mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X} > t,\, |\mathbf{X}| > (j+1)!).
\end{aligned}$$

By arguments similar to those for case (i), we obtain

$$t^{\alpha}|\mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X} > t) - \mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X}_0 > t)| \leqslant 2(2^{-n} + j^{-\alpha}).$$

It follows that

$$\lim_{t\to\infty} t^{\alpha}\mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X} > t) = \lim_{t\to\infty} t^{\alpha}\mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X}_0 > t) = \lim_{t\to\infty} t^{\alpha}\mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X}_1 > t),$$

which proves (4).

Finally, we find subsequences (u_n) and (v_n) satisfying (5) and (6). Take $S \in \mathcal{B}(\mathbb{S})$ with $\sigma_0(\partial S) = \sigma_1(\partial S) = 0$. Put

$$c_n = \sum_{k=1}^{2n} 2^k \quad \text{and} \quad d_n = \sum_{k=1}^{2n+1} 2^k.$$

Note that for $c_n! < |\mathbf{y}| \leqslant (c_n+1)!$ we have $g(\mathbf{y}) = g_{b(2n+1)}(\mathbf{y}) = g_0(\mathbf{y})$, whereas for $d_n! < |\mathbf{y}| \leqslant (d_n+1)!$ we have $g(\mathbf{y}) = g_{b(2n+2)}(\mathbf{y}) = g_1(\mathbf{y})$. It follows that, with $u_n = c_n!$,

$$\begin{aligned}
u_n^{\alpha}\mathrm{P}(|\mathbf{X}| > u_n,\, \mathbf{X}/|\mathbf{X}| \in S) &= u_n^{\alpha}\mathrm{P}(c_n! < |\mathbf{X}| \leqslant (c_n+1)!,\, \mathbf{X}/|\mathbf{X}| \in S) \\
&\quad + u_n^{\alpha}\mathrm{P}(|\mathbf{X}| > (c_n+1)!,\, \mathbf{X}/|\mathbf{X}| \in S).
\end{aligned}$$

Since the second term is less than or equal to

$$u_n^{\alpha}\mathrm{P}(|\mathbf{X}| > (c_n+1)!) = (c_n!)^{\alpha}[(c_n+1)!]^{-\alpha} \to 0,$$

as $n \to \infty$, it follows that

$$\begin{aligned}
\lim_{n\to\infty} u_n^{\alpha}\mathrm{P}(|\mathbf{X}| > u_n,\, \mathbf{X}/|\mathbf{X}| \in S) &= \lim_{n\to\infty} u_n^{\alpha}\big(u_n^{-\alpha} - [(c_n+1)!]^{-\alpha}\big)\sigma_0(S) \\
&= \sigma_0(S).
\end{aligned}$$

By a similar argument, with $v_n = d_n!$,

$$\lim_{n\to\infty} v_n^{\alpha}\mathrm{P}(|\mathbf{X}| > v_n,\, \mathbf{X}/|\mathbf{X}| \in S) = \sigma_1(S).$$

Thus, we have found sequences (u_n) and (v_n) satisfying (5) and (6) and the counterexample is complete.

3. Vectors with non-negative components

In this section we construct a counterexample corresponding to (B) in the case of integer-valued indices of regular variation.

The following construction of $\mathbf{X}_0$ and $\mathbf{X}_1$ was given in Basrak *et al*. (2002a) for $\alpha = 2$ but can, as we will see, be extended to any positive integer α. Take $\alpha \in \{1, 2, \ldots\}$ and let Θ_0, Θ_1 be two $[0, \pi/2]$-valued random variables with unequal distributions satisfying

$$\mathrm{E}(\cos^k\Theta_0 \sin^{\alpha-k}\Theta_0) = \mathrm{E}(\cos^k\Theta_1 \sin^{\alpha-k}\Theta_1), \qquad k = 0, 1, \ldots, \alpha. \tag{7}$$

Let $R \sim \text{Pareto}(\alpha)$, that is, $\mathrm{P}(R > x) = x^{-\alpha}$ for $x \geqslant 1$. Suppose R is independent of Θ_i, $i = 0, 1$ and put $\mathbf{X}_i \stackrel{d}{=} (R\cos\Theta_i,\ R\sin\Theta_i)^{\mathrm{T}}$. For $\mathbf{x} \in [0, \infty)^2 \backslash \{0\}$ we have

$$\begin{aligned} t^\alpha \mathrm{P}(\mathbf{x}^{\mathrm{T}}\mathbf{X}_i > t) &= t^\alpha \mathrm{P}(x_1 R\cos\Theta_i + x_2 R\sin\Theta_i > t) \\ &= t^\alpha \int_1^\infty \mathrm{P}(x_1\cos\Theta_i + x_2\sin\Theta_i > t/r)\alpha r^{-\alpha-1}\mathrm{d}r \\ &= \int_0^{t^\alpha} \mathrm{P}((x_1\cos\Theta_i + x_2\sin\Theta_i)^\alpha > v)\mathrm{d}v \\ &= \sum_{k=1}^{\alpha} \tbinom{\alpha}{k} x_1^k x_2^{\alpha-k}\mathrm{E}(\cos^k\Theta_i \sin^{\alpha-k}\Theta_i) \end{aligned}$$

for t sufficiently large. We can now construct the vector $\mathbf{X}$ as in Section 2.2, with $\mathbf{x} \in \mathbb{S}_+ = \mathbb{S} \cap [0, \infty)^2$ and the new densities g_0 and g_1 of $\mathbf{X}_0$ and $\mathbf{X}_1$. It remains to show that we can find unequal distributions of the $[0, \pi/2]$-valued random variables Θ_0 and Θ_1 satisfying (7). Let Θ_0 have density f_0 satisfying, for some $\varepsilon > 0$, $f_0(\theta) > \varepsilon$ for all $\theta \in [0, \pi/2]$. We will show that the density f_1 of Θ_1 can be chosen as $f_1(\theta) = f_0(\theta) + vf(\theta)$, where $v \in (0, \varepsilon)$ and f is chosen such that $\sup_{\theta\in[0,\pi/2]}|f(\theta)| = 1$, $\int_0^{\pi/2} f(\theta)\mathrm{d}\theta = 0$, and (7) holds. Let

$$A := \operatorname{span}\{1,\ \sin^\alpha(\theta), \cos(\theta)\sin^{\alpha-1}(\theta), \ldots, \cos^\alpha(\theta)\} \subset \mathbb{C}_2([0, \pi/2]),$$

where $\mathbb{C}_2([0, \pi/2])$ is the space of real-valued continuous functions on $[0, \pi/2]$ equipped with the inner product $(h_1, h_2) = \int_0^{\pi/2} h_1(s)h_2(s)\mathrm{d}s$, which makes it an inner product space. For any non-zero $\widetilde{f} \notin A$ with $\widetilde{f} \in \mathbb{C}_2([0, \pi/2])$, we can choose

$$f := \frac{\widetilde{f} - \mathrm{Proj}_A(\widetilde{f})}{\sup_{\theta\in[0,\pi/2]}|\{\widetilde{f} - \mathrm{Proj}_A(\widetilde{f})\}(\theta)|}.$$

Then $f \perp A$, f_1 is a density function and (7) holds. Since A is a finite-dimensional subspace of the infinite-dimensional space $\mathbb{C}_2([0, \pi/2])$ it is clear that its orthogonal complement is non-empty. Hence, the density f_1 above exists.

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